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On Principal n-ideals of a Distributive nearlattice

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ON PRINCIPAL n-IDEALS OF A DISTRIBUTIVE NEARLATTICE

A THESIS

SUBMITTED TO THE UNIVERSITY OF RAJSHAHI

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

IN MATHEMATICS

BY

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In the
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DEDICATED TO

MY PARENTS WHO HAVE PROFOUNDLY INFLUENCED MY LIFE

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It is certified that the thesis entitled **"On Principal n-ideals of a Distributive nearlattice"** submitted by **Md. Sazuwar Raihan,** in fulfillment of all requirements for the degree of **Doctor of Philosophy** in Mathematics, University of Rajshahi, has been completed under our supervision. We believe that this research work is an original one and it has not been submitted elsewhere for any degree.

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10/5/06

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STATEMENT OF ORIGINALITY

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University, and to the best of my knowledge and belief, does not contain material previously published or written by another person except where due reference is made in the text.

Md. Sazuwar Raihan

(Md. Sazuwar Raihan)

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ABSTRACT

This thesis studies some of the nature of Principal n-ideal of a distributive nearlattice. The n-ideals of a lattice have been studied by several authors including [4] and [36]. For a fixed element n of a lattice L, a convex sublattice of L containing n is called an n-*ideal* of L. If a lattice L has a '0', then replacing n by 0, an n-ideal becomes an ideal and if L has a '1', then it becomes a filter by replacing n by 1. Thus the idea of n-ideals is a kind of generalization of both ideals and filters of lattices. Thus the study on n-ideals generalized many results on lattice theory involving ideals and filters. Then many authors have extended the concept of n-ideals in a nearlattice. A *nearlattice* is a meet semilattice with the property that any two elements possessing a common upper bound, have a supremum. For a fixed element n of a nearlattice S, a convex subnearlattice of S containing n is called an n-*ideal* of S. For two n-ideals I and J of a nearlattice S, [30] has given a neat description of $I \vee J$, while the set theoretic intersection is the infimum. Hence the set of all n-ideals of a nearlattice S, denoted by $I_n(S)$ is an algebraic lattice. An n-ideal generated by a finite number of elements $a_1, a_2, \ldots, a_n$ is called a *finitely generated* n-*ideal*, denoted by $< a_1, a_2, \ldots, a_n >_n$, while the n-ideal generated by a single element a is called a *principal* n-*ideal*, denoted by $< a >_n$. The set of finitely generated n-ideals and the set of principal n-ideals are denoted by $F_n(S)$ and $P_n(S)$ respectively.

In this thesis, we devote ourselves in studing several properties of a distributive nearlattice S. Replacing n by 0, $P_n(S)$ becomes the set of all principal ideals of S which is isomorphic to S itself. Thus all the results on $P_n(S)$ are generalizations of the corresponding results of S. In this thesis our results generalize many results on semi-Boolean, generalized Stone, normal,

relatively normal, m-normal and relatively m-normal nearlattices. We have also generalized some results on annulets and α-ideals in terms of n-ideals.

In chapter 1, we discuss some fundamental properties of n-ideals which are basic to this thesis. Here we give an explicit description of $F_n(S)$ and $P_n(S)$ which are essential for the development of the thesis. $F_n(S)$ is not a lattice for a general nearlattice S. It is merely a join semilattice. But if S is distributive and n is medial, then $F_n(S)$ is a lattice. For a neutral element $n \in S$, if n is medial, then $P_n(S)$ is a meet semilattice. Moreover if n is sesquimedial, then $P_n(S)$ is a nearlattice. What is more, for a central element $n \in S$, $P_n(S) \cong (n]^d \times [n)$. Thus when n is a central element, then $P_n(S)$ is sectionally complemented if and only if the intervals [a, n] and [n, b] are complemented for each $a, b \in S$. In this chapter we also discuss on prime n-ideals. We include a proof of the generalization of Stone's separation theorem. We also include a result that for a central element n of S, $P_n(S)$ is semi-Boolean if and only if the set of prime n-ideals P(S) of S is unordered by set inclusion.

Chapter 2 discusses on minimal prime n-ideals of a nearlattice and on normal nearlattices. A distributive nearlattice S with 0 is called *normal* if its every prime ideal contains a unique minimal prime ideal. We give some characterizations on minimal prime n-ideals which are essential for the further development of this chapter. We prove that if n is a central element of S, then $P_n(S)$ is normal if and only if for any two minimal prime n-ideals P and Q of S, $P \vee Q = S$, which is equivalent to the condition that for all $x, y \in S$, $<x>_n \cap <y>_n = \{n\}$ implies $<x>_n^* \vee <y>_n^* = S$.

In chapter 3, we introduce the notion of relative n-annihilators $< a, b >^n$. We have included several characterizations of $< a, b >^n$. We have also given some characterizations of distributive and modular nearlattices in terms of relative n-annihilators. We have characterized those $P_n(S)$ which are relatively normal, which are generalizations of several results of [56] on relatively normal nearlattices. Among many results we have shown that for a central element n, $P_n(S)$ is relatively normal if and only if any two in-comparable prime n-ideals of S are co-maximul. What is more, this is also equivalent to the condition, $< <a>_n, <b>_n > \vee < <b>_n, <a>_n > =S$ for all a, b∈S.

Pseudo-complemented distributive lattices satisfying Lee's identities form equational subclasses denoted by B_m , $-1 \leq m \leq \omega$. Cornish and Mandelker have studied distributive lattices analogues to B_1-lattices and relatively B_1-lattices. Cornish, Beazer and Davey have each independently obtained several characterizations of (sectionally) B_m-lattices and relatively B_m-lattices. Recently [56] has extended these concepts and studied the n-normal and relatively n-normal near lattices. In chapter 4, we generalize the results of [56] by studying principal n-ideals which form a (sectionally) m-normal and a relatively m-normal nearlattice. We show that for a central element n, $P_n(S)$ is (sectionally) m-normal if and only if for any $x_0,x_1,--,x_m \in S$ with $m(x_i, n, x_j) = n$ $(i \neq j)$ which is also equivalent to the condition that for any m+1 distinct minimal prime n-ideals $P_0,---,P_m$ of S, $P_0 \vee ----- \vee P_m=S$. In this chapter, we also show that $P_n(S)$ is relatively m-normal if and only if any m+1 pairwise incomparable prime n-ideals are co-maximal.

In chapter 5, we introduce the concepts of n-annulets and α-n-ideals in S. Then we generalize several results on annulets and α-ideals given in [12] and [52]. We have shown that the set of n-annulets of S, $A_n(S)$ is a join semilattice of $I_n(S)$ if and only if $P_n(S)$ is normal. We have also shown that $A_n(S)$ is relatively complemented if and only if $P_n(S)$ is sectionally quasicomplemented. Finally, we have given a characterization for $P_n(S)$ to be generalized Stone in terms of $A_n(S)$.

In section 2, we have shown that the n-ideal n(P) where P is a prime n-ideal is an α-n-ideal. Moreover, all the minimal prime n-ideals are α-n-ideals. We have shown that $P_n(S)$ is disjunctive if and only if each n-ideal is an α-n-ideal. Also, $P_n(S)$ is sectionally quasicomplemented if and only if each prime α-n-ideal is a minimal prime n-ideal. We conclude the thesis by proving that $P_n(S)$ is generalized Stone if and only if each prime n-ideal contains a unique prime α-n-ideal.

CHAPTER 1

CHAPTER 1

NEARLATTICES AND NORMAL NEARLATTICES

1.1 Preliminaries

In this section it is intended only to outline and fix the notation for some of the concepts of nearlattices which are basic to this thesis. We also formulate some results on arbitrary nearlattices for later use. For the background material in lattice theory we refer the reader to the text of G. Birkhoff [9], G. Grätzer [21], D. E. Rutherford [58], V. K. Khanna [32] and G. Szäsz [64].

By a *nearlattice* S we will mean a (lower) semi-lattice which has the property that any two elements possessing a common upper bound, have a supremum. Cornish and Hickman, in their paper [14], referred this property as the upper bound property, and a semi-lattice of this nature as a semi-lattice with the upper bound property. These types of semi-lattices have been studied extensively by [14], [15], [16], [28], [30], [42] and [43]. They have noticed that the behaviour of such a semi-lattice is closer to that of a lattice than an ordinary semi-lattice. So they preferred to use the term ' nearlattice ' in place of semi-lattice with the upper bound property.

Of course, a nearlattice with a largest element is a *lattice*. Since any semi-lattice satisfying the descending chain condition has the upper bound property, all finite semi-lattices are nearlattices.

Now we give an example of a meet semi-lattice which is not a nearlattice.

Example. In $\mathfrak{R}^2$ consider the set

$S = \{(0, 0)\} \cup \{(1, 0)\} \cup \{(0, 1)\} \cup \{(1, y): y > 1\}$ shown by the figure 1.1

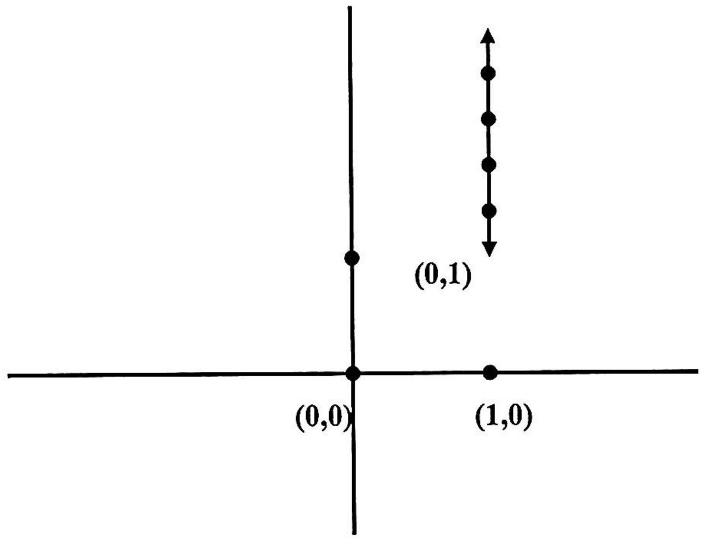

Figure 1.1

Define the partial ordering $\leq$ on S by $(x, y) \leq (x_1, y_1)$ if and only if $x \leq x_1$ and $y \leq y_1$. Observe that $(S;\leq)$ is a meet semi-lattice. Both (1, 0) and (0, 1) have common upper bounds. In fact $\{(1, y): y > 1\}$ are common upper bounds of them. But the supremum of (1, 0) and (0, 1) does not exist. Therefore $(S; \leq)$ is not a nearlattice.

The upper bound property appears in G. Grätzer and Lakser [24], while Rozen [57] shows that it is the result of placing certain associativity conditions on the partial join operation. Moreover, Evans in a more recent paper [18] referred nearlattices as conditional lattices. By a conditional lattice he means a (lower) semi-lattice S with the condition that for each $x \in S$, $\{y \in S: y \leq x\}$ is a lattice and it is very easy to check that this condition is equivalent to the upper bound property of S.

Whenever a nearlattice has a least element we will denote it by 0. If $x_1, x_2,\dots,x_n$ are elements of a nearlattice then by $x_1 \vee x_2 \vee \dots \vee x_n$, we mean that the supremum of $x_1, x_2,\dots,x_n$ exists and $x_1 \vee x_2 \vee \dots \vee x_n$ is the symbol denoting this supremum.

A non-empty subset K of a nearlattice S is called a *subnearlattice* of S if for any $a,b \in K$, both $a \wedge b$ and $a \vee b$ (whenever it exists in S) belong to K ($\wedge$ and $\vee$ are taken in S) and the $\wedge$ and $\vee$ are the restrictions of the $\wedge$ and $\vee$ of S to K. Moreover, a subnearlattice K of a nearlattice S is called a *sub-lattice* of S if $a \vee b \in K$ for all $a, b \in K$.

A nearlattice S *is* called *modular* if for any $a, b, c \in S$ with $c \leq a$, $a \wedge (b \vee c) = (a \wedge b) \vee c$ whenever $b \vee c$ exists.

By [51], a nearlattice S is *modular* if and only if for all $t, x, y, z \in S$ with $z \leq x$, $x \wedge ((t \wedge y) \vee (t \wedge z)) = (x \wedge t \wedge y) \vee (t \wedge z)$.

A nearlattice S is called *distributive* if for any $x, x_1, x_2,\dots, x_n$, $x \wedge (x_1 \vee x_2 \vee \dots \vee x_n) = (x \wedge x_1) \vee (x \wedge x_2) \vee \dots \vee (x \wedge x_n)$ whenever $x_1 \vee x_2 \vee \dots \vee x_n$ exists.

Notice that the right hand expression always exists by the upper bound property of S. By [51], a nearlattice S is *distributive* if and only if for all $t, x, y, z \in S$, $t \wedge ((x \wedge y) \vee (x \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.

Lemma 1.1.1. *A nearlattice* S is *distributive* (*modular*) *if and only* (x] = {y ∈ S: y ≤ x} is *a distributive* (*modular*) *lattice for each* x ∈ S. □

Consider the following lattices:

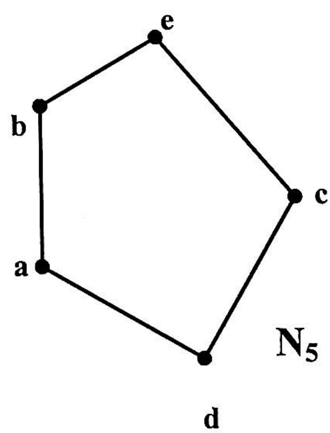

Figure 1.2

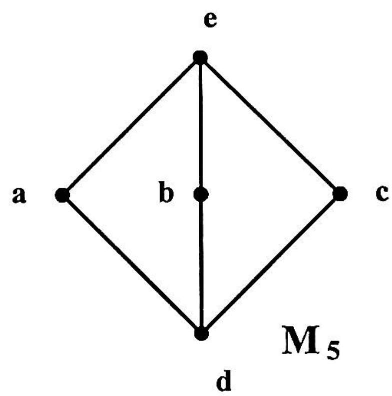

Figure 1.3

Cornish and Hickman in [14] has given the following extension of a very fundamental result of lattice theory.

Theorem 1.1.2. *A nearlattice* S is *distributive if and only if* S does *not contain a sub-lattice isomorphic to* N_5 or M_5. □

Following result is also an extension of a fundamental of lattice theory, which is due to [14].

Theorem 1.1.3. *A nearlattice* S is *modular if and only if* S does *not contain a sub-lattice isomorphic to* N_5. □

In this context it should be mentioned many lattice theorists e.g. R. Balbes [5], J. C. Varlet [66], R. C. Hickman [27], [28] and K. P. Shum [62] have worked with a class of semi-lattices S which has the property that for each $x, a_1, a_2, \ldots, a_r \in S$ if $a_1 \vee a_2 \vee \ldots \vee a_r$ exists then $(x \wedge a_1) \vee (x \wedge a_2) \vee \ldots \vee (x \wedge a_r)$ exists and equals $x \wedge (a_1 \vee a_2 \vee \ldots \vee a_r)$. [5] called them as prime semi-lattices while [27] referred them as weakly distributive semi-lattices.

Hickman in [28] has defined a ternary operation j by $j(x, y, z) = (x \wedge y) \vee (y \wedge z)$, on a nearlattice S (which exists by the upper bound property of S). In fact he has shown that (also see Lyndon [38], Theorem 4), the resulting algebras of the type (S; j) form a variety, which is referred to as the variety of join algebras and following are its defining identities.

(i) $j(x, x, x) = x$.

(ii) $j(x, y, x) = j(y, x, y)$.

(iii) $j(j(x, y, x), z, j(x, y, x)) = j(x, j(y, z, y), x)$.

(iv) $j(x, y, z) = j(z, y, x)$.

(v) $j(j(x, y, z), j(x, y, x), j(x, y, z)) = j(x, y, x)$.

(vi) $j(j(x, y, x), y, z) = j(x, y, z)$.

(vii) $j(x, y, j(x, z, x)) = j(x, y, x)$.

(viii) $j(j(x, y, j(w, y, z)), j(x, y, z), j(x, y, j(x, y, z))) = j(x, y, z)$.

We call a nearlattice S a *medial nearlattice* if for all $x, y, z \in S$, $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists.

For a (lower) semi-lattice S, if $m(x, y, z)$ exists for all $x, y, z \in S$, then it is not hard to see that S has upper bound property and hence is a nearlattice. Distributive medial nearlattices were first studied by Sholander in

[60] and [61], and recently by Evans in [18]. Sholander preferred to call these as median semi-lattices. There he showed that every medial nearlattice S can be characterized by means of an algebra (S; m) of type < 3 >, known as median algebra, satisfying the following two identities:

(i) m(a, a, b) = a;

(ii) m(m(a, b, c), m(a, b, d), e) = m(m(c, d, e), a, b).

A nearlattice S is said to have the *three property* if for any a, b, c$\in$S, $a \vee b \vee c$ exists whenever $a \vee b$, $b \vee c$ and $c \vee a$ exists.

Nearlattices with the three property were discussed by Evan's in [18], where he referred it as strong conditional lattice.

Following result shows that the Evan's strong conditional lattices are precisely the medial nearlattices.

The equivalence of (i) and (iii) of the following lemma is trivial, while the proof of (i)$\Leftrightarrow$(ii) is inductive.

Lemma 1.1.4. (Evan's [18]). *For a nearlattice* S *the following conditions are equivalent.*

(i) S has *the three property.*

(ii) *Every pair of a finite number* n ($\geq$ 3) *of elements of* S *possess a supremum ensures the existence of the supremum of all the n elements.*

(iii) S *is medial.* □

A family $\mathcal{A}$ of subsets of a set A is called a *closure system* on A if (i) $A \in \mathcal{A}$ and

(ii) $\mathcal{A}$ is closed under arbitrary intersection.

Suppose $\mathcal{B}$ is a subfamily of $\mathcal{A}$. $\mathcal{B}$ is called a *directed system* if for any $X, Y \in \mathcal{B}$ there exists Z in $\mathcal{B}$ such that $X, Y \subseteq Z$.

If $\cup\{X: X \in \mathcal{B}\} \in \mathcal{A}$ for every directed system $\mathcal{B}$ contained in the closure system $\mathcal{A}$, then $\mathcal{A}$ is called *algebraic*. When ordered by set inclusion, an closure system forms an algebraic lattice.

A non-empty subset H of a nearlattice S is called *hereditary* if for any $x \in S$ and $y \in H$, $x \leq y$ implies $x \in H$. The set H(S) of all hereditary subsets of S is a complete distributive lattice when partially ordered by set inclusion, where the meet and join in H(S) are given by set theoretic intersection and union respectively.

1.2 Ideals of Nearlattices

A non-empty subset I of a nearlattice S is called an *ideal* if it is hereditary and closed under existent finite suprema.

We denote the set of all ideals S by I(S). If S has a smallest element 0 then I(S) is an algebraic closure system on S, and is consequently an algebraic lattice. However, if S does not possess smallest element then we can only assert that $I(S) \cup \{\phi\}$ is an algebraic closure system. For any subset K of a nearlattice S, (K] denotes the ideal generated by K.

Infimum of two ideals of a nearlattice is their set theoretic intersection. In a general nearlattice the formula for the supremum of two ideals is not very easy. We start this section with the following lemma which gives the formula for the supremum of two ideals. It is in fact exercise 22 of Grätzer [20, P-54] for partial lattices.

Lemma 1.2.1. *Let* I and J be *ideals of a nearlattice* S. *Let* $Ao = I \cup J$, $A_n = \{x \in S : x \leq y \vee z;\ y \vee z$ *exists and* $y, z \in A_{n-1}\}$, *for* $n = 1,2\ldots\ldots\ldots$, *and* $K = \bigcup_{n=0}^{\infty} A_n$. *Then* $K = I \vee J$. □

This will be needed for further development of the thesis.

Lemma 1.2.2. *Let* K be *a non-empty subset of a nearlattice* S. *Then* $(K] = \bigcup_{m=0}^{\infty} A_m$, *where* $A_o = \{t \in S : t = (k_1 \wedge t) \vee (k_2 \wedge t)$ *for some* $k_1, k_2 \in K\}$ *and* $A_m = \{t \in S : t = (a_1 \wedge t) \vee (a_2 \wedge t)$ *for some* $a_1, a_2 \in A_{m-1}\}$. □

Cornish and Hickman in [14, Theorem 1.1] has given the following result for distributive nearlattices.

Theorem 1.2.3. *The following conditions on a nearlattice* S are *equivalent.*

(i) S is *distributive.*

(ii) *For any* $H \in H(S)$, $(H] = \{h_1 \vee h_2 \vee \ldots \vee h_n : h_1, \ldots, h_n \in H\}$.

(iii) *For any* $I, J \in I(S)$, $I \vee J = \{a_1 \vee a_2 \vee \ldots \vee a_n : a_1, a_2, \ldots, a_n \in I \cup J\}$.

(iv) $I(S)$ *is a distributive lattice.*

(v) *The map* $H \to (H]$ is *a lattice homomorphism of* $H(S)$ *onto* $I(S)$ (*which preserves arbitrary suprema*). □

Let $I_f(S)$ from hence forth denote the set of all finitely generated ideals of a nearlattice S. Of course, $I_f(S)$ is an upper subsemilattice of $I(S)$. Also for any $x_1, x_2, \ldots, x_m \in S$, $(x_1, \ldots, x_m]$ is clearly the supremum of $(x_1] \vee (x_2] \vee \ldots \vee (x_m]$.

When S is distributive,

$(x_1, \ldots, x_m] \cap (y_1, \ldots, y_n] = ((x_1] \vee \ldots \vee (x_m]) \cap ((y_1] \vee \ldots \vee (y_n]) = \vee_{i,j} (x_i \wedge y_j]$ for any $x_1, \ldots, x_m, y_1, \ldots, y_n \in S$ (by Theorem 1.2.3.) and so $I_f(S)$ is a distributive sub-lattice of $I(S)$, c.f. Cornish and Hickman [14].

A nearlattice S is said to be *finitely smooth* if the intersection of two finitely generated ideals is itself finitely generated.

For example, (i) distributive nearlattices,

(ii) finite nearlattices,

(iii) lattices, are finitely smooth.

Hickman in [28] exhibited a nearlattice which is not finitely smooth.

By Theorem 1.2.3, a nearlattice S is distributive if and only if I(S) is distributive. But for modular nearlattices, we do not get the similar result. [56] has proved that for a nearlattice S, if I(S) is modular then S is also modular. [56] has also provided the example to show that its converse may not be true.

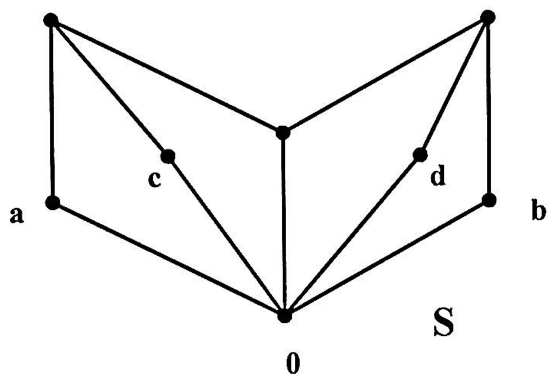

Figure 1.4

Notice that in S, (r] is modular for each r∈S. But in I(S), {(0], (a], (a, d], (b, c], S} is a pentagonal sub-lattice.

A *filter* F of a nearlattice S is a non-empty subset of S such that if f_1, f_2∈F and x∈S with $f_1 \leq x$, then both $f_1 \wedge f_2$ and x are in F.

A filter G is called a *prime filter* if G ≠ S and at least one of $x_1, x_2, \ldots, x_n$ is in G whenever $x_1 \vee x_2 \vee \ldots \vee x_n$ exists and in G.

An ideal P in a nearlattice S is called a *prime ideal* if P ≠ S and $x \wedge y \in P$ implies x∈P *or* y∈P.

It is not hard to see that a filter F of a nearlattice S is prime if and only if S-F is a prime ideal.

The set of filters of a lattice is an upper semi-lattice; yet it is not a lattice in general, as there is no guarantee that the intersection of two filters is non-empty. The join $F_1 \vee F_2$ of two filters is given by $F_1 \vee F_2 = \{s \in S: s \geq f_1 \wedge f_2$ for some $f_1 \in F_1, f_2 \in F_2\}$. The smallest filter containing a subset H is denoted by [H). Moreover, the description of the join of filters shows that for all a, b∈ S, $[a) \vee [b) = [a \wedge b)$.

A subnearlattice K of a nearlattice S is called a *convex subnearlattice* if $a \leq c \leq b$ with a, b∈K, c∈S implies c∈K.

Now, we study some properties of convex subnearlattices of a nearlattice.

Theorem 1.2.4. *In a nearlattice* S, *suppose* K *is a convex subnearlattice. Then* $[K) = \{x \in S: x \geq k$ *for some* $k \in K\}$. □

In a lattice L, it is well known that for a convex sub-lattice C of L, C = (C] ∩ [C). Following figure 1.5 shows that for a convex subnearlattice C in a general nearlattice, this may not be true.

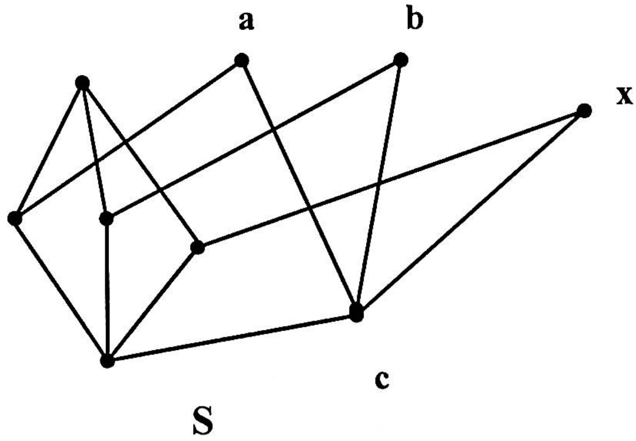

Figure 1.5

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Here $C = \{a, b, c\}$ is a convex subnearlattice of S.Observe that $(C] = \{a, b, c, x\}$, hence $(C] \cap [C) = [C) \neq C$. But this result holds when the nearlattice S is distributive.

To prove this we need the following lemma.

Lemma 1.2.5. *Suppose* C *is a convex sub-nearlattice of a distributive nearlattice* S. *Then*

$$(C] = \{x \in S : x = (x \wedge c_1) \vee (x \wedge c_2) \vee \ldots \vee (x \wedge c_n) \text{ for } \textit{some } c_1, c_2, \ldots, c_n \in C\}.$$

Proof. Let $x, y \in$ R.H.S. such that $x \vee y$ exists. Then $x = (x \wedge p_1) \vee (x \wedge p_2) \vee \ldots \vee (x \wedge p_m)$ and $y = (y \wedge q_1) \vee (y \wedge q_2) \vee \ldots \vee (y \wedge q_n)$ for some $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n \in C$.

Thus $x \vee y = (x \wedge p_1) \vee (x \wedge p_2) \vee \ldots \vee (x \wedge p_m) \vee (y \wedge q_1) \vee (y \wedge q_2) \vee \ldots \vee (y \wedge q_n)$

$$\leq ((x \vee y) \wedge p_1) \vee \ldots \vee ((x \vee y) \wedge p_m) \vee ((x \vee y) \wedge q_1) \vee \ldots \vee ((x \vee y) \wedge q_n)$$

$\leq x \vee y$ implies

$$x \vee y = ((x \vee y) \wedge p_1) \vee \ldots \vee ((x \vee y) \wedge p_m) \vee ((x \vee y) \wedge q_1) \vee \ldots \vee ((x \vee y) \wedge q_n)$$

Therefore $x \vee y \in$ R.H.S.

If $x \in$ R.H.S. and $t \in S$ with $t \leq x$ then

$x = (x \wedge p_1) \vee (x \wedge p_2) \vee \ldots \vee (x \wedge p_m)$ for some $p_1, p_2, \ldots, p_m \in C$.

Thus $t = t \wedge x$

$$= t \wedge [(x \wedge p_1) \vee (x \wedge p_2) \vee \ldots \vee (x \wedge p_m)]$$

$= (t \wedge p_1) \vee (t \wedge p_2) \vee ... \vee (t \wedge p_m)$, as S is distributive, which implies $t \in$R.H.S, and so R.H.S. is an ideal. For any $c \in C$, $c = c \wedge c$ implies $c \in$ R.H.S. Hence R.H.S. is an ideal containing C.

Finally, suppose that I is any ideal containing C. Then for any $x \in$R.H.S. implies $x = (x \wedge p_1) \vee (x \wedge p_2) \vee ... \vee (x \wedge p_m)$ for some $p_1, p_2, ..., p_m \in C$. Then $(x \wedge p_1), (x \wedge p_2), ..., (x \wedge p_m) \in I$ and hence $x \in I$. Therefore, R.H.S. = (C]. □

Thus we have the following result.

Theorem 1.2.6. *For a convex sub-nearlattice* C of *a distributive nearlattice* S, $(C] \cap [C) = C$.

Proof. Obviously, $C \subseteq (C] \cap [C)$.

For the reverse inclusion let $x \in (C] \cap [C)$.

Then $x \in [C)$ implies $x \geq c$ for some $c \in C$. By above lemma, $x \in (C]$ implies $x = (x \wedge c_1) \vee (x \wedge c_2) \vee ... \vee (x \wedge c_n)$ for some $c_1, c_2, ..., c_n \in C$. Then $c \wedge c_i \leq x \wedge c_i \leq c_i$ and so by convexity of C, $x \wedge c_i \in C$ for each $i = 1, 2, 3, ..., n$. Hence $x \in C$. Therefore, $(C] \cap [C) = C$. □

In case of a convex sub-lattice C of a lattice it is well known that $x \in (C]$ implies $x \leq c$ for some $c \in C$. Again, $x \in [C)$ implies $x \geq c_1$for some $c_1 \in C$ and so by convexity of C, $C = (C] \cap [C)$. But in a general nearlattice $x \in (C]$ does not necessarily imply that $x \leq c$ for some $c \in C$.

Following figure 1.6 shows that this is not true even in a distributive nearlattice, although C = (C] ∩ [C) holds there by Theorem 1.2.6.

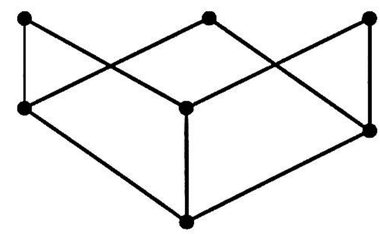

Figure 1.6

Here clearly S is distributive. Let C = {a, b, c}. Here (C] = S. Thus x ∈ (C] but x is not less than equal to c for any c∈C. Because of this fact it is very difficult to study the convex sub-nearlattices of a nearlattice. But it becomes much easier incase of a medial nearlattice.

Recall that a nearlattice S is *medial* if for all x, y, z∈S, $m(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ exists in S.

The following result is due to [43]. But we prefer to include it proof for the convenience of the reader.

Theorem 1.2.7. *If* C *is a convex sub-nearlattice in a medial nearlattice* S *then* x∈(C] *implies* $x \le c$ *for some* c∈C. *Hence* C = (C] ∩ [C).

Proof. By Lemma 1.2.2, $(C] = \bigcup_{m=0}^{\infty} A_m$ where A_m's are defined as in the lemma.

If $x \in A_0$, then $x = (x \wedge c_1) \vee (x \wedge c_2)$ for some $c_1, c_2 \in C$. Observe that $c_1 \wedge c_2 \le$

$(x \wedge c_1) \vee (c_1 \wedge c_2) \leq c_1$ implies $(x \wedge c_1) \vee (c_1 \wedge c_2) \in C$, by convexity. Similarly, $(x \wedge c_2) \vee (c_1 \wedge c_2) \in C$. Thus $m(c_1, x, c_2) = (x \wedge c_1) \vee (x \wedge c_2) \vee (c_1 \wedge c_2) \in C$ and so $x \leq c$ where $c = (x \wedge c_1) \vee (x \wedge c_2) \vee (c_1 \wedge c_2)$. Now we use the method of induction. Suppose $x \in A_{m-1}$ implies $x \leq c$ for some $c \in C$. Let $y \in A_m$, then $y = (y \wedge a_1) \vee (y \wedge a_2)$ for some $a_1, a_2 \in A_{m-1}$. Now $a_1, a_2 \in A_{m-1}$ implies $a_1 \leq p$ and $a_2 \leq q$ for some $p, q \in C$. Thus, $y \leq (y \wedge p) \vee (y \wedge q) \leq y$, and so $y = (y \wedge p) \vee (y \wedge q)$. This implies $y \in A_o$, and so $y \leq c$ for some $c \in C$. This completes the proof. □

From the above theorem we have the following corollary.

Corollary 1.2.8. *For a convex sub-nearlattice* C *of a medial nearlattice* S, $(C] = A_m$ *for each* $m = 0, 1, 2, \ldots$ *where* A_m *are defined as in Lemma* 1.2.2. *In other words,*

$$(C] = \{t \in S : t = (t \wedge c_1) \vee (t \wedge c_2) \text{ for some } c_1, c_2 \in C\}. \ \square$$

1.3 n-Ideals of Nearlattices

The n-ideals of a lattice have been studied extensively by [4], [35], [36], [46], [47] and [48] in different contexts. The idea of n-ideals was introduced by Cornish and Noor in [16]. The n-ideals have also been used in [43] and [44].

For a fixed element n of a lattice L, a convex sub-lattice containing n is called an n-*ideal.* If L has '0', then replacing n by '0' an n-ideal becomes an ideal. Similarly, if L has '1', an n-ideal becomes a filter by replacing n by '1'. Thus the idea of n-ideals is a kind of generalization of both ideals and filters of lattice. So any result involving n-ideals of a lattice L will give a generalization work of the results on ideals and filters of L.

For a fixed element n of a nearlattice S, a convex sub-nearlattice of S containing n is called an n-*ideal* of S.

The set of all n-ideals of a lattice L is denoted by $I_n(L)$, which is an algebraic lattice under set inclusion. Moreover, {n} and L are respectively the smallest and largest elements of $I_n(L)$. For two n-ieals I and J of a lattice L,
$I \vee J = \{x \in L: i_1 \wedge j_1 \leq x \leq i_2 \vee j_2$ for some $i_1, i_2 \in I$ and $j_1, j_2 \in J\}$

But for nearlattices it is not so easy to define the supremum of two n-ideals. For two n-ideals I and J of a nearlattice S, [30] has given a neat description of $I \vee J$, while the set theoretic intersection is the infimum. Hence the set of all n-ideals of a nearlattice S is a lattice which is denoted by $I_n(S)$.
$I \vee J = \{x: i \wedge j \leq x = \vee^{p}_{r=1} (x \wedge a_r)$ for some positive integer p where $i, j \in I \cup J\}$.

An n-ideal generated by a finite number of elements $a_1, a_2, \ldots, a_n$ is called a *finitely generated* n-*ideal*, denoted by $< a_1, a_2, \ldots, a_n >_n$. The set of finitely generated n-ideals is denoted by $F_n(S)$.

Clearly, $< a_1, a_2, \ldots, a_m >_n = < a_1 > \vee < a_2 > \vee \ldots \vee < a_m >$.

An n-ideal generated by a single element a is called a *principal* n-*ideal*, denoted by $< a >_n$. The set of principal n-ideals is denoted by $P_n(S)$.

Standard and neutral elements (ideals) in lattices have been studied by several authors including [21], [22] and [25]. Then [15] has studied them very extensively in nearlattices. By [15] an element s *of a nearlattice* S is called *standard* if for all $t, x, y \in S$,

$$t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s).$$

The element s is called *neutral* if

(i) s is standard and

(ii) for all $x, y, z \in S$, $s \wedge [(x \wedge y) \vee (x \wedge z)]$
$= (s \wedge x \wedge y) \vee (s \wedge x \wedge z)$.

In a distributive nearlattice every element is neutral and hence standard.

Following result is due to [30] which gives a description of finitely generated n-ideals of a nearlattice.

Proposition 1.3.1. *Let* S *be a nearlattice and* $n \in S$. *For* $a_1, a_2, \ldots, a_m \in S$,

(i) $< a_1, a_2, \ldots, a_m >_n \subseteq \{y \in S: (a_1] \cap \ldots \cap (a_m] \cap (n] \subseteq (a_1] \vee \ldots \vee (a_m] \vee (n]\}$;

(ii) $< a_1, a_2, \ldots, a_m >_n = \{y \in S: a_1 \wedge \ldots \wedge a_m \wedge n \leq y = (y \wedge a_1) \vee \ldots \vee (y \wedge a_m) \vee (y \wedge n)\}$, *provided S is distributive*;

(iii) *For any* $a \in S$, $< a >_n = \{y \in S: a \wedge n \leq y = (y \wedge a) \vee (y \wedge n)\}$
$= \{y \in S: y = (y \wedge a) \vee (y \wedge n) \vee (a \wedge n)\}$, whenever n is *standard in* S;

(iv) *When* S *is a lattice, each finitely generated* n-*ideal is two generated. In dead,* $< a_1, a_2, \ldots, a_m >_n = < a_1 \wedge a_2 \wedge \ldots \wedge a_m \wedge n, \; a_1 \vee a_2 \vee \ldots \vee a_m \vee n >_n$.

(v) *When* S *is a lattice,* $F_n(S)$ is *a lattice and its members are simply the intervals* [a, b] *such that* $a \leq n \leq b$, *and for each interval,* $[a, b] \vee [a_1, b_1] = [a \wedge a_1, b \vee b_1]$ *and* $[a, b] \cap [a_1, b_1] = [a \vee a_1, b \wedge b_1]$.

An element n in a nearlattice S is called a *medial element* if $m(x, n, y) = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$ exists for all $x, y \in S$. Of course, in a medial nearlattice every element is medial.

An element n in a nearlattice S is called *sesquimedial* if for all $x, y, z \in S$, $([(x \wedge n) \vee (y \wedge n)] \wedge [(y \wedge n) \vee (z \wedge n)]) \vee (x \wedge y) \vee (y \wedge z)$ exists in S. It is very easy to see that every sesquimedial element is medial and in a medial nearlattice every element is sesquimedial. For detailed literature on these elements see [16] and [43].

An element n of a nearlattice S is called an *upper element* if $x \vee n$ exists for all $x \in S$. Every upper element is of course a sesquimedial element.

An element n is called a *central element* of S if it is neutral, upper complemented in each interval containing it. Cornish and Noor have given a nice description of central elements of nearlattices in [15].

Following result is due to [30].

Lemma 1.3.2. *Let* I *and* J *be* n-*ideals of a nearlattice* S. *Suppose* $A_0 = I \vee J$, $A_m = \{x \in S: i \wedge j \leq x \leq i_1 \vee j_1$, *where* $i_1 \vee j_1$ *exists and* $i, i_1, j, j_1 \in A_{m-1}$ *for* $m = 1,2,\ldots$. *Then* $I \vee J = \bigcup_{m=0}^{\infty} A_m$. □

Theorem 1.3.3. *If* n *is a medial element of a nearlattice* S, *then for* n-*ideals* I *and* J *of* S, $I \cap J = \{m(i, n, j): i \in I, j \in J\}$.

Following results are due to [30].

Theorem 1.3.4. *If* n is *a standard and medial element of a nearlattice* S, *then* $P_n(S)$ *is a meet semi-lattice. In fact, for all* $a, b \in S$, $<a>_n \cap <b>_n = <m(a, n, b)>_n$.

Moreover, when n is neutral and sesquimedial, then $P_n(S)$ is also a nearlattice.

Corollary 1.3.5. *If* n *is neutral and sesquimedial in a nearlattice* S, *then any finitely generated* n-*ideal which is contained in a principal* n-*ideal is principal.*

It should be noted that the set of finitely generated n-ideals $F_n(S)$ is merely a join semi-lattice for general nearlattices. We have stated in section 2 of this chapter that the intersection of two finitely generated ideals of a nearlattice is not necessarily finitely generated. Similarly, the intersection of two finitely generated n-ideals is not necessarily finitely generated. Thus $F_n(S)$ is not a lattice for a general nearlattice S. But if S is distributive and n is medial then

$F_n(S)$ is a lattice. In fact, we have the following result due to [30].

Theorem 1.3.6. *Let* S be *a nearlattice with a neutral and medial element* n. *Then the following conditions are equivalent.*

(i) S is *distributive.*

(ii) $I_n(S)$ *is a distributive lattice.*

(iii) $F_n(S)$ *is a distributive lattice.*

The following result which will be needed in proving several results of this thesis.

Theorem 1.3.7. *Let* S be *a distributive nearlattice with an upper element* n *and let* I, J *be two* n*-ideals of* S. *Then for any* $x \in I \vee J$, $x \vee n = i \vee j$ *and* $x \wedge n = i' \wedge j'$ *for some* $i, i' \in I, j, j' \in J$ *with* $i, j \geq n$ *and* $i', j' \leq n$.

Proof. Let $x \in I \vee J$. Then $x \in (I \cup J]$. Then by Lemma 1.2.5, $x = (x \wedge c_1) \vee .... \vee (x \wedge c_r)$ for some $c_1, c_2, ..., c_r \in I \cup J$.Thus $x \vee n = (x \wedge c_1) \vee ... \vee (x \wedge c_r) \vee n$. Without loss of generality, suppose $c_k \in I$ for some $k = 1, 2, ..., r$.

Then $x \vee n = [(x \vee n) \wedge (c_k \vee n)] \vee n$.

Now $n \leq (x \vee n) \wedge (c_k \vee n) \leq c_k \vee n$ implies $(x \vee n) \wedge (c_k \vee n) \in I$ by convexity, and $n \in J$. Hence $(c_k \vee n)] \vee n = i_k \vee j_k$ for some $i_k \in I, j_k \in J, i_k \geq n$, $j_k \geq n$ for each k. Therefore, $x \vee n = i \vee j$ for some $i \in I, j \in J, i \geq n, j \geq n$. A dual proof of above shows that $x \wedge n = i' \wedge j'$ for some $i' \in I, j' \in J$ with $i', j' \leq n$. □

1.4 Prime n-Ideals of a Nearlattice

For a medial element n, an n-ideal P of a nearlattice S is called a *prime n-ideal* if $P \neq S$ and $m(x, n, y) \in P$ $(x, y \in S)$ implies either $x \in P$ *or* $y \in P$. The set of all prime n-ideals of S is denoted by P(S).

Following result gives a clear idea about prime n-ideals.

Theorem 1.4.1. *If* n is *a medial element and* P *is a prime* n-*ideal of a nearlattice* S. *Then* P contains *either* (n] or [n), *but not both.*

Proof. Suppose P is a prime n-ideal and $(n] \not\subset P$. Then there exists $r < n$ such that $r \notin P$. Now let $s \in [n)$. Then $m(r, n, s) = (r \wedge n) \vee (n \wedge s) \vee (r \wedge s) = r \vee n \vee r = n \in P$. That is, $m(r, n, s) \in P$. Since P is prime, this implies $s \in P$ and so $[n) \subseteq P$. Similarly, if $[n) \not\subset P$, then we can show that $(n] \subseteq P$. Finally if P contains both (n] and [n) then by convexity of P,

P = S which is impossible. □

Moreover we have,

Theorem 1.4.2. *Let* n be *a medial element of a nearlattice* S. *Then every prime* n-*ideal* P of S is *either an ideal or a filter. If it is an ideal, then it is also a prime ideal. If it is a filter, then it is a prime filter.*

Proof. By Theorem 1.4.1, P contains either (n] or [n). Suppose $(n] \subseteq P$. Since any convex sub-nearlattice containing an ideal is clearly an ideal, so P is an ideal. Now let $a \wedge b \in P$ $(a, b \in S)$. Then $a \wedge b \wedge n \in P$. Then by convexity of P, $a \wedge b \wedge n \leq (a \wedge n) \vee (b \wedge n) \leq n$ implies that $(a \wedge n) \vee (b \wedge n) \in P$.

Hence $m(a, n, b) = (a \wedge b) \vee (a \wedge n) \vee (b \wedge n) \in P$. Since P is a prime n-ideal, so either $a \in P$ or $b \in P$. Therefore, P is a prime ideal.

Now suppose $[n) \subseteq P$. Let $x \in P$ and $t \geq x$. Since $(n] \not\subset P$, so there exists $y < n$ such that $y \notin P$. Then $x \wedge n \in P$ and $x \wedge n \leq (t \wedge n) \vee y \leq n$ implies that $(t \wedge n) \vee y \in P$ by convexity.

Now, $m(t, n, y) = (t \wedge n) \vee (n \wedge y) \vee (y \wedge t)$

$= (t \wedge n) \vee y \vee (y \wedge t)$

$= (t \wedge n) \vee y \in P$.

Since P is prime, so $t \in P$. Therefore, P is a filter.

Now, let $a \vee b$ exists and $a \vee b \in P$ $(a, b \in S)$.

So $n \wedge (a \vee b) = (a \wedge n) \vee (b \wedge n) \in P$. But $(a \wedge n) \vee (b \wedge n) = m(a \wedge n, n, b \wedge n)$. Thus, either $a \wedge n \in P$ or $b \wedge n \in P$ as P is a prime n-ideal. Since P is a filter, $a \wedge n \in P$ implies $a \in P$ and $b \wedge n \in P$ implies $b \in P$. Therefore, P is a prime filter. □

Following results are trivial.

Lemma 1.4.3. *For a medial element* n, *any prime ideal* P *containing* n *of a nearlattice* S *is a prime* n-*ideal.* □

Lemma 1.4.4. *Let* n *be a neutral and medial element of a nearlattice* S. *Then any prime filter* Q *containing* n *is a prime* n-*ideal.* □

Proposition 1.4.5. *In a distributive nearlattice* S, *if* I *is an* n-*ideal and* D *is a convex sub-nearlattice with* $I \cap D = \phi$, *then either* $(I] \cap D = \phi$ or $[I) \cap D = \phi$.

Proof. Suppose $(I] \cap D \neq \phi$. Let $x \in (I] \cap D$. This implies $x \in D$ and $x = (x \wedge i_1) \vee ... \vee (x \wedge i_n)$. Again let $y \in [I) \cap D$. This implies $y \in D$ and $y \geq i$ for some $i \in I$.

$$\therefore y \wedge x = y \wedge [(x \wedge i_1) \vee ... \vee (x \wedge i_n)]$$
$$= (x \wedge y \wedge i_1) \vee ... \vee (x \wedge y \wedge i_n)$$
$$\leq (y \wedge i_1) \vee ... \vee (y \wedge i_n)$$
$$\leq y$$

This implies $(y \wedge i_1) \vee ... \vee (y \wedge i_n) \in D$. Now $i \wedge i_1 \leq y \wedge i_1 \leq i_1$. Since i_1, $i \wedge i_1 \in I$. Then by convexity $y \wedge i_1 \in I$. Similarly, $y \wedge i_2$, $y \wedge i_3, ..., y \wedge i_n \in I$.This implies $(y \wedge i_1) \vee ... \vee (y \wedge i_n) \in I$. This implies $I \cap D \neq \phi$ which contradicts the fact that $I \cap D = \phi$. Hence $(I] \cap D = \phi$. Dually we can show that if $I \cap D = \phi$, then $[I) \cap D = \phi$. Therefore, if $I \cap D = \phi$, then either $(I] \cap D = \phi$ or $[I) \cap D = \phi$. □

In lattice theory, Stone's separation theorem is a well known result. Following result is an extension of that theorem for nearlattices which is due to [14].

Theorem 1.4.6. *Let* I *be an ideal and* D *be a convex sub-nearlattice of a distributive nearlattice* S *with* $I \cap D = \phi$. *Then there exists a prime ideal* $P \supseteq I$ *such that* $P \cap D = \phi$. □

Now we generalize above result for n-ideals.

Theorem 1.4.7. *Let* S *be a distributive nearlattice and* n *be a medial element of* S. *Let* I *be an* n-*ideal and* D *be a convex subnearlattice with* $I \cap D = \phi$. *Then there exists a prime* n-*ideal* P *of* S *such that* $P \supseteq I$ *and* $P \cap D = \phi$.

Proof. Since $I \cap D = \phi$, so by Proposition 1.4.5, either $(I] \cap D = \phi$ or $[I) \cap D = \phi$. If $(I] \cap D = \phi$, then there exists a prime ideal $P \supseteq (I]$ such that $P \cap D = \phi$. Since $n \in P$, so by Lemma 1.4.3, P is a prime n-ideal.

On the other hand if $[I) \cap D = \phi$, Then by dually there exists a prime filter $Q \supseteq [I)$ such that $Q \cap D = \phi$. Since $n \in Q$, so by Lemma 1.4.4, Q is a prime n-ideal. This completes the proof. □

Following Corollary trivially follows from above results.

Corollary 1.4.8. *Let* I *be an* n-*ideal of a distributive nearlattice* S *with* n *as a medial element and* $a \in S$ *such that* $a \notin I$. *Then there exists a prime* n-*ideal* P *of* S *such that* $P \supseteq I$ *and* $a \notin P$. □

Thus we have the following extension of a well known result in terms of n-ideals. We omit the proof as it is very trivial.

Corollary 1.4.9. *Let* n *be a medial element of a distributive nearlattice* S. *Then every* n-*ideal* I *of* S *is the intersection of all prime* n-*ideals containing it.* □

1.5 Semi-Boolean algebras and Principal n-Ideals

An interesting class of distributive nearlattice is provided by those semi-lattices in which each principal ideal is a Boolean algebra. These semi-lattices have been studied by Abbott [1], [2], [3] under the name of semi-Boolean algebras and mainly from the view of Abbott's implication algebras. An implication algebra is a groupoid (I; .) satisfying:

(i) $(a\,b)\,a = a$.

(ii) $(a\,b)\,b = (b\,a)\,a$.

(iii) $a\,(b\,c) = b\,(a\,c)$.

Abbott shows in [1, P-227-236] that each implication algebra determines a semi-Boolean algebra and conversely each semi-Boolean algebra determines an implication algebra.

Recall that according to [14], a semi-lattice S is a *semi-Boolean algebra* if and only if the following conditions are satisfied.

(i) S has the upper bound property.

(ii) S is distributive.

(iii) S has a 0 and for any $x \in S$, $(x]^* = \{y \in S: y \wedge x = 0\}$ is an ideal and $(x] \vee (x]^* = S$.

A nearlattice S is *relatively complemented* if each interval $[x, y]$ in S is complemented. That is, for $x \leq t \leq y$ there exists a s in $[x, y]$ such that $t \wedge s = x$ and $t \vee s = y$. A nearlattice S with 0 is called *sectionally complemented,* if the interval $[0, x]$ is complemented for each $x \in S$. Of course, every relatively complemented nearlattice S with 0 is sectionally complemented. Thus a nearlattice S with 0 is *semi-Boolean* if and only if it distributive and sectionally complemented.

In section 3 of this chapter, we have defined the principal n-ideal $< a >_n$, generated by $a \in S$. Then set of principal n-ideals of a nearlattice S is denoted $P_n(S)$. By Proposition 1.3.1, if n is a standard element of a nearlattice, then for any $a \in S$, $< a >_n = \{y \in S: a \wedge n \leq y = (y \wedge a) \vee (y \wedge n)\}$.

By Theorem 1.3.4, we know that when n is standard and medial, then the set of principal n-ideals $P_n(S)$ is a meet semi-lattice and $< a >_n \cap < b >_n = < m(a, n, b) >_n$ for all $a, b \in S$. Also by Corollary 1.3.5, when n is neutral and sesquimedial, then $P_n(S)$ is in fact a nearlattice. Moreover,[50] has proved the following result.

Theorem 1.5.1. *If* S *is a nearlattice and* n *is a neutral element of* S, *then* $P_n(S)$ *is also a nearlattice.* □

In section 3 of this chapter, we have defined the central element of a nearlattice. The following theorem gives a characterization of a central element of a nearlattice which is due to [50]. In fact, this plays the vital role in this thesis, as it will be used in proving most of the important results for the rest of the thesis. We prefer to include the proof of this theorem for the convenience of the reader.

Theorem 1.5.2. *For an element* n *of a nearlattice* S, *the following are equivalent.*

(i) n *is central in* S.

(ii) n *is upper and the map* $\varphi: P_n(S) \rightarrow (n]^d \times [n)$ *defined by* $\varphi(<a>_n) = (a \wedge n, a \vee n)$ *is an isomorphism, where* $(n]^d$ *represents the dual of the lattice* $(n]$.

Proof. **(i)⇒(ii).** Suppose n is central in S. Then of course, n is upper by definition. Now let $<a>_n \subseteq <b>_n$, then $[a \wedge n, a \vee n] \subseteq [b \wedge n, b \vee n]$. Thus $b \wedge n \leq a \wedge n$ and $a \vee n \leq b \vee n$. This implies $a \wedge n \leq_d b \wedge n$ and $a \vee n \leq b \vee n$ and so $(a \wedge n, a \vee n) \leq (b \wedge n, b \vee n)$ in $(n]^d \times [n)$.

Thus $\varphi(<a>_n) \subseteq \varphi(<b>_n)$.

Again, let $\varphi(<a>_n) \subseteq \varphi(<b>_n)$, then $(a \wedge n, a \vee n) \leq (b \wedge n, b \vee n)$ in $(n]^d \times [n)$. Thus $a \wedge n \leq_d b \wedge n$ in $(n]^d$ and $a \vee n \leq b \vee n$ in $[n)$, this implies $b \wedge n \leq a \wedge n$ and $a \vee n \leq b \vee n$ in S, and so $[a \wedge n, a \vee n] \subseteq [b \wedge n, b \vee n]$. Hence $<a>_n \subseteq <b>_n$. Therefore, φ is an order isomorphism if we can show that φ is onto.

Let $(p, q) \in (n]^d \times [n)$. This implies $p \leq n \leq q$. Since n is central so there exists r such that $r \wedge n = p$, $r \vee n = q$. This implies $(p, q) = (r \wedge n, r \vee n) = \varphi(<r>_n)$. Thus φ is onto and so (ii) holds.

(ii)⇒(i). Suppose (ii) holds, let $a \leq n \leq b$ $(a, b \in S)$. Then $(a, b) \in (n]^d \times [n)$. Since $\varphi: P_n(S) \rightarrow (n]^d \times [n)$ is an isomorphism, so there exists $<c>_n \in P_n(S)$ such that $\varphi(<c>_n) = (c \wedge n, c \vee n) = (a, b)$. This implies that c is the relative complement of n in $a \leq n \leq b$. Therefore, n is central. □

Following results are easy consequences of the above theorem.

Corollary 1.5.3. *Let* S *be a nearlattice and* $n \in S$ *be a central element. Then* $P_n(S)$ *is sectionally complemented if and only if the intervals* $[a, n]$ *and* $[n, b]$ *are complemented for each* $a, b \in S$.

We know by Theorem 1.3.6, that if n is medial in a distributive nearlattice S, then $I_n(S)$ is also distributive and hence $P_n(S)$ (if it is a nearlattice) is also distributive.

Corollary 1.5.4. *If* n *is a central element of a distributive nearlattice* S, *then* $P_n(S)$ *is semi-Boolean if and only if the intervals* [a, n] *and* [n, b] *are complemented for each* a, b∈S (a ≤ n ≤ b). □

Following results are due to [56]. These will be needed for further development of the thesis.

Lemma 1.5.5. *If* S_1 *is a nearlattice of a distributive nearlattice* S *and* P_1 *is a prime ideal (filter) in* S_1, *then there exists a prime ideal (filter)* P *in* S *such that* $P_1 = P \cap S_1$. □

In lattice theory it is well known that [20, Theorem 22, P-76] a distributive lattice L with 0 and 1 is Boolean if and only if its set of prime ideals is unordered by set inclusion.

Following result is due to [53] which has generalized above result for a distributive nearlattice with 0.

Theorem 1.5.6. *If* S *is a distributive nearlattice with* 0, *then* S *is semi-Boolean if and only if its set of prime ideals (filters) is unordered by set inclusion.* □

The next theorem gives a generalization of above theorem which is due to [45].

Theorem 1.5.7. *Let* S *be a distributive nearlattice and* $n \in S$ *be a central element. Then the following conditions are equivalent.*

(i) $P_n(S)$ *is semi-Boolean.*

(ii) *The set of prime* n-*ideals* P(S) *of* S *is unordered by set inclusion.* □

CHAPTER 2

CHAPTER 2

SOME GENERALIZATION WORK ON NORMAL NEARLATTICES

Introduction.

Minimal prime ideals and Normal lattices have been studied by many authors including [11], [10], [13] and [33]. On the other hand, [56], [49] and [54] have given the concept of normal nearlattices and generalized many results of [11] for nearlattices. In this chapter we devoted ourselves in further studies in this area. We generalized several results of [56] and [49] in terms of n-ideals when n is a central element.

Here we introduce the concept of minimal prime n-ideals and generalize some of the results on minimal prime n-ideals. These results are used to generalize several important results on normal nearlattices in terms of n-ideals.

A prime n-ideal P is said to be a *minimal prime* n-*ideal* belonging to n-ideal I if

(i) $I \subseteq P$ and

(ii) There exists no prime n-ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$

A prime n-ideal P of a nearlattice S is called a *minimal prime* n-*ideal* if there exists no prime n-ideal Q such that $Q \neq P$ and $Q \subseteq P$. Thus a minimal prime n-ideal is a minimal prime n-ideal belonging to $\{n\}$.

A distributive nearlattice S with 0 is called a *normal nearlattice* if its every prime ideal contains a unique minimal prime ideal.

Since the lattice of n-ideals $I_n(S)$ of a distributive nearlattice S is a distributive algebraic lattice, so $I_n(S)$ is pseudo-complemented. If n is a medial element, then for any n-ideals J of a distributive nearlattice S, we define $J^* = \{x \in S: m(x, n, j) = n \text{ for all } j \in J\}$. Obviously, J^* is an n-ideal and $J \cap J^* = \{n\}$. We call J^*, the annihilator n-ideal of J which is the pseudo-complement of J in $I_n(S)$.

From chapter one we know that for a distributive medial nearlattice S with an element n, $P_n(S)$ is a distributive medial nearlattice with the smallest element $\{n\}$. Since $P_n(S)$ may not have a largest element, so we can not talk on pseudo-complementation on $P_n(S)$. So for any $< a >_n \in P_n(S)$, $< a >_n^*$ represents the pseudo-complement of $< a>_n$ in $I_n(S)$. Moreover, if $< a >_n \subset < b >_n$, then $< a >_n^+$ denotes the relative pseudo-complement of $< a>_n$ in $[\{n\}, < b >_n]$, which may not be a principal n-ideal. Moreover, for any n-ideals $\{n\} \subseteq J \subseteq I$, J^+ denotes the relative pseudo-complement of J in $[\{n\}, I]$.

By [56] for a prime ideal P of a distributive nearlattice S with 0, 0(P) is used to denote the set $\{y \in S: y \wedge x = 0 \text{ for some } x \in S\text{-}P\}$. It is easy to show that 0(P) is an ideal contained in P.

Two prime n-ideals P and Q of a nearlattice S are called *co-maximal* if $P \vee Q = S$.

In section 1, we have studied minimal prime n-ideals of S. There we have given some characterization of minimal prime n-ideals. These results give nice generalizations of several results on minimal prime ideals which will be used to prove some important results in section 2.

In section 2, we have given several characterizations of those $P_n(S)$ which are normal medial nearlattices in terms of n-ideals. We have also discussed on $0(P)$ and $n(P)$ and given some properties of $n(P)$. Moreover, we have proved that $P_n(S)$ is normal if and only if each prime n-ideal contains a unique minimal prime n-ideal, when n is central.

2.1 Minimal Prime n-Ideals.

Recall that a prime n-ideal P is a *minimal prime* n-*ideal* belonging to n-ideal I if

(i) $I \subseteq P$ and

(ii) There exists no prime n-ideal Q such that $Q \neq P$ and $I \subseteq Q \subseteq P$.

Thus a prime n-ideal P of S is a minimal prime n-ideal if there exists no prime n-ideal Q such that $Q \neq P$ and $Q \subseteq P$. In other words, minimal prime n-ideal is a minimal prime n-ideal belonging to $\{n\}$.

Recall that an element n of a nearlattice S is medial if m(x, n, y) exists for all $x, y \in S$. Since for the definition of a prime n-ideal of S, the medial property of n is essential, so in talking about prime n-ideals of S we will always assume n as a medial element. We start this section with the following result which is a generalization of a well known result in lattice theory.

Lemma 2.1.1. *Let* S *be a nearlattice with a medial element* n. *Then every prime* n-*ideal contains a minimal prime* n-*ideal.*

Proof. Let P be a prime n-ideal of S and let χ be the set of all prime n-ideals Q contained in P. Then χ is non-void, since $p \in \chi$ .If C is a chain in χ and $Q = \cap\ (X: X \in C)$, then Q is a non-empty because $n \in Q$ and Q is an n-ideal, in fact, Q is prime.

Indeed, if $m(a, n,b) \in Q$ for some $a, b \in S$, then $m(a, n, b) \in X$ for all $X \in C$. Since X is prime, either $a \in X$ or $b \in X$. Thus, either $Q = \cap\ (X: a \in X)$ or

$Q = \cap(X: b \in X)$, proving that $a \in Q$ or $b \in Q$. Therefore, we can apply to χ the dual form of Zorn's lemma to conclude the existence of a minimal member of χ. □

If S is a distributive nearlattice with $n \in S$, then we already know that $F_n(S)$ is a distributive lattice with $\{n\}$ as the smallest element. So we can talk on the sectionally pseudo-complementedness of $F_n(S)$. $F_n(S)$ is called *sectionally pseudo-complemented* if each interval $[\{n\}, < a_1,\ldots,a_r >_n]$ is pseudo-complemented. That is for $\{n\} \subseteq < b_1,\ldots,b_s >_n \subseteq < a_1,\ldots,a_r >_n$, relative pseudo-complement $< b_1,\ldots,b_s >_n^+$ in $[\{n\}, < a_1,\ldots,a_r >_n]$ belongs to $F_n(S)$.

Now we give a characterization of minimal prime n-ideals of a distributive nearlattice S, when $F_n(S)$ is sectionally pseudo-complemented. To do this we establish the following lemmas.

Lemma 2.1.2. *Let* S *be a distributive nearlattice and* $n \in S$ *be a medial element. Then for any* $I, J \in I_n(S)$, $(I \cap J)^* \cap I = J^* \cap I$.

Proof. Since $I \cap J \subseteq J$. so R.H.S. $\subseteq$ L.H.S.

To prove the reverse inclusion, let $x \in$L.H.S. Then $x \in I$ and $m(x, n, t) = n$ for all

$t \in I \cap J$. Since $x \in I$, so $m(x, n, j) \in I \cap J$. Thus, $m(x, n, m(x, n, j)) = n$. But it can be easily seen that $m(x, n, m(x, n, j)) = m(x, n, j)$. This implies $m(x, n, j) = n$ for all

$j \in J$. Hence $x \in$R.H.S, and so L.H.S. $\subseteq$ R.H.S. Thus $(I \cap J)^* \cap I = J^* \cap I$. □

Lemma 2.1.3. *Suppose* n *is a medial element of a nearlattice* S. *If* $I \subseteq J$, $I, J \in I_n(S)$, *then*

(i) $I^+ = I^* \cap J$ and

(ii) $I^{++} = I^{**} \cap J$.

Proof. (i) is trivial. For (ii), using (i), we have,

$I^{++} = (I^+)^* \cap J = (I^* \cap J)^* \cap J$. Thus by Lemma 2.1.2, $I^{++} = I^{**} \cap J$. □

A characterization of minimal prime ideals in lattices is given in [20, Theorem 1.3]. [31] has generalized the result for nearlattices. Recently [46] has provided a characterization of the minimal prime n-ideals in lattices. Here we generalize the result for nearlattices.

Theorem 2.1.4. *Let* n *be a sesquimedial element of a distributive nearlattice* S. *Suppose* $F_n(S)$ *is a sectionally pseudo-complemented distributive nearlattice and* P *is a prime* n*-ideal of* S. *Then the following conditions are equivalent.*

(i) P *is minimal.*

(ii) $x \in P$ *implies* $<x>_n^* \not\subseteq P$.

(iii) $x \in P$ *implies* $<x>_n^{**} \subseteq P$.

(iv) $P \cap D(<t>_n) = \phi$ *for all* $t \in S - P$, *where* $D(<t>_n) = \{x \in <t>_n : <x>_n^+ = \{n\}\}$.

Proof. **(i)⇒(ii).** Suppose P is minimal. If (ii) fails, then there exists $x \in P$ such that $<x>_n^* \subseteq P$. Since P is a prime n-ideal, so by Theorem 2.1.2, P is a prime ideal or a prime filter. Suppose P is a prime ideal. Let $D = (S - P) \vee [x)$. We claim that

$n \notin D$. If $n \in D$, then $n = q \wedge x$ for some $q \in S - P$. Then

$<q>_n \cap <x>_n = <(q \wedge x) \vee (q \wedge n) \vee (x \wedge n)>_n = \{n\}$ implies

$<q>_n \subseteq <x>_n^* \subseteq P$. Thus $q \in P$, which is a contradiction. Hence $n \notin D$.

Then by Theorem 1.4.7, there exists a prime n-ideal Q with $Q \cap D = \phi$. Then

$Q \subseteq P$ as $Q \cap (S - P) = \phi$ and $Q \neq P$, since $x \notin Q$. But this contradicts the minimality of P. Hence $<x>_n^* \subseteq P$.

Similarly, we can prove that $<x>_n^* \subseteq P$ if P is a prime filter.

(ii)⇒(iii). Suppose (ii) holds and $x \in P$. Then $<x>_n^* \not\subset P$. Since

$<x>_n^* \cap <x>_n^{**} = \{n\} \subseteq P$, and P is prime, so $<x>_n^{**} \subseteq P$.

(iii)⇒(iv). Suppose (iii) holds and $t \in S - P$.

Let $x \in P \cap D(<t>_n)$. Then $x \in P$, $x \in D(<t>_n)$. Thus, $<x>_n^+ = \{n\}$ and so

$<x>_n^{++} = <t>_n$. By (iii), $x \in P$ implies $<x>_n^{**} \subseteq P$. Also by Lemma 2.1.3,

$<x>_n^{++} = <x>_n^{**} \cap <t>_n$. Hence $<x>_n^{**} \cap <t>_n = <t>_n$ and so

$<t>_n \subseteq <x>_n^{++} \subseteq P$. That is, $t \in P$, which is a contradiction.

Therefore, $P \cap D(<t>_n) = \phi$ for all $t \in S - P$.

(iv)⇒(i). Suppose P is not minimal. Then there exists a prime n-ideal $Q \subset P$. Let

$x \in P - Q$. Since $<x>_n \cap <x>_n^* = \{n\} \subseteq Q$, so $<x>_n^* \subseteq Q \subset P$. Thus,

$<x>_n \vee <x>_n^* \subseteq P$. Choose any $t \in S - P$. Then $<t>_n \cap (<x>_n \vee <x>_n^*) \subseteq P$.
Now $<t>_n \cap (<x>_n \vee <x>_n^*) = (<t>_n \cap <x>_n) \vee (<t>_n \cap (<x>_n^*)$

$$= <m(t, n, x)>_n \vee ((<t>_n \cap (<x>_n)^* \cap <t>_n) \text{ (by Lemma2.1.2)}$$

$$= <m(t, n, x)>_n \vee (<m(t, n, x)>_n^* \cap <t>_n)$$

$$= <m(t, n, x)>_n \vee <m(t, n, x)>_n^+ \text{ (by Lemma 2.1.3.)}$$

where $<m(t, n, x)>_n^+$ is the relative pseudo-complement of $<m(t, n, x)>_n$ in

$<t>_n$. Since $F_n(S)$ is sectionally psuedo-complemented, $<m(t, n, x)>_n^+$ is finitely generated and so $<m(t, n, x)>_n \vee <m(t, n, x)>_n^+$ is a finitely generated n-ideal contained in $<t>_n$. Therefore by Theorem 1.3.7,

$<m(t, n, x)>_n \vee <m(t, n, x)>_n^+ = <r>_n$ for some $r \in <t>_n$. Moreover, $<r>_n^+ =$
$<m(t, n, x)>_n^+ \cap <m(t, n, x)>_n^{++} = \{n\}$.Thus, $r \in P \cap D(<t>_n)$, which is a contradiction. Therefore, P must be minimal. □

2.2 Nearlattices whose Principal n-Ideals form Normal Nearlattices.

Recall that a distributive nearlattice S with 0 is *normal* if every prime ideal of S contains a unique minimal prime ideal.

We already know from Theorem 1.5.2 that for a central element $n \in S$, $P_n(S) \cong (n]^d \times [n)$.

Thus we have the following result.

Lemma 2.2.1. *Suppose* n *is a central element of a distributive nearlattice* S. *Then* $P_n(S)$ *is normal if and only if* $(n]^d$ *and* $[n)$ *are normal.* □

A distributive lattice L with 1 is called *dual normal* if its every prime filter is contained in a unique ultra-filter (maximal and proper). In a general lattice, this condition is also equivalent to the condition of normality, that is, every prime ideal contains a unique minimal prime ideal. Thus obviously the concept of dual normality coincides with the normality in case of bounded distributive lattices.

Therefore from above lemma $P_n(S)$ is normal if and only if [n) is a normal nearlattice and (n] is a dual normal lattice. Following theorem is needed to prove the main results of this chapter.

Theorem 2.2.2. *Suppose* S *be a distributive nearlattice and* $n \in S$. *Let* $x, y \in S$ *with* $<x>_n \cap <y>_n = \{n\}$. *Then the following conditions are equivalent.*

(i) $<x>_n^* \vee <y>_n^* = S$.

(ii) *For any* $t \in S$, $<m(x, n, t)>_n^+ \vee <m(y, n, t)>_n^+ = <t>_n$ *where* $<m(x, n, t)>_n^+$ *denote the relative pseudo-complement of* $<m(x, n, t)>_n$ *in* $[\{n\}, <t>_n]$.

Proof. **(i)⇒(ii).** Suppose (i) holds. Then for any $t \in S$,

$<m(x, n, t)>_n^+ \vee <m(y, n, t)>_n^+ = (<x>_n \cap <t>_n)^+ \vee (<y>_n \cap <t>_n)^+$

$=((<x>_n \cap <t>_n)^* \cap <t>_n) \vee ((<y>_n \cap <t>_n)^* \cap <t>_n)$ [by Lemma 2.1.3.]

$= (<x>_n^* \cap <t>_n) \vee (<y>_n^* \cap <t>_n)$ [by Lemma 2.1.2.]

$= (<x>_n^* \vee <y>_n^*) \cap <t>_n$

$= S \cap <t>_n$

$= <t>_n$.

Hence (ii) holds.

(ii)⇒(i). Suppose (ii) holds and $t \in S$.

By (ii), $<m(x, n, t)>_n^+ \vee <m(y, n, t)>_n^+ = <t>_n$. Then using Lemmas 2.1.2. and 2.1.3. and the calculation of (i)⇒(ii) above, we get,

$(<x>_n^* \vee <y>_n^*) \cap <t>_n = <t>_n$. This implies $<t>_n \subseteq <x>_n^* \vee <y>_n^*$ and so $t \in <x>_n^* \vee <y>_n^*$. Therefore, $<x>_n^* \vee <y>_n^* = S$. □

Cornish in [11] has given some characterizations of normal lattices. Then [49] extended those results for nearlattices. [49] has given the following characterizations for normal nearlattices.

Theorem 2.2.3. *Let* S *be a distributive nearlattice with* 0. *Then the following conditions are equivalent.*

(i) *Any two distinct minimal prime ideals are co-maximal.*

(ii) S *is normal.*

(iii) 0(P) *is a prime ideal for each prime ideal* P.

(iv) *For all* $x, y \in S$, $x \wedge y = 0$ *implies* $(x]^* \vee (y]^* = S$.

(v) $(x \wedge y]^* = (x]^* \vee (y]^*$. □

Now we generalize a part of the above result in terms of n-ideals.

Theorem 2.2.4. *Let* S *be a distributive nearlattice and* n *be a central element of* S. *The following conditions are equivalent.*

(i) $P_n(S)$ *is normal.*

(ii) *Every prime* n-*ideal of* S *contains a unique minimal prime* n-*ideal.*

(iii) *For any two minimal prime* n-*ideals* P *and* Q *of* S, $P \vee Q = S$.

Proof. (i)⇒(ii). Let $P_n(S)$ be normal, since $P_n(S) \cong (n]^d \times [n)$, so both $(n]^d$ and $[n)$ are normal. Suppose P is any prime n-ideal of S. Then by Theorem 1.4.1, either $P \supseteq (n]$ or $P \supseteq [n)$. Without loss of generality, suppose $P \supseteq (n]$. Then by Theorem 1.4.2, P is prime ideal of S. Hence by Lemma 1.5.5, $P_1 = P \cap [n)$ is a prime ideal of [n). Since [n) is normal, so by definition P_1 contains a unique minimal prime ideal R_1 of [n). Therefore, P contains a unique minimal prime ideal R of S where $R_1 = R \cap [n)$. Since $n \in R_1$ so $n \in R$ and hence R is a minimal prime n-ideal of S. Thus (ii) holds.

(ii)⇒(i). Suppose (ii) holds. Let P_1 be a prime ideal in [n). Then by Lemma 1.5.5, $P_1 = P \cap [n)$ for some prime ideal P of S. Since $n \in P_1 \subseteq P$, so P is prime n-ideal.

Therefore, P contains a unique minimal prime n-ideal R of S. Thus by Lemma 1.5.5 P_1 contains the unique minimal prime ideal $R_1 = R \cap [n)$ of $[n)$. Hence by definition $[n)$ is normal. Similarly, we can prove that $(n]^d$ is also normal. Since $P_n(S) \cong (n]^d \times [n)$, so $P_n(S)$ is normal.

(ii)⇔(iii) is trivial by Stone's separation Theorem.

Recall that for a prime ideal P of a distributive nearlattice S with 0, [56] has defined $0(P) = \{x \in S: x \wedge y = 0$ for some $y \in S - P\}$. Clearly, $0(P)$ is an ideal and $0(P) \subseteq P$. [56] has shown that $0(P)$ is the intersection of all the minimal prime ideals of S which are contained in P.

For a prime n-ideal P of a distributive nearlattice S, we write $n(P) = \{y \in S: m(y, n, x) = n$ for some $x \in S - P\}$. Clearly, $n(P)$ is an n-ideal and $n(P) \subseteq P$.

Lemma 2.2.5. *Let* n *be a medial element of a distributive nearlattice* S *and* P *be a prime* n-*ideal in* S. *Then each minimal prime* n-*ideal belonging to* n(P) *is contained in* P.

Proof. Let Q be a minimal prime n-ideal belonging to n(P). If $Q \not\subset P$, then choose $y \in Q - P$. Since Q is a prime n-ideal, so by Theorem 1.4.2, we know th Q is either an ideal or a filter. Without loss of generality, suppose Q is an ideal. Now let $T = \{t \in S: m(y, n, t) \in n(P)\}$. We shall show that $T \not\subset Q$.

If not, let $D = (S - Q) \vee [y)$. Then $n(P) \cap D = \phi$.

For otherwise, $y \wedge r \in n(P)$ for some $r \in S - Q$. Then by convexity, $y \wedge r \leq m(y, n, r) \leq (y \wedge r) \vee n$ implies $m(y,n,r) \in n(P)$.

Hence $r \in T \subseteq Q$,which is a contradiction. Thus by Theorem 1.4.7, there exists a prime n-ideal R containing n(P) disjoint to D. Then $R \subseteq Q$.
Moreover, $R \neq Q$ as $y \notin R$, this shows that Q is not a minimal prime n-ideal belonging to n(P), which is a contradiction. Therefore, $T \not\subset Q$. Hence there exists $z \notin Q$ such that $m(y, n, z) \in n(P)$. Thus $m(m(y, n, z), n, x) = n$ for some $x \in S - P$.
It is easy to see that $m(m(y, n, z), n, x) = m(m(y, n, x), n, z)$.
Hence $m(m(y, n, x), n, z) = n$. Since P is prime and $y, x \notin P$ so $m(y, n, x) \notin P$.
Therefore, $z \in n(P) \subseteq Q$, which is a contradiction. Hence $Q \subseteq P$. □

Proposition 2.2.6. *If* n *is a medial element of a distributive nearlattice* S *and* P *is a prime* n*-ideal in* S, *then* n(P) *is the intersection of all minimal prime* n*-ideals contained in* P.

Proof. Clearly, n(P) is contained in any prime n-ideal which is contained in P. Hence n(P) is contained in the intersection of all minimal prime n-ideals contained in P. Since S is distributive, so by Corollary 1.4.9, n(P) is the intersection of all minimal prime n-ideals belonging to it. Since each prime n-ideal contains a minimal prime n-ideal, above remarks and Lemma 2.2.5.establish the proposition.

□

Thus we have the following result which gives a generalization of Theorem 2.2.3.

Theorem 2.2.7. *Let* S *be a distributive nearlattice and let* n be *central element in* S. *Then the following conditions are equivalent.*

(i) $P_n(S)$ *is normal.*

(ii) *Every prime* n*-ideal contains a unique minimal prime* n*-ideal.*

(iii) *For each prime* n*-ideal* P, n(P) *is prime* n*-ideal.*

(iv) *For all* $x, y \in S$, $<x>_n \cap <y>_n = \{n\}$

implies $<x>_n^* \vee <y>_n^* = S$.

(v) *For all* $x, y \in S$, $(<x>_n \cap <y>_n)^* = <x>_n^* \vee <y>_n^*$.

Proof. (i)⇒(ii) holds by Theorem 2.2.4.

(ii)⇒(iii) is a direct consequence of Proposition 2.2.6.

(iii)⇒(iv). Suppose (iii) holds.

Consider $x, y \in S$ with $<x>_n \cap <y>_n = \{n\}$.

If $<x>_n^* \vee <y>_n^* \neq S$, then by Theorem 1.4.7, there exists a prime n-ideal P such that $<x>_n^* \vee <y>_n^* \subseteq P$, then $<x>_n^* \subseteq P$ and $<y>_n^* \subseteq P$ imply $x \notin n(P)$ and $y \notin n(P)$. But n(P) is prime and so $m(x, n, y) = n \in n(P)$ is contradictory. Therefore, $<x>_n^* \vee <y>_n^* = S$.

(iv)⇒(v). Obviously, $<x>_n^* \vee <y>_n^* \subseteq (<x>_n \cap <y>_n)^*$.

Conversely, let $w \in (<x>_n \cap <y>_n)^*$. Then, $<w>_n \cap <x>_n \cap <y>_n = \{n\}$

or, $<m(w, n, x)>_n \cap <y>_n = \{n\}$

So by (iv), $<m(w, n, x)>_n^* \vee <y>_n^* = S$.

So, $w \in <m(w, n, x)>_n^* \vee <y>_n^*$.

Therefore, $w \wedge n, w \vee n \in <m(w, n, x)>_n^* \vee <y>_n^*$. Here $w \vee n$ exists as n is an upper element. Then by Theorem 1.3.7, $w \vee n = r \vee s$ for some $r \in <m(w, n, x)>_n^*$ and $s \in <y>_n^*$ with $r, s \geq n$.

Now $r \in <m(w, n, x)>_n^*$ implies

$r \wedge [(w \wedge n) \vee (w \wedge x) \vee (x \wedge n)] \vee (r \wedge n) \vee [(w \wedge n) \vee (x \wedge n) \vee (w \wedge x)] \wedge n = n$.

Observe that above left hand expression exists as S is medial. That is, $(r \wedge w \wedge n) \vee (r \wedge w \wedge x) \vee (r \wedge x \wedge n) \vee (r \wedge n) \vee (w \wedge n) \vee (x \wedge n) = n$, and so $(r \wedge w \wedge x) \vee n = n$. This implies $(r \vee n) \wedge (w \vee n) \wedge (x \vee n) = n$, so $(r \vee n) \wedge (x \vee n) = n$ as $r \vee n \leq w \vee n$. Thus, $(r \wedge x) \vee n = n$. Hence $(r \wedge x) \vee (x \wedge n) \vee (r \wedge n) = n$, which implies $r \in <x>_n^*$.

Therefore, $w \vee n \in \langle x \rangle_n^* \vee \langle y \rangle_n^*$.

A dual proof of above shows that $w \wedge n \in \langle x \rangle_n^* \vee \langle y \rangle_n^*$. So by convexity, $w \in \langle x \rangle_n^* \vee \langle y \rangle_n^*$. Therefore, $(\langle x \rangle_n \cap \langle y \rangle_n)^* \subseteq \langle x \rangle_n^* \vee \langle y \rangle_n^*$, and so $(\langle x \rangle_n \cap \langle y \rangle_n)^* = \langle x \rangle_n^* \vee \langle y \rangle_n^*$, which is (v).

(v)⇒(iv). Let $\langle x \rangle_n \cap \langle y \rangle_n = \{n\}$, for some $x, y \in S$.

By (v), $S = \{n\}^* = (\langle x \rangle_n \cap \langle y \rangle_n)^* \subseteq \langle x \rangle_n^* \vee \langle y \rangle_n^*$.

Thus (iv) holds.

(iv)⇒(i). Consider $[n)$. Let $x, y \in [n)$ with $x \wedge y = n$.

Then $\langle x \rangle_n \cap \langle y \rangle_n = \{n\}$. Thus, by (iv), $\langle x \rangle_n^* \vee \langle y \rangle_n^* = S$.

This implies
$$[n) = (\langle x \rangle_n^* \vee \langle y \rangle_n^*) \cap [n) \\ = (\langle x \rangle_n^* \cap [n)) \vee (\langle y \rangle_n^* \cap [n)) \\ = \langle x \rangle_n^+ \vee \langle y \rangle_n^+.$$

Notice that both $\langle x \rangle_n$ and $\langle y \rangle_n$ are ideals in $[n)$ and $\langle x \rangle_n^+, \langle y \rangle_n^+$ are annihilator ideals of $\langle x \rangle_n$ and $\langle y \rangle_n$ respectively in $[n)$. This implies by Theorem 2.2.3 that $[n)$ is a normal nearlattice. A dual proof of above shows that $[n)^d$ is also a normal nearlattice. Therefore $P_n(S)$ is also normal as $P_n(S) \cong (n]^d \times [n)$. □

CHAPTER 3

CHAPTER 3

ON NEARLATTICES S WHOSE $P_n(S)$ FORM RELATIVELY NORMAL NEARLATTICES

Introduction.

Relative annihilators in lattices and semi-lattices have been studied by many authors including [39], [65], [48] and [51]. Also [11] has used the annihilators in studing relative normal lattices. In this chapter we introduce the notion of relative annihilators around a fixed element n of a nearlattice S which is used to generalize several results on relatively nearlattices.

In chapter 2, we have already discussed on relative annihilators. For $a, b \in S$, $< a, b >$ denotes the relative annihilator. That is $< a, b > = \{x \in S : x \wedge a \leq b\}$. In presence of distributivity, it is easy to show that each relative annihilator is an ideal. Also note that $< a, b > = < a, a \wedge b >$. For detailed literature on this see [39] and [51]. Again for $a, b \in L$, where L is a lattice, recall from chapter two, that $< a, b >_d = \{x \in L : x \vee a \geq b\}$ is a relative dual annihilator. In presence of distributivity of L, $< a, b >_d$ is a dual ideal (filter).

In case of a nearlattice it is not possible to define a dual relative annihilator ideal for any a and b. But if n is an upper element of S, then $x \vee n$ exists for all $x \in S$ by the upper bound property of S. As we have mentioned in chapter 2, then for any $a \in (n]$, we can talk about dual relative annihilator ideal of the form $< a, b >_d$ for any $b \in S$. That is, for any $a \leq n$ in S, $< a, b >_d = \{x \in S: x \vee a \geq b\}$.

For $a, b \in S$ and an upper element $n \in S$,
we define, $< a, b >^n = \{x \in S: m(a, n, x) \in < b >_n\}$

$$= \{x \in S: \ b \wedge n \leq \ m(a, n, x) \leq b \vee n\}.$$

We call $< a, b >^n$ the annihilator of a relative to b around the element n or simply a relative n-annihilator. It is easy to see that for all $a, b \in S$, $< a, b >^n$ is always a convex subset containing n. In presence of distributivity, it can easily be seen that $< a, b >^n$ is an n-ideal. If $0 \in S$, then putting n =0, we have, $< a, b >^n = < a, b >$.

For two n-ideals A and B of a nearlattice S, $< A, B >$ denotes $\{x \in S: m(a, n, x) \in B$ for all $a \in A\}$, when n is a medial element. In presence of distributivity, clearly $< A, B >$ is an n-ideal. Moreover, we can easily show that $< a, b >^n = < <a>_n, <b>_n >$.

A distributive nearlattice S is called a *relatively normal nearlattice* if each closed interval [x, y] with x < y ($x, y \in S$) is a normal lattice. We have already mentioned in Chapter 2 that the concept of normality in a bounded distributive lattice is self dual. So the concept of relative normality in a nearlattice is also self dual.

In section 1 of this chapter, we have given several characterizations of $< a, b>^n$. We have also given some characterizations of distributive and modular nearlattices in terms of relative n-annihilators.

In section 2, we have characterized those $P_n(S)$ which are relatively normal in terms of n-ideals and relative n-annihilators. These results are certainly generalizations of several results on relatively normal nearlattices given by [54]. At the end, we have shown that, for a central element n, $P_n(S)$ is relatively normal if and only if any two incomparable prime n-ideals of S are co-maximal.

3.1 Relative Annihilators around a central element of a Nearlattice.

We start with the following characterization of $< a, b >^n$.

Theorem 3.1.1. *Let* S *be a nearlattice with a central element* n. *Then for all* a, b∈S, *the following conditions are equivalent.*

(i) $<a, b>^n$ *is an* n-*ideal.*

(ii) $< a \wedge n, b \wedge n >_d$ *is a filter and* $< a \vee n, b \vee n >$ *is an ideal.*

Proof. (I)⇒(ii). Suppose (i) holds. Let $x, y \in < a \vee n, b \vee n >$ and $x \vee y$ exists. Then $x \wedge (a \vee n) \leq (b \vee n)$. Thus $(x \wedge (a \vee n)) \vee n \leq (b \vee n)$, then by the neutrality of n, $(x \vee n) \wedge (a \vee n) \leq (b \vee n)$.

Also $m(x \vee n, n, a) = (x \vee n) \wedge (a \vee n) \leq b \vee n$. This implies $x \vee n \in < a, b >^n$. Similarly, $y \vee n \in < a, b >^n$. Since $< a, b >^n$ is an n-ideal, so $x \vee y \vee n \in < a, b >^n$. This implies $m(x \vee y \vee n, n, a) \leq b \vee n$. That is, $(x \vee y \vee n) \wedge (a \vee n) \leq b \vee n$ and so $(x \vee y) \wedge (a \vee n) \leq b \vee n$. Therefore, $x \vee y \in < a \vee n, b \vee n >$.

Moreover, for $x \in < a \vee n, b \vee n >$ and $t \leq x$ ($t \in S$).

Obviously, $t \wedge (a \vee n) \leq b \vee n$, and so $t \in < a \vee n, b \vee n >$.

Hence $< a \vee n, b \vee n >$ is an ideal.

A dual proof of above shows that $< a \wedge n, b \wedge n >_d$ is a filter.

(ii)⇒(i). Suppose (ii) holds and $x, y \in < a, b >^n$.

Then $b \wedge n \leq (x \wedge a) \vee (x \wedge n) \vee (a \wedge n) \leq b \vee n$, and $b \wedge n \leq (y \wedge a) \vee (y \wedge n) \vee (a \wedge n) \leq b \vee n$. So, $b \vee n \leq [(x \wedge a) \vee (x \wedge n) \vee (a \wedge n)] \wedge n = (x \wedge n) \vee (a \wedge n)$. This implies $x \wedge n \in < a \wedge n, b \wedge n >_d$. Similarly,

$y \wedge n \in \langle a \wedge n, b \wedge n \rangle_d$. Since $\langle a \wedge n, b \wedge n \rangle_d$ is a filter, so we have, $x \wedge y \wedge n \in \langle a \wedge n, b \wedge n \rangle_d$. Thus, $(x \wedge y \wedge n) \vee (a \wedge n) \geq (b \wedge n)$.
But $m(x \wedge y \wedge n, n, a) = (x \wedge y \wedge n) \vee (a \wedge n) \geq (b \wedge n)$, and so $x \wedge y \wedge n \in \langle a, b \rangle^n$. Again, by neutrality of n, $(x \vee n) \wedge (a \vee n) = (x \wedge a) \vee n \leq (b \vee n)$. Similarly, $(y \vee n) \wedge (a \vee n) \leq (b \vee n)$.
Thus $((x \wedge y) \vee n) \wedge (a \vee n) \leq (b \vee n)$.
But $((x \wedge y) \vee n) \wedge (a \vee n) = m((x \wedge y) \vee n, n, a)$, as n is neutral.
There fore, $(x \wedge y) \vee n \in \langle a, b \rangle^n$ and so by the convexity of $\langle a, b \rangle^n$, $x \wedge y \in \langle a, b \rangle^n$.
A dual proof of above also that $x \vee y \in \langle a, b \rangle^n$. Clearly, $\langle a, b \rangle^n$ contains n. Therefore, $\langle a, b \rangle^n$ is an n –ideal. □

Proposition 3.1.2. *Let* S *be a nearlattice with a central element* n. *Then for all* a, b∈S, *the following conditions hold.*

(i) $\langle a \vee n, b \vee n \rangle$ *is an ideal if and only if* [n) *is a distributive subnearlattice of* S.

(ii) $\langle a \wedge n, b \wedge n \rangle_d$ *is a filter if and only if* (n] *is a distributive sublattice of* S.

Proof. Suppose for all a, b∈S, $\langle a \vee n, b \vee n \rangle$ is an ideal. Thus for all p, q∈[n), $\langle p, q \rangle \cap [n)$ is an ideal in the subnearlattice [n).Then by [39], [n) is distributive.
Conversely, suppose [n) is distributive. Let $x, y \in \langle a \vee n, b \vee n \rangle$ and $x \vee y$ exists. Then $x \wedge (a \vee n) \leq b \vee n$. Since n is neutral, so $(x \vee n) \wedge (a \vee n) = [x \wedge (a \vee n)] \vee n \leq b \vee n$ implies that $x \vee n \in \langle a \vee n, b \vee n \rangle$.
Similarly, $y \vee n \in \langle a \vee n, b \vee n \rangle$. Then $(x \vee y) \wedge (a \vee n)$
$\leq (x \vee y \vee n) \wedge (a \vee n) = [(x \vee n) \wedge (a \vee n)] \vee [(y \vee n) \wedge (a \vee n)]$
as [n) is distributive.

$\leq (b \vee n)$.

Therefore, $x \vee y \in \langle a \vee n, b \vee n \rangle$. Since $\langle a \vee n, b \vee n \rangle$ has always the hereditary property, so $\langle a \vee n, b \vee n \rangle$ is an ideal.

(ii) can be proved dually. □

By Theorem 3.1.1. and above result and using Theorem 1.5.2, we have the following result.

Theorem 3.1.3. *Let* S *be a nearlattice with a central element* n. *Then for all* $a, b \in S$, $\langle a, b \rangle^n$ is an n-ideal if and only if $P_n(S)$ is distributive nearlattice. □

Recall that a nearlattice S is distributive if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ provided $y \vee z$ exists.[48] has given an alternative definition of distributivity of S. A nearlattice S is distributive if and only if for all $t, x, y, z \in S$, $t \wedge ((x \wedge y) \vee (x \wedge z)) = (t \wedge x \wedge y) \vee (t \wedge x \wedge z)$.

Similarly, by [51], a nearlattice S is modular if and only if for all $t, x, y, z \in S$ with $z \leq x$, $x \wedge ((t \wedge y) \vee (t \wedge z)) = (x \wedge t \wedge y) \vee (t \wedge z)$.

Since for a sesquimedial element n, S is distributive if and only if $P_n(S)$ is distributive, we have the following Corollary, which is a generalization of [39, Theorem 1] and a result of [46]. This also generalizes a result of [4, theorem 3.1.3.].

Corollary 3.1.4. *Suppose* S *is a nearlattice. Then for a central element* $n \in S$, $\langle a, b \rangle^n$ *is an* n-*ideal for all* $a, b \in S$ *if and only if* S *is distributive.* □

[39] gave a characterization of distributive lattices in terms of relative annihilators. Then [51] extended the result for nearlattices. [48] generalized the

result for n-ideals in lattices. Following result gives a generalization of that result for n-ideals in nearlattices.

Theorem 3.1.5. *Let* n *be a central element of a nearlattice* S. *Then the following conditions are equivalent.*

(i) S *is distributive.*

(ii) $< a \vee n, b \vee n >$ *is an ideal and* $< a \wedge n, b \wedge n >_d$ *is a filter whenever* $< b >_n \subseteq < a >_n$.

Proof. (i)⇒(ii). Suppose (i) holds. That is, S is distributive. Then by Corollary 3.1.4, $< a, b >^n$ is an n-ideal for all $a, b \in S$. Thus by Theorem 3.1.1, (ii) holds.

(ii)⇒(i). Suppose (ii) holds and let $x, y, z \in [n)$ and $y \vee z$ exists.

Clearly, $(x \wedge y) \vee (x \wedge z) \leq x$. So, $< x, (x \wedge y) \vee (x \wedge z) >$ is an ideal as $< (x \wedge y) \vee (x \wedge z) >_n \subseteq < x >_n$. Since $x \wedge y \leq (x \wedge y) \vee (x \wedge z)$, so $y \in < x, (x \wedge y) \vee (x \wedge z) >$. Similarly, $z \in < x, (x \wedge y) \vee (x \wedge z) >$.

Hence $y \vee z \in < x, (x \wedge y) \vee (x \wedge z) >$ and so $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. This implies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and so [n) is distributive. Using the other part of (ii) we can similarly show that (n] is also distributive. Thus by theorem 1.5.2, $P_n(S)$ is distributive and so S is distributive. □

Theorem 3.1.6. *Let* n *be a central element of a nearlattice* S. *Then the following conditions are equivalent.*

(i) $P_n(S)$ *is modular.*

(ii) *For* $a, b \in S$ *with* $< b >_n \subseteq < a >_n$, $x \in < b >_n$ and $y \in < a, b >^n$ *imply* $x \wedge y$, $x \vee y \in < a, b >^n$ *if* $x \vee y$ *exists in* S.

Proof. (i)⇒(ii). Suppose $P_n(S)$ is modular. Then by Theorem 1.5.2, [n) and (n] are modular. Here $< b >_n \subseteq < a >_n$, so $a \wedge n \leq b \wedge n \leq n \leq b \vee n \leq a \vee n$.

Since $x \in < b >_n$, so $b \wedge n \leq x \leq b \vee n$.

Hence $a \wedge n \leq b \wedge n \leq x \wedge n \leq x \vee n \leq b \vee n \leq a \vee n$.

Now, $y \in \langle a, b \rangle^n$ implies $m(y, n, a) \in \langle b \rangle_n$.

Thus, $(y \wedge a) \vee (y \wedge n) \vee (a \wedge n) \leq b \vee n$, and so by the neutrality of n,

$((y \wedge a) \vee (y \wedge n) \vee (a \wedge n)) \vee n = (y \vee n) \wedge (a \vee n) \leq b \vee n$.

Thus, using the modularity of [n) and the existence of $x \vee y$,

$$m(x \vee y \vee n, n, a) = (x \vee y \vee n) \wedge (a \vee n)$$
$$= [(a \vee n) \wedge (y \vee n)] \vee (x \vee n) \text{ as } x \vee n \leq b \vee n \leq a \vee n.$$

This implies $m(x \vee y \vee n, n, a) \leq b \vee n$ and so $x \vee y \vee n \in \langle a, b \rangle^n$. Since n is neutral, so $a \wedge n \leq b \wedge n \leq x \wedge n$ implies that

$$b \wedge n \leq (x \wedge n) \vee (y \wedge n) \vee (a \wedge n)$$
$$= ((x \vee y) \wedge n) \vee (a \wedge n)$$
$$= m((x \vee y) \wedge n, n, a)$$
$$\leq b \vee n.$$

Therefore, $(x \vee y) \wedge n \in \langle a, b \rangle^n$. Hence by convexity of $\langle a, b \rangle^n$, $x \vee y \in \langle a, b \rangle^n$.

Again, using the modularity of (n], a dual proof of above shows that $x \wedge y \in \langle a, b \rangle^n$. Hence (ii) holds.

(ii)⇒(i). Suppose (ii) holds. Let $x, y, z \in [n)$ with $x \leq z$ and whenever $x \vee y$ exists. Then $x \vee (y \wedge z) \leq z$. This implies $\langle x \vee (y \wedge z) \rangle_n \subseteq \langle z \rangle_n$.

Now, $x \leq x \vee (y \wedge z)$ implies $x \in \langle x \vee (y \wedge z) \rangle_n$.

Again, $y \wedge z \leq x \vee (y \wedge z)$ implies $m(y, n, z) = y \wedge z \in \langle x \vee (y \wedge z) \rangle_n$.

Hence $y \in \langle z, x \vee (y \wedge z) \rangle^n$. Thus by (ii), $x \vee y \in \langle z, x \vee (y \wedge z) \rangle^n$. That is, $(x \vee y) \wedge z \leq x \vee (y \wedge z)$ and so $(x \vee y) \wedge z = x \vee (y \wedge z)$. Therefore, [n) is modular.

Similarly, using the condition (ii) we can easily show that (n] is also modular. Hence by Theorem 1.5.2, $P_n(S)$ is modular. □

We conclude the section with the following characterization of minimal prime n-ideals belonging to an n-ideal. Since the proof of this is almost similar to Theorem 2.1.4, we omit the proof.

Theorem 3.1.7. *Let* S *be a distributive nearlattice and* P *be a prime* n-*ideal of* S *belonging to an* n-*ideal* J. *Then the following conditions are equivalent.*

(i) P *is minimal prime* n-*ideal belonging to* J.

(ii) $x \in P$ *implies* $<<x>_n, J> \not\subseteq P$. □

3.2 Some characterizations of those $P_n(S)$ which are Relatively Normal Nearlattices.

Recall that a distributive nearlattice S is relatively normal if each interval $[x, y]$ in S ($x, y \in S$) $x < y$, is normal.

When n is a sesquimedial element of a distributive nearlattice S, then $P_n(S)$ is also a distributive nearlattice. Thus, $P_n(S)$ is a relatively normal nearlattice if each interval $[<a>_n, <b>_n]$ in $P_n(S)$ is normal.

The following result will be needed for the further development of this chapter, which is due to [59].

Theorem 3.2.1. *Let* S *be a distributive nearlattice with an upper element* n. *Then the following conditions hold.*

(i) $<<x>_n \vee <y>_n, <x>_n> = <<y>_n, <x>_n>$.

(ii) $<<x>_n, J> = \vee_{y \in J} <<x>_n, <y>_n>$, *the supremum of* n-*ideals* $<<x>_n, <y>_n>$ *in the lattice of* n-*ideals of* S, *for any* $x \in S$ *and any* n-*ideal* J. □

Following lemma is dual of [11, Lemma 3.6] and is very easy to prove. So we prefer to omit the proof.

Lemma 3.2.2. *Let* L *be distributive lattice. Then the following conditions hold.*

(i) $<x \wedge y, x>_d = <y, x>_d$.

(ii) $<[x), F>_d = \vee_{y \in F} <x, y>_d$, *where* F *is a filter of* L.

(iii) $\{<x, a>_d \vee <y, a>_d\} \cap [a, b] = \{<x, a>_d \cap [a, b]\} \vee \{<y, a>_d \cap [a, b]\}$, *where* $[a, b]$ *represents any interval in* L. □

Lemma 3.2.3. and Lemma 3.2.4. are essential for the proof of our main result of this section, which are also due to [59].

Lemma 3.2.3. *Let* S *be a distributive nearlattice with an upper element* n. *Suppose* a, b, c$\in$S.

(i) *If* a,b, c $\geq$ n, *then* $<<m(a, n, b)>_n, <c>_n> = <<a>_n, <c>_n> \vee <<b>_n, <c>_n>$ *is equivalent to* $<a \wedge b, c> = <a, c> \vee <b, c>$.

(ii) *If* a, b, c $\leq$ n, *then* $<<m(a, n, b)>_n, <c>_n> = <<a>_n, <c>_n> \vee <<b>_n, <c>_n>$ *is equivalent to* $<a \vee b, c>_d = <a, c>_d \vee <b, c>_d$. □

Lemma 3.2.4. *Let* S *be a distributive nearlattice with an upper element* n. *Suppose* a, b, c$\in$S.

(i) *If* a, b, c $\geq$ n *and* $a \vee b$ *exists, then*
$<<c>_n, <a>_n \vee <b>_n> = <<c>_n, <a>_n> \vee <<c>_n, <b>_n>$ *is equivalent to* $<c, a \vee b> = <c, a> \vee <c, b>$.

(ii) *If* a, b, c $\leq$ n, *then* $<<c>_n, <a>_n \vee <b>_n> = <<c>_n, <a>_n> \vee <<c>_n, <b>_n>$ *is equivalent to* $<c, a \wedge b>_d = <c, a>_d \vee <c, b>_d$. □

The following result is due to [59], which a generalization of [11, Lemma-3.6]. This plays an important role in proving our main result in this section.

Theorem 3.2.5. *Let* S *be a distributive nearlattice. Then the following conditions hold.*

(i) $<<x>_n \vee <y>_n, <x>_n> = <<y>_n, <x>_n>$.

(ii) $<<x>_n, J> = \vee_{y \in J} <<x>_n, <y>_n>$, *the supremum of* n-*ideals* $<<x>_n, <y>_n>$ *in the lattice of* n-*ideals of* L, *for any* $x \in L$ *and any* n-*ideal* J.

Following lemma will be needed for further development of this chapter. This is in fact, the dual of [11, Lemma-3.6] and very easy to proof. So we prefer to omit the proof.

Lemma 3.2.6. *Let* L *be a distributive lattice. Then the following hold.*

(i) $<x \wedge y, x>_d = <y, x>_d$.

(ii) $<[x), F>_d = \vee_{y \in F} <x, y>_d$, *where* F *is filter of* L.

(iii) $\{<x, a>_d \vee <y, a>_d\} \cap [a, b] = \{<x, a>_d \cap [a, b]\} \vee \{<y, a>_d \cap [a, b]\}$. □

Recall that a distributive lattice L with 1 is a dual normal lattice if L^d is a normal lattice. In other words, a distributive lattice L with 1 is called dual normal if every prime filter of L is contained in a unique ultra filter (maximal and proper) of L. As we mentioned earlier that this condition in a lattice is self dual. Thus for a bounded distributive lattice, the concept of normality and dual normality coincides.

Following technique of the proof of [11, Theorem 2.4], we can similarly prove the following result, which gives some characterization of dual normal lattices. These results are in fact, the dual result of Theorem 2.2.3.

Theorem 3.2.7. *Let* L *be a distributive lattice with* 1. *Then the following conditions are equivalent.*

(i) L *is dual normal.*

(ii) *Each prime filter of* L *is contained in a unique ultra-filter (maximal and proper).*

(iii) *For each* x, y∈L, $[x \vee y)^{*d} = [x)^{*d} \vee [y)^{*d}$.

(iv) *If* $x \vee y = 1$, x, y∈L, *then* $[x)^{*d} \vee [y)^{*d} = L$. □

Corollary 3.2.8. *Let* L *be a bounded distributive lattice. Then the following conditions are equivalent.*

(i) L *is normal.*

(ii) *For each* x, y∈L, $(x \wedge y]^{*} = (x]^{*} \vee (y]^{*}$.

(iii) *If* $x \wedge y = 0$, *then* $(x]^{*} \vee (y]^{*} = L$.

(iv) *For each* x, y∈L, $[x \vee y)^{*d} = [x)^{*d} \vee [y)^{*d}$.

(v) *If* $x \vee y = 1$, *then* $[x)^{*d} \vee [y)^{*d} = L$. □

Ayub in [4, Theorem 3.2.7.] has given a nice characterization of relatively normal lattices in terms of dual relative annihilators, which is in fact, the dual of [11, Theorem 3.7]. As we have mentioned earlier that in nearlattices the idea of dual relative annihilators is not always possible. But when n is an upper element in S then $x \vee n$ exists for all x∈S. Thus for any a ∈ (n], $x \vee a$ exists for x∈S. Hence we can define $< a, b >_d$ for all a∈(n] and b∈S.

When n is a central element in S, then by Theorem 1.5.2,

$P_n(S) \cong (n]^d \times [n)$.

Thus we have the following result.

Proposition 3.2.9. *For a distributive nearlattice* S *with a central* n, $P_n(S)$ *is relatively normal if and only if* (n] *and* [n) *are relatively normal.* □

Now we prove the following important result.

Theorem 3.2.10. *Let* n *be a central element of a distributive nearlattice* S *such that* (n] *is relatively normal. Let* a, b, c∈(n] *be arbitrary elements and* A, B *be arbitrary filters on* (n]. *Then the following conditions are equivalent.*

(i) (n] *is relatively normal.*

(ii) $< a, b >_d \vee < b, a >_d = (n]$.

(iii) $< c, a \wedge b >_d = < c, a >_d \vee < c, b >_d$.

(iv) $< [c), A \vee B >_d = < [c), A >_d \vee < [c), B >_d$.

(v) $< a \vee b, c >_d = < a, c >_d \vee < b, c >_d$.

Proof. (i)⇒(ii). Suppose (i) holds. Let $z \in (n]$ be arbitrary. Consider the interval $I = [z, a \vee b \vee z]$. Then $a \vee b \vee z$ is the largest element of I. Since by (i), I is normal, then by Theorem [11,Theorem2.4], there exists $r, s \in I$ such that $a \vee s = a \vee b \vee z = b \vee z \vee r$ and $z = s \wedge r$. Now, $a \vee s \geq b$ implies $s \in < a, b >_d$ and $b \vee r = b \vee z \vee r = a \vee b \vee z \geq a$ implies $r \in < b, a >_d$. Hence (ii) holds.

(ii)⇒(iii). Suppose (ii) holds. In (iii), R.H.S. ⊆ L.H.S. is obvious.

Let $z \in < c, a \wedge b >_d$, then $z \vee c \geq a \wedge b$. Since (ii) holds, so $z = x \wedge y$ where $x \in < a, b >_d$ and $y \in < b, a >_d$. Then $x \vee a \geq b$ and $y \vee b \geq a$.

Thus, $x \vee c = x \vee z \vee c$

$\geq x \vee (a \wedge b)$

$= (x \vee a) \wedge (x \vee b) \geq b$, which implies $x \in < c, b >_d$.

Similarly, $y \in < c, a >_d$. Hence $z = x \wedge y \in < c, a >_d \vee < c, b >_d$ and so $< c, a \wedge b >_d \subseteq < c, a >_d \vee < c, b >_d$. Thus (iii) holds.

(iii)⇒(iv) follows from Lemma 3.2.2 (ii).

(iv)⇒(iii) is trivial.

(iii)⇒(ii) follows from Lemma 3.2.2 (i) by putting $c = a \wedge b$.

(ii)⇒(v). Suppose (ii) holds. Let $z \in \langle a \vee b, c \rangle_d$. Then by (ii), $z = x \wedge y$, where $x \vee a \geq b$ and $y \vee b \geq a$. Also $x \vee a = x \vee a \vee b \geq z \vee a \vee b \geq c$. This implies $x \in \langle a, c \rangle_d$. Similarly, $y \in \langle b, c \rangle_d$.
Hence $z = x \wedge y \in \langle a, c \rangle_d \vee \langle b, c \rangle_d$ and
so $\langle a \vee b, c \rangle_d \subseteq \langle a, c \rangle_d \vee \langle b, c \rangle_d$.
Since the reverse inequality is obvious, so (v) holds.

(v)⇒(i). Consider an interval [a, b] in (n]. For $x \in [a, b]$, $a < b$, let $[x]^{od} = \{y \in [a, b]: y \vee x = b\}$. Clearly $[x]^{od} = \langle x, a \rangle_d \cap [a, b]$.
Then Lemma 3.2.6. for any $x, y \in [a, b]$, we have,
$[\langle x, a \rangle_d \vee \langle y, a \rangle_d] \cap [a, b] = (\langle x, a \rangle_d \cap [a, b]) \vee (\langle y, a \rangle_d \cap [a, b])$.
Then by (v), $\langle x \vee y, a \rangle_d \cap [a, b] = [x]^{od} \vee [y]^{od}$, which implies $[x \vee y]^{od} = [x]^{od} \vee [y]^{od}$. Therefore, by Corollary 3.2.8, [a, b] is normal.
Therefore (n] is relatively normal. □

Now we prove our main results of this chapter, which are generalizations of [11, Theorem 3.7], [39, Theorem 5] and a result of [17], also see [51]. These give characterizations of those $P_n(S)$ which are relatively normal.

Theorem 3.2.11. *Let* n *be a central element of a distributive nearlattice. Suppose* A, B *are two* n*-ideals of* S. *Then for all* a, b, $c \in S$ *the following conditions are equivalent.*

(i) $P_n(S)$ *is relatively normal.*

(ii) $\langle\langle a \rangle_n, \langle b \rangle_n\rangle \vee \langle\langle b \rangle_n, \langle a \rangle_n\rangle = S$.

(iii) $\langle\langle c \rangle_n, \langle a \rangle_n \vee \langle b \rangle_n\rangle = \langle\langle c \rangle_n, \langle a \rangle_n\rangle \vee \langle\langle c \rangle_n, \langle b \rangle_n\rangle$, *whenever* $a \vee b$ *exists.*

(iv) $<<c>_n, A \vee B> = <<c>_n, A> \vee <<c>_n, B>$.

(v) $<<m(a, n, b)>_n, <c>_n> = <<a>_n, <c>_n> \vee$
$<<b>_n, <c>_n>$.

Proof. **(i)⇒(ii).** Let $z \in S$. Consider the interval $I = [<a>_n \cap <b>_n \cap <z>_n, <z>_n]$ in $P_n(S)$. Then $<a>_n \cap <b>_n \cap <z>_n$ is the smallest element of the interval I. By (i), I is normal. Then by Theorem 5.2.5, there exist principal n-ideals $<p>_n, <q>_n \in I$ such that, $<a>_n \cap <z>_n \cap <p>_n = <a>_n \cap <b>_n \cap <z>_n = <b>_n \cap <z>_n \cap <q>_n$ and
$<z>_n = <p>_n \vee <q>_n$. Now, $<a>_n \cap <p>_n = <a>_n \cap <p>_n \cap <z>_n$

$= <a>_n \cap <b>_n \cap <z>_n \subseteq <b>_n$

implies $<p>_n \subseteq <<a>_n, <b>_n>$.

Also, $<b>_n \cap <q>_n = <b>_n \cap <z>_n \cap <q>_n$

$= <a>_n \cap <b>_n \cap <z>_n \subseteq <a>_n$.

implies $<q>_n \subseteq <<b>_n, <a>_n>$

Thus $<z>_n \subseteq <<a>_n, <b>_n> \vee <<b>_n, <a>_n>$ and
so $z \in <<a>_n, <b>_n> \vee <<b>_n, <a>_n>$.

Hence $<<a>_n, <b>_n> \vee <<b>_n, <a>_n> = S$.

(ii)⇒(iii). Suppose (ii) holds and $a \vee b$ exists. For (iii), R.H.S. $\subseteq$ L.H.S. is obvious. Now, let $z \in <<c>_n, <a>_n \vee <b>_n>$.

Then $z \vee n \in <<c>_n, <a>_n \vee <b>_n>$ and

$m(z \vee n, n, c) \in <a>_n \vee <b>_n$.

That is, $m(z \vee n, n, c) \in [a \wedge b \wedge n, a \vee b \vee n]$.

This implies $(z \vee n) \wedge (c \vee n) \leq a \vee b \vee n$.

Now, by (ii), $z \vee n \in <<a>_n, <b>_n> \vee <<b>_n, <a>_n>$.

So $z \vee n \leq (p \vee n) \vee (q \vee n)$ for some $p \vee n \in <<a>_n, <b>_n>$ and
$q \vee n \in <<b>_n, <a>_n>$.

Hence, $z \vee n = ((z \vee n) \wedge (p \vee n)) \vee ((z \vee n) \wedge (q \vee n)) = r \vee t$ (say).

Now, $m(p \vee n, n, a) = (p \vee n) \wedge (a \vee n) \leq (b \vee n)$.

So $b \wedge n \leq r \wedge (a \vee n) \leq b \vee n$. Hence, $r \wedge (c \vee n) = r \wedge (z \vee n) \wedge (c \vee n)$

$$\leq r \wedge (a \vee b \vee n)$$

$$= (r \wedge (a \vee n)) \vee (r \wedge (b \vee n))$$

$$\leq (b \vee n).$$

This implies $r \in << c >_n, < b >_n >$. Similarly, $t \in << c >_n, < a >_n >$.

Hence $z \vee n \in << c >_n, < a >_n > \vee << c >_n, < b >_n >$.

Again, $z \in << c >_n, < a >_n \vee < b >_n >$ implies

$z \wedge n \in << c >_n, < a >_n \vee < b >_n >$. Then a dual calculation of above shows that

$z \wedge n \in << c >_n, < a >_n > \vee << c >_n, < b >_n >$. Thus by convexity,

$z \in << c >_n, < a >_n > \vee << c >_n, < b >_n >$ and so L.H.S. $\subseteq$ R.H.S. Hence (iii) holds.

(iii)⇒(iv). Suppose (iii) holds. In (iv), R.H.S. $\subseteq$ L.H.S. is obvious.

Now let $x \in << c >_n, A \vee B >$. Then $x \vee n \in << c >_n, A \vee B >$.

Thus $m(x \vee n, n, c) \in A \vee B$. Now $m(x \vee n, n, c) = (x \vee n) \wedge (n \vee c) \geq n$ implies $m(x \vee n, n, c) \in (A \vee B) \cap [n)$. Hence by Theorem 3.2.1(ii),

$x \vee n \in << c >_n, (A \cap [n)) \vee (B \cap [n)) >$

$= \vee_{r \in (A \cap [n)) \vee (B \cap [n))} << c >_n, < r >_n >$. But by Theorem 1.3.7,

$r \in (A \cap [n)) \vee (B \cap [n))$ implies $r = s \vee t$ for some $s \in A$, $t \in B$ and

$s, t \geq n$. Then by (iii), $<< c >_n, < r >_n > = << c >_n, < s \vee t >_n >$

$$= << c >_n, < s >_n \vee < t >_n >$$

$$= << c >_n, < s >_n > \vee << c >_n, < t >_n >$$

$$\subseteq << c >_n, A > \vee << c >_n, B >$$

Hence $x \vee n \in << c >_n, A > \vee << c >_n, B >$. Also $x \in << c >_n, A \vee B >$ implies

$x \wedge n \in << c >_n, A \vee B >$.

Since $m(x \wedge n, n, c) = (x \wedge n) \vee (x \wedge c) \leq n$,

so $x \wedge n \in << c >_n, (A \vee B) \cap (n] >$.

Then, by Theorem 3.2.1(ii),

$$x \wedge n \in \langle \langle c \rangle_n, (A \cap (n]) \vee (B \cap (n]) \rangle$$

$$= \vee_{l \in (A \cap (n]) \vee (B \cap (n])} \langle \langle c \rangle_n, \langle l \rangle_n.$$

Again, using Theorem 1.3.7, we see that $l = p \wedge q$ where $p \in A$, $q \in B$ and $p, q \leq n$. Then by (iii),

$$\langle \langle c \rangle_n, \langle l \rangle_n \rangle = \langle \langle c \rangle_n, \langle p \wedge q \rangle_n \rangle$$

$$= \langle \langle c \rangle_n, \langle p \rangle_n \vee \langle q \rangle_n \rangle$$

$$= \langle \langle c \rangle_n, \langle p \rangle_n \rangle \vee \langle \langle c \rangle_n, \langle q \rangle_n \rangle$$

$$\subseteq \langle \langle c \rangle_n, A \rangle \vee \langle \langle c \rangle_n, B \rangle$$

Hence $x \wedge n \in \langle \langle c \rangle_n, A \rangle \vee \langle \langle c \rangle_n, B \rangle$. Therefore, by convexity, $x \in \langle \langle c \rangle_n, A \rangle \vee \langle \langle c \rangle_n, B \rangle$ and so L.H.S. $\subseteq$ R.H.S. Thus (iv) holds.

(iv)⇒(iii) is trivial.

(ii)⇒(v). Suppose (ii) holds. In (v), R.H.S. $\subseteq$ L.H.S. is obvious.

Now let $z \in \langle \langle m(a, n, b) \rangle_n, \langle c \rangle_n \rangle$ which implies $z \vee n \in \langle \langle m(a, n, b) \rangle_n, \langle c \rangle_n \rangle$.

By (ii), $z \vee n \in \langle \langle a \rangle_n, \langle b \rangle_n \rangle \vee \langle \langle b \rangle_n, \langle a \rangle_n \rangle$. Then by Theorem 1.3.7, $z \vee n = x \vee y$ for some $x \in \langle \langle a \rangle_n, \langle b \rangle_n \rangle$ and $y \in \langle \langle b \rangle_n, \langle a \rangle_n \rangle$ and $x, y \geq n$. Thus, $\langle x \rangle_n \cap \langle a \rangle_n \subseteq \langle b \rangle_n$ and so $\langle x \rangle_n \cap \langle a \rangle_n = \langle x \rangle_n \cap \langle a \rangle_n \cap \langle b \rangle_n \subseteq \langle z \vee n \rangle_n \cap \langle a \rangle_n \cap \langle b \rangle_n$

$$= \langle z \vee n \rangle_n \cap \langle m(a, n, b) \rangle_n$$

$$\subseteq \langle c \rangle_n.$$

This implies $x \in \langle \langle a \rangle_n, \langle c \rangle_n \rangle$. Similarly, $y \in \langle \langle b \rangle_n, \langle c \rangle_n \rangle$ and so $z \vee n \in \langle \langle a \rangle_n, \langle c \rangle_n \rangle \vee \langle \langle b \rangle_n, \langle c \rangle_n \rangle$. Similarly, a dual calculation above shows that $z \wedge n \in \langle \langle a \rangle_n, \langle c \rangle_n \rangle \vee \langle \langle b \rangle_n, \langle c \rangle_n \rangle$. Thus by convexity, $z \in \langle \langle a \rangle_n, \langle c \rangle_n \rangle \vee \langle \langle b \rangle_n, \langle c \rangle_n \rangle$ and so L.H.S. $\subseteq$ R.H.S. Hence (v) holds.

(v)⇒(i). Suppose (v) holds. Let $a, b, c \geq n$.

By (v), $<<m(a, n, b)>_n, <c>_n> = <<a>_n, <c>_n> \vee <<b>_n, <c>_n>$. But by Lemma 3.2.3(i), this is equivalent to $<a \wedge b, c> = <a, c> \vee <b,c>$. Then by [56, Theorem 3.3.5], this shows that [n) is relatively normal. Similarly, for a, b, c $\leq$ n, using Lemma 3.2.3(ii) and Theorem 3.2.10, we find that (n] is relatively normal. Therefore by Theorem 1.5.2, $P_n(S)$ is relatively normal.

Finally we need to prove that **(iii)⇒(i).**

Suppose (iii) holds. Let a, b, c$\in S \cap$ [n).

By (iii), $<<c>_n, <a>_n \vee <b>_n> = <<c>_n, <a>_n \vee <<c>_n, <b>_n$, when ever a $\vee$ b exists.

But by Lemma 3.2.4(i), this is equivalent to

$$<c, a \vee b> = <c, a> \vee <c, b>.$$

Then by [11, Theorem 3.7], this shows that [n) is relatively normal.

Similarly, for a, b,c $\leq$ n, using the Lemma 3.2.4(ii) and Theorem 3.2.10, we find that (n] is relatively normal. Therefore by Theorem 1.5.2, $P_n(S)$ is relatively normal. □

By [11], [39], and [17] we know that a lattice is relatively normal if and only if any two incomparable prime ideals are co-maximal. [54] extended this result for nearlattices. We conclude this chapter by proving the following result, which is a generalization of [56, Theorem 3.3.10].

Theorem 3.2.12. *Let* S *be a distributive nearlattice. If* n *is central in* S, *then the following conditions are equivalent.*

(i) $P_n(S)$ *is relatively normal.*

(ii) *Any two incomparable prime* n*-ideals* P *and* Q *are co-maximal*, i.e. $P \vee Q = S$.

Proof. (i)⇒(ii). Suppose (i) holds. Let P and Q be two incomparable prime n-ideals of S. Then there exist a, $b \in S$ such that $a \in P\text{-}Q$ and $b \in Q\text{-}P$.

Then $< a >_n \subseteq P\text{-}Q$ and $< b >_n \subseteq Q\text{-}P$. Since by (i), $P_n(S)$ is relatively normal, so by Theorem 5.2.8,

$<< a >_n, < b >_n > \vee << b >_n, < a >_n > = S$.

But as P, Q are prime, so it is easy to see that $<< a >_n, < b >_n > \subseteq Q$ and $<< b >_n, < a >_n > \subseteq P$. Therefore, $S \subseteq P \vee Q$ and so $P \vee Q = S$. Thus (ii) holds.

(ii)⇒(i). Suppose (ii) holds. Let P_1 and Q1 be two incomparable prime ideals of [n). Then by Lemma 1.5.5, there exist two incomparable prime ideals P and Q of S such that $P_1 = P \cap [n)$ and $Q_1 = Q \cap [n)$. Since $n \in P_1$ and $n \in Q_1$, so by Lemma 1.4.3, P and Q are in fact two incomparable prime n-ideals of S. Then by (ii), $P \vee Q = S$.

Therefore, $P_1 \vee Q_1 = (P \vee Q) \cap [n)$

$$= S \cap [n)$$

$$= [n).$$

Thus by [56, Theorem 3.3.10], [n) is relatively normal.

Similarly, considering two prime filters of (n] and proceeding as above and using the dual result of [56, Theorem 3.3.10] we find that (n] is relatively normal. Therefore, by Theorem 1.5.2, $P_n(S)$ is relatively normal. □

CHAPTER 4

CHAPTER 4

CHARACTERIZATION OF THOSE $P_n(S)$ WHICH FORM m-NORMAL AND RELATIVELY m-NORMAL NEARLATTICES

Introduction.

Lee in [37] also see Lakser [34] has determined the lattice of all equational subclasses of the class of all pseudo-complemented distributive lattices. They are given by $B_{-1} \subset B_0 \subset \ldots. \subset B_m \subset \ldots. \subset B\omega$,where all the inclusions are proper and $B\omega$ is the class of all pseudo-complemented distributive lattices, B_{-1} consists of all one element algebra, B_0 is the variety of Boolean algebras while B_m, for $-1 \leq m < \omega$ consists of all algebras satisfying the equation $(x_1 \wedge x_2 \wedge \ldots. \wedge x_m)^* \vee \bigvee_{i=1}^{n} (x_1 \wedge x_2 \wedge \ldots.\wedge x_{i-1} \wedge x_i^* \wedge x_{i+1} \wedge \ldots.. \wedge x_m)^* = 1$ where x^* denotes the pseudo-complemented of x. Thus B_1 consists of all Stone algebras.

He also generalized Grätzer and Schmidt's theorem by proving that for $-1 \leq m < \omega$ the mth variety consists of all lattices such that each prime ideal contains at most m minimal prime ideals.

Cornish in [11] and Mandelker in [39] have studied distributive lattices analogues to B_1-lattices and relatively B_1-lattices. Cornish [13], Beazer [7] and Davey [17] have each independently given several characterizations of (sectionally) B_m and relatively B_m-lattices. Moreover, Grätzer and Lakser in [23] and [24] have obtained some results on this topic.

Cornish in [13] have studied distributive lattices (without pseudo-complementation) analogues to B_m-lattices and relatively B_m-lattices. These are known as m-normal and relatively m-normal lattices. Then [42] and [56] extended the concept for nearlattices.

By [13], a distributive nearlattice S with 0 is called m-*normal* if each prime ideal of L contains at most m-minimal prime ideal.

A distributive nearlattice S is called *relatively* m-*normal* if each interval $[x, y]$, $x, y \in S$ is m-normal.

In section 1 we will study principal n-ideals which form a (sectionally) m-normal nearlattice. We will include several characterizations which generalize several results of [13], [17], [42] and [56].We shall show that for an element $n \in S$, $P_n(S)$ is m-normal if and only if for any $x_1, x_2, \ldots, x_m \in S$, $< x_o >_n^* \vee \ldots.. \vee < x_m >_n^* = S$ which is also equivalent to the condition that for any m+1 distinct minimal prime n-ideals $P_o, \ldots, P_m$ of S, $P_o \vee \ldots \vee P_m = S$.

In section 2 we will study those $P_n(S)$ which are relatively in m-normal. Here we will include a number of characterizations of those $P_n(S)$ which are m-normal nearlattices and these are generalizations of results of [13], [17] and[56]. We shall show that for a central element n, $P_n(S)$ is relatively m-normal if and only if any m+1 pairwise incomparable prime n-ideals are co-maximal.

4.1 Nearlattices whose $P_n(S)$ form m-Normal Nearlattices.

The following result is due to [13, Lemma 2.2]. This follows from the corresponding result for commutative semi-groups due to Kist [33].This is also true in case of a distributive nearlattice.

Lemma 4.1.1. *Let* M *be a prime ideal containing an ideal* J *in a distributive medial nearlattice. Then* M *is a minimal prime ideal belonging to* J *if and only if for all* $x \in M$, *there exists* $x' \notin M$ *such that* $x \wedge x' \in J$. □

Now we generalize this result for n-ideals.

Lemma 4.1.2. *Let* n *be a medial element and* M *be a prime* n*-ideal containing an* n*-ideal* J. *Then* M *is a minimal prime* n*-ideal belonging to* J *if and only if for all* $x \in M$ *there exists* $x' \notin M$ *such that* $m(x, n, x') \in J$.

Proof. Let M be a minimal prime n-ideal belonging to J and $x \in M$. Then by Theorem 3.1.8, $<< a >_n, J > \not\subset M$. So there exists x' with $m(x, n, x') \in J$ such that $x' \notin M$.

Conversely, suppose $x \in M$, then there exists $x' \notin M$ such that $m(x, n, x') \in J$. This implies $x' \notin M$, but $x' \in << x >_n, J >$, that is $<< x >_n, J > \not\subset M$. Hence by Theorem 3.1.7, M is a prime n-ideal belonging to J. □

Davey in [17, Corollary 2.3] used the following result in proving several equivalent conditions on B_m-lattices. On the other hand, Cornish in [13] has used this result in studing n-normal lattices. [56] generalized the result for nearlattices.

Proposition 4.1.3. *Let* $M_o,\dots, M_n$ *be* $n+1$ *distinct minimal prime ideals of a distributive nearlattice* S. *Then there exists* $a_o, a_1,\dots, a_n \in S$ *such that* $a_i \wedge a_j \in J\ (i \neq j)$ *and* $a_j \notin M_j,\ j = 0, 1,\dots, n.$ □

The following result is a generalization of above result in terms of n-ideals.

Proposition 4.1.4. *Let* S *be a distributive nearlattice and* $n \in S$ *is medial. Suppose* $M_o,\dots, M_m$ *be* $m+1$ *distinct minimal prime* n*-ideals containing* n*-ideal* J. *Then there exists* $a_o, a_1,\dots, a_n \in S$ *such that* $m(a_i, n, a_j) \in J\ (i \neq j)$ *and* $a_j \notin M_j\ (j = 0, 1,\dots, m)$.

Proof. For $n = 1$. Let $x_o \in M_1 - M_o$ and $x_1 \in M_o - M_1$. Then by Lemma 4.1.1, there exists $x_1' \notin M_o$ such that $m(x_o, n, x_1') \in J$. Hence $a_1 = x_1$, $a_o = m(x_o, n, x_1')$ are the required elements.

Observe that $m(a_o, n, a_1) = m(m(x_o, n, x_1'), n, x_1)$

$$= (x_o \wedge x_1 \wedge x_1') \vee (x_o \wedge n) \vee (x_1 \wedge n) \vee (x_1' \wedge n)$$

$$= (x_o \wedge m(x_1, n, x_1')) \vee (x_o \wedge n) \vee (m(x_1, n, x_1') \wedge n)$$

$$= m(x_o, n, m(x_1, n, x_1'))$$

Now, $m(x_1, n, x_1') \wedge n \leq m(x_o, n, m(x_1, n, x_1'))$

$$\leq m(x_1, n, x_1') \vee n$$

and $m(x_1, n, x_1') \in J$, so by convexity $m(a_o, n, a_1) \in J$.

Assume that, the result is true for $n = m-1$, and let $M_o,\dots,M_m$ be $m+1$ distinct minimal prime n-ideals. Let $b_j\ (j = 0, 1,\dots, m-1)$ satisfy

$m(b_i, n, b_j) \in J\ (i \neq j)$ and $b_j \notin M_j$. Now choose $b_m \in M_m - \bigcup_{j=0}^{m-1} M_j$ and by Lemma 4.1.2, let $b_{m'}$ satisfy $b_{m'} \notin M_m$ and $m(b_m, n, b_{m'}) \in J$. Clearly, $a_j = m(b_j, n, b_m)\ (j = 0,\dots, m-1)$ and $a_m = b_{m'}$, establish the result. □

Let J be an n-ideal of a distributive lattice L. A set of elements $x_0, \ldots, x_n \in L$ is said to be *pairwise* in J if $m(x_i, n, x_j) = n$ for all $i \neq j$.

The next result is due to [56, Lemma 3.4.1]. In case of lattice it was proved by [13, Lemma 2.3] which was suggested by Hindman in [29, Theorem 1.8].

Lemma 4.1.5. *Let* J *be an ideal in a distributive nearlattice* S. *For a given positive integer* $n \geq 2$, *the following conditions are equivalent.*

(i) *For any* $x_1, \ldots, x_n \in S$ *which are 'pairwise in* J' *that is* $x_i \wedge x_j \in J$ *for any* $i \neq j$, *there exists* k *such that* $x_k \in J$.

(ii) *For any ideals* $J_1, \ldots, J_n$ *in* S *such that* $J_i \cap J_j \subseteq J$ *for any* $i \neq j$, *there exists* k *such that* $J_k \subseteq J$.

(iii) J *is the intersection of at most* n-1 *distinct prime ideals.* □

Our next result is a generalization of above result. This result will be needed in proving the next theorem which is the main result of this section. In fact, the following lemma is very useful in studing those $P_n(S)$ which are m-normal.

Lemma 4.1.6. *Let* J *be an* n-*ideal in a distributive nearlattice* S *and* $n \in S$ *is medial. For a given positive integer* $m \geq 2$, *the following conditions are equivalent.*

(i) *For any* $x_1, \ldots, x_n \in S$ *with* $m(x_i, n, x_j) \in J$ (*that is, they are pairwise in* J) *for any* $i \neq j$, *there exists* k *such that* $x_k \in J$.

(ii) *For any* n- *ideals* $J_1, \text{---}, J_m$ *in* S *such that* $J_i \cap J_j \subseteq J$ *for any* $i \neq j$, *there exists* k *such that* $J_k \subseteq J$.

(iii) J *is the intersection of at most* m-1 *distinct prime* n- *ideals.*

Proof. (i) and (ii) are easily seen to be equivalent.

(iii)⇒(i). Suppose $P_1, P_2, \ldots, P_k$ are k $(1 \leq k \leq m\text{-}1)$ distinct prime n-ideals such that $J = P_1 \cap P_2 \cap \ldots \cap P_k$. Let $x_1, x_2, \ldots, x_m$ S be such that $m(x_i, n, x_j) \in J$ for all $i \neq j$. Suppose no element x_i is a member of J.Then for each r $(1 \leq r \leq k)$ there is at most one i $(1 \leq i \leq m)$ such that $x_i \in P_r$. Since $k < m$, there is some i such that $x_i \in P_1 \cap P_2 \cap \ldots \cap P_k$.

We need to show **(i)⇒(iii).** Suppose (i) holds for m = 2, then it implies that J is a prime n-ideal. Then (iii) is trivially true. Thus we may assume that there is a largest integer t with $2 \leq t < m$ such that the condition (i) does not hold for J (consequently condition (i) holds for t+1, t+2,…, m). Then for some $2 \leq t < m$ we may suppose that there exist elements $a_1, a_2, \ldots, a_t \in L$ such that $m(a_i, n, a_j) \in J$ for $i \neq j$, $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, t$, yet $a_1, a_2, \ldots, a_t \notin J$.

As S is a distributive lattice, $<< a_i >_n, J>$ is an n-ideal for any $i \in \{1, 2, \ldots, t\}$. Each $<< a_i >_n, J>$ is in fact a prime n-ideal. Firstly $<< a_i >_n, J> \neq S$, since $a_i \notin J$. Secondly, suppose that b and c are in S and $m(b, n, c) \in << a_i >_n, J>$. Consider the set of t+1 elements $\{a_1, a_2, \ldots, a_{i-1}, m(b, n, a_i), m(c, n, a_i), a_{i+1}, \ldots, a_t\}$. This set is pairwise in J and so, either $m(b, n, a_i) \in J$ or $m(c, n, a_i) \in J$. Since condition (i) holds for t+1. That is, $b \in << a_i >_n, J>$ or $c \in << a_i >_n, J>$ and so $<< a_i >_n, J>$ is prime. Clearly, $J \subseteq \bigcap_{1 \leq i \leq t} << a_i >_n, J>$. If $w \in \bigcap_{1 \leq i \leq t} << a_i >_n, J>$. Then $w, a_1, a_2, \ldots, a_t$ are pairwise in j and so $w \in J$. Hence $J = \bigcap_{1 \leq i \leq t} << a_i >_n, J>$ is the intersection of $t < m$ prime n-ideals. □

An ideal $J \neq S$ satisfying the equivalent conditions of Lemma 4.1.5. is called an m-*prime ideal.*

Similarly, an n-ideal $J \neq S$ satisfying the equivalent conditions of Lemma 4.1.6. is called an m-*prime* n-*ideal.*

Now we generalize a result of Davey in [17, Proposition 3.1.].

Theorem 4.1.7. *Let* J *be an* n-*ideal of a distributive nearlattice* S *and* n *be a central element of* S. *Then the following conditions are equivalent.*

(i) *For any* m+1 *distinct prime* n-*ideals* $P_o, P_1, \ldots, P_m$ *belonging to* J, $P_o \vee P_1 \vee \ldots \vee P_m = S$.

(ii) *Every prime* n-*ideal containing* J *contains at most m distinct minimal prime* n-*ideals belonging to* J.

(iii) *If* $a_o, a_1, \ldots, a_m \in S$ *with* $m(a_i, n, a_j) \in J$ $(i \neq j)$ *then* $\underset{j}{\vee} << a_j >_n, J > = S$.

Proof. (i)⇒(ii) is obvious.

(ii)⇒(iii). Assume $a_o, a_1, \ldots, a_m \in S$ with $m(a_i, n, a_j) \in J$ and $\underset{j}{\vee} << a_j >_n, J > \neq S$. It follows that $a_j \notin J$, for all j. Then by Theorem 1.4.7, there exists a prime n-ideal P such that $\underset{j}{\vee} << a_j >_n, J > \subseteq P$. But by Theorem 1.4.2, we know that P is either a prime ideal or a prime filter.

Suppose P is a prime ideal. For each j, let $F_j = \{x \wedge y : x \geq a_j, x, y \geq n, y \notin P\}$. Let $x_1 \wedge y_1, x_2 \wedge y_2 \in F_j$.

Then $(x_1 \wedge y_1) \wedge (x_2 \wedge y_2) = (x_1 \wedge x_2) \wedge (y_1 \wedge y_2)$.

Now, $x_1 \wedge x_2 \geq a_j$ and $y_1 \wedge y_2 = m(y_1, n, y_2)$. So $t \geq x \wedge y$ implies $t = (t \vee x) \wedge (t \vee y)$. Since $y \notin P$, so $t \vee y \notin P$. Hence $t \in F_j$, and so F_j is a dual ideal.

We now show that $F_j \cap J = \phi$, for all $j = 0, 1, 2, \ldots, m$. If not let $b \in F_j \cap J$, then $b = x \wedge y$, $x \geq a_j$, $x, y \geq n$, $y \notin P$. Hence $m(a_j, n, y) = (a_j \wedge n) \vee n \vee (a_j \wedge y)$ $= (a_j \wedge y) \vee n = (a_j \vee n) \wedge (y \vee n)$. But $(a_j \vee n) \wedge (y \vee n) \in F_j$ and

$n \leq (a_j \wedge y) \vee n \leq b$ implies $m(a_j, n, y) \in J$. Therefore, $m(a_j, n, y) \in F_j \cap J$. Again, $m(a_j, n, y) \in J$ with $y \notin P$ implies $<< a_j >_n, J> \not\subset P$, which is a contradiction. Hence $F_j \cap J = \phi$ for all j. For each j, let P_j be a minimal prime n-ideal belonging to J and $F_j \cap P_j = \phi$. Let $y \in P_j$. If $y \notin P$, then $y \vee n \notin P$.
Then $m(a_j, n, y \vee n) = (a_j \vee n) \wedge (y \vee n) \in F_j$.
But $m(a_j, n, y \vee n) \in < y \vee n >_n \subseteq < y >_n \subseteq P_j$, which is a contradiction.
So $y \in P$. Therefore $P_j \subseteq P$, and $a_j \notin P_j$. For if $a_j \in P_j$, then $a_j \vee n \in P_j$. Now, $a_j \vee n = (a_j \vee n) \wedge (a_j \vee n \vee y) \in F_j$ for any $y \notin P$. This implies $P_j \cap F_j \neq \phi$, which is a contradiction. So, $a_j \notin P_j$. But $m(a_i, n, a_j) \in J \subseteq P_j$ $(i \neq j)$ which implies $a_i \in P_j$ $(i \neq j)$ as P_j is prime. It follows that P_j form a set of m+1 distinct minimal prime n-ideals belonging to J and contained in P. This contradicts (ii). Therefore, $\bigvee_j << a_j >_n, J> = S$.

Similarly, if P is filter, then a dual proof of above also shows that
$\bigvee_j << a_j >_n, J> = S$, and hence (iii) holds.

Finally, we need to show **(iii)⇒(i)**. Let $P_o, P_1, \ldots, P_m$ be m+1 distinct minimal prime n-ideals belonging to J. Then by Proposition 4.1.4, there exists $a_o, a_1, \ldots, a_m \in S$ such that $m(a_i, n, a_j) \in J$ $(i \neq j)$ and $a_j \notin P_j$. This implies $<< a_j >_n, J> \subseteq P_j$ for all j. Then by (iii),

$<< a_o >_n, J> \vee << a_1 >_n, J> \vee \ldots \vee << a_m >_n, J> \subseteq P_o \vee P_1 \vee \ldots \vee P_m$,

which implies $P_o \vee P_1 \vee \ldots \vee P_m = S$. □

The following result is due to [56, Theorem 3.4.2], also see [42], which is a generalization of a result in [13]. For lattices this result characterizes the distributive lattices analogues to B_m-lattices.

Beazer [7], Davey [17] have each independently obtained a version of this result. Grätzer and Lakser in [23] (also see [20, Lemma-2 page-169]) have shown that condition (iii) of the theorem is equivalent to Lee's condition which characterize the nth variety for $0 < n < \omega$, of distributive lattices. Thus, this theorem should be compared with Lee's Theorem 2 of [37].

Recall that for a prime ideal P of a distributive nearlattice S, $0(P) = \{x: x \wedge y = 0 \text{ for some } y \in S - P\}$, which is an ideal contained in P.

Theorem 4.1.8. *Let* S *be a distributive nearlattice with* 0 *and* n *be a central element of* S. *Then the following conditions are equivalent.*

(i) *For any* m+1 *distinct minimal prime ideals* $P_o, P_1, \ldots, P_m$, $P_o \vee P_1 \vee \ldots. \vee P_m = S$.

(ii) *Every prime ideals contains at most* m *minimal prime ideals.*

(iii) *For any* $x_o, x_1, \ldots, x_m \in S$ *such that* $x_i \wedge x_j = 0$ *for* $(i \neq j)$, $i = 0, 1, 2, \ldots, m$, $j = 0, 1, 2, \ldots, m$, $(x_o]^* \vee (x_1]^* \vee \ldots \vee (x_m]^* = S$.

(iv) *For each prime ideal* P, 0(P) *is* m+1 *prime.* □

Our next result is a nice extension of above result in terms of n-ideals. Recall that for a prime n-ideal P of S, $n(P) = \{x \in S: m(x, n, y) = n$ for some $y \in S\text{-}P\}$. Of course, n(P) is an ideal and $n(P) \subseteq P$.

Theorem 4.1.9. *Let* S *be a distributive nearlattice with a central element* n. *Then the following conditions are equivalent.*

(i) *For any* m+1 *distinct minimal prime* n-*ideals* $P_o, P_1, \ldots, P_m$, $P_o \vee P_1 \vee \ldots. \vee P_m = S$.

(ii) *Every prime* n- *ideal contains at most* m *minimal prime* n- *ideals.*

(iii) *For any* $a_o, a_1, \ldots, a_m \in S$ *with* $m(a_i, n, a_j) = n$ *for* $(i \neq j)$, $i = 0, 1, 2, \text{---}, m$, $j = 0, 1, 2, \text{---}, m$, $< a_o >_n^* \vee < a_1 >_n^* \vee \ldots \vee < a_m >_n^* = S$.

(iv) *For each prime* n- *ideal* P, n(P) *is an* m+1- *prime* n-*ideal.*

Proof. **(i)⇒(ii)**, **(ii)⇒(iii)** and **(iii)⇒(i)** easily hold by Theorem 4.1.7, replacing J by {n}.

To complete the proof we need to show that **(iv)⇒(iii)** and **(ii)⇒(iv)**.

(iv)⇒(iii). Suppose (iv) holds and $x_o, x_1, \ldots, x_m$ are m+1 elements of S such that $m(x_i, n, x_j) = n$ for $(i \neq j)$. Suppose that $<x_o>_n^* \vee \ldots \vee <x_m>_n^* \neq S$. Then by Theorem 1.4.7, there is a prime n-ideal P such that $<x_o>_n^* \vee \ldots \vee <x_m>_n^* \subseteq P$. Hence $x_o, x_1, \ldots, x_m \in S - n(P)$. This contradicts (iv) by Lemma 4.1.6, since $m(x_i, n, x_j) = n \in n(P)$ for all $i \neq j$. Thus (iii) holds.

(ii)⇒(iv). This follows immediately from Proposition 2.2.6. and Lemma 4.1.6. above. □

Following result is due to [56, Theorem 3.4.5].

Proposition 4.1.10. *Let* S *be a distributive nearlattice with* 0. *If the equivalent conditions of Theorem* 4.1.8. *hold, then for any* m+1 *elements* $x_o, x_1, \ldots, x_m$, $(x_o \wedge x_1 \wedge \ldots \wedge x_m]^* = \underset{0 \le i \le n}{\vee} (x_o \wedge x_1 \wedge \ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots \wedge x_m]^*$. □

Proposition 4.1.11. *Let* S *be a distributive medial nearlattice and* $n \in S$ *is a central element. If the equivalent conditions of Theorem* 4.1.9. *hold, then for any* m+1 *elements* $x_o, x_1, \ldots, x_m$; $(<x_o>_n \cap <x_1>_n \cap \ldots \cap <x_m>_n)^* = \underset{0 \le i \le n}{\vee} (<x_o>_n \cap <x_1>_n \cap \ldots \cap <x_{i-1}>_n \cap <x_{i+1}>_n \cap \ldots \cap <x_m>_n)^*$.

Proof. Let $<b_i>_n = <x_o>_n \cap <x_1>_n \cap \ldots \cap <x_{i-1}>_n \cap <x_{i+1}>_n \cap \ldots \cap <x_m>_n$ for each $0 \le i \le m$. Suppose $x \in (<x_o>_n \cap <x_1>_n \cap \ldots \cap <x_m>_n)^*$. Then $<x>_n \cap <x_o>_n \cap <x_1>_n \cap \ldots \cap <x_m>_n = \{n\}$. For all $i \neq j$, $(<x>_n \cap <b_i>_n) \cap (<x>_n \cap <b_j>_n) = \{n\}$.

So $(<x>_n \cap <b_o>_n)^* \vee \ldots \vee (<x>_n \cap <b_m>_n)^* = S$.

Thus $x \in (<x>_n \cap <b_o>_n)^* \vee \ldots \vee (<x>_n \cap <b_m>_n)^*$. Hence by Theorem 1.3.7, $x \vee n = a_o \vee \ldots \vee a_m$ where $a_i \in (<x>_n \cap <b_i>_n)^*$ and $a_i \geq n$ for $i = 0,1,\ldots,m$. Then $x \vee n = (a_o \wedge (x \vee n)) \vee \ldots \vee (a_m \wedge (x \vee n))$.

Now $a_i \in (<x>_n \cap <b_i>_n)^*$ implies $<a_i>_n \cap <x>_n \cap <b_i>_n = \{n\}$. Then by a routine calculation we find that $(a_i \wedge x \wedge b_i) \vee n = n$

Thus $<a_i \wedge (x \vee n)>_n \cap <b_i>_n = [n, (a_i \wedge x \wedge b_i) \vee n] = \{n\}$ implies that $a_i \wedge (x \vee n) \in <b_i>_n^*$ and so $x \vee n \in <b_o>_n^* \vee <b_1>_n^* \vee \ldots \vee <b_m>_n^*$. By a dual proof of above and using Theorem 1.3.7, we can easily show that $x \wedge n \in <b_o>_n^* \vee <b_1>_n^* \vee \ldots \vee <b_m>_n^*$.

Thus by convexity, $x \in <b_o>_n^* \vee <b_1>_n^* \vee \ldots \vee <b_m>_n^*$ This proves that L.H.S. $\subseteq$ R.H.S. The reverse inclusion is trivial. □

Theorem 4.1.12. *Let* S *be a distributive nearlattice and* $n \in S$ *is central. Then the following conditions are equivalent.*

(i) $P_n(S)$ *is* m-*normal.*

(ii) *Every prime* n-*ideal contains at most* m *minimal prime* n-*ideals.*

(iii) *For any* m+1 *distinct minimal prime* n-*ideals* $P_o, \ldots, P_m$; $P_o \vee \ldots \vee P_m = S$.

(iv) *If* $m(a_i, n, a_j) = n$, *this implies* $<a_o>_n^* \vee \ldots \vee <a_m>_n^* = S$.

(v) *For each prime* n-*ideal* P, n(P) *is an* m+1 *prime* n-*ideal.*

Proof. (i)⇒(ii). Let $P_n(S)$ be m-normal, since n is central, so by Theorem 1.5.2 both $(n]^d$ and $[n)$ are m-normal. Suppose P is any prime n-ideal of S. Then by Theorem 1.4.1, either $P \supseteq (n]$ or $P \supseteq [n)$. Without loss of generality, suppose $P \supseteq [n)$. Then by Theorem 1.4.2, P is prime ideal of S. Hence by Lemma 1.5.5, $P_1 = P \cap [n)$ is a prime ideal of $[n)$. Since $[n)$ is m-normal, so by definition P_1 contains at most m minimal prime ideals $R_1, R_2, \ldots, R_m$ of $[n)$. Therefore, P contains at most m minimal prime ideals $T_1, T_2, \ldots, T_m$ of S where

$R_1 = T_1 \cap [n)$, $R_2 = T_2 \cap [n)$,..., $R_m = T_m \cap [n)$. Since $n \in R_1,\ldots, R_m$, $n \in T_1,\ldots, T_m$, hence $T_1,\ldots, T_m$ are minimal prime n-ideals of S. Thus (ii) holds.

(ii)⇒(i). Suppose (ii) holds. Let P_1 be a prime ideal in [n). Then by Lemma 1.5.5, $P_1 = P \cap [n)$ for some prime ideal P of S. Since $n \in P_1 \subseteq P$, so P is prime n-ideal. Therefore, P contains at most m minimal prime n-ideals $R_1,\ldots, R_m$ of S. Thus by Lemma 1.5.5, P_1 contains at most m minimal prime ideals $T_1 = R_1 \cap [n)$, $T_2 = R_2 \cap [n)$,......., $T_m = R_m \cap [n)$ of [n). Hence by definition, [n) is m-normal. Similarly, we can prove that $(n]^d$ is also m-normal. Hence by Theorem 1.5.2, $P_n(S)$ is normal.

(ii)⇔ (iii) easily hold by Theorem 4.1.7 replacing J by {n}.Other conditions follow from Theorem 4.1.9. □

4.2 Generalizations of some results on Relatively m-Normal Nearlattices

Several characterizations on relative B_m- lattices have been given by Davey in [17]. Also Cornish have studied these lattices in [13] under the name of relatively m-normal lattices. Then [56] have given the concept of relatively m-normal lattices.

Recall that a distributive nearlattice S is called *relatively* m-*normal* if each interval [x, y], x, y$\in$S is m-normal.

Following result gives some characterizations of $P_n(S)$ which are relatively m-normal nearlattices which is a generalization of [56, Theorem 3.5.1 and Theorem 3.5.2]. This also generalizes an analogues result in [17].

Theorem 4.2.1. *Let* S *be a distributive nearlattice with* n *as a central element of* S. *Then the following conditions are equivalent.*

(i) $P_n(S)$ *is relatively* m-*normal.*

(ii) *For all* $x_o, x_1, \ldots, x_m \in S$

$$<<x_1>_n \cap <x_2>_n \ldots \cap <x_m>_n, <x_o>_n> \vee <<x_o>_n \cap <x_2>_n \ldots\ldots \cap <x_m>_n, <x_1>_n> \vee \ldots \vee <<x_o>_n \cap \ldots \cap <x_{m-1}>_n, <x_m>_n> = S.$$

(iii) *For all* $x_o, x_1, \ldots, x_m, z \in S$

$$<<x_o>_n \cap <x_1>_n \cap \ldots \cap <x_m>_n, <z>_n> = <<x_1>_n \cap \ldots \cap <x_m>_n, <z>_n> \vee <<x_o>_n \cap \ldots \cap <x_m>_n, <z>_n> \vee \ldots \vee <<x_o> \cap <x_1>_n \cap \ldots \cap <x_{m-1}>_n, <z>_n>>.$$

(iv) *For any* m+1 *pairwise incomparable prime* n-*ideals* $P_o, P_1, \ldots, P_m$, $P_o \vee P_1 \vee \ldots\ldots \vee P_m = S$.

(v) *Any prime* n-*ideal contains at most* m *mutually incomparable prime* n-*ideals*.

Proof. (i)⇒(ii). Let $z \in S$, consider the interval

$I = [\langle x_o \rangle_n \cap \langle x_1 \rangle_n \cap \ldots \cap \langle x_m \rangle_n \cap \langle z \rangle_n, \langle z \rangle_n]$ in $P_n(S)$. Then $\langle x_o \rangle_n \cap \langle x_1 \rangle_n \cap \ldots \cap \langle x_m \rangle_n \cap \langle z \rangle_n$ is the smallest element of the interval I. For $0 \le i < m$, the set of elements $\langle t_i \rangle_n = \langle x_o \rangle_n \cap \langle x_1 \rangle_n \cap \ldots \cap \langle x_{i-1} \rangle_n \cap \langle x_{i+1} \rangle_n \cap \ldots \cap \langle x_m \rangle_n \cap \langle z \rangle_n$ are obviously pairwise disjoint in the interval I. Since I is m-normal, so by Theorem 4.1.12, $\langle t_o \rangle_n^+ \vee \ldots \vee \langle t_m \rangle_n^+ = \langle z \rangle_n$. So by Theorem 1.3.7, $z \vee n = p_o \vee \ldots \vee p_m$ where $p_i \ge n$. Thus, $\langle p_o \rangle_n \cap \langle t_o \rangle_n = \langle p_1 \rangle_n \cap \langle t_1 \rangle_n = \ldots = \langle p_m \rangle_n \cap \langle t_m \rangle_n =$ The smallest element of $I = \langle x_o \rangle_n \cap \langle x_1 \rangle_n \cap \ldots \cap \langle x_m \rangle_n \cap \langle z \rangle_n$.

Now, $\langle p_o \rangle_n \cap \langle t_o \rangle_n = \langle x_o \rangle_n \cap \langle x_1 \rangle_n \cap \ldots \cap \langle x_m \rangle_n \cap \langle z \rangle_n$ which implies $\langle p_o \rangle_n \cap \langle t_o \rangle_n \subseteq \langle x_o \rangle_n$.

Again, $\langle p_o \rangle_n \cap \langle t_o \rangle_n = \langle p_o \rangle_n \cap \langle x_1 \rangle_n \cap \ldots \cap \langle x_m \rangle_n \cap \langle z \rangle_n$

$= \langle p_o \rangle_n \cap \langle x_1 \rangle_n \cap \ldots \cap \langle x_m \rangle_n$, as $\langle p_o \rangle_n \subseteq \langle z \rangle_n$

This implies $\langle p_o \rangle_n \cap \langle x_1 \rangle_n \cap \ldots \cap \langle x_m \rangle_n \subseteq \langle x_o \rangle_n$

and so, $\langle p_o \rangle_n \in \langle \langle x_1 \rangle_n \cap \ldots \cap \langle x_m \rangle_n, \langle x_o \rangle_n \rangle$

$\langle p_1 \rangle_n \in \langle \langle x_o \rangle_n \cap \langle x_2 \rangle_n \cap \ldots \cap \langle x_m \rangle_n, \langle x_1 \rangle_n \rangle$

....

....

$\langle p_m \rangle_n \in \langle \langle x_o \rangle_n \cap \langle x_1 \rangle_n \cap \ldots \cap \langle x_{m-1} \rangle_n, \langle x_m \rangle_n \rangle$.

Therefore, $z \vee n \subseteq \langle \langle x_1 \rangle_n \cap \ldots \cap \langle x_m \rangle_n, \langle x_o \rangle_n \rangle$

$\vee \langle \langle x_o \rangle_n \cap \langle x_2 \rangle_n \cap \ldots \cap \langle x_m \rangle_n, \langle x_1 \rangle_n \rangle$

$\vee \ldots \vee \langle \langle x_o \rangle_n \cap \langle x_1 \rangle_n \cap \ldots \cap \langle x_{m-1} \rangle_n, \langle x_m \rangle_n \rangle$.

By a dual proof of above we can easily show that

$z \wedge n \subseteq \langle \langle x_1 \rangle_n \cap \ldots \cap \langle x_m \rangle_n, \langle x_o \rangle_n \rangle$

$\vee \langle \langle x_o \rangle_n \cap \langle x_2 \rangle_n \cap \ldots \cap \langle x_m \rangle_n, \langle x_1 \rangle_n \rangle$

$$\vee \ldots \vee << x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_{m-1} >_n, < x_m >_n >.$$

Hence by convexity,

$$z \in << x_1 >_n \cap \ldots \cap < x_m >_n, < x_o >_n >$$

$$\vee << x_o >_n \cap < x_2 >_n \cap \ldots \cap < x_m >_n, < x_1 >_n >$$

$$\vee \ldots \vee << x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_{m-1} >_n, < x_m >_n >.$$

This implies (ii) holds.

(ii)⇒(iii). Suppose $b \in << x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_m >_n, < z >_n >$.

Then by (ii) and Theorem 1.3.7, $b \vee n = s_o \vee s_1 \vee \ldots \vee s_m$, for some

$$s_o \in << x_1 >_n \cap \ldots \cap < x_m >_n, < x_o >_n >$$

$$s_1 \in << x_o >_n \cap < x_2 >_n \cap \ldots \cap < x_m >_n, < x_1 >_n >$$

....

....

$$s_m \in << x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_{m-1} >_n, < x_m >_n >$$

and so $s_i \geq n$, $i = 0,1,2,\ldots,m$.

Thus $< x_1 >_n \cap \ldots \cap < x_m >_n \cap < s_o >_n \subseteq < x_o >_n$

$$< x_o >_n \cap < x_2 >_n \cap \ldots \cap < x_m >_n \cap < s_1 >_n \subseteq < x_1 >_n$$

....

....

$$< x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_{m-1} >_n \cap < s_m > \subseteq < x_m >_n.$$

This implies $< x_1 >_n \cap \ldots \cap < x_m >_n \cap < s_o >_n$

$$= < x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_m >_n \cap < s_o >_n$$

$$\subseteq < x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_m >_n \cap < b \vee n >_n \subseteq < z >_n .$$

Hence, so $s_o \in << x_1 >_n \cap \ldots \cap < x_m >_n, < z >_n >$

Similarly, $s_1 \in << x_o >_n \cap < x_2 >_n \cap \ldots \cap < x_m >_n, < z >_n >$

....

....

$$s_m \in << x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_{m-1} >_n, < z >_n >.$$

Therefore, $b \vee n \in << x_1 >_n \cap \ldots \cap < x_m >_n, < z >_n >$

$\vee << x_o >_n \cap < x_2 >_n \cap \ldots \cap < x_m >_n, < z >_n >$

$\vee \ldots. \vee << x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_{m-1} >_n, < z >_n >.$

The dual proof of above gives

$b \wedge n \in << x_1 >_n \cap \ldots. \cap < x_m >_n, < z >_n >$

$\vee << x_o >_n \cap < x_2 >_n \cap \ldots \cap < x_m >_n, < z >_n >$

$\vee \ldots. \vee << x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_{m-1} >_n, < z >_n >.$

Thus by convexity,

$b \in << x_1 >_n \cap \ldots. \cap < x_m >_n, < z >_n >$

$\vee << x_o >_n \cap < x_2 >_n \cap \ldots \cap < x_m >_n, < z >_n >$

$\vee \ldots. \vee << x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_{m-1} >_n, < z >_n >.$

Therefore, $<< x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_m >_n, < z >_n > \subseteq$

$<< x_1 >_n \cap \ldots. \cap < x_m >_n, < z >_n >$

$\vee << x_o >_n \cap < x_2 >_n \cap \ldots \cap < x_m >_n, < z >_n >$

$\vee \ldots. \vee << x_o >_n \cap < x_1 >_n \cap \ldots \cap < x_{m-1} >_n, < z >_n >.$

Since the reverse inequality always holds, so (iii) holds.

(iii)⇒(i). Suppose $n \leq b \leq d$.

Let $x_o, x_1, \ldots, x_m \in [b, d]$ such that $x_i \wedge x_j = b$, for all $i \neq j$.

Let $t_o = x_1 \vee x_2 \vee \ldots\ldots\ldots. \vee x_m$

$t_1 = x_o \vee x_2 \vee \ldots\ldots\ldots.. \vee x_m$

....

....

$t_m = x_o \vee x_1 \vee \ldots\ldots\ldots.. \vee x_{m-1}$

Clearly, $n \leq b \leq t_i \leq d$ and

$x_o = t_1 \wedge t_2 \wedge \ldots\ldots\ldots.. \wedge t_m$

$x_1 = t_o \wedge t_2 \wedge \ldots\ldots\ldots.. \wedge t_m$

....

....

$x_m = t_o \wedge t_1 \wedge \ldots\ldots\ldots.. \wedge t_{m-1}.$

Then $[b, d] \cap \{<<x_o>_n, <b>_n> \vee \ldots\ldots \vee <<x_m>_n, <b>_n>\}$

$= [b, d] \cap \{<<t_1>_n \cap <t_2>_n \cap \ldots\ldots\ldots \cap <t_m>_n, <b>_n>$

$\vee <<t_o>_n \cap <t_2>_n \cap \ldots\ldots\ldots\ldots \cap <t_m>_n, <b>_n>$

$\vee \ldots\ldots \vee <<t_o>_n \cap <t_1>_n \cap \ldots\ldots\ldots \cap <t_{m-1}>_n, <b>_n>\}$

$= [b, d] \cap \{<<t_o>_n \cap <t_1>_n \cap \ldots\ldots\ldots \cap <t_m>_n, <b>_n>\}$

$= [b, d] \cap <<b>_n, <b>_n>$

$= [b, d] \cap S$

$= [b, d]$.

That is [b, d] is m-normal. Hence [n) is relatively m-normal. A dual proof of above shows that (n] is relatively dual m-normal. Since $P_n(S) \cong (n]^d \times [n)$ so, $P_n(S)$ is relatively m-normal.

(ii)⇒(iv). Suppose (ii) holds. Let $P_o, P_1, \ldots, P_m$ be m+1 pairwise incomparable prime n-ideals. Then, there exists $x_o, x_1, \ldots, x_m \in S$ such that $x_i \in P_j - \bigcup_{i=1}^{n} P_j$. Then by (ii),

$<<x_1>_n \cap <x_2>_n \cap \ldots. \cap <x_m>_n, <x_o>_n>$

$\vee <<x_o>_n \cap <x_2>_n \cap \ldots \cap <x_m>_n, <x_1>_n> \vee \ldots\ldots.$

$\vee <<x_o>_n \cap <x_1>_n \cap \ldots \cap <x_{m-1}>_n, <x_m>_n> = S.$

Let $t_o \in <<x_1>_n \cap <x_2>_n \cap \ldots \cap <x_m>_n, <x_o>_n>$,

then $<t_o>_n \cap <x_1>_n \cap <x_2>_n \cap \ldots \cap <x_m>_n \subseteq <x_o>_n \subseteq P_o$.

Now, $x_i \notin P_o$ for $i = 1, 2, \ldots\ldots, m$.

Thus $<x_1>_n \cap <x_2>_n \cap \ldots. \cap <x_m>_n \not\subset P_o$ as P_o is prime. This implies $<t_o>_n \subseteq P_o$, and so $t_o \in P_o$.

Therefore, $<<x_1>_n \cap <x_2>_n \cap \ldots \cap <x_m>_n, <x_o>_n> \subseteq P_o$.

Similarly, $<<x_o>_n \cap <x_2>_n \cap \ldots \cap <x_m>_n, <x_1>_n> \subseteq P_1$

$<<x_o>_n \cap <x_1>_n \cap \ldots \cap <x_m>_n, <x_2>_n> \subseteq P_2$

....

....

$<<x_o>_n \cap <x_1>_n \cap \ldots \cap <x_{m-1}>_n, <x_m>_n> \subseteq P_m$.

Hence $P_o \vee P_1 \vee P_2 \vee \ldots\ldots\ldots\ldots\ldots\ldots \vee P_m = S$.

(iv)⇒(v) is trivial by Stone's separation theorem.

(iv)⇒(i). Let any m+1 pairwise incomparable prime n-ideals of S are co-maximal. Consider the interval [b, d] in S with $d \geq n$, let $P_o', P_1', \ldots, P_m'$ be m+1 distinct minimal prime ideals of [b, d].Then by Lemma 1.5.5, there exist prime ideals $P_o, P_1, \ldots, P_m$ of S such that $P_o' = P_o \cap [b, d], \ldots\ldots, P_m' = P_m \cap [b, d]$. Since each P_i is an ideal, so $b \in P_i$. More over, $n \leq b$ implies that $n \in P_i$. Therefore each P_i is a prime n-ideal by Lemma 1.4.3, $i = 1, 2, \ldots\ldots, m$. Since $P_o', P_1', \ldots, P_m'$ are incomparable, so $P_o, P_1, \ldots, P_m$ are also incomparable. Now by (iv), $P_o \vee P_1 \vee \ldots\ldots \vee P_m = S$. Hence

$$\begin{aligned} P_o' \vee P_1' \vee \ldots \vee P_m' &= (P_o \vee P_1 \vee \ldots \vee P_m) \cap [b, d] \\ &= S \cap [b, d] \\ &= [b, d]. \end{aligned}$$

Therefore, by Theorem 4.1.12, [b, d] is m-normal. Hence [n) is relatively m-normal.

A dual proof of above shows that (n] is relatively (dual) m-normal. Since $P_n(S) \cong (n]^d \times [n)$, so $P_n(S)$ is relatively m-normal.

We conclude this chapter with the following result which is also a generalization of [17, Theorem 3.4.].

Theorem 4.2.2. *Let* S *be a distributive nearlattice with* $n \in S$ *as an upper element. Then the following conditions are equivalent.*

(i) $P_n(S)$ *is relatively* m-*normal.*

(ii) *If* $b, a_o, a_1, \ldots, a_m \in S$ *with* $m(a_i, n, a_j) \in \langle b \rangle_n$ $(i \neq j)$, *then*

$$\langle \langle a_o \rangle_n, \langle b \rangle_n \rangle \vee \ldots\ldots\ldots\ldots \vee \langle \langle a_m \rangle_n, \langle b \rangle_n \rangle = S.$$

Proof. **(i)⇒(ii)**. By Theorem 4.2.1(v), any prime n-ideal containing b contains at most m minimal prime n-ideals belonging to $< b >_n$. Hence by Theorem 4.1.7, with $J = < b >_n$, we have,

$<< a_o >_n, < b >_n > \vee \ldots\ldots\ldots\ldots \vee << a_m >_n, < b >_n > = S$. Thus (ii) holds.

(ii)⇒(i). Consider $b \in [n)$ with $b \le c$. Let $a_o, a_1, \ldots, a_m \in [b, c]$ with

$a_i \wedge a_j = b$ $(i \ne j)$, then by $m(a_i, n, a_j) = b \in < b >_n$. Then by (ii),

$<< a_o >_n, < b >_n > \vee \ldots\ldots \vee << a_m >_n, < b >_n > = S$.

So, $[b, c] = (<< a_o >_n, < b >_n > \cap [b, c]) \vee \ldots\ldots \vee (<< a_m >_n, < b >_n > \cap [b, c])$

$= < a_o, b >_{[b, c]} \vee \ldots\ldots \vee < a_m, b >_{[b, c]}$.

Hence by Theorem 4.1.8, [b, c] is m-normal. Therefore, [n) is relatively m-normal.

A dual proof of above shows that (n] is relatively dual m-normal. Therefore, by Theorem 1.5.2, $P_n(S)$ is relatively m-normal. □

CHAPTER 5

CHAPTER 5

n-ANNULETS AND α-n-IDEAL OF A DISTRIBUTIVE NEARLATTICE

Introduction.

Annulets and α-ideals in a distributive lattice with 0 have been studied by W. H. Cornish in [12]. In a distributive lattice L with 0, the set of ideals of the form $(x]^*$ can be made into a lattice $A_0(L)$, called the lattice of annulets of L. $A_0(L)$ is a sub-lattice of the Boolean algebra of all annihilator ideals in L. While the lattice of annulets is no more than the dual of the so-called lattice of filets (carriers) as studied in l-groups and abstractly for distributive lattices in [6]. From the basic theorem of [11] it follows that $A_0(L)$ is a sub-lattice of the lattice of all ideals of L if and only if each prime ideal in L contains a unique minimal prime ideal.

Subramanian [63] studied h-ideals with respect to the space of maximal l-ideals in an f-ring. Of course Cornish's α -ideals and his h-ideals were both suggested by the z-ideals of Gilman and Jerison [19]. On the other hand, Bigard [8] has studied α -ideals in the context of lattice ordered groups.

By [12], for an ideal J in L we define $\alpha(J) = \{(x]^* : x \in J\}$. Also for a filter F in $A_0(L)$, $\overleftarrow{\alpha}(F) = \{x \in L: (x]^* \in F\}$. It is easy to see that $\alpha(J)$ is a filter in $A_0(L)$ and $\overleftarrow{\alpha}(F)$ is an ideal in L. An ideal J in L is called an

α-*ideal* if $\overleftarrow{\alpha}\,\alpha(J) = J$. Recently [52] has studied the annulets and α-ideals in a distributive nearlattice with 0 and generalized most of the results of [12]. By [52], the set of all annulets of a distributive nearlattice S with 0 is denoted by $A_o(S)$, which is a dual sub-nearlattice (join sub-semilattice with the lower bound property) of A(S), the Boolean algebra of annihilator ideals of S. Also by [52], for an ideal J of a distributive nearlattice S with 0, we have $\alpha(J) = \{(x]^*: x \in J\}$ and $\overleftarrow{\alpha}(F) = \{x \in S: (x]^* \in F\}$, where F is a filter in $A_o(S)$. [52] have shown that $\alpha(J)$ is a filter in $A_o(S)$, while $\overleftarrow{\alpha}(F)$ is an ideal. By [52] an ideal I of a distributive nearlattice S is called an α-*ideal* if $\overleftarrow{\alpha}\,\alpha(I) = I$.

In this chapter we have generalized these concepts around a central element n of a distributive nearlattice. We have introduced the notion of n-annulets and α-n-ideals in S. For an element n of a distributive nearlattice S, the lattice of n-ideals $I_n(S)$ is a distributive algebraic nearlattice, and so it is pseudo-complemented. We denote the set of annihilator n-ideals (the n-ideals J such that $J = J^{**}$) by $S_n(S)$.

By [20, Theorem 4, P-54], $(S_n(S); \cap, \veebar, *, \{n\}, S)$ is a Boolean algebra which is not necessarily a sublattice of $I_n(S)$.

We denote the set of all n-ideals of the form $< x >_n^*$ by $A_n(S)$. We call the n-ideals of the form $< x >_n^*$ by n-annulets. For a central element n this is a join sub semi-lattice of $S_n(S)$. In fact it has the lower bound property. The set of all n-annulets is denoted by $A_n(S)$.

In section 1, we have studied n-annulets when n is central and generalized several results of [52]. We have proved that $A_n(S)$ is a join sub semi-lattice of $I_n(S)$ if and only if $P_n(S)$ is normal. We have also shown that $A_n(S)$ is relatively complemented if and only if $P_n(S)$ is sectionally quasi-complemented. Finally, we have given a characterization for $P_n(S)$ to be generalized Stone in terms of $A_n(S)$.

In section 2, we have introduced the notion of α-n-ideals. We have shown that the n-ideal n(P) where P is a prime n-ideal is an α-n-ideal. Moreover, all the minimal prime n-ideals are α-n-ideals. Then we have generalized all the results of [12] and [52] in terms of α-n-ideals. We have shown that $P_n(S)$ is disjunctive if and only if each n-ideal is an α-n-ideal. Also, $P_n(S)$ is sectionally quasi-complemented if and only if each prime α-n-ideal is a minimal prime n-ideal. We conclude the thesis by characterizing $P_n(S)$ to be generalized Stone in terms of α-n-ideals.

5.1 n-Annulets of a distributive Nearlattice

For a distributive nearlattice S with 0, I(S), the lattice of ideals of S is pseudo-complemented. Recall that an ideal J of L is an annihilator ideal $J = J^{**}$. The pseudo-complement of an ideal J is the annihilator ideal $J^* = \{x \in L:\ x \wedge j = 0 \ \text{ for all } j \in J\}$.It is well known by that the set of annihilator ideals A(S) is a Boolean algebra, where the supremum of J and K in A(S) is given by $J \veebar K = (J^* \cap K^*)^*$. Ideals of the form $(x]^*$ $(x \in S)$ are called the *annulets* of S. Thus for two annulets $(x]^*$ and $(y]^*$, $(x]^* \veebar (y]^* = ((x]^{**} \cap (y]^{**})^* = ((x \wedge y]^{**})^* = (x \wedge y]^*$. Hence the set of annulets $A_o(S)$ is join sub-semi-lattice of A (S). In fact [52] has shown that $A_o(S)$ has the lower bound property. Therefore it is a dual subnearlattice of A(S). In general A(S) and $A_o(S)$ are not join subsemi-lattices of I(S). It is mentioned in [52] that $(x]^* \cap (y]^*$ may not exists in $A_o(S)$. So $A_o(S)$ is not necessarily a meet semilattice. But for x, $y \in S$ if x v y exists then $(x]^* \cap (y]^* = (x \vee y]^* \in A_o(S)$.

For a distributive nearlattice S with $n \in S$, the lattice of n-ideals $I_n(S)$ is a distributive algebraic lattice with $\{n\}$ and S as the smallest and largest elements respectively. If n is a medial element, then for an n-ideal J of S, the pseudo-complement is the annihilator n-ideal $J^* = \{x \in S:\ m(x, n, j) = n \text{ for all } j \in J\}$. We denote the set of annihilator n-ideals by $S_n(S)$, where the supemum of J and K in $S_n(S)$ is given by $J \veebar K = (J^* \cap K^*)^*$. Recall that the n-ideals of the form $< x >_n^*$ $(x \in S)$ are the n-annulets of S. We denote the set of n-annulets of S by $A_n(S)$. Thus for two n-annulets $< x >_n^*$ and $< y >_n^*$,
$< x >_n^* \veebar < y >_n^* = (< x >_n^{**} \cap < y >_n^{**})^* = (< x >_n \cap\ y >_n)^{***}$

$= \langle m(x, n, y) \rangle_n^*$.

Thus for a medial element n, $A_n(S)$ is a join sub-semi-lattice of $S_n(S)$. $S_n(S)$ is a Boolean algebra with $\{n\}^* = S$ as the largest element and $S^* = \{n\}$ as the smallest element. Of course, $S_n(S)$ is not necessarily a sub-lattice of $I_n(S)$.

We start this section with the following result.

Proposition 5.1.1. *Let* S *be a distributive nearlattice with* n *as a medial element. Then the set of* n*-annulets* $A_n(S)$ *of* S *is a dual nearlattice and it is a dual sub-nearlattice of the Boolean algebra* $\{S_n(S)\ ;\ \cap, \veebar, *, \{n\}, S\}$. *Moreover,* $A_n(S)$ *has the same largest element* $S = \{n\}^*$ *as* $S_n(S)$.

Proof. We have already shown that $A_n(S)$ is a join subsemilattice of $S_n(S)$. Now suppose $\langle x \rangle_n^* \supseteq \langle t \rangle_n^*$ and $\langle y \rangle_n^* \supseteq \langle t \rangle_n^*$ for some $x, y, t \in S$. Then $\langle x \rangle_n^* \cap \langle y \rangle_n^* = (\langle x \rangle_n^* \cap \langle y \rangle_n^*) \veebar \langle t \rangle_n^*$

$$= (\langle x \rangle_n^* \veebar \langle t \rangle_n^*) \cap (\langle y \rangle_n^* \veebar \langle t \rangle_n^*)$$

$$= (\langle x \rangle_n^{**} \cap \langle t \rangle_n^{**})^* \cap (\langle y \rangle_n^{**} \cap \langle t \rangle_n^{**})^*$$

$$= (\langle x \rangle_n \cap \langle t \rangle_n)^{***} \cap (\langle y \rangle_n \cap \langle t \rangle_n)^{***}$$

$$= \langle m(x, n, t) \rangle_n^* \cap \langle m(y, n, t) \rangle_n^*$$

$$= (\langle m(x, n, t) \rangle_n \vee \langle m(y, n, t) \rangle_n)^*.$$

Since $\langle m(x, n, t) \rangle_n \vee \langle m(y, n, t) \rangle_n \subseteq \langle t \rangle_n$, so by [30, Corollary 1.7] $\langle m(x, n, t) \rangle_n \vee \langle m(y, n, t) \rangle_n = \langle r \rangle_n$ for some $r \in S$. Therefore, $A_n(S)$ is a dual nearlattice. Since $S = \langle n \rangle_n^* \in A_n(S)$, so it has the same largest element as $S_n(S)$. □

Proposition 5.1.2. *Let* S *be a distributive nearlattice with* n *as a medial element. Then* $A_n(S)$ *has a smallest element (then of course it is a lattice) if and only if* S *possess an element* d *such that* $< d >_n^* = \{n\}$.

Proof. If there is an element $d \in S$ with $< d >_n^* = \{n\}$, then clearly $\{n\}$ is the smallest element in $A_n(S)$.

Conversely, if $A_n(S)$ has a smallest element $< d >_n^*$. Then for any $x \in S$,

$< x >_n^* = < x >_n^* \veebar < d >_n^* = < m(x, n,d) >_n^*$.

Thus for any $y \in < d >_n^*$ implies $< y >_n^* = < m(y, n, d) >_n^* = < n >_n^* = S$, so that $y = n$. Therefore, $< d >_n^* = \{n\}$. □

Now we generalize [12, Proposition 2.2] and [52, Theorem 1.3].

Theorem 5.1.3. *Let* S *be a distributive nearlattice with* n *as a central element. Then* $P_n(S)$ *is normal if and only if* $A_n(S)$ *is a dual sub-nearlattice of* $I_n(S)$.

Proof. Let $< x >_n^*, < y >_n^* \in A_n(S)$. Then by Theorem 2.2.7, $P_n(S)$ is normal if and only if $< x >_n^* \vee < y >_n^* = (< x >_n \cap < y >_n)^*$

$= (< x >_n \cap < y >_n)^{***}$

$= (< x >_n^{**} \cap < y >_n^{**})^*$

$= < x >_n^* \veebar < y >_n^*$.

That is, $\veebar$ in $A_n(S)$ is same as $\vee$ in $I_n(S)$, which proves the proposition. □

A distributive nearlattice S with 0 is called *disjunctive* if for $0 \le a < b$ $(a, b \in L)$ there is an element $x \in S$ such that $a \wedge x = 0$ where

$0 < x \leq b$. This is also known as *sectionally semi-complemented distributive nearlattice*. It is easy to check that S is disjunctive if and only if $(a]^* = (b]^*$ implies $a = b$ for any $a, b \in S$.

Similarly, a distributive lattice L with 1 is called *dual disjunctive* if for $c < d \leq 1$ ($c, d \in L$), there is an element $y \in L$ such that $d \vee y = 1$ where $c \leq y < 1$. Since for a central element n of S, $P_n(S) \cong (n]^d \times [n)$, so $P_n(S)$ is disjunctive if and only if $(n]$ is a dual disjunctive lattice and $[n)$ is a disjunctive nearlattice. By [59], we know that $P_n(S)$ is disjunctive if and only if $< a >_n = < a >_n^{**}$ for each $a \in S$.

Following result is a generalization of [12, Proposition 1.3] and [52, Proposition 1.4].

Proposition 5.1.4. *For a distributive nearlattice* S *with a central element* n, *if* $P_n(S)$ *is disjunctive and normal, then* $P_n(S)$ *is dual isomorphic to* $A_n(S)$. *Hence* $0, 1 \in S$ (*then of course* S *is a lattice*) *if and only if there is an element* $d \in S$ *such that* $< d >_n^* = \{n\}$.

Proof. Define φ: $P_n(S) \to A_n(S)$ by $\varphi(< a >_n) = < a >_n^*$. Then for $< a >_n, < b >_n \in P_n(S)$, $\varphi(< a >_n \cap < b >_n) = \varphi(< m(a, n, b) >_n)$

$$= < m(a, n, b) >_n^*$$

$$= (< a >_n \cap < b >_n)^*$$

$$= < a >_n^* \vee < b >_n^*$$

$$= \varphi(< a >_n) \vee \varphi(< b >_n), \text{ as } P_n(S) \text{ is normal.}$$

Moreover, suppose $< a >_n \vee < b >_n$ exists in $P_n(S)$ and $< a >_n \vee < b >_n = < c >_n$. Then $\varphi(< a >_n \vee < b >_n) = \varphi(< c >_n) = < c >_n^* = (< a >_n \vee < b >_n)^* = < a >_n^* \cap < b >_n^*$.

Therefore, φ is a dual homomorphism from $P_n(S)$ onto $A_n(S)$. Now let $\varphi(<a>_n) = \varphi(<b>_n)$. Then $<a>_n^* = <b>_n^*$, and so $<a>_n^{**} = <b>_n^{**}$. Thus by [59], $<a>_n = <b>_n$ as $P_n(S)$ is disjunctive. Therefore, $P_n(S)$ is dual isomorphic to $A_n(S)$.

Finally, if $0, 1 \in S$, then $[0,1]$ is the largest element of $P_n(S)$ as n is central. Thus from the dual isomorphism, $A_n(S)$ has a smallest element. Then by Proposition 5.1.1, there is an element $d \in S$ such that $<d>_n^* = \{n\}$. Conversely, if for some $d \in S$, $<d>_n^* = \{n\}$, then $A_n(S)$ has a smallest element and so $P_n(S)$ has a largest element which implies $0, 1 \in S$. □

Following result is a generalization of [12, Proposition 2.5].

Proposition 5.1.5. *For a generalized Stone nearlattice* S, $A_o(S)$ *is a relatively complemented dual sub-nearlattice of* I(S).

Proof. From [52(a), Theorem 1.7] a generalized Stone nearlattice S is normal. So by [52, Theorem 1.3], $A_o(S)$ is dual sub-nearlattice of I(S). We therefore write $\veebar$ as $\vee$. As $A_o(S)$ is distributive with largest element S, $A_o(S)$ will be relatively complemented if and only if each interval of the form $[I, S]$, $I \in A_o(S)$ is complemented. Thus let $J = [(x]^*, S]$ be an interval in Ao(S) and let $(y]^* \in J$. As S is generalized Stone $(y]^* \vee (y]^{**} = S$ and $(y]^* \cap (y]^{**} = (0]$ always holds. Hence $((x] \cap (y]^*) \vee ((x] \cap (y]^{**}) = (x]$ and $((x] \cap (y]^*) \cap (x] \cap (y]^{**} = (0]$. Thus by [30, Theorem 1.9], $(x] \cap (y]^* = (a]$ for some $a \in S$. As $a \leq x$ $(x]^* \subseteq (a]^*$, and so$(a]^* \in J$. Also $(a] \subseteq (y]^*$, so $(y]^{**} \subseteq (a]^*$. Thus $(a]^* \vee (y]^* = S$. Now $(a]^* \cap (y]^* \cap (x] = (a]^* \cap (a] = (0]$. This implies $(a]^* \cap (y]^* \subseteq (x]^*$.

But $(x]^* \subseteq (a]^*, (y]^*$. Hence $(x]^* = (a]^* \cap (y]^*$. Therefore, $(a]^*$ is the required complement of $(y]^*$ in the interval J. □

Now we generalize the above result for n-annulets.

Theorem 5.1.6. *Let* n *be a central element of a distributive nearlattice* S. *If* $P_n(S)$ *is generalized Stone, then* $A_n(S)$ *is a relatively complemented dual sub-nearlattice of* $I_n(S)$.

Proof. Since every generalized Stone nearlattice is normal so by Proposition 5.1.3, $A_n(S)$ is a dual sub-nearlattice of $I_n(S)$. We therefore replace $\vee$ for $\underline{\vee}$. Since $A_n(S)$ is a distributive dual nearlattice with largest element S, so $A_n(S)$ will be relatively complemented if and only if each interval of the form $[I, S]$, $I \in A_n(S)$, is complemented. Thus let $J = [< x >_n^*, S]$ be an interval in $A_n(S)$ and let $< y >_n^* \in J$. As $P_n(S)$ is generalized Stone, so by [59, Theorem 2.4.6], $< y >_n^* \vee < y >_n^{**} = S$ and $< y >_n^* \cap < y >_n^{**} = \{n\}$ always holds. Hence $(< x >_n \cap < y >_n^*) \vee (< x >_n \cap < y >_n^{**}) = < x >_n$ and $(< x >_n \cap < y >_n^*) \cap (< x >_n \cap < y >_n^{**}) = \{n\}$. Then by [30, Theorem 1.9] both $< x >_n \cap < y >_n^*$ and $< x >_n \cap < y >_n^{**}$ are principal.

Let $< x >_n \cap < y >_n^* = < a >_n$. Then $< a >_n \leq < x >_n$ and so $< x >_n^* \subseteq < a >_n^*$. Thus $< a >_n^* \in J$. Also, $< a >_n \subseteq < y >_n^*$ implies $< y >_n^{**} \subseteq < a >_n^*$, and so $< a >_n^* \vee < y >_n^* = S$. Now $< a >_n^* \cap < y >_n^* \cap < x >_n = < a >_n^* \cap < a >_n = \{n\}$ implies $< a >_n^* \cap < y >_n^* \subseteq < x >_n^*$. But $< x >_n^* \subseteq < y >_n^*, < a >_n^*$. Hence $< a >_n^* \cap < y >_n^* = < x >_n^*$, and so $< a >_n^*$ is the required relative complement of $< y >_n^*$ in J. □

Consider the interval $I = [n, x]$, $n < x$ in a distributive nearlattice. For any $a \in I$, we define $(a]^{+} = \{s \in I: s \wedge a = n\}$. This is of course an ideal in I and is the annihilator of a with respect to I. Dually for $b \in J = [y, n]$, we define $[b)^{+d} = \{t \in J: t \vee b = n\}$. It is easy to check that this is a filter in J and is the dual annihilator of b with respect to J. Clearly, both I and J are also n-ideals.

Similarly, we define

(i) For any $x \in (n]$, $[x)^{+d} = \{t \leq n: t \vee x = n\}$ and

(ii) For any $x \in [n)$, $(x]^{+} = \{t \geq n: t \wedge x = n\}$.

Following lemmas are needed for the proof of next two results.

Lemma 5.1.7. *Let* S *be a distributive nearlattice and* $n \in S$ *be an upper element of* S. *Let* $a < n < b$, *then for any* $x \in S$,

(i) $< x >_n^{*} \cap [a, n] = [a \vee (x \wedge n))^{+d}$, *dual annihilator with respect to* $[a, n]$.

(ii) $< x >_n^{*} \cap [n, b] = ((x \vee n) \wedge b]^{+}$, *annihilator with respect to* $[n, b]$.

Proof. Let $p \in < x >_n^{*} \cap [a, n]$. Then $a \leq p \leq n$ and $m(p, n, x) = n$. Thus, $n = (p \vee x) \wedge (p \vee n) \wedge (x \vee n) = (p \vee x) \wedge n = p \vee (x \wedge n)$ $= p \vee (a \vee (x \wedge n))$, and so $p \in [a \vee (x \wedge n))^{+d}$. Here $p \vee x$ exists by the upper bound property of S and n is an upper.

Conversely, let $p \in [a \vee (x \wedge n))$. Then $p \vee a \vee (x \wedge n) = n$, and so $p \vee (x \wedge n) = n$ as $a \leq p \leq n$. Thus, $n = (p \vee x) \wedge (p \vee n)$ $= (p \vee x) \wedge n = (p \vee x) \wedge n \wedge (x \vee n) = (p \vee x) \wedge (p \vee n) \wedge (x \vee n)$ $= m(p, n, x)$, which implies $p \in < x >_n^{*}$, and so $p \in < x >_n^{*} \cap [a, n]$.

This implies (i).

A dual proof of (i) proves (ii).

Similarly, we have the following result.

Lemma 5.1.8. *Let* S *be a distributive nearlattice and* n *be an upper element of* S. *Then for any* $x \in S$,

(i) $<x>_n^* \cap (n] = [x \wedge n)^{+d}$ *in* $(n]$ *and*

(ii) $<x>_n^* \cap [n) = (x \vee n]^+$ *in* $[n)$. □

Lemma 5.1.9. *Let* S *be a distributive nearlattice and* n *be an upper element of* S.

(i) *Suppose* $I = [n, x]$; $n < x$. *Then for any* $a, b \in I$,

$(a]^+ \subseteq (b]^+$ *implies* $<a>_n^* \subseteq <b>_n^*$.

(ii) *Suppose* $J = [y, n]$; $y < n$. *Then for any* $a, b \in J$,

$[a)^{+d} \subseteq [b)^{+d}$ *implies* $<a>_n^* \subseteq <b>_n^*$.

Proof. (i) Let $p \in <a>_n^*$. Then $m(p, n, a) = n$ which implies $(a \wedge p) \vee n = n$. Now $(p \vee n) \wedge x \in I$, and $a \wedge [(p \vee n) \wedge x]$ $= (a \wedge p \wedge x) \vee (a \wedge x \wedge n) = (a \wedge p) \vee n = n$. This implies $(p \vee n) \wedge x \in (a]^+$ $\subseteq (b]^+$, and so $(p \vee n) \wedge x \wedge b = n$. Thus, $(p \vee n) \wedge b = n$. Therefore, $n = (p \vee n) \wedge b = (p \vee n) \wedge (b \vee n) \wedge (p \vee b) = m(p, n, b)$, and so $p \in <b>_n^*$. Hence $<a>_n^* \subseteq <b>_n^*$.

A dual proof of above proves (ii). □

A nearlattice S with 0 is called *quasi-complemented* if for each $x \in S$ there exists $x' \in S$ such that $x \wedge x' = 0$ and $(x \vee x']^* = (0]$, that is, $(x]^* \cap (x']^* = (0]$. This also equivalent to the condition that for each $x \in S$ there exists

$x' \in S$ such that $(x]^{**} = (x']^{*}$.

Dually we can define a dual quasi-complemented lattice L with 1. Since for a central element n of S, $P_n(S) \cong (n]^d \times [n)$, so we have,

Corollary 5.1.10. *If* S *is a nearlattice with* n *as a central element of* S, *then*

(i) $P_n(S)$ *is quasi-complemented if and only if* (n] *is dual quasi-complemented and* [n) *is quasi-complemented.*

(ii) $P_n(S)$ *is sectionally quasi-complemented if and only if* (n] *is sectionally dual quasi-complemented and* [n) *is sectionally quasi-complemented.* □

The following theorem is a generalization of [52, Theorem 1.7].

Theorem 5.1.11. *Let* S *be distributive nearlattice with* n *as a central element. Then* $A_n(S)$ *is relatively complemented if and only if* $P_n(S)$ *is sectionally quasi-complemented.*

Proof. Suppose $A_n(S)$ is relatively complemented. Let $I = [n, x]$. Consider $a \in I$. Then $<x>_n^* \subseteq <a>_n^* \subseteq \{n\}^* = S$. Since $[<x>_n^*, S]$ is complemented in $A_n(S)$, there exists $w \in S$ such that $<a>_n^* \cap <w>_n^* = <x>_n^*$ and $<a>_n^* \veebar <w>_n^* = S$.

Now, $S = <a>_n^* \veebar <w>_n^* = (<a>_n^{**} \cap <w>_n^{**})^*$

$= (<a>_n \cap <w>_n)^{***} = (<a>_n \cap <w>_n)^*$. This implies

$<a>_n \cap <w>_n = \{n\}$, and so $<a>_n \cap <w>_n \cap <x>_n = \{n\}$.

But $< w >_n \cap < x >_n = < (w \wedge x) \vee (w \wedge n) \vee (x \wedge n) >_n = < (w \wedge x) \vee n >_n$.
Thus, $n = (a \wedge n) \vee (a \wedge ((w \wedge x) \vee n)) \vee ((w \wedge x) \vee n)$
$= a \wedge ((w \wedge x) \vee n)$, where $(w \wedge x) \vee n \in I$. On the other hand,
$< a >_n^* \cap < w >_n^* = < x >_n^*$ implies $< a >_n^* \cap < w >_n^* \cap < x >_n = \{n\}$ and so by Lemma 5.1.7, $(a]^+ \cap ((w \wedge x) \vee n]^+ = \{n\}$. This implies I is quasi-complemented and so [n) is sectionally quasi-complemented. A dual proof of above shows that (n] is sectionally dual quasi-complemented, and so by Corollary 5.1.10, $P_n(S)$ is sectionally quasi-complemented.

Conversely, suppose $P_n(S)$ is sectionally quasi-complemented. Since $A_n(S)$ is distributive, it remains to prove that the interval
$[< x >_n^*, S]$ is complemented for each $x \in S$. Let $< y >_n^* \in [< x >_n^*, S]$.
Then $< y >_n^* = < x >_n^* \veebar < y >_n^* = < m(x, n, y) >_n^*$. Now consider
$I = [n, x \vee n]$ in [n). Then $(x \vee n) \wedge (y \vee n) \in I$. Since by Corollary 5.1.10,
I is quasi-complemented, so there exists $w \in I$ such that
$w \wedge (x \vee n) \wedge (y \vee n) = n$ and $(w]^+ \cap ((x \vee n) \wedge (y \vee n)]^+ =$
$\{n\} = (x \vee n]^+$. Thus by Lemma 5.1.9, $< w \vee ((x \vee n) \wedge (y \vee n)) >_n^*$
$= < x \vee n >_n^*$, and so $< w >_n^* \cap < (x \vee n) \wedge (y \vee n)) >_n^* = < x \vee n >_n^*$.
Dually considering the interval $[x \wedge n, n]$ in (n] and using some argument there exists $v \in [x \wedge n, n]$ such that $v \vee (x \wedge n) \vee (y \wedge n) = n$
and $< w >_n^* \cap < (x \wedge n) \vee (y \wedge n)) >_n^* = < x \wedge n >_n^*$.
Then $[v, w]^* \cap < y >_n^* = [v, w]^* \cap < m(x, n, y) >_n^* =$
$[v, w]^* \cap [m(x, n, y) \wedge n, m(x, n, y) \vee n]^* = < v >_n^* \cap < w >_n^* \cap$
$< m(x, n, y) \wedge n >_n^* \cap < m(x, n, y) \vee n >_n^* = < v >_n^* \cap < w >_n^* \cap$
$< (x \wedge n) \vee (y \wedge n)) >_n^* \cap < (x \vee n) \wedge (y \vee n)) >_n^* =$
$< x \wedge n >_n^* \cap < x \vee n >_n^* = [x \wedge n, x \vee n]^* = < x >_n^*$.
Also, $[v, w]^* \veebar < y >_n^* = [v, w]^* \veebar < m(x, n, y) >_n^*$
$= ([v, w] \cap [(x \wedge n) \vee (y \wedge n), (x \vee n) \wedge (y \vee n)])^*$

$= [v \vee (x \wedge n) \vee (y \wedge n), w \wedge (x \vee n) \wedge (y \vee n)]^*$

$= \{n\}^* = S.$

Since n is central, so $[v, w] = \langle t \rangle_n$ where $\langle t \rangle_n^* \in [\langle x \rangle_n^*, S]$,

which is the required relative complement of $\langle y \rangle_n^*$. □

In [52, Theorem 1.5] Cornish has proved that if S is a distributive nearlattice with 0, then S is quasi-complemented if and only if $A_0(S)$ is a Boolean sub-algebra of $A(S)$. But we are unable to get such a result for $A_n(S)$, when $P_n(S)$ is quasi-complemented. We could not prove that there exists $d \in S$ with $\langle d \rangle_n^* = \{n\}$, when $P_n(S)$ is quasi-complemented. We leave it to the reader as an open question. Does $A_n(S)$ possess a smallest element when $P_n(S)$ is a quasi-complemented nearlattice with n as a central element?

But following the same technique of proof of Theorem 5.1.11, we can establish the following result.

Theorem 5.1.12. *Let* S *be distributive nearlattice with* n *as a central element. Then,*

(i) $P_n(S)$ *is Boolean implies* $P_n(S)$ *is quasi-complemented.*

(ii) *If* $P_n(S)$ *is quasi-complemented and* $A_n(S)$ *has a smallest element, then* $A_n(S)$ *is Boolean.* □

By [52(a), Theorem 2.3] we know that a distributive nearlattice with 0 is a generalized Stone nearlattice if and only if it is both normal and sectionally quasi-complemented. So we conclude this section with the following result. This also gives a nice characterization of $P_n(S)$, which are generalized Stone.

Theorem 5.1.13. *Let* S *be a distributive nearlattice with* n *as a central element. Then* $P_n(S)$ *is generalized Stone if and only if* $A_n(S)$ *is a relatively complemented dual sub-lattice of* $I_n(S)$.

Proof. Suppose $P_n(S)$ is generalized Stone. Then it is normal and sectionally quasi-complemented. Thus by Proposition 5.1.3. and Theorem 5.1.11, $A_n(S)$ is a relatively complemented sub-lattice of $I_n(S)$.

Conversely, if $A_n(S)$ is a relatively complemented dual sub-nearlattice of $I_n(S)$, then again by Proposition 5.1.3. and Theorem 5.1.11, $P_n(S)$ is normal and sectionally quasi-complemented. Therefore, by [52(a), Theorem 2.3], $P_n(S)$ is generalized Stone. □

5.2 α-n-Ideals in a distributive Nearlattice

Recall that for an ideal J in a distributive nearlattice S with 0, $\alpha(J) = \{(x]^*: x \in J\}$, which is a filter in $A_o(S)$, and conversely $\alpha^{\leftarrow}(F) = \{x \in S: (x]^* \in F\}$ is an ideal in S, when F is a filter in $A_o(S)$. Clearly for any ideal I, $I \subseteq \alpha^{\leftarrow}\alpha(I)$. An ideal I is called an *α-ideal if* $I = \alpha^{\leftarrow}\alpha(I)$.

Now for any n-ideal J in a distributive nearlattice S with a central element n, we define $\alpha(J) = \{< x >_n^*: x \in J\}$ and conversely $\alpha^{\leftarrow}(F) = \{x \in S: < x >_n^* \in F\}$, where F is any filter in $A_n(S)$. Note that here $A_n(S)$ is a dual nearlattice (i.e. a join semi-lattice with the lower bound property). A non-empty subset F of a dual nearlattice A is called a *filter if*

(i) *for any* $x, y \in F$ *if* $x \wedge y$ *exists then* $x \wedge y \in F$ *and*

(ii) $x \in F$ *and* $t \geq x$ $(t \in A)$ *implies* $t \in F$.

We start this section with the following result which is a generalization of [52, Proposition2.1].

Proposition 5.2.1. *Let* L *be a distributive nearlattice and* $n \in S$ *is central. Then*

(a) *For any n-ideal* J, $\alpha(J)$ *is a filter in* $A_n(S)$.

(b) $\alpha^{\leftarrow}(F)$ *is an n-ideal in* S, *when* F *is any filter of* $A_n(S)$.

(c) *If* I_1, I_2 *are n-ideals, then* $I_1 \subseteq I_2$ *implies that* $\alpha(I_1) \subseteq \alpha(I_2)$; *and if* F_1, F_2 *are filters in* $A_n(S)$, *then* $F_1 \subseteq F_2$ *implies* $\alpha^{\leftarrow}(F_1) \subseteq \alpha^{\leftarrow}(F_2)$.

(d) *For any filter* F *of* $A_n(S)$, $\alpha\,\alpha^{\leftarrow}(F) = F$.

(e) *The map* $I \rightarrow \alpha^{\leftarrow}\alpha(I) = \alpha^{\leftarrow}(\alpha(I))$ *is a closure operation* *on the lattice of* n-*ideals. That is,*

(i) $\alpha^{\leftarrow}\alpha\,(\alpha^{\leftarrow}\alpha(I)) = \alpha^{\leftarrow}\alpha(I)$.

(ii) $I \subseteq \alpha^{\leftarrow}\alpha(I)$.

(iii) $I \subseteq J$ *implies* $\alpha^{\leftarrow}\alpha(I) \subseteq \alpha^{\leftarrow}\alpha(J)$; *for any* n-*ideals* I *and* J *in* S.

Proof. (a) Let $<x>_n^*, <y>_n^* \in \alpha(J)$ with $x, y \in J$.

Suppose $<x>_n^* \cap <y>_n^*$ exists in $A_n(S)$ and suppose $<x>_n^* \cap <y>_n^* = <t>_n^*$ for some $t \in S$. Then from the calculation of the proof of Proposition 5.1.1, $<x>_n^* \cap <y>_n^* = (<m(x, n, t)>_n \vee <m(y, n, t)>_n)^*$

$= ([(x \wedge n) \vee (t \wedge n), (x \wedge t) \vee n] \vee [(y \wedge n) \vee (t \wedge n), (y \wedge t) \vee n])^*$

$= [(x \wedge y \wedge n) \vee (t \wedge n), (x \wedge t) \vee (y \wedge t) \vee n]^* = <r>_n^*$

as n is central, where $r \wedge n = (x \wedge y \wedge n) \vee (t \wedge n)$

and $r \vee n = (x \wedge t) \vee (y \wedge t) \vee n$.

Observe that $x \wedge y \wedge n, n \in J$. So by convexity $x \wedge y \wedge n \le r \wedge n \le n$ implies $r \wedge n \in J$. Again, $x \vee n \in J$. So $n \le (x \wedge t) \vee n \le x \vee n$ implies $(x \wedge t) \vee n \in J$. Similarly, $(y \wedge t) \vee n \in J$. Therefore, $r \vee n = (x \wedge t) \vee (y \wedge t) \vee n \in J$, and so again by convexity of J, $r \in J$. Hence $<x>_n^* \cap <y>_n^* \in \alpha(J)$. Now suppose $<x>_n^* \in \alpha(J)$ with $x \in J$ and $<s>_n^* \supseteq <x>_n^*$ for some $<s>_n^* \in A_n(S)$. Then $<s>_n^* = <s>_n^* \underline{\vee} <x>_n^* = <m(s, n, x)>_n^*$, and $x \wedge n \le m(s, n, x) \le x \vee n$ implies by convexity that $m(s, n, x) \in J$.

Hence $<s>_n^* \in \alpha(J)$, and so $\alpha(J)$ is a filter.

(b) Since S is the largest element of $A_n(S)$, so $S \in F$. Then $S = \{n\}^*$ implies $n \in \alpha^{\leftarrow}(F)$. Let $x < t < y$ with $x, y \in \alpha^{\leftarrow}(F)$.

Now $x < t$ implies $<t \wedge n>_n \subseteq <x \wedge n>_n \subseteq <x>_n$ and so $<x>_n^* \subseteq <t \wedge n>_n^*$.

Similarly, $t < y$ implies $<y>_n^* \subseteq <t \vee n>_n^*$. Thus $<t \wedge n>_n^*, <t \vee n>_n^* \in F$ as F is a filter. Therefore, $<t>_n^* = (<t \wedge n>_n \vee <t \vee n>_n)^*$

$= \langle t \wedge n \rangle_n^* \cap \langle t \vee n \rangle_n^* \in F$. This implies $t \in \alpha^{\leftarrow}(F)$ and $\alpha^{\leftarrow}(F)$ is convex. Now let x, $y \in \alpha^{\leftarrow}(F)$. Then $\langle x \rangle_n^*, \langle y \rangle_n^* \in F$. Now $\langle x \wedge n \rangle_n^* \supseteq \langle x \rangle_n^*$ and $\langle y \wedge n \rangle_n^* \supseteq \langle y \rangle_n^*$ imply $\langle x \wedge n \rangle_n^*, \langle y \wedge n \rangle_n^* \in F$.

Thus $\langle x \wedge y \wedge n \rangle_n^* = (\langle x \wedge n \rangle_n \vee \langle y \wedge n \rangle_n)^* = \langle x \wedge n \rangle_n^* \cap \langle y \wedge n \rangle_n^*$ $\in F$ and so $x \wedge y \wedge n \in \alpha^{\leftarrow}(F)$. Similarly, $\langle x \vee n \rangle_n^* \supseteq \langle x \rangle_n^*$ and $\langle y \vee n \rangle_n^* \supseteq \langle y \rangle_n^*$ imply $\langle x \vee n \rangle_n^*, \langle y \vee n \rangle_n^* \in F$, and so $x \vee n$, $y \vee n \in \alpha^{\leftarrow}(F)$. Then by convexity $x \wedge y \wedge n \le x \wedge y \le x \vee n$ implies $x \wedge y \in \alpha^{\leftarrow}(F)$. Also if $x \vee y$ exists, then $\langle x \vee y \vee n \rangle_n^* =$ $(\langle x \vee n \rangle_n \vee \langle y \vee n \rangle_n)^* = \langle x \vee n \rangle_n^* \cap \langle y \vee n \rangle_n^* \in F$. This implies $x \vee y \vee n \in \alpha^{\leftarrow}(F)$. Then $x \wedge y \wedge n \le x \vee y \le x \vee y \vee n$ implies $x \vee y \in \alpha^{\leftarrow}(F)$, and hence $\alpha^{\leftarrow}(F)$ is an n-ideal.

(c) This is trivial.

(d) Suppose $\langle x \rangle_n^* \in F$. Then $x \in \alpha^{\leftarrow}(F)$ and so $\langle x \rangle_n^* \in \alpha(\alpha^{\leftarrow}(F))$. Therefore, $F \subseteq \alpha(\alpha^{\leftarrow}(F))$. Conversely, let $\langle x \rangle_n^* \in \alpha(\alpha^{\leftarrow}(F))$. Then $\langle x \rangle_n^* = \langle y \rangle_n^*$ for some $y \in \alpha^{\leftarrow}(F)$. Thus $\langle y \rangle_n^* \in F$, and so $\langle x \rangle_n^* \in F$. That is $\alpha(\alpha^{\leftarrow}(F)) \subseteq F$, and so $F = \alpha(\alpha^{\leftarrow}(F))$.

(e) (i) Since $\alpha(I)$ is a filter in $A_n(S)$, so by (d) $\alpha\alpha^{\leftarrow}(\alpha(I)) = \alpha(I)$.

Therefore, $\alpha^{\leftarrow}(\alpha\alpha^{\leftarrow}(\alpha(I))) = \alpha^{\leftarrow}\alpha(I)$;

That is, $\alpha^{\leftarrow}\alpha(\alpha^{\leftarrow}\alpha(I)) = \alpha^{\leftarrow}\alpha(I)$.

(ii) is obvious.

(iii) follows from (c). □

An n-ideal I is called an α-n-*ideal* if $\alpha^{\leftarrow}\alpha(I) = I$. Thus α-n-ideals are simply the closed elements with respect to the closure operation of Proposition 5.2.1.

Following result is a generalization of [52, Proposition 2.3] in terms of n-ideals.

Proposition 5.2.2. *Let* S *be a distributive nearlattice with* n *as a central element. Then* α-n-*ideals of* S *form a complete distributive nearlattice isomorphic to the nearlattice of filters, ordered by set inclusion of* $A_n(S)$.

Proof. Let $\{I_i\}$ be any class of α- n-ideals of S. Then $\alpha^{\leftarrow}\alpha(I_i) = I_i$ for all i. By Proposition 5.2.1, $\cap I_i \subseteq \alpha^{\leftarrow}\alpha(\cap I_i)$. Again, $\alpha^{\leftarrow}\alpha(\cap I_i) \subseteq \alpha^{\leftarrow}\alpha(I_i) = I_i$ for each i. Thus $\alpha^{\leftarrow}\alpha(\cap I_i) \subseteq \cap I_i$, and so $\alpha^{\leftarrow}\alpha(\cap I_i) = \cap I_i$. Hence $\cap I_i$ is an α-n-ideal. Therefore, by [20, Lemma 14, P-29], set of

α-n-ideals is a complete nearlattice, and it is distributive as S is so.

Now α is onto and both α, $\alpha^{\leftarrow}$ are isotone by proposition 5.2.1(c). Moreover, for α-n-ideals I, $\alpha^{\leftarrow}\alpha(I) = I$ and by Proposition 5.2.1 (d),

$\alpha\alpha^{\leftarrow}(F) = F$ for any filter F of $A_n(S)$. Therefore the map α is an isomorphism from the nearlattice of α-n-ideals to the nearlattice of filters of $A_n(S)$. □

Following theorem gives a nice characterization of α-n-ideals which also generalizes [12, Proposition 3.3] and [52, Proposition 2.5].

Theorem 5.2.3. *For a central element* n *of a distributive nearlattice* S, *the following conditions are equivalent.*

(i) I *is an* α-n-*ideal.*

(ii) *For* $x, y \in S$, $<x>_n^* = <y>_n^*$ *and* $x \in I$ *implies* $y \in I$.

(iii) $I = \bigcup_{x \in I} <x>_n^{**}$ *where* $\cup$ *is set theoretic union.*

Proof. **(i)⇒(ii).** Suppose I is an α-n-ideal. Then $\alpha^{\leftarrow}\alpha(I) = I$. Let

$x, y \in S$ with $<x>_n^* = <y>_n^*$ and $x \in I$. Then $<x>_n^* \in \alpha(I)$ and so $<y>_n^* \in \alpha(I)$. This implies $y \in \alpha^{\leftarrow}(\alpha(I)) = I$.

(ii)⇒(i). Suppose (ii) holds and I is any n-ideal. $I \subseteq \alpha^{\leftarrow}\alpha\ (I)$ always holds. Thus suppose $x \in \alpha^{\leftarrow}\alpha\ (I)$. Then $<x>_n^* \in \alpha(I)$. This implies $<x>_n^* = <y>_n^*$ for some $y \in I$. Then by (ii), $x \in I$. Therefore $\alpha^{\leftarrow}\alpha\ (I) \subseteq I$ and so $\alpha^{\leftarrow}\alpha\ (I) = I$; in other words I is an α-n-ideal.

(ii) ⇒ (iii). Clearly $I \subseteq \bigcup_{x \in I} <x>_n^{**}$. Now let $x \in I$ and $y \in <x>_n^{**}$.

Then $<x>_n^* \subseteq <y>_n^*$. Thus $<y>_n^* = <x>_n^* \veebar <y>_n^* = <m(x, n, y)>_n^*$. Since $x \in I$, so by convexity $x \wedge n \leq m(x, n, y) \leq x \vee n$ implies $m(x,n,y) \in I$. Hence by (ii) $y \in I$ which implies $<x>_n^{**} \subseteq I$ and so $\bigcup_{x \in I} <x>_n^{**} \subseteq I$.

Therefore, (iii) holds.

(iii)⇒(ii). Suppose (iii) holds and $<x>_n^* = <y>_n^*$ with $x \in I$.

Then $<x>_n^{**} = <y>_n^{**}$. This implies $y \in <y>_n^{**} = <x>_n^{**}$. Hence by (iii), $y \in \bigcup_{x \in I} <x>_n^{**} = I$ and so (ii) holds. □

By [12] and [52], we know that every minimal prime n-ideal is an α-ideal. Here we extend the result.

Proposition 5.2.4. *For a central element* n *of a distributive nearlattice every minimal prime* n-*ideal is an* α-n-*ideal.*

Proof. Let P be a minimal prime n-ideal. Suppose $x \in \alpha^{\leftarrow}\alpha\ (P)$. Then $<x>_n^* \in \alpha(P)$. So $<x>_n^* = <y>_n^*$ for some $y \in P$. Since P is minimal so by Theorem 2.1.4, $<y>_n^{**} \subseteq P$. Thus, $<x>_n^{**} \subseteq P$. This implies $x \in <x>_n^{**} \subseteq P$. Therefore, $\alpha^{\leftarrow}\alpha\ (P) \subseteq P$. Since the reverse inclusion is trivial, so $\alpha^{\leftarrow}\alpha\ (P) = P$. Hence P is α-n-ideal. □

Recall that for prime n-ideal P of a distributive nearlattice S, $n(P) = \{y \in S: m(y, n, x) = n \text{ for some } x \in S - P\}$. Clearly n(P) is an n-ideal and $n(P) \subseteq P$. We already know from [12] that 0(P) is an α-ideal. Following result is an extension of it.

Proposition 5.2.5. *For a prime* n-*ideal* P, n(P) *is an* α-n-*ideal.*

Proof. Let $x \in \alpha^{\leftarrow}\alpha(n(P))$. Then $\langle x \rangle_n^* \in \alpha(n(P))$. Thus $\langle x \rangle_n^* = \langle y \rangle_n^*$ for some $y \in n(P)$. Then $m(y, n, t) = n$ for some $t \in S - P$. This implies $\langle y \rangle_n \cap \langle t \rangle_n = \{n\}$ and so $\langle t \rangle_n \subseteq \langle y \rangle_n^* = \langle x \rangle_n^*$. Therefore, $\langle x \rangle_n^{**} \subseteq \langle t \rangle_n^*$. Thus, $x \in \langle x \rangle_n^{**} \subseteq \langle t \rangle_n^*$ which implies $m(x, n, t) = n$ and so $x \in n(P)$. Hence $\alpha^{\leftarrow}\alpha(n(P)) \subseteq P$. Since the reverse inclusion is trivial, so $\alpha^{\leftarrow}\alpha(n(P)) = n(P)$. Therefore, n(P) is an α-n-ideal. □

Following lemma is needed to prove our next theorem. Latif [35] and Akhter [59] have proved that for a central element n of S, $P_n(S)$ is disjunctive if and only if $\langle x \rangle_n = \langle x \rangle_n^{**}$ for each $x \in S$. Here is a slight improvement of that result.

Lemma 5.2.6. *For a central element* n *of a distributive nearlattice* S, $P_n(S)$ *is disjunctive if and only if* $\langle x \rangle_n^* = \langle y \rangle_n^*$ *implies* $\langle x \rangle_n = \langle y \rangle_n$ *for some* $x, y \in S$.

Proof. Suppose $\langle x \rangle_n^* = \langle y \rangle_n^*$ implies $\langle x \rangle_n = \langle y \rangle_n$. Since for $x \in S$ $\langle x \rangle_n \subseteq \langle x \rangle_n^{**}$ always holds, so suppose $y \in \langle x \rangle_n^{**}$. Then $\langle y \rangle_n^* \supseteq \langle x \rangle_n^*$. Thus, $\langle x \rangle_n^* = \langle x \rangle_n^* \cap \langle y \rangle_n^*$ $= [x \wedge y \wedge n, x \vee y \vee n]^* = \langle t \rangle_n^*$, as n is central. Then by the given condition, $\langle x \rangle_n = \langle t \rangle_n$. Thus $\langle x \rangle_n = [x \wedge y \wedge n, x \vee y \vee n]$ and so by convexity, $y \in \langle x \rangle_n$.

Therefore, $< x >_n^{**} \subseteq < x >_n$, and so $< x >_n =< x >_n^{**}$.

Hence by [59], $P_n(S)$ is disjunctive.

Conversely, let $P_n(S)$ be disjunctive. Then for each $x \in S$, $< x >_n =< x >_n^{**}$.

Therefore, for $x, y \in S$, $< x >_n^{*} = < y >_n^{*}$ implies $< x >_n^{**} = < y >_n^{**}$, and so $< x >_n = < y >_n$. □

Now we given a generalization of [52, Proposition 2.6].

Theorem 5.2.7. *Let* S *be a distributive nearlattice with* n *as a central element. Then the following conditions are equivalent.*

(i) *Each prime* n-*ideal is an* α-n-*ideal.*

(ii) *Each* n-*ideal is an* α-n-*ideal.*

(iii) $P_n(S)$ *is disjunctive.*

Proof. (i)⇒(ii). Suppose I is an n-ideal. Then by [59],

$I = \cap \{P: P \supseteq I, P \text{ prime n-ideals}\}$. Then $\alpha^{\leftarrow}\alpha (I) = \alpha^{\leftarrow}\alpha[\cap\{P\ P \supseteq I\}]$

$= \cap \{\alpha^{\leftarrow}\alpha(P): P \supseteq I\} = \cap \{P: P \supseteq I\} = I$ (by (i)).

Therefore, (ii) holds.

(ii)⇒(i) is trivial.

(ii)⇒(iii). Suppose $< x>_n^{*} = < y >_n^{*}$ for some $x, y \in S$.

Since by (ii), $< x >_n$ is an α-n-ideal and $x \in < x >_n$, so by Theorem 5.2.3, $y \in < x >_n$. Thus $< y >_n \subseteq < x >_n$. Similarly, $< x >_n \subseteq < y >_n$.

Therefore, $< x >_n = < y >_n$, and so by Lemma 5.2.6, $P_n(S)$ is disjunctive.

(iii)⇒(ii). Let I be an n-ideal. Suppose $x \in \alpha^{\leftarrow}\alpha (I)$. Then $< x >_n^{*} \in \alpha(I)$ and so $< x>_n^{*} = < y >_n^{*}$ for some $y \in I$. Thus by (iii), $< x >_n = < y >_n$, which implies $x \in I$. Therefore, $\alpha^{\leftarrow}\alpha (I) \subseteq I$. Since the reverse inclusion is trivial, so $\alpha^{\leftarrow}\alpha (I) = I$ and I is an α-n-ideal. □

Proposition 5.2.2. implies that there is an order isomorphism between the prime α-n-ideals of S and the prime filters of $A_n(S)$. It is not hard to show that each α-n-ideal is an intersection of prime α-n-ideals.

Following theorem is a generalization of [12, Theorem 3.6] and [52, Theorem 2.9]. For this we need the following lemma. It was proved for bounded lattices in [41] and announced in general in [40] ; an explicit proof is given in [26, P-276].

Lemma 5.2.8. *A distributive nearlattice with* 0 is *relatively complemented if and only if its every prime filter is an ultra-filter* (*proper and maximal*).

Theorem 5.2.9. *Let* S *be a distributive nearlattice with* n *as a central element. Then the following conditions are equivalent.*

(i) $P_n(S)$ *is sectionally quasi-complemented.*

(ii) *Each prime* α-n-*ideal is a minimal prime* n-*ideal.*

(iii) *Each* α-n-*ideal is an intersection of minimal prime* n-*ideals. Moreover, the above conditions are equivalent to* $P_n(S)$ *being quasi-complemented if and only if there is an element* $d \in S$ *such that* $< d >_n^* = \{n\}$.

Proof. (i)⇒(ii). Suppose $P_n(S)$ is sectionally quasi-complemented. Then by Theorem 5.1.10, $A_n(S)$ is relatively complemented. Hence its every prime filter is an ultra-filter. Then by Proposition 5.2.2. each prime α-n-ideal is a minimal prime n-ideal.

(ii)⇒(iii). From the isomorphism between the prime α-n-ideals of S and the prime filters of $A_n(S)$. We see that each α-n-ideal is an intersection of prime α-n-ideals. This shows (ii)⇒(iii).

(iii)⇒(ii) is obvious.

(ii)⇒(i). suppose (ii) holds. Then by Proposition 5.2.2, each prime filter of $A_n(S)$ is maximal. Then by Lemma 5.2.8, $A_n(S)$ is relatively complemented, and so by Theorem 5.1.11, $P_n(S)$ is sectionally quasi-complemented. □

We conclude the thesis with the following result which is a generalization of [12, Theorem 3.7] and [52, Theorem 2.9].

Theorem 5.2.10. *Let* S *be a distributive nearlattice and* $n \in S$ *is central. Then* $P_n(S)$ *is generalized Stone if and only if each prime* n-*ideal contains a unique prime* α-n-*ideal.*

Proof. Since minimal prime n-ideals are α-n-ideals, so by a given condition, every prime n-ideal contains a unique minimal prime n-ideal. Hence by Theorem 2.2.7, $P_n(S)$ is normal. Also, by the given condition each prime α-n-ideal contains a unique prime α-n-ideal. Since each minimal prime n-ideal is a prime α-n-ideal, so each prime α-n-ideal is itself a minimal prime n-ideal. Hence by Theorem 5.2.9, $P_n(S)$ is sectionally quasi-complemented. Therefore, by [52(a), Theorem 2.3], $P_n(S)$ is generalized Stone.

Conversely, if $P_n(S)$ is generalized Stone, then by [52(a), Theorem 2.1.7], $P_n(S)$ is normal, and so by Theorem 2.2.7, each prime n-ideal contains a unique minimal prime n-ideal. Thus the result follows as each minimal prime n-ideal is a prime α-n-ideal. □

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