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On Some New Approximate Solutions of Fourth Order Nonlinear Differential Equations

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ON SOME NEW APPROXIMATE SOLUTIONS OF FOURTH ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Declaration

The thesis entitled 'On Some New Approximate Solutions of Fourth Order Nonlinear Differential Equations' is written by me and has been submitted in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi, Bangladesh. Here I confirm that this research work is an original one and it has not been submitted elsewhere for any degree.

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CERTIFICATE

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Abstract

Most of the perturbation methods are developed to find periodic solutions of nonlinear systems; transients are not considered. First Krylov and Bogoliubov introduced a perturbation method to discuss the transients in the second order autonomous systems with small nonlinearities. The method is well known as an "asymptotic averaging method" in the theory of nonlinear oscillations. Later, the method has been amplified and justified by Bogoliubov and Mitropolskii. In this dissertation, we have modified and extended the Krylov-Bogoliubov-Mitropolskii (KBM) method to investigate some fourth order nonlinear systems.

First a fourth order over-damped nonlinear autonomous differential system is considered and a new perturbation solution is developed. Then a method is developed to find asymptotic solution of damped oscillatory nonlinear systems. We then again solve the fourth order over-damped nonlinear systems under some special conditions. Later, unified KBM method is used to obtain the approximate solution of the fourth order ordinary differential equation with small nonlinearities, when a pair of eigen-values of the unperturbed equation is a multiple (*i. e.*, double, triple etc.) of the other pair or pairs. In case of oscillatory processes some of the natural frequencies of the unperturbed equation may be in integral ratio and thus internal resonance is introduced, which is an interesting and important part of nonlinear vibrations. Modified and compact form of KBM method is used to find approximate solutions of fourth order nonlinear systems with large damping. The methods are illustrated by several examples.

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Introduction

Numerous physical, mechanical, chemical, biological, biochemical and some economic laws and relations appear mathematically in the form of differential equations which are linear or nonlinear, autonomous or non-autonomous. Practically, all differential equations involving physical phenomena are nonlinear. Methods of solutions of linear differential equations are comparatively easy and well established. On the contrary, the techniques of solutions of nonlinear differential equations are less available and in general, linear approximations are frequently used. The method of small oscillations is a well-known example of the linearizatoin of problems, which are essentially nonlinear. With the discovery of numerous phenomena of self-excitation of circuits containing nonlinear conductors of electricity, such as electron tubes, gaseous discharge, etc., and in many cases of nonlinear mechanical vibrations of special types, the method of small oscillations becomes inadequate for their analytical treatment. There exists an important difference between the phenomena which oscillate in steady state and the phenomena governed by linear differential equations with constant coefficients, e. g., oscillations of a pendulum with small amplitudes, in that the amplitude of the ultimate stable oscillation seems to be entirely independent of the initial conditions, whereas in oscillations governed by a linear differential equation with constant coefficients, it depends upon the initial conditions.

Van der pol first paid attention to the new (self-excitation) oscillation and indicated that their existence is inherent in the nonlinearity of the differential equations characterizing the process. This nonlinearity appears, thus, as the very essence of these phenomena and by linearizing the differential equation in the sense of the method of small oscillations, one simply eliminates the possibility of investigating such problems. Thus it is necessary to deal with the nonlinear problems directly instead of evading them by dropping the nonlinear terms. To solve nonlinear differential equations there exist some methods. Among the methods, the method of perturbations, *i. e.*, asymptotic expansions in terms of a small parameter, are foremost. According to these techniques, the solutions are presented by the first two terms to avoid rapidly growing algebraic complexity. Although these perturbation expansions may be divergent, they can be more useful for qualitative and quantitative representations than the expansions that are uniformly convergent.

Now, the perturbation methods are used widely in science to obtain approximate solutions based on known exact solutions to nearby problems. Such asymptotic techniques can also be used to provide initial guesses for numerical approximations, and they can now be generated through clever use of symbolic computation. The perturbation method is most effectively used to analyze problems in fluid and solid mechanics, control theory and celestial mechanics, and a variety of nonlinear oscillation, nonlinear wave propagation, and reaction-diffusion systems arising in numerous physical and biological contexts.

In this dissertation, we shall discuss problems that can be described by the dynamical systems of the fourth order nonlinear autonomous differential equations with small nonlinearities by use of the Krylov-Bogoliubov-Mitropolskii (KBM) method. An important approach to study such nonlinear oscillatory problems is the small parameter expansion. Two widely spread methods in this theory are mainly used; one is averaging, particularly the KBM technique and the other is multi-time scale method. According to the KBM technique the solution starts with the solution of linear equation, termed as generating solution, assuming that, in the nonlinear case, the amplitude and phase of the solution of the linear differential equation are time-dependent functions rather than constants. This method introduces an additional condition on the first derivative of the generating solution for determining the solution of a second order equation. Originally, the method was developed (by Krylov-Bogoliubov) to obtain the periodic solutions of second order nonlinear differential equations. Now, the method is used to obtain oscillatory, damped oscillatory and non-oscillatory solutions of second, third, fourth etc. order nonlinear differential equations by imposing some restrictions to make the solutions uniformly valid.

Most of the authors, found solutions of second order nonlinear systems. Only a small number of authors investigated solutions, considering a fourth order nonlinear differential equation. In this dissertation, fourth order nonlinear differential equations, describing oscillatory, damped oscillatory and non-oscillatory systems are considered and their solutions are investigated.

It is customary in the KBM method that correction terms (*i.e.*, the terms with small parameter) in the solution is free from the first harmonics. KBM demanded that such asymptotic solutions are free from secular terms. These assumptions are certainly valid for the second and third order equations. But for the fourth order equation the correction terms sometimes contain secular terms, although the solution is generated by the classical KBM asymptotic method. As a result, the traditional solutions fail to explain the real situation of the systems. In order to prevent the appearances of secular terms and thus to obtain the desired results, we need to impose some additional conditions. The main objective of this dissertation is to find out these limitations and determine the proper solutions under some special conditions. The results may be used in mechanics, physics, chemistry, plasma physics, circuit and control theory, population dynamics etc.

Chapter 1

The Survey and the Proposal

1.1 The Survey

The characteristics of nonlinear differential equations are peculiar. But mathematical formulations of many physical problems often result in differential equations that are nonlinear. However, in many cases it is possible to replace a nonlinear differential equation with a related linear differential equation that approximates the actual equation closely enough to give useful results. Often such linearizatoin is not possible or feasible; when it is not, the original nonlinear equation itself must be tackled.

During the last several decades a number of Russian scientists, like, Mandelstam and Papalexi [44], Andronov [7,8], Krylov and Bogoliubov [34], Bogoliubov and Mitropolskii [13] worked jointly and investigated nonlinear mechanics. Among them, Krylov and Bogoliubov are certainly to be found most active.

Krylov and Bogoliubov considered primarily equations of the form

$$\ddot{x} + \omega^2 x = \varepsilon f(t, x, \dot{x}, \varepsilon) \tag{1.1}$$

where ε is a small positive quantity and f is a power series in ε, whose coefficients are polynomials in $x, \dot{x}, \sin t, \cos t$. In fact, generally f contains neither ε nor t. Similar equations are well known in astronomy and have been the object of systematic investigations by Lindstedt [41,42], Gylden [32], Liapounoff [39] and, above all by Poincare [65]. In general sense, it seems that, Krylov and Bogoliubov apply the same methods. However, the applications in which they view are quite different, being mainly in engineering, technology or physics, notably electrical circuit theory. The method has also been used in plasma physics, theory of oscillations and control theory.

In the treatment of nonlinear oscillations, by perturbation method, Lindstedt [41,42], Gylden [32], Liapounoff [39], Poincare [65], discussed only periodic solutions, transients were not considered. Krylov and Bogoliubov (KB) first discussed transient response. The method of KB starts with the solution of the linear equation, assuming that, in the nonlinear case, the amplitude and phase in the solution of the linear equation are time dependent functions rather than constants. This procedure introduces an additional condition on the first derivative of the assumed solution for determining the solution.

Extensive uses have been made and some important works are done by Stoker [97], McLachlan [45], Minorsky [48], Nayfeh [55,56], Bellman *et al* [12].

Most probably, Poisson initiated approximate solutions of nonlinear differential equations around 1830 and the technique was introduced by Liouville [43]. Duffing [29] investigated many significant results concerning the periodic solutions of the equation

$$\ddot{x} + 2k\dot{x} + x = -\varepsilon x^3 \tag{1.2}$$

Somewhat different nonlinear phenomena occur when the amplitude of the dependent variable of a dynamical system is less or greater than unity. The damping is negative when the amplitude is less than unity and the damping is positive when the amplitude is greater than unity. The governing equation, like these phenomena is

$$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0 \qquad (1.3)$$

This equation is known as Van der Pol [98] equation. This equation has a very extensive field of application in connection with self-excited oscillations in electron-tube circuits.

Since, in general, f contains neither ε nor t, the equation (1.1) therefore takes the form

$$\ddot{x} + \omega^2 x = \varepsilon f(x, \dot{x}) \qquad (1.4)$$

The method of KB is very similar to that of Van der Pol and related to it. Van der Pol applies the method of variation of constants to the basic solution $x = a\cos\omega t + b\sin\omega t$ of $\ddot{x} + \omega^2 x = 0$, on the other hand KB apply the same method to the basic solution $x = a\cos(\omega t + \varphi)$ of the same equation. Thus in the KB method the varied constants are a and φ, while in the Van der Pol's method the constants are a and b. The method of KB seems more interesting from the point of view of applications, since it deals directly with the amplitude and phase of the quasi-harmonic oscillation.

If $\varepsilon = 0$, then the equation (1.4) reduces to linear equation and its solution is

$$x = a\cos(\omega t + \varphi) \qquad (1.5)$$

where a and φ are arbitrary constants to be determined from initial conditions.

If $\varepsilon \neq 0$, but is sufficiently small, then KB assumed that the solution is still given by (1.5) with the derivative of the form

$$\dot{x} = -a\omega\sin(\omega t + \varphi) \qquad (1.6)$$

where a and φ are functions of t, rather than being constants. Thus the solution of the equation (1.4) is of the form

$$x = a(t)\cos(\omega t + \varphi(t)) \tag{1.7}$$

and the derivative of the solution is of the form

$$\dot{x} = -a(t)\omega\sin(\omega t + \varphi(t)) \tag{1.8}$$

Differentiating the assumed solution (1.7) with respect to t, gives

$$\dot{x} = \dot{a}\cos\psi - a\,\omega\sin\psi - a\dot{\varphi}\sin\psi, \qquad \psi = \omega t + \varphi \tag{1.9}$$

Therefore,

$$\dot{a}\cos\psi - a\dot{\varphi}\sin\psi = 0 \tag{1.10}$$

by using (1.6).

Again differentiating (1.8) with respect to t, gives

$$\ddot{x} = -\dot{a}\omega\sin\psi - a\,\omega^2\cos\psi - a\,\omega\dot{\varphi}\cos\psi \tag{1.11}$$

Substituting (1.11) into the equation (1.4) and using equations (1.7)-(1.8), gives

$$\dot{a}\,\omega\sin\psi + a\omega\dot{\varphi}\cos\psi = -\varepsilon f(a\cos\psi, -a\omega\sin\psi) \tag{1.12}$$

Solving (1.10) and (1.12) for $\dot{a}$ and $\dot{\varphi}$ yields

$$\begin{aligned} \dot{a} &= -\varepsilon f(a\cos\psi, -a\omega\sin\psi)\sin\psi/\omega \\ \dot{\varphi} &= -\varepsilon f(a\cos\psi, -a\omega\sin\psi)\cos\psi/a\omega \end{aligned} \tag{1.13}$$

Thus instead of the single differential equation (1.4) of the second order in the unknown x, we obtain two differential equations of the first order in the unknowns a and φ. Since $\dot{a}$ and $\dot{\varphi}$ are proportional to the small parameter ε; a and φ are slowly varying functions of the time with the period $T = 2\pi/\omega$ and, as a first approximation, they are constants.

Expanding $f(a\cos\psi, -a\omega\sin\psi)\sin\psi$ and $f(a\cos\psi, -a\omega\sin\psi)\cos\psi$ in Fourier series in the total phase ψ, the first approximate solution of (1.4), by averaging (1.13) over one period is

$$\begin{aligned} \langle\dot{a}\rangle &= -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\sin\psi d\psi \\ \langle\dot{\varphi}\rangle &= -\frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi, -a\omega\sin\psi)\cos\psi d\psi \end{aligned} \tag{1.14}$$

where a and φ are independent of time under the integrals.

KB called their method asymptotic in the sense that $\varepsilon \to 0$. An asymptotic series itself is not convergent, but for a fixed number of terms the approximate solution tends to the exact solution as ε tends to zero. It is noted that the term asymptotic is frequently used in the theory of oscillation, also in the sense that $\varepsilon \to \infty$. But in this case the mathematical method is quite different.

Later, this technique has been amplified and justified mathematically by Bogoliubov and Mitropolskii [13], and extended to non-stationary vibrations by Mitropolskii [49]. They assumed the solution of the nonlinear differential equation (1.4) in the form

$$x = a\cos\psi + \varepsilon u_1(a,\psi) + \varepsilon^2 u_2(a,\psi) + \ldots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \tag{1.15}$$

where u_k, $k = 1, 2, \ldots n$ are periodic functions of ψ with a period 2π, and the quantities a and ψ are functions of time t, defined by

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots + \varepsilon^n A_n(a) + O\left(\varepsilon^{n+1}\right) \\ \dot{\psi} &= \omega + \varepsilon B_1(a) + \varepsilon^2 B_2(a) + \ldots + \varepsilon^n B_n(a) + O\left(\varepsilon^{n+1}\right) \end{aligned} \tag{1.16}$$

The functions u_k, A_k and B_k, $k = 1, 2, \ldots, n$ are to be chosen in such a way that the equation (1.15), after replacing a and ψ by the functions defined in equation (1.16), is a solution of the equation (1.4). Since there are no restrictions in choosing the functions A_k and B_k, that generate the arbitrariness in the definitions of the functions u_k [14]. To remove this arbitrariness, the following additional conditions are imposed

$$\begin{aligned} &\int_0^{2\pi} u_k(a,\psi) \cos\psi \, d\psi = 0, \\ &\int_0^{2\pi} u_k(a,\psi) \sin\psi \, d\psi = 0, \end{aligned} \tag{1.17}$$

These conditions guarantee the absence of secular terms in all successive approximations.

Differentiating (1.15) twice with respect to t, substituting x and the derivatives $\dot{x}, \ddot{x}$, utilizing the relations in (1.16), and equating the coefficients of ε^k, $k = 1, 2, \ldots, n$ results a recursive system

$$\omega^2 \left(\frac{\partial^2 u_k}{\partial \psi^2} + u_k \right) = f^{(k-1)}(a,\psi) + 2\omega(a B_k \cos\psi + A_k \sin\psi), \tag{1.18}$$

where

$$f^{0}(a,\psi) = f(a\cos\psi, \quad -\omega a\sin\psi),$$

$$f^{(1)}(a,\psi) = u_1 f_x(a\cos\psi, \quad -\omega a\sin\psi) + \left(A_1\cos\psi - aB_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi}\right)$$

$$\times f_{\dot{x}}(a\cos\psi, \quad -\omega a\sin\psi) + \left(aB_1^{\,2} - A_1\frac{dA_1}{da}\right)\cos\psi + \left(2A_1B_1 - aA_1\frac{dB_1}{da}\right)\sin\psi \qquad (1.19)$$

$$-2\omega\left(A_1\frac{\partial^2 u_1}{\partial a\partial\psi} + B_1\frac{\partial^2 u_1}{\partial\psi^2}\right).$$

It is obvious that f^{k-1} is a periodic function of the variable ψ with period 2π, which depends also on the amplitude a. Therefore, f^{k-1} as well as u_k can be expanded in a Fourier series as

$$f^{(k-1)}(a,\psi) = g_0^{\,(k-1)}(a) + \sum_{n=1}^{\infty} g_n^{\,(k-1)}(a)\cos n\psi + h_n^{\,(k-1)}(a)\sin n\psi$$
$$u_k(a,\psi) = v_0^{\,(k-1)}(a) + \sum_{n=1}^{\infty} v_n^{\,(k-1)}(a)\cos n\psi + w_n^{\,(k-1)}(a)\sin n\psi, \qquad (1.20)$$

where

$$g_0^{\,(k-1)} = \frac{1}{2\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, \quad -\omega a\sin\psi)\,d\psi,$$
$$g_n^{\,(k-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, \quad -\omega a\sin\psi)\cos n\psi\,d\psi, \qquad (1.21)$$
$$h_n^{\,(k-1)} = \frac{1}{\pi}\int_0^{2\pi} f^{(k-1)}(a\cos\psi, \quad -\omega a\sin\psi)\sin n\psi\,d\psi, \qquad n \geq 1$$

Here $v_1^{\,(k-1)} = w_1^{\,(k-1)} = 0$ for all values of k, since both integrals of (1.17) vanish.

Substituting these values into equation (1.18), it becomes

$$\omega^2 v_0^{(k-1)}(a) + \sum_{n=1}^{\infty} \omega^2 (1-n^2)\left[v_n^{(k-1)}(a)\cos n\psi + w_n^{(k-1)}(a)\sin n\psi\right]$$
$$= g_0^{(k-1)}(a) + \left(g_1^{(k-1)}(a) + 2\omega a B_k\right)\cos\psi + \left(h_1^{(k-1)}(a) + 2\omega B\right)\sin\psi \tag{1.22}$$
$$+ \sum_{n=2}^{\infty}\left[g_n^{(k-1)}(a)\cos n\psi + h_n^{(k-1)}(a)\sin n\psi\right]$$

Now equating the coefficients of harmonics of the same order, gives

$$g_1^{(k-1)}(a) + 2\omega a B_k = 0, \qquad h_1^{(k-1)}(a) + 2\omega A_k = 0,$$
$$v_0^{(k-1)}(a) = \frac{g_0^{(k-1)}(a)}{\omega^2}, \quad v_n^{(k-1)}(a) = \frac{g_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \tag{1.23}$$
$$w_n^{(k-1)}(a) = \frac{h_n^{(k-1)}(a)}{\omega^2(1-n^2)}, \quad n \geq 1$$

These are the sufficient conditions to obtain the desired order of approximation. For the first order approximation, we have

$$A_1 = -\frac{h_1^{(1)}(a)}{2\omega} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, \quad -\omega a\sin\psi)\sin\psi\, d\psi,$$
$$B_1 = -\frac{g_1^{(1)}(a)}{2\omega a} = -\frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi, \quad -\omega a\sin\psi)\cos\psi d\psi. \tag{1.24}$$

Therefore, the variational equations in (1.16) become

$$\dot{a} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} f(a\cos\psi, \quad -\omega a\sin\psi)\sin\psi\, d\psi,$$
$$\dot{\psi} = \omega - \frac{\varepsilon}{2\pi\omega a}\int_0^{2\pi} f(a\cos\psi, \quad -\omega a\sin\psi)\cos\psi\, d\psi. \tag{1.25}$$

The equations of (1.25) are similar to the equations in (1.14). Thus the first order solution obtained by Bogoliubov and Mitropolskii [13] is identical to the original solution obtained by KB [34]. In the second case, higher order solution can be found easily. The correction term u_1 is obtained from (1.23) as

$$u_1 = \frac{g_0^{(1)}(a)}{\omega^2} + \sum_{n=2}^{\infty} \frac{g_n^{(1)}(a)\cos n\psi + h_n^{(1)}(a)\cos n\psi}{\omega^2(1-n^2)}. \tag{1.26}$$

The solution (1.15) together with u_1 is known as the first order improved solution in which a and ψ are the solutions of the equation (1.25). If the values of the functions A_1 and B_1 are substituted from (1.24) in the second relation of (1.19), the function $f^{(1)}$, and in the similar way, the unknown functions A_2, B_2 and u_2 can be found. Thus the determination of the higher order approximation is complete.

Volosov [99,100], Museenkov [54] and Zebreiko [101] also obtained higher order effects.

The KB method has been extended by Kruskal [33] to solve the fully nonlinear differential equation

$$\ddot{x} = F(x, \dot{x}, \varepsilon) \tag{1.27}$$

The solution of this equation is based on recurrent relations and is given as the power series of the small parameter.

Cap [28] has studied nonlinear systems of the form

$$\ddot{x} + \omega^2 f(x) = \varepsilon F(x, \dot{x}) \tag{1.28}$$

He solved this equation by using elliptical functions in the sense of Krylov and Bogoliubov.

Later, the method of Krylov-Bogoliubov-Mitropolskii (KBM) has been extended by Popov [66] to damped nonlinear systems

$$\ddot{x} + 2k\dot{x} + \omega^2 x = \varepsilon\, f(x, \dot{x}) \tag{1.29}$$

where $-2k\dot{x}$ is the linear damping force and $0 < k < \omega$. It is noteworthy that, because of the importance of the method [66] in the physical systems, involving damping force, Mendelson [46] and Bojadziev [24] rediscovered Popov's results. In case of damped nonlinear systems the first equation of (1.16) has been replaces by

$$\dot{a} = -ka + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots + \varepsilon^n A_n(a) + O\left(\varepsilon^{n+1}\right) \tag{1.16a}$$

Murty, Deekshatulu and Krishna [52] found a hyperbolic type asymptotic solution of an over-damped system represented by the nonlinear differential equation (1.29) in the sense of KBM method; *i. e.*, in this case $k > \omega$. They used hyperbolic function, $\cosh\varphi$ or $\sinh\varphi$ instead of the harmonic function, $\cos\varphi$, which is used in [13,34,46,66]. In case of oscillatory or damped oscillatory process $\cos\varphi$ may be used arbitrarily for all kinds of initial conditions. But in case of non-oscillatory systems $\cosh\varphi$ or $\sinh\varphi$ should be used depending on the given set of initial conditions [25,52,53]. Murty and Deekshatulu [51] found another asymptotic solution of the over-damped system represented by the equation (1.29), by the method of variation of parameters. Shamsul [87] extended the KBM method to find solutions of over-damped nonlinear systems, when one root becomes much smaller than the other root. Murty [53] has presented a unified KBM method for solving the nonlinear systems represented by the equation (1.29). Bojadziev and Edwards [25] investigated solutions of oscillatory and non-oscillatory systems represented by (1.29) when k and ω are slowly varying functions of time t. Arya and Bojadziev [9,10] examined damped oscillatory systems and time-dependent oscillating systems with slowly varying parameters and delay. Shamsul, Feruj and Shanta [78] extended the Krylov-

Bogoliubov-Mitropolskii method to certain non-oscillatory nonlinear systems with varying coefficients. Later, Shamsul [89] unified the KBM method for solving n-th order nonlinear differential equation with varying coefficients. Sattar [70] has developed an asymptotic method to solve a critically damped nonlinear system represented by (1.29). He has found the asymptotic solution of the system (1.29) in the form

$$x = a(1+\psi) + \varepsilon u_1(a,\psi) + \ldots + \varepsilon^n u_n(a,\psi) + O(\varepsilon^{n+1}), \tag{1.30}$$

where a is defined in the equation (1.16a) and ψ is defined by

$$\dot{\psi} = 1 + \varepsilon C_1(a) + \ldots + \varepsilon^n C_n(a) + O(\varepsilon^{n+1}), \tag{1.16b}$$

Shamsul [75] has developed an asymptotic method for the second-order over-damped and critically damped nonlinear systems. Shamsul [84,90] has also extended the KBM method for certain non-oscillatory nonlinear systems when the eigen-values of the unperturbed equation are real and non-positive. Shamsul [77] has presented a new perturbation method based on the work of Krylov-Bogoliubov-Mitropolskii to find approximate solutions of nonlinear systems with large damping. Later, he extended the method to an n-th order nonlinear differential systems [81]. Shamsul, Bellal and Shanta [79] investigated perturbation solution of a second order time-dependent nonlinear system based on the modified Krylov-Bogoliubov-Mitropolskii method.

Making use of the KBM method, Bojadziev [15] has investigated nonlinear damped oscillatory systems with small time lag. Bojadziev [20] has also found solutions of damped forced nonlinear vibrations with small time delay. Bojadziev [22], Bojadziev and Chan [23] applied the KBM method to problems of population dynamics. Bojadziev [24] used the KBM method to investigate nonlinear biological and biochemical systems. Lin and Khan [40] have also used the KBM method to some biological problems. Proskurjakov [67], Bojadziev, Lardner and Arya [16] have investigated periodic solutions of nonlinear systems by the KBM and Poincare method, and compared the two solutions. Bojadziev and Lardner [17,18] have investigated monofrequent oscillations in mechanical systems including the case of internal resonance, governed by hyperbolic differential equation with small nonlinearities. Bojadziev and Lardner [19] have also investigated hyperbolic differential equations with large time delay. Freedman, Rao and Lakshami [30] used the KBM method to study stability, persistence and extinction in a prey-predator system with discrete and continuous time delay. Freedman and Ruan [31] used the KBM method in three-species chain models with group defense.

Most probably, Osiniskii [57], first extended the KBM method to a third order nonlinear differential equation

$$\dddot{x} + k_1 \ddot{x} + k_2 \dot{x} + k_3 x = \varepsilon f(x, \dot{x}, \ddot{x}) \tag{1.31}$$

where ε is a small positive parameter and f is a nonlinear function. Osiniski assumed the asymptotic solution in the form

$$x = a + b\cos\psi + \varepsilon u_1(a,b,\psi) + \ldots + \varepsilon^n u_n(a,b,\psi) + O(\varepsilon^{n+1}), \tag{1.32}$$

where each u_k, $k = 1,2,\ldots,n$ is a periodic function of ψ with period 2π and, a, b and ψ are functions of time t, given by

$$\begin{aligned} \dot{a} &= -\lambda a + \varepsilon A_1(a) + \varepsilon^2 A_2(a) + \ldots + \varepsilon^n A_n(a) + O\left(\varepsilon^{n+1}\right) \\ \dot{b} &= -\mu b + \varepsilon B_1(b) + \varepsilon^2 B_2(b) + \ldots + \varepsilon^n B_n(b) + O\left(\varepsilon^{n+1}\right) \\ \dot{\psi} &= \omega + \varepsilon C_1(b) + \varepsilon^2 C_2(b) + \ldots + \varepsilon^n C_n(b) + O\left(\varepsilon^{n+1}\right) \end{aligned} \tag{1.33}$$

where $-\lambda$, $-\mu \pm \omega$ are the eigen-values of the equation (1.31) when $\varepsilon = 0$.

Osiniskii [58] has also extended the KBM method to a third order nonlinear partial differential equation with internal friction and relaxation. Mulholland [50] studied nonlinear oscillations governed by a third order differential equation. Lardner and Bojadziev [36] investigated nonlinear damped oscillations governed by a third order partial differential equation. They introduced the concept of "couple amplitude" where the unknown functions A_k, B_k and C_k depend on both the amplitudes a and b. Rauch [68] has studied oscillations of a third order nonlinear autonomous system. Bojadziev [26], Bojadziev and Hung [27] used the method of KBM to investigate a 3-dimensional time dependent differential system. Sattar [71] has extended the KBM asymptotic method for three-dimensional over-damped nonlinear systems. Shamsul and Sattar [73] developed a method to solve third order critically damped nonlinear systems. Shamsul [82] redeveloped the method presented in [73] to find approximate solutions of critically damped nonlinear systems in the presence of different damping forces. Later, he unified the KBM method for solving critically damped nonlinear systems [96]. Shamsul and Sattar [76] studied time dependent third order oscillating systems with damping based on an extension of the asymptotic method of Krylov-Bogoliubov-Mitropolskii. Shamsul [85,87], Shamsul, Bellal and Ali Akbar [94] have developed a simple method to obtain the time response of second order over-damped nonlinear systems together with slowly varying coefficients under some special conditions. Later, Shamsul [83], Shamsul and Bellal [88] have extended the method [85,87] to obtain the time response of n-th order ($n \geq 2$), over-damped systems. Shamsul [86] has also developed a method for obtaining non-oscillatory solution of third order nonlinear systems. Shamsul and Sattar [74] presented a unified KBM method for solving third order nonlinear systems. Shamsul [80] has also presented a unified Krylov-Bogoliubov-Mitropolskii method, which is not the formal form of the original KBM method, for solving n-th order nonlinear systems. The solution contains some unusual variables. Yet this solution is very important. Shamsul [92] has also presented a modified and compact form of Krylov-Bogoliubov-Mitropolskii unified method for solving an n-th order nonlinear differential equation. The formula presented in [92] is compact, systematic and practical, and easier than that of [80]. Shamsul [93] developed a general formula based on the extended Krylov-Bogoliubov-Mitropolskii method, for obtaining asymptotic solution of an n-th order time dependent quasi-linear differential equation with damping. Bojadziev [26], Bojadziev and Hung [27] used at least two trial solutions to investigate time dependent differential systems; one is for the resonant case and the other is for the non-resonant case. But Shamsul [93] used only one set of variational equations, arbitrarily for both resonant and non-resonant cases. Shamsul, Ali Akbar and Zahurul [95] presented a general form of the KBM method for solving nonlinear partial differential equations. Raymond and Cabak [69] examined the effects of internal resonance on impulsive forced nonlinear systems with two-degree-of-freedom. Lewis [37,38] investigated stability for an autonomous second-order two-degree-of-freedom system and for a control surface with structural nonlinearities in surface flow.

Andrianov and Awrejcewicz [6], Awrejcewicz and Andrianov [11] present some new trends of asymptotic techniques in application to nonlinear dynamical systems in terms of summation and interpolation methods. In this dissertation, we shall not discuss this technique.

O'Malley [59] found an asymptotic solution of a semiconductor device problem involving reverse bias. O'Malley [60,61,63,64] presented singular perturbation method for ordinary differential equations with matching and used this singular perturbation method to stiff differential equations. He [62] also presented exponential asymptotic for boundary layer resonance and dynamical metastability.

Ali Akbar *et al* [1,2] found an asymptotic solution of the fourth order over-damped and under-damped nonlinear systems based on the work of [80]. The authors [3,4] developed a simple technique for obtaining certain over-damped solution of an n-th order nonlinear differential equation under some special conditions including the case of internal resonance. Ali Akbar *et al* [5] also developed perturbation theory for the fourth order nonlinear systems with large damping.

1.2 The Proposal

We propose a perturbation system of the fourth order nonlinear differential equation

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = \varepsilon\, f(x, \dot{x}, \ddot{x}, \dddot{x}) \qquad (1.34)$$

where ε is a small positive parameter and f is a given nonlinear function.

In **Chapter 2** a new asymptotic solution is investigated for the fourth order over-damped nonlinear systems. A perturbation method for the fourth order damped nonlinear systems is developed in **Chapter 3**. **Chapter 4** contains asymptotic solutions for the fourth order over-damped nonlinear systems under some special conditions. Unified KBM method for solving the fourth order nonlinear differential equations with internal resonance is developed in **Chapter 5**, and finally, perturbation method for the fourth order nonlinear systems with large damping is presented in **Chapter 6**.

Chapter 2

A New Technique for Fourth Order Over-damped Nonlinear Systems

2.1 Introduction

Among the methods used to study nonlinear systems with a small non-linearity, Krylov-Bogoliubov-Mitropolskii [13,34] method is particularly convenient and is the most widely used technique to obtain the approximate solution. Originally, the method developed for systems with periodic solutions, was later extended by Popov [66] and Meldelson [46] for damped nonlinear oscillations. Murty *et al* [52] extended the method to solve over-damped nonlinear systems. Murty *et al* [51] developed a method of variation of parameters to obtain the time response of a second order nonlinear over-damped system with a small nonlinearity based on the work of Krylov-Bogoliubov-Mitropolskii. Murty [53] has presented a unified KBM method for solving second order nonlinear systems. Shamsul [75] developed a new perturbation technique based on the work of Krylov-Bogoliubov-Mitropolskii to find approximate solutions, both of over-damped and critically damped nonlinear systems. Shamsul [87] extended the method of Krylov-Bogoliubov-Mitropolskii to solve certain over-damped nonlinear systems. Sattar [71] has studied a third order over-damped nonlinear system. Shamsul [85] developed a method to obtain the time response of third order over-damped nonlinear systems for some special conditions. Later, Shamsul [83] extended the method to n-th order over-damped nonlinear systems. Shamsul and Sattar [73] developed a method to solve third order critically damped nonlinear equations. Shamsul and Sattar [74] has presented a unified KBM method for solving third order nonlinear systems. Recently, Shamsul [80,81] has presented a unified KBM method for solving an n-th order nonlinear differential equation and a perturbation theory for n-th order nonlinear systems with large damping.

The method, presented in [52] is too much laborious and cumbersome. In this Chapter, a new technique, for a fourth order over-damped system is found. The determination of the solution is very simple and easier than Murty *et al* [52]. An example is solved to show the precision and exactness of the method. The results obtained by the present method compare very well with those obtained by the numerical method and those presented by Murty *et al* [52].

2.2 The Method

Consider a weakly nonlinear over damped system governed by the differential equation

$$x^{(4)} + k_1 \dddot{x} + k_2 \ddot{x} + k_3 \dot{x} + k_4 x = \varepsilon f(x, \dot{x}, \ddot{x}, \dddot{x}), \tag{2.1}$$

where $x^{(4)}$ denotes the fourth derivative of x and over dots are used for the first, the second, and the third derivative of x with respect to t, ε is a small parameter, $f(x)$ is the given nonlinear function and k_1, k_2, k_3, k_4 are constants defined by

$$k_1 = \sum_{i=1}^{4} \lambda_i, \ k_2 = \sum_{\substack{i,j=1 \\ i \neq j}}^{4} \lambda_i \lambda_j, \ k_3 = \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{4} \lambda_i \lambda_j \lambda_k \text{ and } k_4 = \prod_{i=1}^{4} \lambda_i . \tag{2.2}$$

Here $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are the real negative eigen-values of the characteristic equation of the unperturbed equation (2.1) for $\varepsilon = 0$. The over-damping force in the system is represented by these real negative eigen-values. The unperturb solution of the equation (2.1) is

$$x(t,0) = \sum_{j=1}^{4} a_{j,0} e^{\lambda_j t} \tag{2.3}$$

where $a_{j,0}$, $j = 1, 2, 3, 4$ are arbitrary constants.

When $\varepsilon \neq 0$, we seek a solution of the nonlinear differential equation (2.1) of the form

$$x(t,\varepsilon)=\sum_{j=1}^{4} a_j(t)e^{\lambda_j t}+\varepsilon u_1(a_1,a_2,a_3,a_4,t)+\varepsilon^2 u_2(a_1,a_2,a_3,a_4,t)+\varepsilon^3\ldots \qquad (2.4)$$

where each a_j, $j=1,2,3,4$ satisfies the differential equation

$$\dot{a}_j(t)=\varepsilon A_j(a_1,a_2,a_3,a_4,t)+\varepsilon^2 B_j(a_1,a_2,a_3,a_4,t)+\varepsilon^3\ldots, \qquad (2.5)$$

Confining only to a first few terms, 1, 2, 3,…,m in the series expansions of (2.4) and (2.5), we evaluate the functions $u_1, u_2, \ldots$ and $A_j, B_j, j=1,2,3,4$, such that $a_j(t)$, appearing in (2.4) and (2.5) satisfy the given differential equation (2.1) with an accuracy of ε^{m+1}. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first [53]. In order to determine these functions it is assumed that the functions $u_1, u_2, \ldots$ do not contain the fundamental terms which are included in the series expansion (2.4) of order ε^0.

Differentiating (2.4) four times with respect to t, substituting x and the derivatives $x^{(4)}$, $\dddot{x}$, $\ddot{x}$, $\dot{x}$ in the original equation (2.1), utilizing the relations in (2.5) and equating the coefficients of ε, we obtain

$$\prod_{j=1}^{4}\left(\frac{d}{dt}-\lambda_j\right)u_1+\sum_{j=1}^{4} e^{\lambda_j t}\left(\prod_{k=1,j\neq k}^{4}\left(\frac{d}{dt}+\lambda_j-\lambda_k\right)\right)A_j=f^{(0)}(a_1,a_2,a_3,a_4,t), \qquad (2.6)$$

where $f^{(0)}=f(x_0)$ and $x_0=\sum_{j=1}^{4} a_j(t)e^{\lambda_j t}$.

In general, the function $f^{(0)}$ can be expanded in a Taylor series as

$$f^{(0)}=\sum_{m_1=-\infty,m_2=-\infty,m_3=-\infty,m_4=-\infty}^{\infty,\infty,\infty,\infty} F_{m_1,m_2,m_3,m_4}\, a_1^{m_1} a_2^{m_2} a_3^{m_3} a_4^{m_4} e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t}.$$

According to our assumption u_1 does not contain the fundamental terms, therefore (2.6) can be separated into five equations for unknown functions u_1 and A_1, A_2, A_3, A_4 (see [80] for details). Substituting the functional values of $f^{(0)}$ and equating the coefficients of $e^{\lambda_j t}$, $j = 1, 2, 3, 4$, we obtain

$$e^{\lambda_1 t}\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_2\right)\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_4\right)A_1$$
$$= F_{m_1,m_2,m_3,m_4} a_1^{m_1} a_2^{m_2} a_3^{m_3} a_4^{m_4} e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t}; \quad m_3 = m_4, \quad m_1 = m_2 + 1 \tag{2.7}$$

$$e^{\lambda_2 t}\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_1\right)\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_4\right)A_2$$
$$= F_{m_1,m_2,m_3,m_4} a_1^{m_1} a_2^{m_2} a_3^{m_3} a_4^{m_4} e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t}; \quad m_3 = m_4, \quad m_1 = m_2 - 1 \tag{2.8}$$

$$e^{\lambda_3 t}\left(\frac{\partial}{\partial t}+\lambda_3-\lambda_1\right)\left(\frac{\partial}{\partial t}+\lambda_3-\lambda_2\right)\left(\frac{\partial}{\partial t}+\lambda_3-\lambda_4\right)A_3$$
$$= F_{m_1,m_2,m_3,m_4} a_1^{m_1} a_2^{m_2} a_3^{m_3} a_4^{m_4} e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t}; \quad m_1 = m_2, \quad m_3 = m_4 + 1 \tag{2.9}$$

$$e^{\lambda_4 t}\left(\frac{\partial}{\partial t}+\lambda_4-\lambda_1\right)\left(\frac{\partial}{\partial t}+\lambda_4-\lambda_2\right)\left(\frac{\partial}{\partial t}+\lambda_4-\lambda_3\right)A_4$$
$$= F_{m_1,m_2,m_3,m_4} a_1^{m_1} a_2^{m_2} a_3^{m_3} a_4^{m_4} e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t}; \quad m_1 = m_2, \quad m_3 = m_4 - 1 \tag{2.10}$$

and

$$\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)\left(\frac{\partial}{\partial t}-\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_4\right)u_1$$
$$= \sum_{m_1=-\infty,m_2=-\infty,m_3=-\infty,m_4=-\infty}^{\infty,\infty,\infty,\infty}{}' F_{m_1,m_2,m_3,m_4} a_1^{m_1} a_2^{m_2} a_3^{m_3} a_4^{m_4} e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t} \tag{2.11}$$

where $\sum'$ exclude those terms for $m_1 = m_2 \pm 1$, $m_3 = m_4 \pm 1$.

The particular solutions of the equations (2.7)-(2.11) give the functions A_1, A_2, A_3, A_4 and u_1. Thus the determination of the first approximate solution is complete.

2.3 Example

Consider the fourth order differential equation

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = \varepsilon x^3 . \qquad (2.12)$$

For equation (2.12), we have, $f^{(0)} = (a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} + a_3 e^{\lambda_3 t} + a_4 e^{\lambda_4 t})^3$.

or,

$$\begin{aligned} f^{(0)} &= a_1^{\ 3} e^{3\lambda_1 t} + a_2^{\ 3} e^{3\lambda_2 t} + a_3^{\ 3} e^{3\lambda_3 t} + a_4^{\ 3} e^{3\lambda_4 t} + 3a_1^{\ 2} a_2 e^{(2\lambda_1+\lambda_2)t} + 3a_1^{\ 2} a_3 e^{(2\lambda_1+\lambda_3)t} \\ &+ 3a_1^{\ 2} a_4 e^{(2\lambda_1+\lambda_4)t} + 3a_2^{\ 2} a_1 e^{(2\lambda_2+\lambda_1)t} + 3a_2^{\ 2} a_3 e^{(2\lambda_2+\lambda_3)t} + 3a_2^{\ 2} a_4 e^{(2\lambda_2+\lambda_4)t} \\ &+ 3a_3^{\ 2} a_1 e^{(2\lambda_3+\lambda_1)t} + 3a_3^{\ 2} a_2 e^{(2\lambda_3+\lambda_2)t} + 3a_3^{\ 2} a_4 e^{(2\lambda_3+\lambda_4)t} + 3a_4^{\ 2} a_1 e^{(2\lambda_4+\lambda_1)t} \\ &+ 3a_4^{\ 2} a_2 e^{(2\lambda_4+\lambda_2)t} + 3a_4^{\ 2} a_3 e^{(2\lambda_4+\lambda_3)t} + 6a_1 a_2 a_3 e^{(\lambda_1+\lambda_2+\lambda_3)t} \\ &+ 6a_1 a_2 a_4 e^{(\lambda_1+\lambda_2+\lambda_4)t} + 6a_1 a_3 a_4 e^{(\lambda_1+\lambda_3+\lambda_4)t} + 6a_2 a_3 a_4 e^{(\lambda_2+\lambda_3+\lambda_4)t} . \end{aligned} \qquad (2.13)$$

Therefore, the equations (2.7)-(2.11) become

$$e^{\lambda_1 t}\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_2\right)\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_3\right)\left(\frac{\partial}{\partial t} + \lambda_1 - \lambda_4\right)A_1 = 3a_1^{\ 2} a_2 e^{(2\lambda_1+\lambda_2)t} + 6a_1 a_3 a_4 e^{(\lambda_1+\lambda_3+\lambda_4)t} \qquad (2.14)$$

$$e^{\lambda_2 t}\left(\frac{\partial}{\partial t} + \lambda_2 - \lambda_1\right)\left(\frac{\partial}{\partial t} + \lambda_2 - \lambda_3\right)\left(\frac{\partial}{\partial t} + \lambda_2 - \lambda_4\right)A_2 = 3a_2^{\ 2} a_1 e^{(2\lambda_2+\lambda_1)t} + 6a_2 a_3 a_4 e^{(\lambda_2+\lambda_3+\lambda_4)t} \qquad (2.15)$$

$$e^{\lambda_3 t}\left(\frac{\partial}{\partial t} + \lambda_3 - \lambda_1\right)\left(\frac{\partial}{\partial t} + \lambda_3 - \lambda_2\right)\left(\frac{\partial}{\partial t} + \lambda_3 - \lambda_4\right)A_3 = 3a_3^{\ 2} a_4 e^{(2\lambda_3+\lambda_4)t} + 6a_1 a_2 a_3 e^{(\lambda_1+\lambda_2+\lambda_3)t} \qquad (2.16)$$

$$e^{\lambda_4 t}\left(\frac{\partial}{\partial t} + \lambda_4 - \lambda_1\right)\left(\frac{\partial}{\partial t} + \lambda_4 - \lambda_2\right)\left(\frac{\partial}{\partial t} + \lambda_4 - \lambda_3\right)A_4 = 3a_4^{\ 2} a_3 e^{(2\lambda_4+\lambda_3)t} + 6a_1 a_2 a_4 e^{(\lambda_1+\lambda_2+\lambda_4)t} \qquad (2.17)$$

and

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial}{\partial t} - \lambda_2\right)\left(\frac{\partial}{\partial t} - \lambda_3\right)\left(\frac{\partial}{\partial t} - \lambda_4\right)u_1 = a_1^{\ 3} e^{3\lambda_1 t} + a_2^{\ 3} e^{3\lambda_2 t} + a_3^{\ 3} e^{3\lambda_3 t} + a_4^{\ 3} e^{3\lambda_4 t} \\ &+ 3a_1^{\ 2} a_3 e^{(2\lambda_1+\lambda_3)t} + 3a_1^{\ 2} a_4 e^{(2\lambda_1+\lambda_4)t} + 3a_2^{\ 2} a_3 e^{(2\lambda_2+\lambda_3)t} + 3a_2^{\ 2} a_4 e^{(2\lambda_2+\lambda_4)t} \\ &+ 3a_3^{\ 2} a_1 e^{(2\lambda_3+\lambda_1)t} + 3a_3^{\ 2} a_2 e^{(2\lambda_3+\lambda_2)t} + 3a_4^{\ 2} a_1 e^{(2\lambda_4+\lambda_1)t} + 3a_4^{\ 2} a_2 e^{(2\lambda_4+\lambda_2)t} . \end{aligned} \qquad (2.18)$$

Solving equations (2.14)-(2.18) and substituting, $\lambda_1 = -\mu_1 + \omega_1$, $\lambda_2 = -\mu_1 - \omega_1$, $\lambda_3 = -\mu_2 + \omega_2$ and $\lambda_4 = -\mu_2 - \omega_2$, we obtain

$$\begin{aligned}
A_1 &= -\frac{3a_1^2 a_2 e^{-2\mu_1 t}}{2(\mu_1 - \omega_1)(3\mu_1 - \mu_2 - \omega_1 + \omega_2)(3\mu_1 - \mu_2 - \omega_1 - \omega_2)} \\
&\quad - \frac{6a_1 a_3 a_4 e^{-2\mu_2 t}}{2(\mu_2 - \omega_1)(\mu_1 + \mu_2 - \omega_1 + \omega_2)(\mu_1 + \mu_2 - \omega_1 - \omega_2)}, \\
A_2 &= -\frac{3a_1 a_2^2 e^{-2\mu_1 t}}{2(\mu_1 + \omega_1)(3\mu_1 - \mu_2 + \omega_1 - \omega_2)(3\mu_1 - \mu_2 + \omega_1 + \omega_2)} \\
&\quad - \frac{6a_1 a_3 a_4 e^{-2\mu_2 t}}{2(\mu_2 + \omega_1)(\mu_1 + \mu_2 + \omega_1 - \omega_2)(\mu_1 + \mu_2 + \omega_1 + \omega_2)}, \\
A_3 &= -\frac{3a_3^2 a_4 e^{-2\mu_2 t}}{2(\mu_2 - \omega_2)(3\mu_2 - \mu_1 - \omega_2 + \omega_1)(3\mu_2 - \mu_1 - \omega_2 - \omega_1)} \\
&\quad - \frac{6a_1 a_2 a_3 e^{-2\mu_1 t}}{2(\mu_1 - \omega_2)(\mu_1 + \mu_2 + \omega_1 - \omega_2)(\mu_1 + \mu_2 - \omega_1 - \omega_2)}, \\
A_4 &= -\frac{3a_3 a_4^2 e^{-2\mu_2 t}}{2(\mu_2 + \omega_2)(3\mu_2 - \mu_1 + \omega_2 - \omega_1)(3\mu_2 - \mu_1 + \omega_2 + \omega_1)} \\
&\quad - \frac{6a_1 a_2 a_3 e^{-2\mu_1 t}}{2(\mu_1 + \omega_2)(\mu_1 + \mu_2 - \omega_1 + \omega_2)(\mu_1 + \mu_2 + \omega_1 + \omega_2)}.
\end{aligned} \tag{2.19}$$

Substituting the values of (2.19) into equation (2.5), we obtain

$$\dot{a}_1 = -\varepsilon\left(\frac{3a_1^2 a_2 e^{-2\mu_1 t}}{2(\mu_1 - \omega_1)(3\mu_1 - \mu_2 - \omega_1 + \omega_2)(3\mu_1 - \mu_2 - \omega_1 - \omega_2)} + \frac{6a_1 a_3 a_4 e^{-2\mu_2 t}}{2(\mu_2 - \omega_1)(\mu_1 + \mu_2 - \omega_1 + \omega_2)(\mu_1 + \mu_2 - \omega_1 - \omega_2)}\right),$$

$$\dot{a}_2 = -\varepsilon\left(\frac{3a_1 a_2^2 e^{-2\mu_1 t}}{2(\mu_1 + \omega_1)(3\mu_1 - \mu_2 + \omega_1 - \omega_2)(3\mu_1 - \mu_2 + \omega_1 + \omega_2)} + \frac{6a_1 a_3 a_4 e^{-2\mu_2 t}}{2(\mu_2 + \omega_1)(\mu_1 + \mu_2 + \omega_1 - \omega_2)(\mu_1 + \mu_2 + \omega_1 + \omega_2)}\right),$$

$$\dot{a}_3 = -\varepsilon\left(\frac{3a_3{}^2 a_4 e^{-2\mu_2 t}}{2(\mu_2-\omega_2)(3\mu_2-\mu_1-\omega_2+\omega_1)(3\mu_2-\mu_1-\omega_2-\omega_1)}\right.$$
$$\left.+\frac{6a_1a_2a_3e^{-2\mu_1 t}}{2(\mu_1-\omega_2)(\mu_1+\mu_2+\omega_1-\omega_2)(\mu_1+\mu_2-\omega_1-\omega_2)}\right), \tag{2.20}$$

$$\dot{a}_4 = -\varepsilon\left(\frac{3a_3 a_4{}^2 e^{-2\mu_2 t}}{2(\mu_2+\omega_2)(3\mu_2-\mu_1+\omega_2-\omega_1)(3\mu_2-\mu_1+\omega_2+\omega_1)}\right.$$
$$\left.+\frac{6a_1a_2a_4e^{-2\mu_1 t}}{2(\mu_1+\omega_2)(\mu_1+\mu_2-\omega_1+\omega_2)(\mu_1+\mu_2+\omega_1+\omega_2)}\right).$$

Substituting $a_1 = \frac{a}{2}e^{\varphi_1}$, $a_2 = \frac{a}{2}e^{-\varphi_1}$, $a_3 = \frac{b}{2}e^{\varphi_2}$ and $a_4 = \frac{-b}{2}e^{-\varphi_2}$ into equation (2.20) and then simplifying, we obtain

$$\dot{a} = \varepsilon\left(l_1 a^3 e^{-2\mu_1 t} + l_2 a b^2 e^{-2\mu_2 t}\right),$$
$$\dot{b} = \varepsilon\left(m_1 b^3 e^{-2\mu_2 t} + m_2 a^2 b e^{-2\mu_1 t}\right),$$
$$\dot{\varphi}_1 = \varepsilon\left(n_1 a^2 e^{-2\mu_1 t} + n_2 b^2 e^{-2\mu_2 t}\right), \tag{2.21}$$
$$\dot{\varphi}_2 = \varepsilon\left(r_1 b^2 e^{-2\mu_2 t} + r_2 a^2 e^{-2\mu_1 t}\right).$$

where

$$l_1 = -\frac{3}{8}\left(\frac{\mu_1\{(3\mu_1-\mu_2)^2+\omega_1{}^2-\omega_2{}^2\}+2(3\mu_1-\mu_2)\omega_1{}^2}{(\mu_1{}^2-\omega_1{}^2)\{(3\mu_1-\mu_2)^2-(\omega_1-\omega_2)^2\}\{(3\mu_1-\mu_2)^2-(\omega_1+\omega_2)^2\}}\right),$$

$$l_2 = \frac{3}{4}\left(\frac{\mu_2\{(\mu_1+\mu_2)^2+\omega_1{}^2-\omega_2{}^2\}+2(\mu_1+\mu_2)\omega_1{}^2}{(\mu_2{}^2-\omega_1{}^2)\{(\mu_1+\mu_2)^2-(\omega_1-\omega_2)^2\}\{(\mu_1+\mu_2)^2-(\omega_1+\omega_2)^2\}}\right),$$

$$m_1 = \frac{3}{8}\left(\frac{\mu_2\{(3\mu_2-\mu_1)^2+\omega_2{}^2-\omega_1{}^2\}+2(3\mu_2-\mu_1)\omega_2{}^2}{(\mu_2{}^2-\omega_2{}^2)\{(3\mu_2-\mu_1)^2-(\omega_2-\omega_1)^2\}\{(3\mu_2-\mu_1)^2-(\omega_2+\omega_1)^2\}}\right),$$

$$m_2 = -\frac{3}{4}\left(\frac{\mu_1\{(\mu_1+\mu_2)^2+\omega_2{}^2-\omega_1{}^2\}+2(\mu_1+\mu_2)\omega_2{}^2}{(\mu_1{}^2-\omega_2{}^2)\{(\mu_1+\mu_2)^2-(\omega_2-\omega_1)^2\}\{(\mu_1+\mu_2)^2-(\omega_2+\omega_1)^2\}}\right), \tag{2.22}$$

$$n_1 = -\frac{3}{8}\left(\frac{\{2\mu_1(3\mu_1-\mu_2)+(3\mu_1-\mu_2)^2+\omega_1{}^2-\omega_2{}^2\}\omega_1}{(\mu_1{}^2-\omega_1{}^2)\{(3\mu_1-\mu_2)^2-(\omega_1-\omega_2)^2\}\{(3\mu_1-\mu_2)^2-(\omega_1+\omega_2)^2\}}\right),$$

$$n_2 = \frac{3}{4}\left(\frac{\{2\mu_2(\mu_1+\mu_2)+(\mu_1+\mu_2)^2+\omega_1{}^2-\omega_2{}^2\}\omega_1}{(\mu_2{}^2-\omega_1{}^2)\{(\mu_1+\mu_2)^2-(\omega_1-\omega_2)^2\}\{(\mu_1+\mu_2)^2-(\omega_1+\omega_2)^2\}}\right),$$

$$r_1 = \frac{3}{8}\left(\frac{\{2\mu_2(3\mu_2-\mu_1)+(3\mu_2-\mu_1)^2+\omega_2{}^2-\omega_1{}^2\}\omega_2}{(\mu_2{}^2-\omega_2{}^2)\{(3\mu_2-\mu_1)^2-(\omega_2-\omega_1)^2\}\{(3\mu_2-\mu_1)^2-(\omega_2+\omega_1)^2\}}\right),$$

$$r_2 = -\frac{3}{4}\left(\frac{\{2\mu_1(\mu_1+\mu_2)+(\mu_1+\mu_2)^2+\omega_2{}^2-\omega_1{}^2\}\omega_2}{(\mu_1{}^2-\omega_2{}^2)\{(\mu_1+\mu_2)^2-(\omega_2-\omega_1)^2\}\{(\mu_1+\mu_2)^2-(\omega_2+\omega_1)^2\}}\right),$$

and

$$\begin{aligned} u_1 &= \frac{1}{16}\sum_{i,j=1,i\neq j}^{2} a^{-i+2j}b^{2i-j}\cosh 3(\omega_i t+\varphi_i)g_{i,j}e^{-3\mu_i t} \\ &+\frac{1}{16}\sum_{i,j=1,i\neq j}^{2} a^{-i+2j}b^{2i-j}\sinh 3(\omega_i t+\varphi_i)h_{i,j}e^{-3\mu_i t} \\ &+\frac{3ab}{16}\sum_{i,j=1,i\neq j}^{2} a^{j-1}b^{i-1}\cosh(2\omega_i t\pm\omega_j t+2\varphi_i\pm\varphi_j)c_{i,j}{}^{\pm}\ e^{-(2\mu_i+\mu_j)t} \\ &+\frac{3ab}{16}\sum_{i,j=1,i\neq j}^{2} a^{j-1}b^{i-1}\sinh(2\omega_i t\pm\omega_j t+2\varphi_i\pm\varphi_j)d_{i,j}{}^{\pm}\ e^{-(2\mu_i+\mu_j)t}, \end{aligned} \tag{2.23}$$

where

$$g_{i,j} = \frac{\mu_i{}^2(3\mu_i-\mu_j)+\mu_i{}^2(9\omega^2{}_i-\omega_j{}^2)+18\mu_i(3\mu_i-\mu_j)\omega_j{}^2+2(3\mu_i-\mu_j)^2\omega_j{}^2}{(\mu_i{}^2-\omega_i{}^2)(\mu_i{}^2-4\omega_i{}^2)\{(3\mu_i-\mu_j)^2-(3\omega_i-\omega_j)^2\}\{(3\mu_i-\mu_j)^2-(3\omega_i+\omega_j)^2\}},$$

$$h_{i,j} = \frac{3\omega_i\{2\mu_i{}^2(3\mu_i-\mu_j)+\mu_i(3\mu_i-\mu_j)^2+\mu_i(21\omega_i{}^2-\omega_j{}^2)-4\omega_i{}^2\mu_j\}}{(\mu_i{}^2-\omega_i{}^2)(\mu_i{}^2-4\omega_i{}^2)\{(3\mu_i-\mu_j)^2-(3\omega_i-\omega_j)^2\}\{(3\mu_i-\mu_j)^2-(3\omega_i+\omega_j)^2\}}, \tag{2.24}$$

$$c^{\pm}{}_{i,j} = \frac{\begin{array}{c}\mu_i{}^2(\mu_1+\mu_2)^2+\mu_i{}^2(12\omega_i{}^2\pm13\omega_1\omega_2+3\omega_j{}^2)+2\mu_1\mu_2(5\omega_1{}^2\pm5\omega_1\omega_2+\omega_j{}^2)\\ +\mu_j{}^2(\omega_i{}^2\pm\omega_1\omega_2)+(\omega_i\pm\omega_j)^2(3\omega_i\pm\omega_j)\omega_i\end{array}}{(\mu_i{}^2-\omega_i{}^2)\{\mu_i{}^2-(\omega_i\pm\omega_j)^2\}\{(\mu_1+\mu_2)^2-(\omega_i\pm\omega_j)^2\}\{(\mu_1+\mu_2)^2-(3\omega_i\pm\omega_j)^2\}},$$

$$d^{\pm}{}_{i,j} = \frac{(2\omega_i\pm\omega_j)\{2\mu_i{}^2(\mu_1+\mu_2)+\mu_i(\mu_1+\mu_2)^2+\mu_i(5\omega_i{}^2\pm6\omega_1\omega_2+\omega_j{}^2)+2\mu_j\omega_i(\omega_i\pm\omega_j)\}}{(\mu_i{}^2-\omega_i{}^2)\{\mu_i{}^2-(\omega_i\pm\omega_j)^2\}\{(\mu_1+\mu_2)^2-(\omega_i\pm\omega_j)^2\}\{(\mu_1+\mu_2)^2-(3\omega_i\pm\omega_j)^2\}}.$$

$i, j = 1, 2.$

Equation (2.21) has no exact solution. Since $\dot{a}$, $\dot{b}$, $\dot{\varphi}_1$ and $\dot{\varphi}_2$ are proportional to the small parameter ε, therefore $\dot{a}$, $\dot{b}$, $\dot{\varphi}_1$ and $\dot{\varphi}_2$ are slowly varying functions of time *t*, with the period T, and as a first approximation, we may consider them as constant (see also [80] for details).

Thus, we obtain

$$
\begin{aligned}
a &= a_0 + \varepsilon\{l_1 a_0^3 (1-e^{-2\mu_1 t})/\mu_1 + l_2 a_0 b_0^2 (1-e^{-2\mu_2 t})/\mu_2\}/2, \\
b &= b_0 + \varepsilon\{m_1 b_0^3 (1-e^{-2\mu_2 t})/\mu_2 + m_2 a_0^2 b_0 (1-e^{-2\mu_1 t})/\mu_1\}/2, \\
\varphi_1 &= \varphi_1(0) + \varepsilon\{n_1 a_0^2 (1-e^{-2\mu_1 t})/\mu_1 + n_2 b_0^2 (1-e^{-2\mu_2 t})/\mu_2\}/2, \qquad (2.25) \\
\varphi_2 &= \varphi_2(0) + \varepsilon\{r_1 b_0^2 (1-e^{-2\mu_2 t})/\mu_2 + r_2 a_0^2 (1-e^{-2\mu_1 t})/\mu_1\}/2.
\end{aligned}
$$

Therefore, we obtain the first approximate solution of equation (2.12) as

$$x = a e^{-\mu_1 t}\cosh(\omega_1 t + \varphi_1) + b e^{-\mu_2 t}\sinh(\omega_2 t + \varphi_2) + \varepsilon u_1, \qquad (2.26)$$

where a, b, φ_1 and φ_2 are given by (2.25) and u_1 is given by (2.23).

2.4 Murty, Deekshatulu and Krishna's Technique [52]

Murty, Deekshatulu and Krishna [52] considered the equation

$$x^{(4)} + K_3 \dddot{x} + K_4 \ddot{x} + K_5 \dot{x} + K_6 x = \varepsilon f(x) \qquad (2.27)$$

Which is similar to the equation (2.1) and these two equations coincide when $k_1 = K_3$, $k_2 = K_4$, $k_3 = K_5$ and $k_4 = K_6$. Murty *et al* [52] found a solution of the equation (2.27) in the form

$$x(t) = a(t)\cosh\psi_1(t) + b(t)\sinh\psi_2(t) + \varepsilon u_1(a,\psi_1) + \varepsilon v_1(b,\psi_2) + \ldots, \qquad (2.28)$$

where $u_1(a,\psi_1)\ldots$, and $v_1(b,\psi_2)\ldots$, are functions of ψ_1 and ψ_2 respectively, and the quantities a, ψ_1, b and ψ_2 are defined by the differential equations

$$\dot{a} = -K_1 a + \varepsilon A_1(a) + \dots,$$
$$\dot{\psi}_1 = -K_2 + \varepsilon C_1(a) + \dots,$$
$$\dot{b} = -K_7 b + \varepsilon B_1(b) + \dots, \tag{2.29}$$
$$\dot{\psi}_2 = -K_8 + \varepsilon D_1(b) + \dots,$$

and

$$K_1 = \frac{\lambda_1 + \lambda_2}{2}, \ K_2 = \frac{\lambda_1 - \lambda_2}{2}, \ K_7 = \frac{\lambda_3 + \lambda_4}{2}, \ K_8 = \frac{\lambda_3 - \lambda_4}{2}. \tag{2.30}$$

Differentiating (2.28) four times with respect to t, using the relations in (2.29), (2.30), substituting x and the derivatives $\dot{x}, \ddot{x}, \dddot{x}$ and $x^{(4)}$ into equation (2.27), expanding the right-hand side of equation (2.27), expressing the powers of $\cosh\psi_1$ and $\sinh\psi_1$ in the resulting expansion in terms of the multiple arguments of $\cosh\psi_1$ and $\sinh\psi_1$, comparing the coefficients of equal power of ε on both sides and finally equating the coefficients of simple and higher arguments of $\cosh\psi_1$, $\sinh\psi_1$, $\cosh\psi_2$ and $\sinh\psi_2$, Murty *et al* [52] obtained

$$m_1 A_1 + m_2 a \frac{dA_1}{dt} + m_3 a^2 \frac{d^2 A_1}{da^2} + m_4 a^3 \frac{d^3 A_1}{da^3} + m_5 a C_1$$
$$+ m_6 a^2 \frac{dC_1}{da} + m_7 a^3 \frac{d^2 C_1}{da^2} = h_1(a,b) \quad for \quad \cosh\psi_1,$$
$$n_1 A_1 + n_2 a \frac{dA_1}{dt} + n_3 a^2 \frac{d^2 A_1}{da^2} + n_4 a C_1 + n_5 a^2 \frac{dC_1}{da} \tag{2.31}$$
$$+ n_6 a^3 \frac{d^2 C_1}{da^2} + n_7 a^4 \frac{d^3 C_1}{da^3} = g_1(a,b) \quad for \quad \sinh\psi_1,$$
$$\beta_1 B_1 + \beta_2 b \frac{dB_1}{db} + \beta_3 b^2 \frac{d^2 B_1}{db^2} + \beta_4 b^3 D_1 + \beta_5 b^2 \frac{dD_1}{db}$$
$$+ \beta_6 b^3 \frac{d^2 D_1}{db^2} + \beta_7 b^4 \frac{d^3 D_1}{db^3} = g_2(a,b) \quad for \quad \cosh\psi_2$$

$$\alpha_1 B_1 + \alpha_2 b \frac{dB_1}{db} + \alpha_3 b^2 \frac{d^2 B_1}{db^2} + \alpha_4 b^3 \frac{d^3 B_1}{db^3} + \alpha_5 b D_1 \tag{2.32}$$
$$+ \alpha_6 b^2 \frac{dD_1}{db} + \alpha_7 b^3 \frac{d^2 D_1}{db^2} = h_2(a,b) \quad for \quad \sinh\psi_2$$

and for terms of higher arguments of $\cosh\psi_1$, $\sinh\psi_1$, $\cosh\psi_2$ and $\sinh\psi_2$, Murty *et al* got

$$[K_1^4(a\frac{\partial u_1}{\partial a}+7a^2\frac{\partial^2 u_1}{\partial a^2}+6a^3\frac{\partial^3 u_1}{\partial a^3}+a^4\frac{\partial^4 u_1}{\partial a^4})$$

$$+4K_1^3K_2(a\frac{\partial^2 u_1}{\partial a\partial\psi_1}+3a^2\frac{\partial^3 u_1}{\partial a^2\partial\psi_1}+a^3\frac{\partial^4 u_1}{\partial a^3\partial\psi_1})$$

$$+6(K_1K_2)^2(a\frac{\partial^3 u_1}{\partial a\partial\psi_1^2}+a^2\frac{\partial^4 u_1}{\partial a^2\partial\psi_1^2})+4K_1K_2^3a\frac{\partial^4 u_1}{\partial a\partial\psi_1^3}+K_2^4\frac{\partial^4 u_1}{\partial\psi_1^4}]$$

$$-K_3[K_1^3(a\frac{\partial u_1}{\partial a}+3a^2\frac{\partial^2 u_1}{\partial a^2}+a^3\frac{\partial^3 u_1}{\partial a^3})+3K_1^2K_2(a\frac{\partial^2 u_1}{\partial a\partial\psi_1}+a^2\frac{\partial^3 u_1}{\partial a^2\partial\psi_1})$$

$$+3K_1K_2^2a\frac{\partial^3 u_1}{\partial a\partial\psi_1^2}+K_2^3\frac{\partial^3 u_1}{\partial\psi_1^3}]+K_4[K_1^2(a\frac{\partial u_1}{\partial a}+a^2\frac{\partial^2 u_1}{\partial a^2})$$

$$+2K_1K_2a\frac{\partial^2 u_1}{\partial a\partial\psi_1}+K_2^2\frac{\partial^2 u_1}{\partial\psi_1^2}]$$

$$-K_5[K_1a\frac{\partial u_1}{\partial a}+K_2\frac{\partial u_1}{\partial\psi_1}]+K_6u_1=\sum_{r=2}^{\infty}[h_r(a)\cosh r\psi_1+g_r(a)\sinh r\psi_1], \tag{2.33}$$

$$[K_7^4(b\frac{\partial v_1}{\partial b}+7b^2\frac{\partial^2 v_1}{\partial b^2}+6b^3\frac{\partial^3 v_1}{\partial b^3}+b^4\frac{\partial^4 v_1}{\partial b^4})$$

$$+4K_7^3K_8(b\frac{\partial^2 v_1}{\partial b\partial\psi_2}+3b^2\frac{\partial^3 v_1}{\partial b^2\partial\psi_2}+b^3\frac{\partial^4 v_1}{\partial b^3\partial\psi_2})$$

$$+6(K_7K_8)^2(b\frac{\partial^3 v_1}{\partial b\partial\psi_2^2}+b^2\frac{\partial^4 v_1}{\partial b^2\partial\psi_2^2})+4K_7K_8^3b\frac{\partial^4 v_1}{\partial b\partial\psi_2^3}+K_8^4\frac{\partial^4 v_1}{\partial\psi_2^4}]$$

$$-K_3[K_7^3(b\frac{\partial v_1}{\partial b}+3b^2\frac{\partial^2 v_1}{\partial b^2}+b^3\frac{\partial^3 v_1}{\partial b^3})+3K_7^2K_8(b\frac{\partial^2 v_1}{\partial b\partial\psi_2}+b^2\frac{\partial^3 v_1}{\partial b^2\partial\psi_2})$$

$$+3K_7K_8^2b\frac{\partial^3 v_1}{\partial b\partial\psi_2^2}+K_8^3\frac{\partial^3 v_1}{\partial\psi_2^3}]+K_4[K_7^2(b\frac{\partial v_1}{\partial b}+b^2\frac{\partial^2 v_1}{\partial b^2})$$

$$+2K_7K_8b\frac{\partial^2 v_1}{\partial b\partial\psi_2}+K_8^2\frac{\partial^2 v_1}{\partial\psi_2^2}]$$

$$-K_5[K_7b\frac{\partial v_1}{\partial b}+K_8\frac{\partial v_1}{\partial\psi_2}]+K_6v_1=\sum_{s=2}^{\infty}[h_s(b)\sinh s\psi_2+g_s(b)\cosh s\psi_2] \tag{2.34}$$

where the coefficients m_i, n_i, α_i, and β_i, $i = 1,2,\ldots,7$ are constants involving the characteristic roots λ_i and are given by

$$\begin{aligned}
m_1 &= -K_1^3 - 6K_1K_2^2 + K_3(K_1^2 + 3K_2^2) - K_1K_4 + K_5,\\
m_2 &= -3K_1^3 - 6K_1K_2^2 + 2K_1^2K_3 - K_1K_4,\\
m_3 &= -4K_1^3 + K_1^2K_3,\\
m_4 &= -K_1^3,\\
m_5 &= -4K_2^3 - 12K_1^2K_2 + 6K_1K_2K_3 - 2K_2K_4,\\
m_6 &= -16K_1^2K_2 + 3K_1K_2K_3,\\
m_7 &= -4K_1^2K_2,\\
n_1 &= -4K_1^3 - 4K_1^2K_2 + 3K_1K_2K_3 - 2K_2K_4,\\
n_2 &= -8K_1^2K_2 + 3K_1K_2K_3,\\
n_3 &= -4K_1^2K_2,\\
n_4 &= -4K_1^3 - 12K_1K_2^2 + 3K_3(K_1^2 + K_2^2) - 2K_1K_4 + K_5,\\
n_5 &= -11K_1^3 - 6K_1K_2^2 + 4K_1^2K_3 - K_1K_4,\\
n_6 &= -7K_1^3 + K_1^2K_3,\\
n_7 &= -K_1^3,\\
\alpha_1 &= -K_7^3 - 6K_7K_8^2 + K_3(K_7^2 + 3K_8^2) - K_7K_4 + K_5, \qquad (2.35)\\
\alpha_2 &= -3K_7^3 - 6K_7K_8^2 + 3K_7^2K_3 - K_7K_4,\\
\alpha_3 &= -4K_7^3 + K_7^2K_3,\\
\alpha_4 &= -K_7^3,\\
\alpha_5 &= -4K_8^3 - 12K_7^2K_8 + 6K_7K_8K_3 - 2K_8K_4,\\
\alpha_6 &= -16K_7^2K_8 + 3K_7K_8K_3,\\
\alpha_7 &= -4K_7^2K_8,\\
\beta_1 &= -4K_8^3 - 4K_7^2K_8 + 3K_7K_8K_3 - 2K_8K_4,\\
\beta_2 &= -8K_7^2K_8 + 3K_7K_8K_3,
\end{aligned}$$

$$\beta_3 = -4K_7^{\,2}K_8,$$

$$\beta_4 = -4K_7^{\,3} - 12K_7K_8^{\,2} + 3K_3(K_7^{\,2} + K_8^{\,2}) - 2K_7K_4 + K_5,$$

$$\beta_5 = -11K_7^{\,3} - 6K_7K_8^{\,2} + 4K_7^{\,2}K_3 - K_7K_4,$$

$$\beta_6 = -7K_7^{\,3} + K_7^{\,2}K_3,$$

$$\beta_7 = -K_7^{\,3},$$

where k_j $j = 1, 2, 3, 4$ and K_l $l = 1, 2, \ldots, 8$ are defined in (2.2) and (2.30) respectively. The constants α_i and β_i are derived from the constants m_i and n_i, $i = 1, 2, \ldots, 7$ respectively, if K_1 and K_2 in them are replaced by K_7 and K_8 respectively.

Functions $h_1, h_r, g_1, g_r, h_2, h_s$ and g_2, g_s, $r, s = 2, 3, \ldots, \infty$ are the coefficients of the fundamental and higher argument terms in $\cosh\psi_1, \sinh\psi_1$ and $\cosh\psi_2, \sinh\psi_2$ respectively, obtained in the Taylor's expansion.

A particular solution of the two sets of simultaneous equations (2.31) and (2.32) give the functions A_1, C_1 B_1 and D_1. Such a solution is not straightforward, since the right-hand sides of (2.31) and (2.32) are functions of both a and b, whereas the left-hand sides involve a alone in equation (2.31) and b alone in equation (2.32). Because of this fact the approximate linear relationship that exists between a and b has been used. Thus Murty *et al* set $\varepsilon = 0$ in equation (2.29) and integrated it to obtain

$$a = a_0 e^{-K_1 t},$$

$$b = b_0 e^{-K_7 t}$$

from which Murty *et al* got

$$a = \left(\frac{a_0}{b_0}\right) b e^{-(K_1 - K_7)t},$$

$$b = \left(\frac{b_0}{a_0}\right) a e^{-(K_7 - K_1)t} \qquad (2.36)$$

Replacing b in terms of a in equation (2.31) and a in terms of b in equation (2.32) in their right-hand sides and using the relations (2.36), Murty *et al* obtained the particular solutions for A_1, B_1, C_1 and D_1 of the resulting equations from (2.31) and (2.32). Particular solutions of equation (2.29) then determine the first approximate solution of (2.27).

For the particular case of (2.27) in which $K_3 = 10, K_4 = 35, K_5 = 50$ and $K_6 = 24$ and characteristic roots $\lambda_1 = -4, \lambda_2 = -1, \lambda_3 = -3$ and $\lambda_4 = -2$, Murty *et al* found a solution [52]

$$x(t) = a(t)\cosh\psi_1(t) + b(t)\sinh\psi_2(t) + \varepsilon u_1(a,\psi_1) + \varepsilon v_1(b,\psi_2),$$

where

$$a(t) = \frac{a_0 e^{-\frac{5}{2}t}}{[1-(2P/5)+(2P/5)e^{-5t}]^{\frac{1}{2}}},$$

$$\psi_1(t) = \psi_{10} - \frac{3t}{2} + \frac{13}{30}\log(1 - \frac{2P}{5} + \frac{2P}{5}e^{-5t}),$$

$$b(t) = \frac{b_0 e^{-\frac{5}{2}t}}{[1-(2Q/5)+(2Q/5)e^{-5t}]^{\frac{1}{2}}},$$

$$\psi_2(t) = \psi_{20} - \frac{t}{2} + \frac{1}{5}\log(1 - \frac{2Q}{5} + \frac{2Q}{5}e^{-5t}), \tag{2.37}$$

$$p_1 = -1440\left[\frac{3}{4} - \frac{3}{2}\left(\frac{b_0}{a_0}\right)^2\right]/64{,}512,$$

$$p_2 = -960\left[\frac{3}{4} - \frac{3}{2}\left(\frac{a_0}{b_0}\right)^2\right]/96{,}768,$$

$$P = \varepsilon\, p_1 a_0^{\,2}, \quad Q = \varepsilon\, p_2 b_0^{\,2},$$

$$\varepsilon v_1 = \varepsilon b^3(g_2 \sinh 3\psi_2 - \cosh 3\psi_2),$$

and u_1 is zero in this example.

2.5 Results and Discussions

To obtain equation (2.36) from equation (2.29) Murty *et al* used the relationship that exists between a and b. But this linear relationship does not present the real situation always. In our new technique, we do not use such relationship.

The over-damped solution (2.26) can be used to obtain damped oscillatory solution, replacing ω_1 by $i\omega_1$ and ω_2 by $i\omega_2$, which is an importance of our new technique. In Chapter 3, we have found damped oscillatory solution by using this technique.

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we sometimes compare the approximate solution to the numerical solution.

Our approximate solution x, evaluated from (2.26) with initial conditions $a_0 = 2$, $b_0 = -1.0$, $\varphi_1(0) = 0.5$, $\varphi_2(0) = 1.0$, $\varepsilon = 0.1$ for various values of t, is presented in the second column of the Table 2.1. Corresponding numerical solution computed by a fourth order Runge-Kutta method is designed by x_1, in the third column of the Table 2.1. Computing result x_{11} of Murty, Deekshatulu and Krishna [52] is given in the fourth column of the Table 2.1.

Table 2.1

t	x	x_1	x_{11}
0.00	1.080100	1.0801	1.0801
0.40	0.412377	0.41248	0.41238
0.80	0.253258	0.25374	0.25326
1.20	0.175572	0.17635	0.17557
1.60	0.121351	0.12219	0.12135
2.00	0.082533	0.08329	0.08253
2.40	0.055564	0.05617	0.05556
2.80	0.037233	0.03769	0.03723
3.20	0.024910	0.02524	0.02491

Since the perturbation and numerical results are almost same; so, in figure they are not distinguishable. For this reason, in Chapter 2 and in Chapter 4, the results are presented Tables instead of Figures.

From Table 2.1, we see that the new asymptotic solution (2.26) and the solution of Murty *et al* [52] are almost the same. But the determination of the solution by Murty *et al* [52] is too much laborious and cumbersome, while our new technique is systematic, simple and easier.

2.6 Conclusion

An asymptotic method, based on the theory of Krylov-Bogoluibov-Mitropolskii, is developed in this Chapter for the transient response of a nonlinear system governed by a fourth order ordinary differential equation, when the four characteristic roots of the corresponding linear equation are all real and negative. The solution is presented as a power series in ε, where ε is a small parameter. The series itself is not convergent, but for a fixed number of terms, the approximate solution tends to the exact solution as ε tends to zero. The results obtained by the present method compare very well with those obtained by the numerical method and is simple and easier than that of Murty *et al* [52].

Chapter 3

Asymptotic Method for Fourth Order Damped Nonlinear Systems

3.1 Introduction

Popov [66] has generalized the Krylov-Bogoliubov-Mitropolskii (KBM) [13,34] method. The method [13,34] is particularly convenient and a widely used technique to obtain the approximate solution. Originally, the method was developed for systems with periodic solutions. Murty *et al* [52] extended this method to over-damped nonlinear systems. Murty [53] also presented a unified KBM method for solving second order nonlinear systems. Mendelson [46] rediscovered Popov's results. Bojadziev and Hung [27] developed a technique based on the KBM method to solve damped oscillations modeled by a 3-dimensional time dependent differential system. Shamsul [77] developed a new perturbation method based on the work of Krylov-Bogoliubov-Mitropolskii to find an approximate solution of nonlinear systems with large damping. Shamsul [81] has also extended the method [77] for n-th order nonlinear systems with large damping. Shamsul and Sattar [76] studied third order time-dependent oscillating systems with large damping. Shamsul, Bellal and Shanta [91] has modified the Krylov-Bogoliubov-Mitropolskii method and applied it to obtain an approximate solution of a second order damped nonlinear differential system with slowly varying coefficients. Recently, Shamsul [93] presented a general formula based on the extended (by Popov [66]) Krylov-Bogoliubov-Mitropolskii method for obtaining asymptotic solution of an n-th order time-dependent quasi-linear differential equation with damping.

In the present Chapter, we develop a new asymptotic method to solve the fourth order damped nonlinear systems.

3.2 The Method

Consider a weakly nonlinear damped oscillatory system governed by the differential equation

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = \varepsilon f(x,\dot{x},\ddot{x},\dddot{x}) \tag{3.1}$$

where $x^{(4)}$ denotes the fourth derivative of x, over dot is used for the first, the second, and the third derivatives of x with respect to t, ε is a small parameter, $f(x)$ is the given nonlinear function and k_1, k_2, k_3, k_4 are constants. Let us consider that $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are four eigen-values of the unperturbed equation of (3.1). For the damped oscillation, all the eigen-values are complex.

Therefore, the unperturbed solution becomes

$$x(t,0) = \sum_{j=1}^{4} a_{j,0} e^{\lambda_j t} \tag{3.2}$$

where $a_{j,0}$, $j = 1, 2, 3, 4$ are arbitrary constants.

When $\varepsilon \neq 0$, we seek a solution of the nonlinear differential equation (3.1) of the form (as described in Chapter 2)

$$x(t,\varepsilon) = \sum_{j=1}^{4} a_j(t) e^{\lambda_j t} + \varepsilon u_1(a_1, a_2, a_3, a_4, t) + \varepsilon^2 u_2(a_1, a_2, a_3, a_4, t) + \varepsilon^3 \ldots, \tag{3.3}$$

where each a_j, $j = 1, 2, 3, 4$ satisfies the differential equation

$$\dot{a}_j(t) = \varepsilon A_j(a_1, a_2, a_3, a_4, t) + \varepsilon^2 B_j(a_1, a_2, a_3, a_4, t) + \varepsilon^3 \ldots. \tag{3.4}$$

Confining only to a first few terms, 1, 2, 3,…,m in the series expansions of (3.3) and (3.4), we evaluate the functions $u_1, u_2, \ldots$ and $A_j, B_j, j = 1, 2, 3, 4$, such that $a_j(t)$, appearing in (3.3) and (3.4) satisfy the given differential equation (3.1) with an accuracy of ε^{m+1}. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first [53]. In order to determine these functions it is assumed that the functions $u_1, u_2, \ldots$ do not contain the fundamental terms, which are included in the series expansion (3.3) of order ε^0.

Differentiating (3.3) four times with respect to t, substituting x and the derivatives $x^{(4)}$, $\dddot{x}$, $\ddot{x}$, $\dot{x}$ in the original equation (3.1), utilizing the relations in (3.4) and equating the coefficients of ε, we obtain

$$\prod_{j=1}^{4}\left(\frac{d}{dt}-\lambda_j\right)u_1+\sum_{j=1}^{4}e^{\lambda_j t}\left(\prod_{k=1,j\neq k}^{4}\left(\frac{d}{dt}+\lambda_j-\lambda_k\right)\right)A_j=f^{(0)}(a_1,a_2,a_3,a_4,t), \tag{3.5}$$

where $f^{(0)}=f(x_0)$ and $x_0=\sum_{j=1}^{4}a_j(t)e^{\lambda_j t}$.

In general, the function $f^{(0)}$ can be expanded in a Taylor series as

$$f^{(0)}=\sum_{m_1=-\infty,m_2=-\infty,m_3=-\infty,m_4=-\infty}^{\infty,\infty,\infty,\infty}F_{m_1,m_2,m_3,m_4}\,a_1^{\,m_1}a_2^{\,m_2}a_3^{\,m_3}a_4^{\,m_4}e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t}.$$

According to our assumption, u_1 does not contain the fundamental terms, therefore equation (3.5) can be separated into five equations for unknown functions u_1 and A_1, A_2, A_3, A_4 (see [80] for details). Substituting the functional values of $f^{(0)}$ and equating the coefficients of $e^{\lambda_j t}$, $j=1, 2, 3, 4$, we obtain

$$\begin{aligned}&e^{\lambda_1 t}\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_2\right)\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_4\right)A_1\\&=F_{m_1,m_2,m_3,m_4}\,a_1^{\,m_1}a_2^{\,m_2}a_3^{\,m_3}a_4^{\,m_4}e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t};\quad m_3=m_4,\quad m_1=m_2+1\end{aligned} \tag{3.6}$$

$$\begin{aligned}&e^{\lambda_2 t}\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_1\right)\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_4\right)A_2\\&=F_{m_1,m_2,m_3,m_4}\,a_1^{\,m_1}a_2^{\,m_2}a_3^{\,m_3}a_4^{\,m_4}e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t};\quad m_3=m_4,\quad m_1=m_2-1\end{aligned} \tag{3.7}$$

$$e^{\lambda_3 t}\left(\frac{\partial}{\partial t}+\lambda_3-\lambda_1\right)\left(\frac{\partial}{\partial t}+\lambda_3-\lambda_2\right)\left(\frac{\partial}{\partial t}+\lambda_3-\lambda_4\right)A_3$$
$$= F_{m_1,m_2,m_3,m_4} a_1^{m_1} a_2^{m_2} a_3^{m_3} a_4^{m_4} e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t}; \quad m_1=m_2, \quad m_3=m_4+1 \tag{3.8}$$

$$e^{\lambda_4 t}\left(\frac{\partial}{\partial t}+\lambda_4-\lambda_1\right)\left(\frac{\partial}{\partial t}+\lambda_4-\lambda_2\right)\left(\frac{\partial}{\partial t}+\lambda_4-\lambda_3\right)A_4$$
$$= F_{m_1,m_2,m_3,m_4} a_1^{m_1} a_2^{m_2} a_3^{m_3} a_4^{m_4} e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t}; \quad m_1=m_2, \quad m_3=m_4-1 \tag{3.9}$$

and

$$\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)\left(\frac{\partial}{\partial t}-\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_4\right)u_1$$
$$= \sum_{m_1=-\infty,m_2=-\infty,m_3=-\infty,m_4=-\infty}^{\infty,\infty,\infty,\infty}{}' F_{m_1,m_2,m_3,m_4} a_1^{m_1} a_2^{m_2} a_3^{m_3} a_4^{m_4} e^{(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4)t} \tag{3.10}$$

where $\sum'$ exclude those terms for, $m_1=m_2\pm1, \quad m_3=m_4\pm1$.

The particular solutions of the equations (3.6)-(3.10) give the functions A_1, A_2, A_3, A_4 and u_1. Thus the determination of the first approximate solution is complete.

3.3 Example

As an example of the above method, consider the fourth order differential equation

$$x^{(4)}+k_1\dddot{x}+k_2\ddot{x}+k_3\dot{x}+k_4 x=\varepsilon x^3. \tag{3.11}$$

For equation (3.11), we have, $f=x^3$ and $f^{(0)}=(a_1e^{\lambda_1 t}+a_2e^{\lambda_2 t}+a_3e^{\lambda_3 t}+a_4e^{\lambda_4 t})^3$.

or,

$$\begin{aligned} f^{(0)} &= a_1^3e^{3\lambda_1 t}+a_2^3e^{3\lambda_2 t}+a_3^3e^{3\lambda_3 t}+a_4^3e^{3\lambda_4 t}+3a_1^2a_2e^{(2\lambda_1+\lambda_2)t}+3a_1^2a_3e^{(2\lambda_1+\lambda_3)t} \\ &+3a_1^2a_4e^{(2\lambda_1+\lambda_4)t}+3a_2^2a_1e^{(2\lambda_2+\lambda_1)t}+3a_2^2a_3e^{(2\lambda_2+\lambda_3)t}+3a_2^2a_4e^{(2\lambda_2+\lambda_4)t} \\ &+3a_3^2a_1e^{(2\lambda_3+\lambda_1)t}+3a_3^2a_2e^{(2\lambda_3+\lambda_2)t}+3a_3^2a_4e^{(2\lambda_3+\lambda_4)t}+3a_4^2a_1e^{(2\lambda_4+\lambda_1)t} \\ &+3a_4^2a_2e^{(2\lambda_4+\lambda_2)t}+3a_4^2a_3e^{(2\lambda_4+\lambda_3)t}+6a_1a_2a_3e^{(\lambda_1+\lambda_2+\lambda_3)t} \\ &+6a_1a_2a_4e^{(\lambda_1+\lambda_2+\lambda_4)t}+6a_1a_3a_4e^{(\lambda_1+\lambda_3+\lambda_4)t}+6a_2a_3a_4e^{(\lambda_2+\lambda_3+\lambda_4)t}. \end{aligned} \tag{3.12}$$

Therefore, equations (3.6)-(3.10) become

$$e^{\lambda_1 t}\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_2\right)\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_1-\lambda_4\right)A_1=3a_1^{\,2}a_2e^{(2\lambda_1+\lambda_2)t}+6a_1a_3a_4e^{(\lambda_1+\lambda_3+\lambda_4)t} \quad (3.13)$$

$$e^{\lambda_2 t}\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_1\right)\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_3\right)\left(\frac{\partial}{\partial t}+\lambda_2-\lambda_4\right)A_2=3a_2^{\,2}a_1e^{(2\lambda_2+\lambda_1)t}+6a_2a_3a_4e^{(\lambda_2+\lambda_3+\lambda_4)t} \quad (3.14)$$

$$e^{\lambda_3 t}\left(\frac{\partial}{\partial t}+\lambda_3-\lambda_1\right)\left(\frac{\partial}{\partial t}+\lambda_3-\lambda_2\right)\left(\frac{\partial}{\partial t}+\lambda_3-\lambda_4\right)A_3=3a_3^{\,2}a_4e^{(2\lambda_3+\lambda_4)t}+6a_1a_2a_3e^{(\lambda_1+\lambda_2+\lambda_3)t} \quad (3.15)$$

$$e^{\lambda_4 t}\left(\frac{\partial}{\partial t}+\lambda_4-\lambda_1\right)\left(\frac{\partial}{\partial t}+\lambda_4-\lambda_2\right)\left(\frac{\partial}{\partial t}+\lambda_4-\lambda_3\right)A_4=3a_4^{\,2}a_3e^{(2\lambda_4+\lambda_3)t}+6a_1a_2a_4e^{(\lambda_1+\lambda_2+\lambda_4)t} \quad (3.16)$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)\left(\frac{\partial}{\partial t}-\lambda_3\right)\left(\frac{\partial}{\partial t}-\lambda_4\right)u_1 &= a_1^{\,3}e^{3\lambda_1 t}+a_2^{\,3}e^{3\lambda_2 t}+a_3^{\,3}e^{3\lambda_3 t}+a_4^{\,3}e^{3\lambda_4 t}\\ &+3a_1^{\,2}a_3e^{(2\lambda_1+\lambda_3)t}+3a_1^{\,2}a_4e^{(2\lambda_1+\lambda_4)t}+3a_2^{\,2}a_3e^{(2\lambda_2+\lambda_3)t}+3a_2^{\,2}a_4e^{(2\lambda_2+\lambda_4)t} \quad (3.17)\\ &+3a_3^{\,2}a_1e^{(2\lambda_3+\lambda_1)t}+3a_3^{\,2}a_2e^{(2\lambda_3+\lambda_2)t}+3a_4^{\,2}a_1e^{(2\lambda_4+\lambda_1)t}+3a_4^{\,2}a_2e^{(2\lambda_4+\lambda_2)t}.\end{aligned}$$

Solving equations (3.13)-(3.17) and substituting, $\lambda_1=-\mu_1+i\omega_1$, $\lambda_2=-\mu_1-i\omega_1$, $\lambda_3=-\mu_2+i\omega_2$ and $\lambda_4=-\mu_2-i\omega_2$, we obtain

$$\begin{aligned}A_1 &= -\frac{3a_1^{\,2}a_2e^{-2\mu_1 t}}{2(\mu_1-i\omega_1)(3\mu_1-\mu_2-i\omega_1+i\omega_2)(3\mu_1-\mu_2-i\omega_1-i\omega_2)}\\ &\quad -\frac{6a_1a_3a_4e^{-2\mu_2 t}}{2(\mu_2-i\omega_1)(\mu_1+\mu_2-i\omega_1+i\omega_2)(\mu_1+\mu_2-i\omega_1-i\omega_2)},\end{aligned}$$

$$\begin{aligned}A_2 &= -\frac{3a_1a_2^{\,2}e^{-2\mu_1 t}}{2(\mu_1+i\omega_1)(3\mu_1-\mu_2+i\omega_1-i\omega_2)(3\mu_1-\mu_2+i\omega_1+i\omega_2)}\\ &\quad -\frac{6a_2a_3a_4e^{-2\mu_2 t}}{2(\mu_2+i\omega_1)(\mu_1+\mu_2+i\omega_1-i\omega_2)(\mu_1+\mu_2+i\omega_1+i\omega_2)},\end{aligned}$$

$$A_3 = -\frac{3a_3^{\,2}a_4e^{-2\mu_2 t}}{2(\mu_2 - i\omega_2)(3\mu_2 - \mu_1 - i\omega_2 + i\omega_1)(3\mu_2 - \mu_1 - i\omega_2 - i\omega_1)} - \frac{6a_1a_2a_3e^{-2\mu_1 t}}{2(\mu_1 - i\omega_2)(\mu_1 + \mu_2 + i\omega_1 - i\omega_2)(\mu_1 + \mu_2 - i\omega_1 - i\omega_2)}, \tag{3.18}$$

$$A_4 = -\frac{3a_3a_4^{\,2}e^{-2\mu_2 t}}{2(\mu_2 + i\omega_2)(3\mu_2 - \mu_1 + i\omega_2 - i\omega_1)(3\mu_2 - \mu_1 + i\omega_2 + i\omega_1)} - \frac{6a_1a_2a_4e^{-2\mu_1 t}}{2(\mu_1 + i\omega_2)(\mu_1 + \mu_2 - i\omega_1 + i\omega_2)(\mu_1 + \mu_2 + i\omega_1 + i\omega_2)}.$$

Substituting the values of (3.18) into equation (3.4), we obtain

$$\dot{a}_1 = -\varepsilon\left(\frac{3a_1^{\,2}a_2e^{-2\mu_1 t}}{2(\mu_1 - i\omega_1)(3\mu_1 - \mu_2 - i\omega_1 + i\omega_2)(3\mu_1 - \mu_2 - i\omega_1 - i\omega_2)} + \frac{6a_1a_3a_4e^{-2\mu_2 t}}{2(\mu_2 - i\omega_1)(\mu_1 + \mu_2 - i\omega_1 + i\omega_2)(\mu_1 + \mu_2 - i\omega_1 - i\omega_2)}\right),$$

$$\dot{a}_2 = -\varepsilon\left(\frac{3a_1a_2^{\,2}e^{-2\mu_1 t}}{2(\mu_1 + i\omega_1)(3\mu_1 - \mu_2 + i\omega_1 - i\omega_2)(3\mu_1 - \mu_2 + i\omega_1 + i\omega_2)} + \frac{6a_2a_3a_4e^{-2\mu_2 t}}{2(\mu_2 + i\omega_1)(\mu_1 + \mu_2 + i\omega_1 - i\omega_2)(\mu_1 + \mu_2 + i\omega_1 + i\omega_2)}\right),$$

$$\dot{a}_3 = -\varepsilon\left(\frac{3a_3^{\,2}a_4e^{-2\mu_2 t}}{2(\mu_2 - i\omega_2)(3\mu_2 - \mu_1 - i\omega_2 + i\omega_1)(3\mu_2 - \mu_1 - i\omega_2 - i\omega_1)} + \frac{6a_1a_2a_3e^{-2\mu_1 t}}{2(\mu_1 - i\omega_2)(\mu_1 + \mu_2 + i\omega_1 - i\omega_2)(\mu_1 + \mu_2 - i\omega_1 - i\omega_2)}\right), \tag{3.19}$$

$$\dot{a}_4 = -\varepsilon\left(\frac{3a_3a_4^{\,2}e^{-2\mu_2 t}}{2(\mu_2 + i\omega_2)(3\mu_2 - \mu_1 + i\omega_2 - i\omega_1)(3\mu_2 - \mu_1 + i\omega_2 + i\omega_1)} + \frac{6a_1a_2a_4e^{-2\mu_1 t}}{2(\mu_1 + i\omega_2)(\mu_1 + \mu_2 - i\omega_1 + i\omega_2)(\mu_1 + \mu_2 + i\omega_1 + i\omega_2)}\right).$$

Substituting $a_1 = \frac{a}{2}e^{i\varphi_1}$, $a_2 = \frac{a}{2}e^{-i\varphi_1}$, $a_3 = \frac{b}{2}e^{i\varphi_2}$ and $a_4 = \frac{-b}{2}e^{-i\varphi_2}$ into equation (3.19) and then simplifying, we obtain

$$\begin{aligned}
\dot{a} &= \varepsilon\left(l_1 a^3 e^{-2\mu_1 t} + l_2 a b^2 e^{-2\mu_2 t}\right),\\
\dot{b} &= \varepsilon\left(m_1 b^3 e^{-2\mu_2 t} + m_2 a^2 b e^{-2\mu_1 t}\right),\\
\dot{\varphi}_1 &= \varepsilon\left(n_1 a^2 e^{-2\mu_1 t} + n_2 b^2 e^{-2\mu_2 t}\right),\\
\dot{\varphi}_2 &= \varepsilon\left(r_1 b^2 e^{-2\mu_2 t} + r_2 a^2 e^{-2\mu_1 t}\right),
\end{aligned} \tag{3.20}$$

where

$$l_1 = -\frac{3}{8}\left(\frac{\mu_1\{(3\mu_1-\mu_2)^2-\omega_1^2+\omega_2^2\}-2(3\mu_1-\mu_2)\omega_1^2}{(\mu_1^2+\omega_1^2)\{(3\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2\}\{(3\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2\}}\right),$$

$$l_2 = -\frac{3}{4}\left(\frac{\mu_2\{(\mu_1+\mu_2)^2-\omega_1^2+\omega_2^2\}-2(\mu_1+\mu_2)\omega_1^2}{(\mu_2^2+\omega_1^2)\{(\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2\}\{(\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2\}}\right),$$

$$m_1 = -\frac{3}{8}\left(\frac{\mu_2\{(3\mu_2-\mu_1)^2-\omega_2^2+\omega_1^2\}-2(3\mu_2-\mu_1)\omega_2^2}{(\mu_2^2+\omega_2^2)\{(3\mu_2-\mu_1)^2+(\omega_2-\omega_1)^2\}\{(3\mu_2-\mu_1)^2+(\omega_2+\omega_1)^2\}}\right),$$

$$m_2 = -\frac{3}{4}\left(\frac{\mu_1\{(\mu_1+\mu_2)^2-\omega_2^2+\omega_1^2\}-2(\mu_1+\mu_2)\omega_2^2}{(\mu_1^2+\omega_2^2)\{(\mu_1+\mu_2)^2+(\omega_2-\omega_1)^2\}\{(\mu_1+\mu_2)^2+(\omega_2+\omega_1)^2\}}\right), \tag{3.21}$$

$$n_1 = -\frac{3}{8}\left(\frac{\{2\mu_1(3\mu_1-\mu_2)+(3\mu_1-\mu_2)^2-\omega_1^2+\omega_2^2\}\omega_1}{(\mu_1^2+\omega_1^2)\{(3\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2\}\{(3\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2\}}\right),$$

$$n_2 = -\frac{3}{4}\left(\frac{\{2\mu_2(\mu_1+\mu_2)+(\mu_1+\mu_2)^2-\omega_1^2+\omega_2^2\}\omega_1}{(\mu_2^2+\omega_1^2)\{(\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2\}\{(\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2\}}\right),$$

$$r_1 = -\frac{3}{8}\left(\frac{\{2\mu_2(3\mu_2-\mu_1)+(3\mu_2-\mu_1)^2-\omega_2^2+\omega_1^2\}\omega_2}{(\mu_2^2+\omega_2^2)\{(3\mu_2-\mu_1)^2+(\omega_2-\omega_1)^2\}\{(3\mu_2-\mu_1)^2+(\omega_2+\omega_1)^2\}}\right),$$

$$r_2 = -\frac{3}{4}\left(\frac{\{2\mu_1(\mu_1+\mu_2)+(\mu_1+\mu_2)^2-\omega_2^2+\omega_1^2\}\omega_2}{(\mu_1^2+\omega_2^2)\{(\mu_1+\mu_2)^2+(\omega_2-\omega_1)^2\}\{(\mu_1+\mu_2)^2+(\omega_2+\omega_1)^2\}}\right),$$

and

$$
\begin{aligned}
u_1 &= \frac{1}{16}\sum_{i,j=1,i\neq j}^{2} a^{-i+2j} b^{2i-j}\cos 3(\omega_i t+\varphi_i)\, g_{i,j}\, e^{-3\mu_i t} \\
&+ \frac{1}{16}\sum_{i,j=1,i\neq j}^{2} a^{-i+2j} b^{2i-j}\sin 3(\omega_i t+\varphi_i)\, h_{i,j}\, e^{-3\mu_i t} \\
&+ \frac{3ab}{16}\sum_{i,j=1,i\neq j}^{2} a^{j-1} b^{i-1}\cos(2\omega_i t\pm\omega_j t+2\varphi_i\pm\varphi_j)\, c_{i,j}^{\pm}\, e^{-(2\mu_i+\mu_j)t} \\
&+ \frac{3ab}{16}\sum_{i,j=1,i\neq j}^{2} a^{j-1} b^{i-1}\sin(2\omega_i t\pm\omega_j t+2\varphi_i\pm\varphi_j)\, d_{i,j}^{\pm}\, e^{-(2\mu_i+\mu_j)t},
\end{aligned}
\tag{3.22}
$$

where

$$
\begin{aligned}
g_{i,j} &= \frac{\mu_i^2(3\mu_i-\mu_j)^2-\mu_i^2(9\omega_i^2-\omega_j^2)-18\mu_i(3\mu_i-\mu_j)\omega_i^2-2(3\mu_i-\mu_j)^2\omega_i^2+2(9\omega_i^2-\omega_j^2)\omega_i^2}{(\mu_i^2+\omega_i^2)(\mu_i^2+4\omega_i^2)\{(3\mu_i-\mu_j)^2+(3\omega_i-\omega_j)^2\}\{(3\mu_i-\mu_j)^2+(3\omega_i+\omega_j)^2\}}, \\
h_{i,j} &= \frac{-3\omega_i\{2\mu_i^2(3\mu_i-\mu_j)+\mu_i(3\mu_i-\mu_j)^2-\mu_i(21\omega_i^2-\omega_j^2)+4\omega_i^2\mu_j\}}{(\mu_i^2+\omega_i^2)(\mu_i^2+4\omega_i^2)\{(3\mu_i-\mu_j)^2+(3\omega_i-\omega_j)^2\}\{(3\mu_i-\mu_j)^2+(3\omega_i+\omega_j)^2\}}, \qquad (3.23) \\
c^{\pm}_{i,j} &= \frac{\begin{array}{c}\mu_i^2(\mu_1+\mu_2)^2-\mu_i^2(12\omega_i^2\pm 13\omega_1\omega_2+3\omega_j^2)-2\mu_1\mu_2(5\omega_i^2\pm 5\omega_1\omega_2+\omega_j^2)\\ -\mu_j^2(\omega_i^2\pm\omega_1\omega_2)+(\omega_i\pm\omega_j)^2(3\omega_i\pm\omega_j)\omega_i\end{array}}{(\mu_i^2+\omega_i^2)\{\mu_i^2+(\omega_i\pm\omega_j)^2\}\{(\mu_1+\mu_2)^2+(\omega_i\pm\omega_j)^2\}\{(\mu_1+\mu_2)^2+(3\omega_i\pm\omega_j)^2\}}, \\
d^{\pm}_{i,j} &= \frac{-(2\omega_i\pm\omega_j)\{2\mu_i^2(\mu_1+\mu_2)+\mu_i(\mu_1+\mu_2)^2-\mu_i(5\omega_i^2\pm 6\omega_1\omega_2+\omega_j^2)-2\mu_j\omega_i(\omega_i\pm\omega_j)\}}{(\mu_i^2+\omega_i^2)\{\mu_i^2+(\omega_i\pm\omega_j)^2\}\{(\mu_1+\mu_2)^2+(\omega_i\pm\omega_j)^2\}\{(\mu_1+\mu_2)^2+(3\omega_i\pm\omega_j)^2\}}.
\end{aligned}
$$

$i, j = 1, 2.$

Equation (3.20) has no exact solution. However, we can integrate equation (3.20) by assuming that a and b are constants in the right hand side of (3.20) (see also [80]) and obtain

$$
\begin{aligned}
a &= a_0+\varepsilon\{l_1 a_0^3(1-e^{-2\mu_1 t})/\mu_1+l_2 a_0 b_0^2(1-e^{-2\mu_2 t})/\mu_2\}/2, \\
b &= b_0+\varepsilon\{m_1 b_0^3(1-e^{-2\mu_2 t})/\mu_2+m_2 a_0^2 b_0(1-e^{-2\mu_1 t})/\mu_1\}/2, \\
\varphi_1 &= \varphi_1(0)+\varepsilon\{n_1 a_0^2(1-e^{-2\mu_1 t})/\mu_1+n_2 b_0^2(1-e^{-2\mu_2 t})/\mu_2\}/2, \\
\varphi_2 &= \varphi_2(0)+\varepsilon\{r_1 b_0^2(1-e^{-2\mu_2 t})/\mu_2+r_2 a_0^2(1-e^{-2\mu_1 t})/\mu_1\}/2.
\end{aligned}
\tag{3.24}
$$

Therefore, we obtain the first order approximate solution of equation (3.11) as

$$x = ae^{-\mu_1 t}\cos(\omega_1 t + \varphi_1) + be^{-\mu_2 t}\sin(\omega_2 t + \varphi_2) + \varepsilon u_1, \tag{3.25}$$

where a, b, φ_1 and φ_2 are given by (3.24) and u_1 is given by (3.22).

3.4 Results and Discussions

We have separated equation (3.5) into five individual equations by balancing harmonic terms in which the variational equations contain the first harmonics and the correction terms contain harmonics with multiple arguments. These assumptions are certainly valid for the second and the third order differential equation. But for the fourth order differential equation these assumptions are not sufficient. As a result, much errors occur. So, to obtain desired result, theory of large damping is needed. In Chapter 6, we discussed the perturbation method for the fourth order nonlinear systems with large damping. Also when one of the eigen-values of the corresponding unperturbed equation is a linear combination of the other eigen-values, both the variational equations and correction terms contain secular type terms. As a result, the solutions fail to give desired results. In these cases, to obtain the desired result, the technique presented in [83,87,88] is needed. In Chapter 4, we have solved the fourth order nonlinear differential equation with the special conditions presented in [83,87,88].

For $k_1 = 0.25$, $k_2 = 0.25$, $\omega_1 = 1.0$, $\omega_2 = 1.7320508$, $\varphi_1 = 0$, $\varphi_2 = 1.570796$ and $\varepsilon = 0.1$, we have calculated x from (3.25), in which a, b, φ_1 and φ_2 are calculated from (3.24) with initial conditions $x(0) = 1.000613$, $\dot{x}(0) = -0.250442$, $\ddot{x}(0) = -1.948415$, $\dddot{x}(0) = 1.496248$ [or $a_0 = 0.5$, $b_0 = 0.5$] and is plotted in Fig. 3.1 (denoted by -o-). A second solution of (3.11) is computed by a fourth order Runge-Kutta formula with a small time increment $\Delta t = 0.05$ and the results are plotted in Fig. 3.1 (denoted by --). From the figure it is clear that the perturbation solutions (3.25) together with (3.24) agree with the numerical solutions.

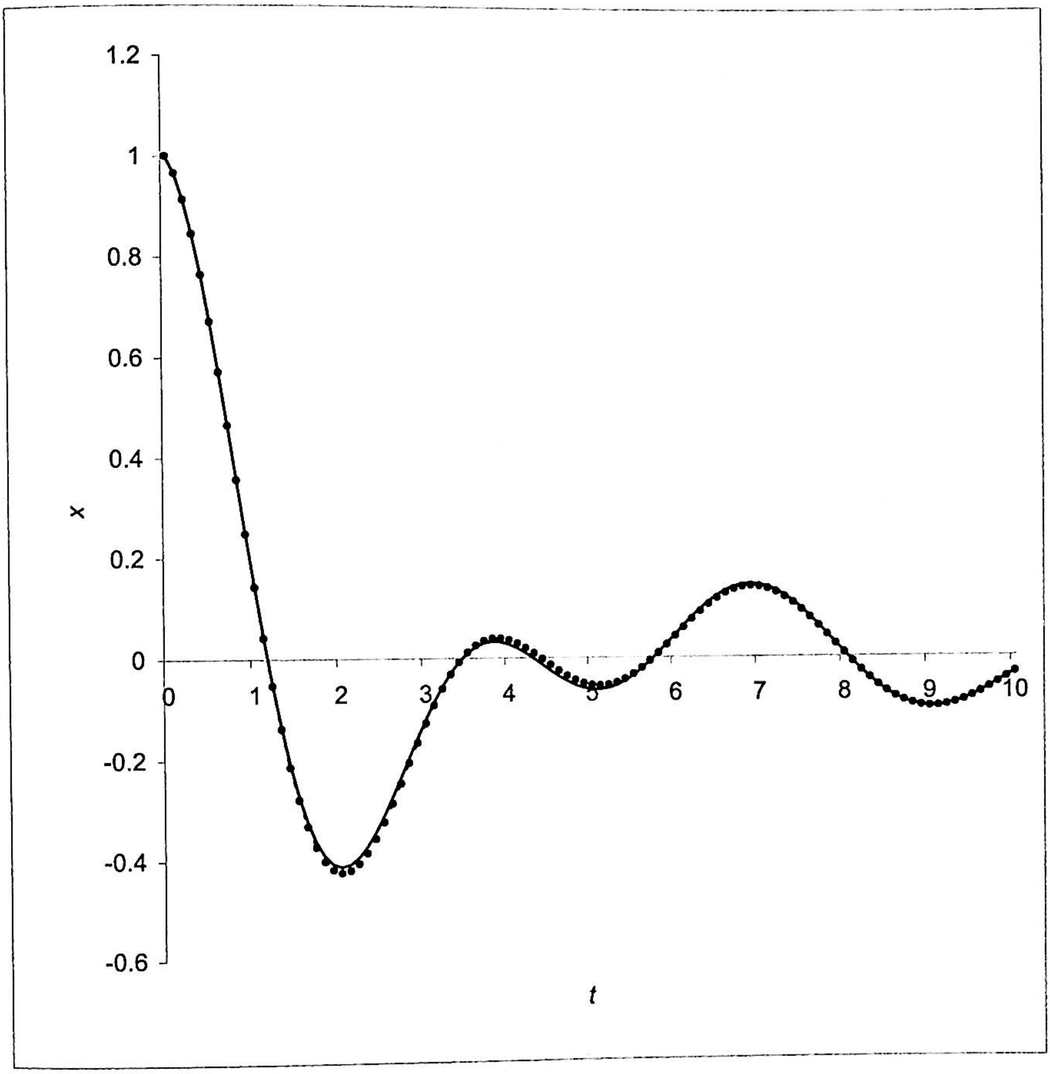

Fig. 3.1 Solution of Eq. (3.11): (i) Perturbation Solution denoted by –o– (ii) Numerical Solution denoted by --. For $k_1 = k_2 = 0.25$, $\omega_1 = 1$, $\omega_2 = \sqrt{3}$, $\varepsilon = 0.1$. Initial conditions $a_0 = b_0 = 0.5$, $\varphi_{1,0} = 0$ and $\varphi_{2,0} = \pi/2$ or $[x(0) = 1.00061$, $\dot{x}(0) = -0.25044$, $\ddot{x}(0) = -1.94842$, $\dddot{x}(0) = 1.49625]$.

3.5 Conclusion

An asymptotic method, based on the theory of Krylov-Bogoluibov-Mitropolskii, is presented for obtaining the transient response of nonlinear systems governed by a fourth order ordinary differential equation with small nonlinearities, when the four eigen-values of the corresponding linear equation are all complex numbers. The results obtained by this method agree with those obtained by the numerical method.

Chapter 4

Solution of Fourth Order Over-damped Nonlinear Systems Under Some Special Conditions

4.1 Introduction

Krylov-Bogoliubov-Mitropolskii (KBM) [13,34] method is particularly convenient and is a widely used technique to obtain the approximate solutions of nonlinear systems with small nonlinearities. The method obtained by Krylov and Bogoliubov [34], originally developed for the systems with periodic solutions, was amplified and justified by Bogoliubov and Mitropolskii [13] and later extended by Popov [66] for damped nonlinear systems. Murty *et al* [52] extended the method to over-damped nonlinear systems. Murty [53] has presented a unified method for solving the second order nonlinear systems. Owing to some limitations of the unified theory of Murty [53], Shamsul [85,87] has further investigated the second and the third order over-damped systems under some special conditions. Sattar [71] early studied a three dimensional over-damped nonlinear system. Shamsul and Sattar [73] developed an asymptotic method to solve third order critically damped nonlinear equations. Recently, Shamsul [83] has generalized the technique presented in [85,87] for *n*-th order over-damped nonlinear systems. Shamsul [80] has also presented a unified method for solving *n*-th order nonlinear systems.

However, when one of the eigen-values is a multiple (*i.e.*, double, triple etc.) of the others, correction terms of the asymptotic solution (found by Murty *et al* [52] or Sattar [71]) contain some secular type terms and are thus unable to give desired results. Shamsul [83,85,87] removed this difficulty and found the desired solutions.

But more serious problem arises in case of the fourth and more than fourth order nonlinear systems (see **Appendix 4.A**), when one obtains an asymptotic solution followed by Murty *et al* [52]. For certain damping effect the variational equations contain some secular type terms when the eigen-values of the corresponding linear equation are in a linear combination.

In this Chapter, we present a solution (based on the works of Shamsul [83,85,87]), which prevent the appearances of the secular type terms in both the variational equations and the correction terms.

4.2 The Method

Let us consider the fourth order nonlinear differential equation

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = \varepsilon f(x, \dot{x}, \ddot{x}, \dddot{x}), \tag{4.1}$$

where $x^{(4)}$ represents the fourth derivative of x, over-dot is used for the first, the second, and the third derivatives of x with respect to t, ε is a small parameter, k_1, k_2, k_3 and k_4 are constants and f is the nonlinear function.

When $\varepsilon = 0$, let us consider that $-\lambda_1, -\lambda_2, -\lambda_3$ and $-\lambda_4$ be the four real negative eigen-values of (4.1). Therefore, the solution of the corresponding linear equation of (4.1) is

$$x(t,0) = \sum_{j=1}^{4} a_{j,0} e^{-\lambda_j t} \tag{4.2}$$

where $a_{j,0}$, $j = 1, 2, 3, 4$ are arbitrary constants.

When $\varepsilon \neq 0$, we seek a solution of the nonlinear differential equation (4.1) in an asymptotic expansion of the form

$$x(t,\varepsilon) = \sum_{j=1}^{4} a_j(t)\, e^{-\lambda_j t} + \varepsilon\, u_1(a_1,a_2,a_3,a_4,t) + \varepsilon^2\, u_2(a_1,a_2,a_3,a_4,t) + \ldots \tag{4.3}$$

where each a_j satisfies the differential equations

$$\dot{a}_j(t) = \varepsilon\, A_j(a_1,a_2,a_3,a_4,t) + \varepsilon^2 B_j(a_1,a_2,a_3,a_4,t) + \ldots \quad ,\ j = 1, 2, 3, 4 \tag{4.4}$$

Confining only to the first few terms 1, 2, ...,m in the series expansions of (4.3) and (4.4), we evaluate the functions A_j, $j = 1, 2, 3, 4$ and u_i, i=1,2, ...,m such that $a_j(t)$, $j = 1, 2, 3, 4$ appearing in (4.3) and (4.4) satisfy the given differential equation (4.1) with an accuracy of order ε^{m+1}. In order to determine these unknown functions it is customary in the KBM method that the correction terms u_i must exclude those terms (known as secular terms) which make them large. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the algebraic complexity for the derivation of the formulae, the solution is, in general, confined to a lower order, usually the first [53,80].

Differentiating (4.3) four times with respect to t, substituting the derivatives $\dot{x}$, $\ddot{x}$, $\dddot{x}$, $x^{(4)}$ and the dependent variable x in the original equation (4.1), utilizing the relations in (4.4) and comparing the coefficients of ε, we obtain

$$\prod_{j=1}^{4}\left(\frac{d}{dt} + \lambda_j\right) u_1 + \sum_{j=1}^{4} e^{-\lambda_j t}\left(\prod_{\substack{k=1\\ j\neq k}}^{4}\left(\frac{d}{dt} - \lambda_j + \lambda_k\right)\right) A_j = f^{(0)}(a_1,a_2,a_3,a_4,t) \tag{4.5}$$

where $f^{(0)} = f(x_0)$ and $x_0 = \sum_{j=1}^{4} a_{j,0} e^{-\lambda_j t}$.

In general, $f^{(0)}$ can be expanded in a Taylor series as

$$f^{(0)} = \sum_{i_1,i_2,i_3,i_4=0}^{\infty} F_{i_1,i_2,i_3,i_4}(a_1,a_2,a_3,a_4,t)\, e^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4)t} \tag{4.6}$$

To solve the equation (4.5) for u_1, A_1, A_2, A_3, A_4, it is assumed that u_1 does not contain the terms involving $e^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4)t}$ where $i_1\lambda_1+\ldots+i_4\lambda_4 \le \frac{1}{4}(i_1+\ldots+i_4)(\lambda_1+\ldots+\lambda_4)$ [83,85,87], so that the coefficients of terms of u_1 do not become large or u_1 does not contain secular type terms $t\,e^{-t}$.

Substituting (4.6) into (4.5), we have

$$\begin{aligned}
& e^{-\lambda_1 t}\left(\frac{d}{dt}-\lambda_1+\lambda_2\right)\left(\frac{d}{dt}-\lambda_1+\lambda_3\right)\left(\frac{d}{dt}-\lambda_1+\lambda_4\right)A_1 \\
& + e^{-\lambda_2 t}\left(\frac{d}{dt}-\lambda_2+\lambda_1\right)\left(\frac{d}{dt}-\lambda_2+\lambda_3\right)\left(\frac{d}{dt}-\lambda_2+\lambda_4\right)A_2 \\
& + e^{-\lambda_3 t}\left(\frac{d}{dt}-\lambda_3+\lambda_1\right)\left(\frac{d}{dt}-\lambda_3+\lambda_2\right)\left(\frac{d}{dt}-\lambda_3+\lambda_4\right)A_3 \\
& + e^{-\lambda_4 t}\left(\frac{d}{dt}-\lambda_4+\lambda_1\right)\left(\frac{d}{dt}-\lambda_4+\lambda_2\right)\left(\frac{d}{dt}-\lambda_4+\lambda_3\right)A_4 \\
& = \sum_{i_1,i_2,i_3,i_4=0}^{\infty} F_{i_1,i_2,i_3,i_4}(a_1,a_2,a_3,a_4,t)\, e^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4)t},
\end{aligned} \tag{4.7}$$

where $i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4 \le \frac{1}{4}(i_1+i_2+i_3+i_4)(\lambda_1+\lambda_2+\lambda_3+\lambda_4)$,

and

$$\begin{aligned}
& \left(\frac{d}{dt}+\lambda_1\right)\left(\frac{d}{dt}+\lambda_2\right)\left(\frac{d}{dt}+\lambda_3\right)\left(\frac{d}{dt}+\lambda_4\right)u_1 \\
& = \sum_{i_1,i_2,i_3,i_4=0}^{\infty} F_{i_1,i_2,i_3,i_4}(a_1,a_2,a_3,a_4,t)\, e^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4)t},
\end{aligned} \tag{4.8}$$

where $i_1\lambda_1 + i_2\lambda_2 + i_3\lambda_3 + i_4\lambda_4 > \frac{1}{4}(i_1 + i_2 + i_3 + i_4)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)$.

Solving (4.8), we obtain

$$u_1 = \sum_{i_1,i_2,i_3,i_4=0}^{\infty} \frac{\sum_{i_1,i_2,i_3,i_4=0}^{\infty} F_{i_1,i_2,i_3,i_4}(a_1,a_2,a_3,a_4,t)\, e^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4)t}}{(i_1\lambda_1 + i_2\lambda_2 + i_3\lambda_3 + i_4\lambda_4 - \lambda_1)\ldots(i_1\lambda_1 + i_2\lambda_2 + i_3\lambda_3 + i_4\lambda_4 - \lambda_4)}. \tag{4.9}$$

However, it is not easy to solve equation (4.7) for the unknown functions A_1, A_2, A_3 and A_4 if the nonlinear function f and the eigen-values $-\lambda_1, -\lambda_2, -\lambda_3$ and $-\lambda_4$ of the corresponding linear equation of (4.1) are not specified. We can find these functions in the forms of

$$A_1 = \sum_{i_1,i_2,i_3,i_4=0}^{\infty} l_{i_1,i_2,i_3,i_4}\, e^{-(-\lambda_1+i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4)t},$$

$$A_2 = \sum_{i_1,i_2,i_3,i_4=0}^{\infty} m_{i_1,i_2,i_3,i_4}\, e^{-(i_1\lambda_1-\lambda_2+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4)t},$$

$$A_3 = \sum_{i_1,i_2,i_3,i_4=0}^{\infty} n_{i_1,i_2,i_3,i_4}\, e^{-(i_1\lambda_1+i_2\lambda_2-\lambda_3+i_3\lambda_3+i_4\lambda_4)t}, \tag{4.10}$$

and

$$A_4 = \sum_{i_1,i_2,i_3,i_4=0}^{\infty} r_{i_1,i_2,i_3,i_4}\, e^{-(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4-\lambda_4)t},$$

such that the unknown coefficients l_{i_1,i_2,i_3,i_4}, m_{i_1,i_2,i_3,i_4}, n_{i_1,i_2,i_3,i_4}, and r_{i_1,i_2,i_3,i_4}, *i. e.*, A_1, A_2, A_3 and A_4 do not become large for any time t.

4.3 Example

As an example of the above method, we consider a Duffing type fourth order nonlinear equation

$$x^{(4)} + k_1\dddot{x} + k_2\ddot{x} + k_3\dot{x} + k_4 x = -\varepsilon\, x^3 \tag{4.11}$$

Murty *et al* [52] found a hyperbolic type asymptotic solution. Shamsul [92] has found an oscillatory solution of the equation.

Here $f = -x^3$ and

$$\begin{aligned}
f^{(0)} &= -(a_1 e^{-3\lambda_1 t} + 3a_1{}^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1{}^2 a_3 e^{-(2\lambda_1+\lambda_3)t} + 3a_2{}^2 a_1 e^{-(2\lambda_2+\lambda_1)t} \\
&+ 6a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 3a_3{}^2 a_1 e^{-(2\lambda_3+\lambda_1)t} + a_2{}^3 e^{-3\lambda_2 t} + 3a_1{}^2 a_4 e^{-(2\lambda_1+\lambda_4)t} \\
&+ 3a_2{}^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + 6a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t} + 3a_2{}^2 a_4 e^{-(2\lambda_2+\lambda_4)t} + 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} \\
&+ 3a_3{}^2 a_2 e^{-(2\lambda_3+\lambda_2)t} + 3a_4{}^2 a_1 e^{-(2\lambda_4+\lambda_1)t} + 6a_2 a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} + a_3{}^3 e^{-3\lambda_3 t} \\
&+ 3a_4{}^2 a_2 e^{-(2\lambda_4+\lambda_2)t} + 3a_3{}^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + 3a_4{}^2 a_3 e^{-(2\lambda_4+\lambda_3)t} + a_4{}^3 e^{-3\lambda_4 t}).
\end{aligned}$$

Therefore, (4.7) and (4.8) respectively become

$$\begin{aligned}
& e^{-\lambda_1 t}\left(\frac{d}{dt} - \lambda_1 + \lambda_2\right)\left(\frac{d}{dt} - \lambda_1 + \lambda_3\right)\left(\frac{d}{dt} - \lambda_1 + \lambda_4\right)A_1 \\
& + e^{-\lambda_2 t}\left(\frac{d}{dt} - \lambda_2 + \lambda_1\right)\left(\frac{d}{dt} - \lambda_2 + \lambda_3\right)\left(\frac{d}{dt} - \lambda_2 + \lambda_4\right)A_2 \\
& + e^{-\lambda_3 t}\left(\frac{d}{dt} - \lambda_3 + \lambda_1\right)\left(\frac{d}{dt} - \lambda_3 + \lambda_2\right)\left(\frac{d}{dt} - \lambda_3 + \lambda_4\right)A_3 \\
& + e^{-\lambda_4 t}\left(\frac{d}{dt} - \lambda_4 + \lambda_1\right)\left(\frac{d}{dt} - \lambda_4 + \lambda_2\right)\left(\frac{d}{dt} - \lambda_4 + \lambda_3\right)A_4 \qquad (4.12) \\
& = -(3a_2{}^2 a_4 e^{-(2\lambda_2+\lambda_4)t} + 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} + 3a_3{}^2 a_2 e^{-(2\lambda_3+\lambda_2)t} \\
& + 3a_4{}^2 a_1 e^{-(2\lambda_4+\lambda_1)t} + 6a_2 a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} + a_3{}^3 e^{-3\lambda_3 t} + 3a_4{}^2 a_2 e^{-(2\lambda_4+\lambda_2)t} \\
& + 3a_3{}^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + 3a_4{}^2 a_3 e^{-(2\lambda_4+\lambda_3)t} + a_4{}^3 e^{-3\lambda_4 t}),
\end{aligned}$$

and

$$\left(\frac{d}{dt}+\lambda_1\right)\left(\frac{d}{dt}+\lambda_2\right)\left(\frac{d}{dt}+\lambda_3\right)\left(\frac{d}{dt}+\lambda_4\right)u_1$$

$$=-(a_1{}^3e^{-3\lambda_1 t}+3a_1{}^2a_2e^{-(2\lambda_1+\lambda_2)t}+3a_1{}^2a_3e^{-(2\lambda_1+\lambda_3)t}+3a_2{}^2a_1e^{-(2\lambda_2+\lambda_1)t}$$
$$+3a_1{}^2a_4e^{-(2\lambda_1+\lambda_4)t}+a_2{}^3e^{-3\lambda_2 t}+6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t}$$
$$+6a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t}+3a_2{}^2a_3e^{-(2\lambda_2+\lambda_3)t}+3a_3{}^2a_1e^{-(2\lambda_3+\lambda_1)t}), \quad (4.13)$$

if $\lambda_1>\lambda_2>\lambda_3>\lambda_4$.

Solving (4.13), we obtain

$$u_1=d_1a_1{}^3e^{-3\lambda_1 t}+d_2a_1{}^2a_2e^{-(2\lambda_1+\lambda_2)t}+d_3a_1{}^2a_3e^{-(2\lambda_1+\lambda_3)t}+d_4a_2{}^2a_1e^{-(2\lambda_2+\lambda_1)t}$$
$$+d_5a_1{}^2a_4e^{-(2\lambda_1+\lambda_4)t}+d_6a_2{}^3e^{-3\lambda_2 t}+d_7a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t}+ \quad (4.14)$$
$$d_8a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t}+d_9a_2{}^2a_3e^{-(2\lambda_2+\lambda_3)t}+d_{10}a_3{}^2a_1e^{-(2\lambda_3+\lambda_1)t},$$

where

$$d_1=-1/[2\lambda_1(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)],$$

$$d_2=-3/[2\lambda_1(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_4)],$$

$$d_3=3/[2\lambda_1(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)(2\lambda_1+\lambda_3-\lambda_4)],$$

$$d_4=-3/[2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_4)],$$

$$d_5=-3/[2\lambda_1(\lambda_1+\lambda_4)(2\lambda_1+\lambda_4-\lambda_2)(2\lambda_1+\lambda_4-\lambda_3)],$$

$$d_6=-1/[2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(3\lambda_2-\lambda_4)],$$

$$d_7=-6/[(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_1+\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)], \quad (4.15)$$

$$d_8=-6/[(\lambda_2+\lambda_4)(\lambda_1+\lambda_4)(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_4-\lambda_3)],$$

and

$$\left(\frac{d}{dt}+\lambda_1\right)\left(\frac{d}{dt}+\lambda_2\right)\left(\frac{d}{dt}+\lambda_3\right)\left(\frac{d}{dt}+\lambda_4\right)u_1$$

$$\begin{aligned}&=-(a_1^{\,3}e^{-3\lambda_1 t}+3a_1^{\,2}a_2e^{-(2\lambda_1+\lambda_2)t}+3a_1^{\,2}a_3e^{-(2\lambda_1+\lambda_3)t}+3a_2^{\,2}a_1e^{-(2\lambda_2+\lambda_1)t}\\&+3a_1^{\,2}a_4e^{-(2\lambda_1+\lambda_4)t}+a_2^{\,3}e^{-3\lambda_2 t}+6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t}\\&+6a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t}+3a_2^{\,2}a_3e^{-(2\lambda_2+\lambda_3)t}+3a_3^{\,2}a_1e^{-(2\lambda_3+\lambda_1)t}),\end{aligned}\tag{4.13}$$

if $\lambda_1>\lambda_2>\lambda_3>\lambda_4$.

Solving (4.13), we obtain

$$\begin{aligned}u_1&=d_1a_1^{\,3}e^{-3\lambda_1 t}+d_2a_1^{\,2}a_2e^{-(2\lambda_1+\lambda_2)t}+d_3a_1^{\,2}a_3e^{-(2\lambda_1+\lambda_3)t}+d_4a_2^{\,2}a_1e^{-(2\lambda_2+\lambda_1)t}\\&+d_5a_1^{\,2}a_4e^{-(2\lambda_1+\lambda_4)t}+d_6a_2^{\,3}e^{-3\lambda_2 t}+d_7a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t}+\\&d_8a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t}+d_9a_2^{\,2}a_3e^{-(2\lambda_2+\lambda_3)t}+d_{10}a_3^{\,2}a_1e^{-(2\lambda_3+\lambda_1)t},\end{aligned}\tag{4.14}$$

where

$$\begin{aligned}d_1&=-1/[2\lambda_1(3\lambda_1-\lambda_2)(3\lambda_1-\lambda_3)(3\lambda_1-\lambda_4)],\\d_2&=-3/[2\lambda_1(\lambda_1+\lambda_2)(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_4)],\\d_3&=3/[2\lambda_1(\lambda_1+\lambda_3)(2\lambda_1+\lambda_3-\lambda_2)(2\lambda_1+\lambda_3-\lambda_4)],\\d_4&=-3/[2\lambda_2(\lambda_1+\lambda_2)(\lambda_1+2\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_4)],\\d_5&=-3/[2\lambda_1(\lambda_1+\lambda_4)(2\lambda_1+\lambda_4-\lambda_2)(2\lambda_1+\lambda_4-\lambda_3)],\\d_6&=-1/[2\lambda_2(3\lambda_2-\lambda_1)(3\lambda_2-\lambda_3)(3\lambda_2-\lambda_4)],\\d_7&=-6/[(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_1+\lambda_3)(\lambda_1+\lambda_2+\lambda_3-\lambda_4)],\\d_8&=-6/[(\lambda_2+\lambda_4)(\lambda_1+\lambda_4)(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_4-\lambda_3)],\end{aligned}\tag{4.15}$$

$$d_9 = -3\,[2\lambda_2(\lambda_2+\lambda_3)(2\lambda_2+\lambda_3-\lambda_1)(2\lambda_2+\lambda_3-\lambda_4)],$$

$$d_{10} = -3\,[2\lambda_3(\lambda_1+\lambda_3)(\lambda_1+2\lambda_3-\lambda_2)(\lambda_1+2\lambda_3-\lambda_4)].$$

To separate the equation (4.12) for A_1, A_2, A_3 and A_4, first we assume that $\lambda_2 \approx 3\lambda_3$ and $\lambda_1 \approx \lambda_2+\lambda_3+\lambda_4$. In this case, we choose

$$e^{-\lambda_1 t}\left(\frac{d}{dt}-\lambda_1+\lambda_2\right)\left(\frac{d}{dt}-\lambda_1+\lambda_3\right)\left(\frac{d}{dt}-\lambda_1+\lambda_4\right)A_1 \tag{4.16}$$
$$= -(3a_4{}^2a_2e^{-(2\lambda_4+\lambda_2)t}+3a_4{}^2a_1e^{-(\lambda_1+2\lambda_4)t}+3a_3{}^2a_2e^{-(2\lambda_3+\lambda_2)t}+6a_1a_3a_4e^{-(\lambda_1+\lambda_3+\lambda_4)t}),$$

$$e^{-\lambda_2 t}\left(\frac{d}{dt}-\lambda_2+\lambda_1\right)\left(\frac{d}{dt}-\lambda_2+\lambda_3\right)\left(\frac{d}{dt}-\lambda_2+\lambda_4\right)A_2 \tag{4.17}$$
$$= -(3a_4{}^2a_3e^{-(2\lambda_4+\lambda_3)t}+3a_3{}^2a_4e^{-(2\lambda_3+\lambda_4)t}+a_3{}^3e^{-3\lambda_3 t}+6a_2a_3a_4e^{-(\lambda_2+\lambda_3+\lambda_4)t}),$$

$$e^{-\lambda_3 t}\left(\frac{d}{dt}-\lambda_3+\lambda_1\right)\left(\frac{d}{dt}-\lambda_3+\lambda_2\right)\left(\frac{d}{dt}-\lambda_3+\lambda_4\right)A_3 = -a_4{}^3e^{-3\lambda_4 t}, \tag{4.18}$$

and

$$e^{-\lambda_4 t}\left(\frac{d}{dt}-\lambda_4+\lambda_1\right)\left(\frac{d}{dt}-\lambda_4+\lambda_2\right)\left(\frac{d}{dt}-\lambda_4+\lambda_3\right)A_4 = -3a_2{}^2a_4e^{-(2\lambda_2+\lambda_4)t}. \tag{4.19}$$

Solving equations (4.16)-(4.19), we obtain

$$A_1 = l_1a_4{}^2a_2e^{(\lambda_1-2\lambda_4-\lambda_2)t}+l_2a_4{}^2a_1e^{-2\lambda_4 t}+l_3a_3{}^2a_2e^{(\lambda_1-2\lambda_3-\lambda_2)t}+l_4a_1a_3a_4e^{-(\lambda_3+\lambda_4)t},$$

$$A_2 = m_1a_4{}^2a_3e^{(\lambda_2-2\lambda_4-\lambda_3)t}+m_2a_3{}^2a_4e^{(\lambda_2-2\lambda_3-\lambda_4)t}+m_3a_3{}^3e^{(\lambda_2-3\lambda_3 t}+m_4a_2a_3a_4e^{-(\lambda_3+\lambda_4)t},$$

$$A_3 = n_1a_4{}^3e^{(\lambda_3-3\lambda_4)t}, \quad A_4 = r_1a_2{}^2a_4e^{-2\lambda_2 t}. \tag{4.20}$$

where

$$l_1 = -3/[2\lambda_4(\lambda_2+\lambda_4)(\lambda_3-\lambda_2-2\lambda_4)],$$

$$l_2 = 3/[(\lambda_1+\lambda_4)(\lambda_2-\lambda_1-2\lambda_4)(\lambda_3-\lambda_1-2\lambda_4)],$$

$$l_3 = -3/[2\lambda_3(\lambda_2+\lambda_3)(\lambda_4-\lambda_2-2\lambda_3)],$$

$$l_4 = -6/[(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_2-\lambda_1-\lambda_3-\lambda_4)],$$

$$m_1 = -3/[2\lambda_4(\lambda_3+\lambda_4)(\lambda_1-\lambda_3-2\lambda_4)],$$

$$m_2 = -3/[2\lambda_3(\lambda_3+\lambda_4)(\lambda_1-2\lambda_3-\lambda_4)], \quad (4.21)$$

$$m_3 = 1/[2\lambda_3(\lambda_1-3\lambda_3)(\lambda_4-3\lambda_3)],$$

$$m_4 = -6/[(\lambda_2+\lambda_3)(\lambda_2+\lambda_4)(\lambda_1-\lambda_2-\lambda_3-\lambda_4)],$$

$$n_1 = 1/[2\lambda_4(\lambda_1-3\lambda_4)(\lambda_2-3\lambda_4)],$$

$$r_1 = 3/[(\lambda_2+\lambda_4)(\lambda_1-2\lambda_2-\lambda_4)(\lambda_3-2\lambda_2-\lambda_4)].$$

Substituting the values of A_1, A_2, A_3 and A_4 from equation (4.20) into equation (4.4), we obtain

$$\dot{a}_1 = \varepsilon\,(l_1 a_4{}^2 a_2 e^{(\lambda_1-2\lambda_4-\lambda_2)t} + l_2 a_4{}^2 a_1 e^{-2\lambda_4 t} + l_3 a_3{}^2 a_2 e^{(\lambda_1-2\lambda_3-\lambda_2)t} + l_4 a_1 a_3 a_4 e^{-(\lambda_3+\lambda_4)t}), \quad (4.22)$$

$$\dot{a}_2 = \varepsilon\,(m_1 a_4{}^2 a_3 e^{(\lambda_2-2\lambda_4-\lambda_3)t} + m_2 a_3{}^2 a_4 e^{(\lambda_2-2\lambda_3-\lambda_4)t} + m_3 a_3{}^3 e^{(\lambda_2-3\lambda_3)t} + m_4 a_2 a_3 a_4 e^{-(\lambda_3+\lambda_4)t}), \quad (4.23)$$

$$\dot{a}_3 = \varepsilon\, n_1 a_4{}^3 e^{(\lambda_3-3\lambda_4)t}, \quad (4.24)$$

$$\dot{a}_4 = \varepsilon\, r_1 a_2{}^2 a_4 e^{-2\lambda_2 t}. \quad (4.25)$$

Now, we can solve equations (4.22)-(4.25) by assuming that a_1, a_2, a_3 and a_4 are constants in the right hand sides of the equations (4.22)-(4.25), since ε is a small quantity.

This assumption was first made by Murty *et al* [52] to solve similar type of nonlinear equations of (4.22)-(4.25). Thus the solutions of equations (4.22)-(4.25) are

$$a_1 = a_{1,0} + \varepsilon\,[l_1 a_{4,0}{}^2 a_{2,0}(e^{(\lambda_1-\lambda_2-2\lambda_4)t} - 1)/(\lambda_1 - \lambda_2 - 2\lambda_4)$$
$$+ l_2 a_{4,0}{}^2 a_{1,0}(e^{-2\lambda_4 t} - 1)/(-2\lambda_4) \quad + l_3 a_{3,0}{}^2 a_{2,0}\, t$$
$$+ l_4 a_{1,0} a_{3,0} a_{4,0}(e^{-(\lambda_3+\lambda_4)t} - 1)/(-\lambda_3 - \lambda_4)],$$

$$a_2 = a_{2,0} + \varepsilon\,[m_1 a_{4,0}{}^2 a_{3,0}(e^{(\lambda_2-\lambda_3-2\lambda_4)t} - 1)/(\lambda_2 - \lambda_3 - 2\lambda_4)$$
$$+ m_2 a_{3,0}{}^2 a_{4,0}(e^{(\lambda_2-2\lambda_3-\lambda_4)t} - 1)/(\lambda_2 - 2\lambda_3 - \lambda_4) \tag{4.26}$$
$$+ m_3 a_{3,0}{}^3 t + m_4 a_{2,0} a_{3,0} a_{4,0}(e^{-(\lambda_3+\lambda_4)t} - 1)/(-\lambda_3 - \lambda_4)],$$

$$a_3 = a_{3,0} + \varepsilon\, n_1 a_{4,0}{}^3 t,$$

$$a_4 = a_{4,0} + \varepsilon\, r_1 a_{2,0}{}^2 a_{4,0}(e^{-2\lambda_2 t} - 1)/(-2\lambda_2).$$

Secondly, we consider the cases $\lambda_2 = 3\lambda_3$ and $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4$. Under these conditions, we choose

$$e^{-\lambda_1 t}\left(\frac{d}{dt} - \lambda_1 + \lambda_2\right)\left(\frac{d}{dt} - \lambda_1 + \lambda_3\right)\left(\frac{d}{dt} - \lambda_1 + \lambda_4\right)A_1$$
$$= -(3a_4{}^2 a_2 e^{-(2\lambda_4+\lambda_2)t} + 6a_2\, a_3\, a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} + 3a_4{}^2 a_1 e^{-(\lambda_1+2\lambda_4)t} \tag{4.27}$$
$$+ 3a_3{}^2 a_2 e^{-(2\lambda_3+\lambda_2)t} + 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t}),$$

$$e^{-\lambda_2 t}\left(\frac{d}{dt} - \lambda_2 + \lambda_1\right)\left(\frac{d}{dt} - \lambda_2 + \lambda_3\right)\left(\frac{d}{dt} - \lambda_2 + \lambda_4\right)A_2 \tag{4.28}$$
$$= -(3a_4{}^2 a_3 e^{-(2\lambda_4+\lambda_3)t} + 3a_3{}^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + a_3{}^3 e^{-3\lambda_3 t}),$$

$$e^{-\lambda_3 t}\left(\frac{d}{dt} - \lambda_3 + \lambda_1\right)\left(\frac{d}{dt} - \lambda_3 + \lambda_2\right)\left(\frac{d}{dt} - \lambda_3 + \lambda_4\right)A_3 = -a_4{}^3 e^{-3\lambda_4 t}, \tag{4.29}$$

$$e^{-\lambda_4 t}\left(\frac{d}{dt}-\lambda_4+\lambda_1\right)\left(\frac{d}{dt}-\lambda_4+\lambda_2\right)\left(\frac{d}{dt}-\lambda_4+\lambda_3\right)A_4=-3a_2{}^2a_4e^{-(2\lambda_2+\lambda_4)t}. \tag{4.30}$$

Solving equations (4.27)-(4.30), we obtain

$$A_1=l_1a_4{}^2a_2e^{(\lambda_1-2\lambda_4-\lambda_2)t}+l_2\,a_2\,a_3\,a_4e^{(\lambda_1-\lambda_2-\lambda_3-\lambda_4)t}+l_3a_4{}^2a_1e^{-2\lambda_4 t}$$
$$+l_4a_3{}^2a_2e^{(\lambda_1-2\lambda_3-\lambda_2)t}+l_5a_1a_3a_4e^{-(\lambda_3+\lambda_4)t},$$

$$A_2=m_1a_4{}^2a_3e^{(\lambda_2-2\lambda_4-\lambda_3)t}+m_2a_3{}^2a_4e^{(\lambda_2-2\lambda_3-\lambda_4)t}+m_3a_3{}^3e^{(\lambda_2-3\lambda_3)t},$$

$$A_3=n_1a_4{}^3e^{(\lambda_3-3\lambda_4)t},\qquad A_4=r_1a_2{}^2a_4e^{-2\lambda_2 t}. \tag{4.31}$$

where

$$l_1=-3/[2\lambda_4(\lambda_2+\lambda_4)(\lambda_3-\lambda_2-2\lambda_4)],$$

$$l_2=6/[(\lambda_2+\lambda_3)(\lambda_2+\lambda_4)(\lambda_3+\lambda_4)],$$

$$l_3=3/[(\lambda_1+\lambda_4)(\lambda_2-\lambda_1-2\lambda_4)(\lambda_3-\lambda_1-2\lambda_4)],$$

$$l_4=-3/[2\lambda_3(\lambda_2+\lambda_3)(\lambda_4-\lambda_2-2\lambda_3)], \tag{4.32}$$

$$l_5=-6/[(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_2-\lambda_1-\lambda_3-\lambda_4)],$$

$$m_1=-3/[2\lambda_4(\lambda_3+\lambda_4)(\lambda_1-\lambda_3-2\lambda_4)],$$

$$m_2=-3/[2\lambda_3(\lambda_3+\lambda_4)(\lambda_1-2\lambda_3-\lambda_4)],$$

$$m_3=1/[2\lambda_3(\lambda_1-3\lambda_3)(\lambda_4-3\lambda_3)],$$

$$n_1=1/[2\lambda_4(\lambda_1-3\lambda_4)(\lambda_2-3\lambda_4)],$$

$$r_1=3/[(\lambda_2+\lambda_4)(\lambda_1-2\lambda_2-\lambda_4)(\lambda_3-2\lambda_2-\lambda_4)].$$

Substituting the values of A_1, A_2, A_3 and A_4 from (4.31) into the equation (4.4), we obtain

$$\dot{a}_1 = \varepsilon(l_1 a_4^{\,2} a_2 e^{(\lambda_1-2\lambda_4-\lambda_2)t} + l_2 a_4^{\,2} a_1 e^{-2\lambda_4 t} + l_3\, a_2\, a_3\, a_4 e^{(\lambda_1-\lambda_2-\lambda_3-\lambda_4)t} + l_4 a_3^{\,2} a_2 e^{(\lambda_1-2\lambda_3-\lambda_2)t} + l_5 a_1 a_3 a_4 e^{-(\lambda_3+\lambda_4)t}), \tag{4.33}$$

$$\dot{a}_2 = \varepsilon(m_1 a_4^{\,2} a_3 e^{(\lambda_2-2\lambda_4-\lambda_3)t} + m_2 a_3^{\,2} a_4 e^{(\lambda_2-2\lambda_3-\lambda_4)t} + m_3 a_3^{\,3} e^{(\lambda_2-3\lambda_3)t}), \tag{4.34}$$

$$\dot{a}_3 = \varepsilon\, n_1 a_4^{\,3} e^{(\lambda_3-3\lambda_4)t}, \tag{4.35}$$

$$\dot{a}_4 = \varepsilon\, r_1 a_2^{\,2} a_4 e^{-2\lambda_2 t}. \tag{4.36}$$

Since ε is a small quantity, we can solve equations (4.33)-(4.36) by assuming that a_1, a_2, a_3 and a_4 are constants in the right hand sides of (4.33)-(4.36). Thus, we obtain

$$\begin{aligned}
a_1 &= a_{1,0} + \varepsilon[l_1 a^2{}_{4,0} a_{2,0}(e^{(\lambda_1-2\lambda_4-\lambda_2)t} - 1)/(\lambda_1 - \lambda_2 - 2\lambda_4) \\
&\quad + l_2\, a_{4,0}^{\,2} a_{1,0}(e^{-2\lambda_4 t} - 1)/(-2\lambda_2) \quad + l_3\, a_{2,0}\, a_{3,0}\, a_{4,0}\, t \\
&\quad + l_4\, a_{3,0}^{\,2} a_{2,0}(e^{(\lambda_1-2\lambda_3-\lambda_2)t} - 1)/(\lambda_1 - \lambda_2 - 2\lambda_3) \\
&\quad + l_5\, a_{1,0} a_{3,0} a_{4,0}(e^{-(\lambda_3+\lambda_4)t} - 1)/(-\lambda_3 - \lambda_4)], \\
a_2 &= a_{2,0} + \varepsilon(m_1 a_{4,0}^{\,2} a_{3,0}(e^{(\lambda_2-2\lambda_4-\lambda_3)t} - 1)/(\lambda_2 - \lambda_3 - 2\lambda_4) \\
&\quad + m_2\, a_{3,0}^{\,2} a_{4,0}(e^{(\lambda_2-2\lambda_3-\lambda_4)t} - 1)/(\lambda_2 - 2\lambda_3 - \lambda_4) + m_3\, a_{3,0}^{\,3}\, t), \\
a_3 &= a_{3,0} + \varepsilon\, n_1 a_{4,0}^{\,3} t, \qquad a_4 = a_{4,0} + \varepsilon\, r_1 a_{2,0}^{\,2} a_{4,0}(e^{-2\lambda_2 t} - 1)/(2\lambda_2).
\end{aligned} \tag{4.37}$$

Therefore, we obtain the first approximate solution of equation (4.11) as

$$x = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} + \varepsilon\, u_1 \tag{4.38}$$

where a_1, a_2, a_3 and a_4 are given by (4.26) when $\lambda_2 \approx 3\lambda_3$ and $\lambda_1 \approx \lambda_2 + \lambda_3 + \lambda_4$ and u_1 is given by (4.14). But when $\lambda_2 = 3\lambda_3$ and $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4$, a_1, a_2, a_3 and a_4 should be computed by (4.37) instead of (4.26). In the second case u_1 is also computed by (4.14). The procedure can be carried out for n-th order in the same way.

4.4 Results and Discussions

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, we sometimes compare the approximate solution to the numerical solution. First of all, we consider the case $\lambda_2 \approx 3\lambda_3$ and $\lambda_1 \approx \lambda_2 + \lambda_3 + \lambda_4$.

For $k_1 = 28/3$, $k_2 = 26$, $k_3 = 68/3$, $k_4 = 5$ and $\varepsilon = 0.1$, we have obtained $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_3 = 1$ and $\lambda_4 = 1/3$. Therefore $\lambda_2 = 3\lambda_3$ and $\lambda_1 \approx \lambda_2 + \lambda_3 + \lambda_4$.

We have computed $x(t)$ by (4.38) in which a_1, a_2, a_3 and a_4 have been evaluated by (4.26) with two sets of initial conditions (i) $x(0) = 0.999952$, $\dot{x}(0) = -2.334831$, $\ddot{x}(0) = 8.782495$, $\dddot{x}(0) = -38.243824$ [or $a_1 = 0.25$, $a_2 = 0.25$, $a_3 = 0.25$, $a_4 = 0.25$] and (ii) $x(0) = 1.000177$, $\dot{x}(0) = -5.667585$, $\ddot{x}(0) = 25.225592$, $\dddot{x}(0) = -113.752365$ [or $a_1 = 0.75$, $a_2 = -0.25$, $a_3 = 0.75$, $a_4 = -0.25$] for various values of t, and the asymptotic solution x has been presented in the second column of the Table 4.1 and Table 4.2 respectively. Corresponding numerical solutions (designated by x^*) have been computed by a fourth order Runge-Kutta method and are given in the third column. Then the percentage errors have been calculated and are given in the fourth column.

From Table 4.1 and Table 4.2, we see that the percentage errors are much smaller than 1%. We also observe that the ratio of the coefficient of x and x^3 in the equation (11) is $\frac{\varepsilon}{5}$ and errors occur in an order of $\frac{\varepsilon^2}{25}$ (see [77] for details). Thus the new asymptotic solutions almost coincide with the numerical solution.

Table 4.1

t	x	x^*	$E\%$
0.0	0.999952	0.999952	0.0000
0.5	0.439283	0.439290	0.0015
1.0	0.285130	0.285132	0.0007
1.5	0.210338	0.210339	0.0004
2.0	0.162867	0.162869	0.0012
2.5	0.129364	0.129365	0.0007
3.0	0.104495	0.104496	0.0009
3.5	0.085445	0.085446	0.0011
4.0	0.070508	0.070509	0.0014

Table 4.2

t	x	x^*	$E\%$
0.0	1.000177	1.000177	0.0000
0.5	-0.134410	-0.134392	-0.0133
1.0	-0.228860	-0.228857	-0.0013
1.5	-0.198857	-0.198857	0.0000
2.0	-0.160484	-0.160484	0.0000
2.5	-0.128923	-0.128923	0.0000
3.0	-0.104468	-0.104468	0.0000
3.5	-0.085499	-0.085499	0.0000
4.0	-0.070572	-0.070572	0.0000

Now, we consider the case $\lambda_2 = 3\lambda_3$ and $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4$. For $k_1 = 26/3$, $k_2 = 208/3$, $k_3 = 178/3$, $k_4 = 13/3$ and $\varepsilon = 0.1$, we have obtained $\lambda_1 = 13/3$, $\lambda_2 = 3$, $\lambda_3 = 1$ and $\lambda_4 = 1/3$. Therefore $\lambda_2 = 3\lambda_3$ and $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4$.

We have again computed $x(t)$ by (4.38), in which a_1, a_2, a_3 and a_4 have been evaluated by (4.37) together with two sets of initial conditions (i) $x(0) = 0.999941$, $\dot{x}(0) = -2.167110$, $\ddot{x}(0) = 7.217916$, $\dddot{x}(0) = -27.282768$ [or $a_1 = 0.25$, $a_2 = 0.25$, $a_3 = 0.25$, $a_4 = 0.25$] and (ii) $x(0) = 1.000175$, $\dot{x}(0) = -5.161273$, $\ddot{x}(0) = 20.513668$, $\dddot{x}(0) = -80.779495$, [or $a_1 = 0.75$, $a_2 = 0.75$, $a_3 = -0.25$, $a_4 = -0.25$] for various values of t, and the results have been presented in the second column of the Table 4.3 and Table 4.4 respectively. Corresponding numerical solutions (designated by x^*) have been computed by a fourth order Runge-Kutta method and given in the third column. Then the percentage errors have been calculated and given in the fourth column. From Table 4.3 and Table 4.4, we find that percentage errors are less than 1%.

Further, we see that the errors occur as the ratio of $\frac{\varepsilon^2}{(\prod \lambda_i)^2}$. Thus the new asymptotic solution is in good agreement with the numerical solution.

Table 4.3

t	x	x^*	$E\%$
0.0	0.999941	0.999941	0.0000
0.5	0.447440	0.447448	0.0027
1.0	0.286720	0.286722	0.0006
1.5	0.210570	0.210571	0.0004
2.0	0.162899	0.162900	0.0006
2.5	0.129370	0.129371	0.0007
3.0	0.104498	0.104499	0.0009
3.5	0.085447	0.085447	0.0000
4.0	0.070509	0.070510	0.0014

Table 4.4

t	x	x^*	$E\%$
0.0	1.000175	1.000175	0.0000
0.5	-0.109503	-0.109483	-0.0182
1.0	-0.223839	-0.223835	-0.0017
1.5	-0.198046	-0.198046	0.0000
2.0	-0.160335	-0.160336	0.0006
2.5	-0.128873	-0.128875	0.0015
3.0	-0.104437	-0.104438	0.0009
3.5	-0.085473	-0.085474	0.0011
4.0	-0.070550	-0.070550	0.0000

In general, KBM are useful when $\varepsilon << 1$. Sometimes solution may be used even $\varepsilon = 1.0$ (see [46,87] for details). For the case $\lambda_2 \approx 3\lambda_3$ and $\lambda_1 \approx \lambda_2 + \lambda_3 + \lambda_4$, we have again computed $x(t)$ by (4.38), a_1, a_2, a_3 and a_4 are computed by (4.26), when $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_3 = 1$, $\lambda_4 = 1/3$ and $\varepsilon = 1.0$ with initial conditions $x(0) = 0.999516$, $\dot{x}(0) = -2.348314$, $\ddot{x}(0) = 8.824944$, $\dddot{x}(0) = -38.104900$ [or $a_1 = 0.25$, $a_2 = 0.25$, $a_3 = 0.25$, $a_4 = 0.25$] for various values of t, and have been presented in the second column of the Table 4.5. The corresponding numerical solutions (designated by x^*) have been computed by a fourth order Runge-Kutta method and are given in the third column of the Table 4.5. Percentage errors have been calculated and are given in the fourth column of the Table 4.5. In this case, theoretically, errors occur 4%, but from Table 4.5, we see that errors occur less than 1%.

Table 4.5

t	x	x^*	$E\%$
0.0	0.999516	0.999516	0.0000
0.5	0.436823	0.436865	0.0096
1.0	0.284190	0.284216	0.0091
1.5	0.210402	0.210446	0.0209
2.0	0.163297	0.163356	0.0361
2.5	0.129848	0.129912	0.0492
3.0	0.104919	0.104981	0.0600
3.5	0.085782	0.085840	0.0675
4.0	0.070765	0.070817	0.0734

The solution presented in this Chapter covers the case when one of the eigen-values of the corresponding unperturbed equation is a linear combination of the others. But this solution cannot be used for oscillatory, damped oscillatory or for large damping effects. In Chapter 5, a unified KBM method is presented for solving fourth order nonlinear differential equations with internal resonance.

4.5 Conclusion

For some particular damping effects in which one of the eigen values of the unperturbed equation is a multiple of the others or there exists a linear combination of the eigen values, the previous asymptotic solution found by Murty *et al* [52] is unable to give correct results. In this case, the over-damped solution, based on the works of Shamsul [83,85,87], is found. The solutions sometimes almost coincide with the numerical solutions when ε is small and sometimes the solution is useful even if $\varepsilon = 1.0$

Appendix 4.A

Murty, Deekshatulu and Krishna's solution [52]

Murty *et al* [52] obtained an asymptotic solution of (4.1) in the form

$$x(t) = \alpha(t)\cosh\psi_1(t) + \beta(t)\sinh\psi_2(t) + \varepsilon u_1(\alpha,\psi_1) + \varepsilon v_1(\beta,\psi_2), \tag{4.A.1}$$

where $\alpha(t)$, $\beta(t)$, $\psi_1(t)$ and $\psi_2(t)$ satisfy the equations

$$\begin{aligned}
\dot{\alpha} &= -K_1\alpha + \varepsilon\tilde{A}_1(\alpha) + \cdots,\\
\dot{\psi}_1 &= -K_2 + \varepsilon\tilde{C}_1(\alpha) + \cdots,\\
\dot{\beta} &= -K_7\beta + \varepsilon\tilde{B}_1(\beta) + \cdots,\\
\dot{\psi}_2 &= -K_8 + \varepsilon\tilde{D}_1(\beta) + \cdots,
\end{aligned} \tag{4.A.2}$$

where $K_1 = \dfrac{\lambda_1+\lambda_2}{2}$, $K_2 = \dfrac{\lambda_1-\lambda_2}{2}$, $K_7 = \dfrac{\lambda_3+\lambda_4}{2}$, $K_8 = \dfrac{\lambda_3-\lambda_4}{2}$.

Murty *et al* [52] determined the unknown functions u_1, v_1, $\tilde{A}_1$, $\tilde{B}_1$, $\tilde{C}_1$ and $\tilde{D}_1$ subject to the conditions that u_1, v_1 exclude the terms $e^{\pm\psi_1}$ and $e^{\pm\psi_2}$. But it is a too much laborious process to determine these unknown functions. In Chapter 2, we rediscovered Murty's *et al* [52] solution following Shamsul's [80] technique under the same assumptions (imposed by Murty *et al* [52]). The method presented in Chapter 2 is equivalent to Murty *et al* [52] technique but simpler than that of [52]. Following the restrictions imposed in [52], the variational equations in Chapter 2 take the form

$$\dot{a}_1 = \varepsilon\left(\frac{3\,a_1^{\,2}a_2\,e^{(\lambda_1+\lambda_2)t}}{2\lambda_1(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_4)} + \frac{6\,a_1a_3a_4\,e^{(\lambda_3+\lambda_4)t}}{(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_1-\lambda_2+\lambda_3+\lambda_4)}\right), \tag{4.A.3}$$

$$m_3 = m_4;\ m_1 = m_2 + 1$$

$$\dot{a}_2 = \varepsilon\left(\frac{3\,a_2^{\,2}a_1\;e^{(\lambda_1+\lambda_2)t}}{2\lambda_2(\lambda_1+2\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_4)} + \frac{6a_2a_3a_4\;e^{(\lambda_3+\lambda_4)t}}{(\lambda_2-\lambda_1+\lambda_3+\lambda_4)(\lambda_2+\lambda_4)(\lambda_2+\lambda_3)}\right), \quad (4.A.4)$$

$$m_3 = m_4;\; m_1 = m_2 - 1$$

$$\dot{a}_3 = \varepsilon\left(\frac{3\,a_3^{\,2}a_4\;e^{(\lambda_3+\lambda_4)t}}{2\lambda_3(\lambda_1-2\lambda_3-\lambda_4)(\lambda_2-2\lambda_3-\lambda_4)} + \frac{6a_1a_2a_3\;e^{(\lambda_1+\lambda_2)t}}{(\lambda_1+\lambda_2+\lambda_3-\lambda_4)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}\right), \quad (4.A.5)$$

$$m_1 = m_2;\; m_3 = m_4 + 1$$

$$\dot{a}_4 = \varepsilon\left(\frac{3\,a_4^{\,2}a_3\;e^{(\lambda_3+\lambda_4)t}}{2\lambda_4(\lambda_1-\lambda_3-2\lambda_4)(\lambda_2-\lambda_3-2\lambda_4)} + \frac{6\,a_1a_2a_4\;e^{(\lambda_1+\lambda_2)t}}{(\lambda_1+\lambda_2-\lambda_3+\lambda_4)(\lambda_1+\lambda_4)(\lambda_2+\lambda_4)}\right). \quad (4.A.6)$$

$$m_1 = m_2;\; m_3 = m_4 - 1$$

Under the transformations, $a_1 = \frac{1}{2}ae^{\varphi_1}$, $a_2 = \frac{1}{2}ae^{-\varphi_1}$, $a_3 = \frac{1}{2}be^{\varphi_2}$ and $a_4 = -\frac{1}{2}be^{-\varphi_2}$, the variational equations are transferred to

$$\begin{aligned}
\dot{a} &= \varepsilon\,(p_1a^3e^{-(\lambda_1+\lambda_2)t} + p_2ab^2e^{-(\lambda_3+\lambda_4)t}),\\
\dot{b} &= \varepsilon\,(p_3b^3e^{-(\lambda_3+\lambda_4)t} + p_4a^2be^{-(\lambda_1+\lambda_2)t}),\\
\dot{\varphi}_1 &= \varepsilon\,(p_5a^2e^{-(\lambda_1+\lambda_2)t} + p_6b^2e^{-(\lambda_3+\lambda_4)t}),\\
\dot{\varphi}_2 &= \varepsilon\,(p_7b^2e^{-(\lambda_3+\lambda_4)t} + p_8a^2e^{-(\lambda_1+\lambda_2)t}),
\end{aligned} \quad (4.A.7)$$

where

$$p_1 = \frac{3}{16\lambda_1(2\lambda_1+\lambda_2-\lambda_3)(2\lambda_1+\lambda_2-\lambda_4)} + \frac{3}{16\lambda_2(\lambda_1+2\lambda_2-\lambda_3)(\lambda_1+2\lambda_2-\lambda_4)},$$

$$p_2 = \frac{-3}{4(\lambda_1+\lambda_3)(\lambda_1+\lambda_4)(\lambda_1-\lambda_2+\lambda_3+\lambda_4)} + \frac{3}{4(\lambda_2+\lambda_4)(\lambda_2+\lambda_3)(\lambda_1-\lambda_2-\lambda_3-\lambda_4)},$$

$$p_3 = \frac{-3}{16\lambda_3(\lambda_1 - 2\lambda_3 - \lambda_4)(\lambda_2 - 2\lambda_3 - \lambda_4)} + \frac{3}{16\lambda_4(\lambda_1 - \lambda_3 - 2\lambda_4)(\lambda_2 - \lambda_3 - 2\lambda_4)},$$

$$p_4 = \frac{3}{4(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4)} + \frac{-3}{4(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4)},$$

$$p_5 = \frac{3}{16\lambda_1(2\lambda_1 + \lambda_2 - \lambda_3)(2\lambda_1 + \lambda_2 - \lambda_4)} + \frac{-3}{16\lambda_2(\lambda_1 + 2\lambda_2 - \lambda_3)(\lambda_1 + 2\lambda_2 - \lambda_4)}, \quad (4.A.8)$$

$$p_6 = \frac{-3}{4(\lambda_1 + \lambda_3)(\lambda_1 + \lambda_4)(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4)} - \frac{-3}{4(\lambda_2 + \lambda_4)(\lambda_2 + \lambda_3)(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)},$$

$$p_7 = \frac{-3}{16\lambda_3(\lambda_1 - 2\lambda_3 - \lambda_4)(\lambda_2 - 2\lambda_3 - \lambda_4)} + \frac{-3}{16\lambda_4(\lambda_1 - \lambda_3 - 2\lambda_4)(\lambda_2 - \lambda_3 - 2\lambda_4)},$$

$$p_8 = \frac{3}{4(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4)} + \frac{3}{4(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_4)(\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4)}.$$

By the substitutions of $\alpha = e^{-K_1 t}a$, $\beta = e^{-K_7 t}b$, $\psi_1 = -K_2 t + \varphi_1$, $\psi_2 = -K_8 t + \varphi_2$, together with $\lambda_{1,2} = K_1 \pm K_2$ and $\lambda_{3,4} = K_7 \pm K_8$, we obtain Murty's [52] solution in the form (4.A.1) and (4.A.2).

Clearly p_2 and p_6 become indefinite when $\lambda_1 = \lambda_2 + \lambda_3 + \lambda_4$, while p_3 and p_7 become indefinite when $\lambda_2 = \lambda_3 + 2\lambda_4$. Thus Murty's first order solution (with out εu_1), as well as the solution presented in Chapter 2 is unable to give the desired results.

Chapter 5

Unified KBM Method for Solving Fourth Order Nonlinear Systems with Internal Resonance

5.1 Introduction

Recently, Shamsul [80] has presented a unified Krylov-Bogoliubov-Mitropolskii (KBM) [13,34] method for solving an n-th order ($n \geq 2$) ordinary differential equation with small nonlinearities. It is noted that the unified technique was introduced by Murty [53] to obtain an approximate solution of a second-order equation when the unperturbed equation has two real or complex or purely imaginary eigen-values. Shamsul [80] has generalized the technique of Murty for the second-, third-, fourth-order etc. equations. Recently, Shamsul [93] has modified and used the formula (derived in [80]) to investigate time-dependent differential systems. Shamsul [89] has also extended the method [80] to similar nonlinear systems with slowly varying coefficients.

On the contrary, Shamsul [83] has observed that the unified solution gives incorrect results when one of the real eigen-values of unperturbed equation becomes a multiple of the others. But more difficulties arise when a pair of eigen-values becomes a multiple of the other pair or pairs or some of the natural frequencies of the unperturbed equation are in integral ratio. Thus the problem appears not only in a nonlinear over-damped process, but also in damped and undamped processes (including the internal resonant vibrations). The former procedure [83] is limited to certain over-damped systems.

In this Chapter, we tackle the general case, which covers undamped, damped and over-damped processes. However, the new unified solution described in this chapter does not cover the situation considered in [83] as well as the solution presented in Chapter 4.

5.2 The Method

Consider a nonlinear differential equation

$$x^{(4)} + k_1 \dddot{x} + k_2 \ddot{x} + k_3 \dot{x} + k_4 x = \varepsilon\, f(x, \dot{x}, \ddot{x}, \dddot{x}), \tag{5.1}$$

where $x^{(4)}$, represents the 4-th derivative of x, over-dot is used for the first, the second and the third derivatives of x with respect to t, ε is a small parameter, k_j, $j = 1, 2, 3, 4$ are constants and f is a nonlinear function.

When $\varepsilon = 0$, equation (5.1) has four eigen-values, say λ_j, $j = 1, 2, 3, 4$ and the solution becomes

$$x(t,0) = \sum_{j=1}^{4} a_{j,0} e^{\lambda_j t}, \tag{5.2}$$

where $a_{j,0}$, $j = 1, 2, 3, 4$ are arbitrary constants.

An asymptotic solution of equation (5.1) has been found in the form [80]

$$x(t,\varepsilon) = \sum_{j=1}^{4} a_j(t) e^{\lambda_j t} + \varepsilon u_1(a_1, a_2, a_3, a_4, t) + O(\varepsilon^2), \tag{5.3}$$

where u_1 is a function of a_1, a_2, a_3, a_4 and t, and each a_j satisfies a differential equation

$$\dot{a}_j = \varepsilon A_j(a_1, a_2, a_3, a_4, t) + O(\varepsilon^2), \tag{5.4}$$

and A_j, $j = 1, 2, 3, 4$ are also function of a_1, a_2, a_3, a_4 and t.

For obtaining a first order solution, Shamsul [80] has presented the following formula

$$\sum_{j=1}^{4}\left(\prod_{k=1,\,k\neq j}^{4}(D-\lambda_k)(e^{\lambda_j t}A_j)\right)+\prod_{j=1}^{4}(D-\lambda_j)u_1=f^{(0)}(a_1,a_2,a_3,a_4,t), \qquad (5.5)$$

where $D=\dfrac{\partial}{\partial t}$, $f^{(0)}=f(x_0,\dot{x}_0,\ddot{x}_0,\dddot{x}_0)$ and $x_0=\sum_{j=1}^{4}a_j(t)e^{\lambda_j t}$. Following [80], equation (5.5) can be separated into five individual equations to determine u_1 and A_j, $j=1,2,3,4$ subject to the condition that u_1 excludes all fundamental terms (see [13,34,53] for details). On the contrary, the special over-damped solution [83] is determined under the restriction that u_1 excludes the terms involving $e^{(i_1\lambda_1+i_2\lambda_2+i_3\lambda_3+i_4\lambda_4)t}$ of $f^{(0)}$, when $\sum_{j=1}^{4}i_j\,|\lambda_j|\leq\sum_{j=1}^{4}i_j\times\sum_{j=1}^{4}|\lambda_j|/4$.

Clearly, solution (5.3) starts containing some unusual variables, a_j, $j=1,2,3,4$ rather than amplitudes and phases. Yet this form is very important. The construction of equation (5.5) is simple and it can be brought to the usual form by the transformation $a_{2l-1}=\frac{1}{2}\alpha_l e^{i\varphi_l}$, $a_{2l}=\frac{1}{2}\alpha_l e^{-i\varphi_l}$, $l=1,2$, where α_l and φ_l are amplitude and phase variables (see [92] for details). The transformed formula [92] is straightforward and it is used to obtain undamped, damped and over-damped solutions quickly. But some special over-damped solutions are directly determined from equation (5.5) [83]. Naturally, it needs two steps to obtain the unified solution (which covers undamped, damped and over-damped cases) from equation (5.5) according to the principle of [80]. First, u_1 and A_j, $j=1,2,3,4$ are obtained from equation (5.5) and then substituting these into equations (5.3)-(5.4) and transforming the variables a_j, $j=1,2,3,4$ by amplitude and phase variables (mentioned above) the desired solution is found. On the contrary, similar unified solution can be found directly form the transformed formula [92]. But the transformed formula [92] is unable to determine the special over-damped solutions [83], when one of the eigen-values is a multiple of the other eigen-value or eigen-values. It is interesting to note that the transformed formula [92] is useful when one pair of eigen-values is a multiple of the other pair or pairs (considered in this Chapter). However, we use both formulae [80,92] for determining the first order solution of a fourth order equation to clear the matter and also compare the two formulae. In this Chapter, we shall discuss briefly the transformed formula [92].

Let us consider the situation when n is an even number and rewrite equation (5.5) as

$$\sum_{l=1}^{2}\left(\prod_{k=1,\, k\neq 2l-1,\, 2l}^{4}(D-\lambda_k)[(D-\lambda_{2l})(e^{\lambda_{2l-1}t}A_{2l-1})+(D-\lambda_{2l-1})(e^{\lambda_{2l}t}A_{2l})]\right)$$
$$+\prod_{j=1}^{4}(D-\lambda_j)u_1 = f^{(0)}(a_1,a_2,a_3,a_4,t). \tag{5.6}$$

With the change of the variables $a_j,\ j=1,2,3,4$ by $a_{2l-1}=\frac{1}{2}\alpha_l e^{i\varphi_l}$, $a_{2l}=\frac{1}{2}\alpha_l e^{-i\varphi_l}$, $l=1,2$, together with the substitutions $\lambda_{2l-1}=-\mu_l+i\omega_l$, $\lambda_{2l}=-\mu_l-i\omega_l$, $A_{2l-1}=\frac{1}{2}(\widetilde{A}_l+i\widetilde{B}_l)$, $A_{2l}=\frac{1}{2}(\widetilde{A}_l-i\widetilde{B}_l)$, equation (5.6) becomes

$$\sum_{l=1}^{2}\left(\prod_{k=1,\, k\neq 2l-1,\, 2l}^{n}(D-\lambda_k)[e^{-\mu_l t}\{\cos\psi_l(D\widetilde{A}_l-2\omega_l\alpha_l\widetilde{B}_l)-\sin\psi_l(2\omega_l\widetilde{A}_l+\alpha_l D\widetilde{B}_l)\}]\right)$$
$$+\prod_{j=1}^{4}(D-\lambda_j)u_1 = f^{(0)}(\alpha_1,\alpha_2,\psi_1,\psi_2,t), \qquad \psi_l=\omega_l t+\varphi_l. \tag{5.7}$$

It is noted that the transformed equations of equation (5.4) for α_l, φ_l is

$$\dot{\alpha}_l=\varepsilon\widetilde{A}_l+O(\varepsilon^2), \quad \dot{\varphi}_l=\varepsilon\widetilde{B}_l+O(\varepsilon^2). \tag{5.8}$$

Equation (5.7) is the transformed formula of equation (5.5), which has been derived in [92] and is used directly to obtain undamped, damped and over-damped systems subject to the condition that u_1 excludes all first harmonic terms or hyperbolic terms (in case of an over-damped system). The method can be carried out for higher order differential equations in the same way.

5.3 Example

First of all, we consider the undamped case of a fourth order differential system as [92]

$$(D^2+\omega_1^2)(D^2+\omega_2^2)x=\varepsilon x^3. \tag{5.9}$$

Here $x_0=a_{1,0}e^{i\omega_1 t}+a_{2,0}e^{-i\omega_1 t}+a_{3,0}e^{i\omega_2 t}+a_{4,0}e^{-i\omega_2 t}$. In accordance with [80], equation (5.5) can be written as

$$\begin{aligned}
&(D^2+\omega_2^2)(D+i\omega_1)(e^{i\omega_1 t}A_1)+(D^2+\omega_2^2)(D-i\omega_1)(e^{-i\omega_1 t}A_2)\\
&+(D^2+\omega_1^2)(D+i\omega_2)(e^{i\omega_2 t}A_3)+(D^2+\omega_1^2)(D-i\omega_2)(e^{-i\omega_2 t}A_4)\\
&+(D^2+\omega_1^2)(D^2+\omega_2^2)u_1=3{a_1}^2a_2e^{i\omega_1 t}+6a_1a_3a_4e^{i\omega_1 t}+3{a_2}^2a_3e^{-i(2\omega_1-\omega_2)t}\\
&+3{a_2}^2a_1e^{-i\omega_1 t}+6a_2a_3a_4e^{-i\omega_1 t}+3{a_1}^2a_4e^{i(2\omega_1-\omega_2)t}\\
&+3{a_3}^2a_4e^{i\omega_2 t}+6a_1a_2a_3e^{i\omega_2 t}+{a_1}^3e^{3i\omega_1 t}\\
&+3{a_4}^2a_3e^{-i\omega_2 t}+6a_1a_2a_4e^{-i\omega_2 t}+{a_2}^3e^{-3i\omega_1 t}\\
&+{a_3}^3e^{3i\omega_2 t}+{a_4}^3e^{-3i\omega_2 t}+3{a_1}^2a_3e^{i(2\omega_1+\omega_2)t}+3{a_2}^2a_4e^{-i(2\omega_1+\omega_2)t}\\
&+3{a_3}^2a_1e^{i(2\omega_2+\omega_1)t}+3{a_4}^2a_2e^{-i(2\omega_2+\omega_1)t}+3{a_3}^2a_2e^{i(2\omega_2-\omega_1)t}+3{a_4}^2a_1e^{-i(2\omega_2-\omega_1)t}.
\end{aligned} \tag{5.10}$$

We can obtain equations for A_1, A_2, A_3, A_4 and u_1 from equation (5.10), subject to the condition that u_1 excludes the terms of $e^{\pm i\omega_1}$, $e^{\pm i\omega_2}$ together with $e^{\pm i3\omega_1}$, $e^{\mp i(2\omega_1-\omega_2)t}$. The later terms are usually included in u_1. But in case of internal resonance [*i.e.*, $3\omega_1-\omega_2=O(\varepsilon)$ for equation (5.9)] u_1 becomes $O(\varepsilon^{-1})$ or $\varepsilon u_1=O(1)$ provided that u_1 includes the terms $e^{\pm i3\omega_1}$ and $e^{\mp i(2\omega_1-\omega_2)t}$ (see **Appendix 5.A**). Therefore, u_1 must exclude the terms $e^{\pm i3\omega_1}$, $e^{\mp i(2\omega_1-\omega_2)t}$ and we have

$$(D^2+\omega_2^2)(D+i\omega_1)(e^{i\omega_1 t}A_1)=3{a_1}^2a_2e^{i\omega_1 t}+6a_1a_3a_4e^{i\omega_1 t}+3{a_2}^2a_3e^{-i(2\omega_1-\omega_2)t}, \tag{5.11}$$

$$(D^2+\omega_2^2)(D-i\omega_1)(e^{-i\omega_1 t}A_2)=3{a_2}^2a_1e^{-i\omega_1 t}+6a_2a_3a_4e^{-i\omega_1 t}+3{a_1}^2a_4e^{i(2\omega_1-\omega_2)t}, \tag{5.12}$$

$$(D^2+\omega_1^2)(D+i\omega_2)(e^{i\omega_2 t}A_3)=3a_3{}^2a_4e^{i\omega_2 t}+6a_1a_2a_3e^{i\omega_2 t}+a_1{}^3e^{3i\omega_1 t}, \tag{5.13}$$

$$(D^2+\omega_1^2)(D-i\omega_2)(e^{-i\omega_2 t}A_4)=3a_4{}^2a_3e^{-i\omega_2 t}+6a_1a_2a_4e^{-i\omega_2 t}+a_2{}^3e^{-3i\omega_1 t}, \tag{5.14}$$

and

$$\begin{aligned}(D^2+\omega_1^2)(D^2+\omega_2^2)u_1 &= a_3{}^3e^{3i\omega_2 t}+a_4{}^3e^{-3i\omega_2 t}+3a_1{}^2a_3e^{i(2\omega_1+\omega_2)t}+3a_2{}^2a_4e^{-i(2\omega_1+\omega_2)t}\\ &+3a_3{}^2a_1e^{i(2\omega_2+\omega_1)t}+3a_4{}^2a_2e^{-i(2\omega_2+\omega_1)t}+3a_3{}^2a_2e^{i(2\omega_2-\omega_1)t}+3a_4{}^2a_1e^{-i(2\omega_2-\omega_1)t}.\end{aligned} \tag{5.15}$$

The particular solutions of equations (5.11)-(5.15) are

$$\begin{aligned}A_1&=\frac{-3(a_1{}^2a_2+2a_1a_3a_4)}{2i\omega_1(\omega_1{}^2-\omega_2{}^2)}+\frac{3a_2{}^2a_3\,e^{-i(3\omega_1-\omega_2)t}}{4i\omega_1(\omega_1-\omega_2)^2},\\ A_2&=\frac{3(a_2{}^2a_1+2a_2a_3a_4)}{2i\omega_1(\omega_1{}^2-\omega_2{}^2)}-\frac{3a_1{}^2a_4\,e^{i(3\omega_1-\omega_2)t}}{4i\omega_1(\omega_1-\omega_2)^2},\\ A_3&=\frac{3(a_3{}^2a_4+2a_1a_2a_3)}{2i\omega_2(\omega_1{}^2-\omega_2{}^2)}-\frac{a_1{}^3e^{i(3\omega_1-\omega_2)t}}{8i\omega_1{}^2(3\omega_1+\omega_2)},\\ A_4&=\frac{-3(a_4{}^2a_3+2a_1a_2a_4)}{2i\omega_2(\omega_1{}^2-\omega_2{}^2)}+\frac{a_2{}^3e^{-i(3\omega_1-\omega_2)t}}{8i\omega_1{}^2(3\omega_1+\omega_2)}.\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}u_1&=\frac{a_3{}^3e^{3i\omega_2 t}+a_4{}^3e^{-3i\omega_2 t}}{8\omega_2^2(9\omega_2^2-\omega_1^2)}+\frac{3a_1{}^2a_3e^{i(2\omega_1+\omega_2)t}+3a_2{}^2a_4e^{-i(2\omega_1+\omega_2)t}}{[(2\omega_1+\omega_2)^2-\omega_1^2][(2\omega_1+\omega_2)^2-\omega_2^2]}\\ &+\frac{3a_3{}^2a_1e^{i(2\omega_2+\omega_1)t}+3a_4{}^2a_2e^{-i(2\omega_2+\omega_1)t}}{[(2\omega_2+\omega_1)^2-\omega_1^2][(2\omega_2+\omega_1)^2-\omega_2^2]}+\frac{3a_3{}^2a_2e^{i(2\omega_2-\omega_1)t}+3a_4{}^2a_1e^{-i(2\omega_2-\omega_1)t}}{[(2\omega_2-\omega_1)^2-\omega_1^2][(2\omega_2-\omega_1)^2-\omega_2^2]}.\end{aligned} \tag{5.17}$$

Substituting the values of $A_1,\cdots,A_4$ from equation (5.16) into equation (5.4), we obtain

$$\dot{a}_1 = \varepsilon\left(\frac{-3({a_1}^2 a_2 + 2a_1 a_3 a_4)}{2i\omega_1({\omega_1}^2 - {\omega_2}^2)} + \frac{3{a_2}^2 a_3\, e^{-i(3\omega_1-\omega_2)t}}{4i\omega_1(\omega_1-\omega_2)^2}\right),$$

$$\dot{a}_2 = \varepsilon\left(\frac{3({a_2}^2 a_1 + 2a_2 a_3 a_4)}{2i\omega_1({\omega_1}^2 - {\omega_2}^2)} - \frac{3{a_1}^2 a_4\, e^{i(3\omega_1-\omega_2)t}}{4i\omega_1(\omega_1-\omega_2)^2}\right),$$

$$\dot{a}_3 = \varepsilon\left(\frac{3({a_3}^2 a_4 + 2a_1 a_2 a_3)}{2i\omega_2({\omega_1}^2 - {\omega_2}^2)} - \frac{{a_1}^3 e^{i(3\omega_1-\omega_2)t}}{8i{\omega_1}^2(3\omega_1+\omega_2)}\right), \tag{5.18}$$

$$\dot{a}_4 = \varepsilon\left(\frac{-3({a_4}^2 a_3 + 2a_1 a_2 a_4)}{2i\omega_2({\omega_1}^2 - {\omega_2}^2)} + \frac{{a_2}^3 e^{-i(3\omega_1-\omega_2)t}}{8i{\omega_1}^2(3\omega_1+\omega_2)}\right),$$

Under the transformations $a_1 = \frac{1}{2}a e^{i\varphi_1}$, $a_2 = \frac{1}{2}a e^{-i\varphi_1}$, $a_3 = \frac{1}{2}b e^{i\varphi_2}$, $a_4 = \frac{1}{2}b e^{-i\varphi_2}$, equation (5.18) readily becomes

$$\dot{a} = \varepsilon {l_1}^* a^2 b \sin(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2),$$

$$\dot{b} = \varepsilon {m_1}^* a^3 \sin(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2),$$

$$\dot{\varphi}_1 = \varepsilon {n_1}^*(a^2 + 2b^2) + \varepsilon {l_1}^* ab\cos(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2), \tag{5.19}$$

$$\dot{\varphi}_2 = \varepsilon {r_1}^*(2a^2 + b^2) - \varepsilon {m_1}^*(a^3/b)\cos(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2),$$

where

$$l_1^* = -3/[16\omega_1(\omega_1-\omega_2)^2], \quad n_1^* = 3/[8\omega_1({\omega_1}^2 - {\omega_2}^2)],$$

$$m_1^* = -1/[32{\omega_1}^2(3\omega_1+\omega_2)], \quad r_1^* = -3/[8\omega_2({\omega_1}^2 - {\omega_2}^2)]. \tag{5.20}$$

By similar transformations (mentioned above), equation (5.17) becomes

$$u_1 = \frac{b^3\cos(3\omega_2 t + 3\varphi_2)}{32\omega_2^2(9\omega_2^2 - \omega_1^2)} + \frac{3a^2 b\cos(2\omega_1 t + \omega_2 t + 2\varphi_1 + \varphi_2)}{4[(2\omega_1+\omega_2)^2 - \omega_1^2][(2\omega_1+\omega_2)^2 - \omega_2^2]}$$

$$+\frac{3ab^2\cos(2\omega_2 t + \omega_1 t + 2\varphi_2 + \varphi_1)}{4[(2\omega_2+\omega_1)^2 - \omega_1^2][(2\omega_2+\omega_1)^2 - \omega_2^2]} + \frac{3ab^2\cos(2\omega_2 t - \omega_1 t + 2\varphi_2 - \varphi_1)}{4[(2\omega_2-\omega_1)^2 - \omega_1^2][(2\omega_2-\omega_1)^2 - \omega_2^2]}. \tag{5.21}$$

Thus, the first order improved solution of equation (5.9) is

$$x = a\cos(\omega_1 t + \varphi_1) + b\cos(\omega_2 t + \varphi_2) + \varepsilon u_1, \tag{5.22}$$

where a, b, φ_1, φ_2 are solutions of equation (5.19) and u_1 is given by equation (5.21).

In general, the first order equations of amplitudes and phases are independent of phase variables (see also [13,34,80,92]), but in equation (5.19), equations for $\dot{a}$, $\dot{b}$, $\dot{\varphi}_1$ and $\dot{\varphi}_2$ depend on both phases. Thus in case of internal resonance [where $3\omega_1 - \omega_2 = O(\varepsilon)$], the first order differential equations of amplitudes and phases are not independent of phases. Clearly equation (5.19) is valid when $a(0) \geq 0$ and $b(0) > 0$. When $b(0) = 0$, φ_2 becomes indefinite provided that $\varepsilon \neq 0$. However, for a very small value of $b(0)$, we can show that the perturbation solution gives the desired results. It is noted that the resonant (whether internal or external [93]) and non-resonant cases are investigated from a single formula equation (5.5), which is an advantage of this method.

Now, we shall apply the transformed equation (5.7) [92] to get the same results as those obtained in equations (5.19)-(5.21). For the system (5.9), equation (5.7) readily becomes

$$\begin{aligned}
&(D^2 + \omega_2^2)[\cos(\omega_1 t + \varphi_1)(D\widetilde{A}_1 - 2\omega_1 a\widetilde{B}_1) - \sin(\omega_1 t + \varphi_1)(2\omega_1 \widetilde{A}_1 + aD\widetilde{B}_1)] \\
&+ (D^2 + \omega_1^2)[\cos(\omega_2 t + \varphi_2)(D\widetilde{A}_2 - 2\omega_2 b\widetilde{B}_2) - \sin(\omega_2 t + \varphi_2)(2\omega_2 \widetilde{A}_2 + bD\widetilde{B}_2)] \\
&(D^2 + \omega_1^2)(D^2 + \omega_2^2)u_1 = \tfrac{3}{4}a^3\cos(\omega_1 t + \varphi_1) + \tfrac{3}{2}ab^2\cos(\omega_1 t + \varphi_1) \\
&+ \tfrac{3}{4}a^2 b\cos(2\omega_1 t - \omega_2 t + 2\varphi_1 - \varphi_2) + \tfrac{3}{4}b^3\cos(\omega_2 t + \varphi_2) + \tfrac{3}{2}a^2 b\cos(\omega_2 t + \varphi_2) \\
&+ \tfrac{1}{4}a^3\cos 3(\omega_1 t + \varphi_1) + \tfrac{1}{4}b^3\cos 3(\omega_2 t + \varphi_2) + \tfrac{3}{4}a^2 b\cos(2\omega_1 t + \omega_2 t + 2\varphi_1 + \varphi_2) \\
&+ \tfrac{3}{4}ab^2\cos(2\omega_2 t + \omega_1 t + 2\varphi_2 + \varphi_1) + \tfrac{3}{4}ab^2\cos(2\omega_2 t - \omega_1 t + 2\varphi_2 - \varphi_1).
\end{aligned} \tag{5.23}$$

Here, we have used the notation $\alpha_1 = a$ and $\alpha_2 = b$. From equation (5.23), we obtain

$$\begin{aligned}
&(D^2 + \omega_2^2)[\cos(\omega_1 t + \varphi_1)(D\widetilde{A}_1 - 2\omega_1 a\widetilde{B}_1) - \sin(\omega_1 t + \varphi_1)(2\omega_1 \widetilde{A}_1 + aD\widetilde{B}_1)] \\
&= \tfrac{3}{4}a^2 b\cos(2\omega_1 t - \omega_2 t + 2\varphi_1 - \varphi_2) + \tfrac{3}{4}a^3\cos(\omega_1 t + \varphi_1) + \tfrac{3}{2}ab^2\cos(\omega_1 t + \varphi_1),
\end{aligned} \tag{5.24}$$

$$(D^2+\omega_1^2)[\cos(\omega_2 t+\varphi_2)(D\widetilde{A}_2-2\omega_2 b\widetilde{B}_2)-\sin(\omega_2 t+\varphi_2)(2\omega_2\widetilde{A}_2+bD\widetilde{B}_2)]$$
$$=\tfrac{3}{4}b^3\cos(\omega_2 t+\varphi_2)+\tfrac{3}{2}a^2 b\cos(\omega_2 t+\varphi_2)+\tfrac{1}{4}a^3\cos 3(\omega_1 t+\varphi_1) \quad (5.25)$$

and

$$(D^2+\omega_1^2)(D^2+\omega_2^2)u_1=\tfrac{1}{4}b^3\cos 3(\omega_2 t+\varphi_2)+\tfrac{3}{4}a^2 b\cos(2\omega_1 t+\omega_2 t+2\varphi_1+\varphi_2)$$
$$+\tfrac{3}{4}ab^2\cos(2\omega_2 t+\omega_1 t+2\varphi_2+\varphi_1)+\tfrac{3}{4}ab^2\cos(2\omega_2 t-\omega_1 t+2\varphi_2-\varphi_1). \quad (5.26)$$

We have already mentioned that u_1 excludes the terms $e^{\pm i3\omega_1}$, $e^{\mp i(2\omega_1-\omega_2)t}$ when $3\omega_1-\omega_2=O(\varepsilon)$. In this situation the real forms of these terms, *i.e.*, $\cos(2\omega_1 t-\omega_2 t+2\varphi_1-\varphi_2)$ and $\cos 3(\omega_1 t+\varphi_1)$ are added respectively to equations (5.24)-(5.25), since $\cos(2\omega_1 t-\omega_2 t)\approx\cos\omega_1 t$ and $\cos 3\omega_1 t\approx\cos\omega_2 t$. Solving equations (5.24)-(5.25), we obtain

$$\begin{aligned}
\widetilde{A}_1&=l_1^* a^2 b\sin(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2),\\
\widetilde{A}_2&=m_1^* a^3\sin(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2),\\
\widetilde{B}_1&=n_1^*(a^2+2b^2)+l_1^* ab\cos(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2),\\
\widetilde{B}_2&=r_1^*(2a^2+b^2)-m_1^*(a^3/b)\cos(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2),
\end{aligned} \quad (5.27)$$

where $l_1^*,\cdots r_1^*$ are given by equation (5.20).

The particular solution of equation (5.26) readily turns out to be the result obtained in equation (5.21). Substituting the values of $\widetilde{A}_1,\cdots,\widetilde{B}_2$ from equation (5.27) into equation (5.8), we obtain the same results obtained in equation (5.19). From equation (5.17) or equation (5.21), we see that u_1 is finite when $3\omega_1\approx\omega_2$, while u_1, obtained in [92], becomes infinite when $3\omega_1\to\omega_2$ (see also **Appendix 5.A**).

Now, we can obtain similar solution of a fourth order equation when the linear damping force acts. In this case equation (5.9) becomes (see also [92])

$$(D^2+2\mu_1 D+\mu_1^2+\omega_1^2)(D^2+2\mu_2 D+\mu_2^2+\omega_2^2)x=\varepsilon x^3. \tag{5.28}$$

In this case equation (5.7) takes the form

$$\begin{aligned}
&[(D+\mu_2)^2+\omega_2^2][e^{-\mu_1 t}\{\cos(\omega_1 t+\varphi_1)(D\widetilde{A}_1-2\omega_1 a\widetilde{B}_1)-\sin(\omega_1 t+\varphi_1)(2\omega_1\widetilde{A}_1+aD\widetilde{B}_1)\}]\\
&+[(D+\mu_1)^2+\omega_1^2][e^{-\mu_2 t}\{\cos(\omega_2 t+\varphi_2)(D\widetilde{A}_2-2\omega_2 b\widetilde{B}_2)-\sin(\omega_2 t+\varphi_2)(2\omega_2\widetilde{A}_2+bD\widetilde{B}_2)\}]\\
&[(D+\mu_1)^2+\omega_1^2][(D+\mu_2)^2+\omega_2^2]u_1=\tfrac{3}{4}a^3e^{-3\mu_1 t}\cos(\omega_1 t+\varphi_1)+\tfrac{3}{2}ab^2e^{-(\mu_1+2\mu_2)t}\cos(\omega_1 t+\varphi_1)\\
&+\tfrac{3}{4}a^2be^{-(2\mu_1+\mu_2)t}\cos(2\omega_1 t-\omega_2 t+2\varphi_1-\varphi_2)+\tfrac{3}{4}b^3e^{-3\mu_2 t}\cos(\omega_2 t+\varphi_2)\\
&+\tfrac{3}{2}a^2be^{-(2\mu_1+\mu_2)t}\cos(\omega_2 t+\varphi_2)+\tfrac{1}{4}a^3e^{-3\mu_1 t}\cos 3(\omega_1 t+\varphi_1)+\tfrac{1}{4}b^3e^{-3\mu_2 t}\cos 3(\omega_2 t+\varphi_2)\\
&+\tfrac{3}{4}a^2be^{-(2\mu_1+\mu_2)t}\cos(2\omega_1 t+\omega_2 t+2\varphi_1+\varphi_2)+\tfrac{3}{4}ab^2e^{-(\mu_1+2\mu_2)t}\cos(2\omega_2 t+\omega_1 t+2\varphi_2+\varphi_1)\\
&+\tfrac{3}{4}ab^2e^{-(\mu_1+2\mu_2)t}\cos(2\omega_2 t-\omega_1 t+2\varphi_2-\varphi_1).
\end{aligned} \tag{5.29}$$

For determining $\widetilde{A}_1, \widetilde{B}_1;\ \widetilde{A}_2, \widetilde{B}_2;\ u_1$ from equation (5.29), we can separate the terms $\cos(\omega_1 t+\varphi_1)$, $\sin(\omega_1 t+\varphi_1)$, $\cos(\omega_2 t+\varphi_2),\cdots$ as mentioned above in which $\mu_1=\mu_2=0$. Therefore, in case of a damped system (*i.e.*, $\mu_1, \mu_2\neq 0$) equations similar to the equations (5.24)-(5.26) take the forms

$$\begin{aligned}
&[(D+\mu_2)^2+\omega_2^2][e^{-\mu_1 t}\{\cos(\omega_1 t+\varphi_1)(D\widetilde{A}_1-2\omega_1 a\widetilde{B}_1)\\
&-\sin(\omega_1 t+\varphi_1)(2\omega_1\widetilde{A}_1+aD\widetilde{B}_1)\}]\\
&=\tfrac{3}{4}a^3e^{-3\mu_1 t}\cos(\omega_1 t+\varphi_1)+\tfrac{3}{2}ab^2e^{-(\mu_1+2\mu_2)t}\cos(\omega_1 t+\varphi_1)\\
&+\tfrac{3}{4}a^2be^{-(2\mu_1+\mu_2)t}\cos(2\omega_1 t-\omega_2 t+2\varphi_1-\varphi_2),
\end{aligned} \tag{5.30}$$

$$\begin{aligned}
&[(D+\mu_1)^2+\omega_1^2][e^{-\mu_2 t}\{\cos(\omega_2 t+\varphi_2)(D\widetilde{A}_2-2\omega_2 b\widetilde{B}_2)\\
&-\sin(\omega_2 t+\varphi_2)(2\omega_2\widetilde{A}_2+bD\widetilde{B}_2)\}]\\
&=\tfrac{3}{4}b^3e^{-3\mu_2 t}\cos(\omega_2 t+\varphi_2)+\tfrac{3}{2}a^2be^{-(2\mu_1+\mu_2)t}\cos(\omega_2 t+\varphi_2)\\
&+\tfrac{1}{4}a^3e^{-3\mu_1 t}\cos 3(\omega_1 t+\varphi_1),
\end{aligned} \tag{5.31}$$

and

$$[(D+\mu_1)^2+\omega_1^2][(D+\mu_2)^2+\omega_2^2]u_1$$
$$=\tfrac{1}{4}b^3e^{-3\mu_2 t}\cos 3(\omega_2 t+\varphi_2)+\tfrac{3}{4}ab^2e^{-(\mu_1+2\mu_2)t}\cos(2\omega_2 t-\omega_1 t+2\varphi_2-\varphi_1)$$
$$+\tfrac{3}{4}a^2be^{-(2\mu_1+\mu_2)t}\cos(2\omega_1 t+\omega_2 t+2\varphi_1+\varphi_2) \tag{5.32}$$
$$+\tfrac{3}{4}ab^2e^{-(\mu_1+2\mu_2)t}\cos(2\omega_2 t+\omega_1 t+2\varphi_2+\varphi_1).$$

The particular solutions of equations (5.30)-(531) are given by

$$\begin{aligned}
\widetilde{A}_1 &= l_1a^3e^{-2\mu_1 t}+l_2ab^2e^{-2\mu_2 t}\\
&\quad +a^2be^{-(\mu_1+\mu_2)t}\{l_3\cos(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2)+l_4\sin(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2)\},\\
\widetilde{A}_2 &= m_1b^3e^{-2\mu_2 t}+m_2a^2be^{-2\mu_1 t}\\
&\quad +a^3e^{-(3\mu_1-\mu_2)t}\{m_3\cos(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2)+m_4\sin(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2)\},\\
\widetilde{B}_1 &= n_1a^2e^{-2\mu_1 t}+n_2b^2e^{-2\mu_2 t}\\
&\quad +abe^{-(\mu_1+\mu_2)t}\{l_4\cos(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2)-l_3\sin(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2)\},\\
\widetilde{B}_2 &= r_1b^2e^{-2\mu_2 t}+r_2a^2e^{-2\mu_1 t}\\
&\quad -(a^3/b)e^{-(3\mu_1-\mu_2)t}\{m_4\cos(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2)-m_3\sin(3\omega_1 t-\omega_2 t+3\varphi_1-\varphi_2)\},
\end{aligned} \tag{5.33}$$

where

$$l_1=-\frac{3\,[\mu_1\{(3\mu_1-\mu_2)^2-\omega_1{}^2+\omega_2{}^2\}-2\omega_1{}^2(3\mu_1-\mu_2)]}{8(\mu_1{}^2+\omega_1{}^2)[(3\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2][(3\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2]},$$

$$l_2=-\frac{3\,[\mu_2\{(\mu_1+\mu_2)^2-\omega_1{}^2+\omega_2{}^2\}-2\omega_1{}^2(\mu_1+\mu_2)]}{4(\mu_2{}^2+\omega_1{}^2)[(\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2][(\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2]},$$

$$l_3=-\frac{3\,[\mu_1\{\mu_1(\mu_1+\mu_2)-(\omega_1-\omega_2)(2\omega_1-\omega_2)\}-(\mu_1+\mu_2)(\omega_1-\omega_2)\omega_1]}{16(\mu_1{}^2+\omega_1{}^2)[(\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2][\mu_1{}^2+(\omega_1-\omega_2)^2]},$$

$$l_4=\frac{3\,[\mu_1\{(\mu_1+\mu_2)(2\omega_1-\omega_2)+\mu_1(\omega_1-\omega_2)\}-\omega_1(\omega_1-\omega_2)^2]}{16(\mu_1{}^2+\omega_1{}^2)[(\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2][\mu_1{}^2+(\omega_1-\omega_2)^2]},$$

$$m_1=-\frac{3\,[\mu_2\{(3\mu_2-\mu_1)^2-\omega_2{}^2+\omega_1{}^2\}-2\omega_2{}^2(3\mu_2-\mu_1)]}{8(\mu_2{}^2+\omega_2{}^2)[(3\mu_2-\mu_1)^2+(\omega_2-\omega_1)^2][(3\mu_2-\mu_1)^2+(\omega_2+\omega_1)^2]},$$

$$m_2 = -\frac{3[\mu_1\{(\mu_1+\mu_2)^2 - \omega_2{}^2 + \omega_1{}^2\} - 2\omega_2{}^2(\mu_1+\mu_2)]}{4(\mu_1{}^2+\omega_2{}^2)[(\mu_1+\mu_2)^2+(\omega_2-\omega_1)^2][(\mu_1+\mu_2)^2+(\omega_2+\omega_1)^2]},$$

$$m_3 = -\frac{\mu_1\{\mu_1(3\mu_1-\mu_2) - 3\omega_1(3\omega_1+\omega_2)\} - 2\omega_1{}^2(3\mu_1-\mu_2)}{16(\mu_1{}^2+\omega_1{}^2)(\mu_1{}^2+4\omega_1{}^2)[(3\mu_1-\mu_2)^2+(3\omega_1+\omega_2)^2]}, \tag{5.34}$$

$$m_4 = \frac{\{4\mu_1(3\mu_1-\mu_2) + 3\mu_1{}^2 - 2\omega_1(3\omega_1+\omega_2\}\omega_1 + \mu_1{}^2\omega_2}{16(\mu_1{}^2+\omega_1{}^2)(\mu_1{}^2+4\omega_1{}^2)[(3\mu_1-\mu_2)^2+(3\omega_1+\omega_2)^2]},$$

$$n_1 = -\frac{3[2\mu_1(3\mu_1-\mu_2) + (3\mu_1-\mu_2)^2 - \omega_1{}^2 + \omega_2{}^2]\omega_1}{8(\mu_1{}^2+\omega_1{}^2)[(3\mu_1-\mu_2)^2+(\omega_1-\omega_2)^2][(3\mu_1-\mu_2)^2+(\omega_1+\omega_2)^2]},$$

$$n_2 = -\frac{3[2\mu_2(\mu_1+\mu_2) + (\mu_1+\mu_2)^2 - \omega_1{}^2 + \omega_2{}^2]\omega_1}{4(\mu_2{}^2+\omega_1{}^2)[(\mu_1+\mu_2)^2+(\omega_1-\omega_2)^2][(\mu_1+\mu_2)^2+(\omega_1+\omega_2)^2]},$$

$$r_1 = -\frac{3[2\mu_2(3\mu_2-\mu_1) + (3\mu_2-\mu_1)^2 - \omega_2{}^2 + \omega_1{}^2]\omega_2}{8(\mu_2{}^2+\omega_2{}^2)[(3\mu_2-\mu_1)^2+(\omega_2-\omega_1)^2][(3\mu_2-\mu_1)^2+(\omega_2+\omega_1)^2]},$$

$$r_2 = -\frac{3[2\mu_1(\mu_1+\mu_2) + (\mu_1+\mu_2)^2 - \omega_2{}^2 + \omega_1{}^2]\omega_2}{4(\mu_1{}^2+\omega_2{}^2)[(\mu_1+\mu_2)^2+(\omega_2-\omega_1)^2][(\mu_1+\mu_2)^2+(\omega_2+\omega_1)^2]}.$$

In this case equation (5.8) becomes

$$\begin{aligned}
\dot{a} &= \varepsilon\,[l_1 a^3 e^{-2\mu_1 t} + l_2 a b^2 e^{-2\mu_2 t} \\
&\quad + a^2 b\, e^{-(\mu_1+\mu_2)t}\{l_3\cos(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2) + l_4 \sin(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2)\}], \\
\dot{b} &= \varepsilon\,[m_1 b^3 e^{-2\mu_2 t} + m_2 a^2 b e^{-2\mu_1 t} \\
&\quad + a^3 e^{-(3\mu_1-\mu_2)t}\{m_3\cos(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2) + m_4 \sin(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2)\}], \\
\dot{\varphi}_1 &= \varepsilon\,[n_1 a^2 e^{-2\mu_1 t} + n_2 b^2 e^{-2\mu_2 t} \\
&\quad + ab e^{-(\mu_1+\mu_2)t}\{l_4\cos(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2) - l_3 \sin(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2)\}], \\
\dot{\varphi}_2 &= \varepsilon\,[r_1 b^2 e^{-2\mu_2 t} + r_2 a^2 e^{-2\mu_1 t} \\
&\quad - (a^3/b) e^{-(3\mu_1-\mu_2)t}\{m_4\cos(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2) - m_3 \sin(3\omega_1 t - \omega_2 t + 3\varphi_1 - \varphi_2)\}],
\end{aligned} \tag{5.35}$$

Therefore, the first order damped solution (without u_1) of equation (5.28) is

$$x = a e^{-\mu_1 t} \cos(\omega_1 t + \varphi_1) + b e^{-\mu_2 t} \cos(\omega_2 t + \varphi_2) \tag{5.36}$$

where a, b, φ_1 and φ_2 are the solutions of equation (5.35).

It is noted that equations (5.33)-(5.34) reduce to equations (5.19)-(5.20) when $\mu_1, \mu_2 \to 0$. In this case $l_4 \to l_1^*$, $m_4 \to m_1^*$, $n_1, \frac{1}{2}n_2 \to n_1^*$, $r_1, \frac{1}{2}r_2 \to r_1^*$ and l_1, l_2, l_3, m_1, m_2, m_3 vanish.

5.4 Results and Discussions

A general formula of KBM method is extended in case of internal resonant vibrations, in which the natural frequencies are in integral multiple [72]. It is noted that two separate trial solutions are needed to study resonant (internal) and non-resonant vibrations in accordance with previous investigations (see [18] for details). But in this method, we have used a single trial solution for resonant (both internal and external [93], see also **Appendix 5.B**) and non-resonant vibrations. The determination of the solution is simple. Damped, undamped and over-damped processes are treated in a unified approach.

The integration of the transformed form of equation (5.4), *i.e.*, equation (5.8) may be done by using well-known techniques of calculus [47]. Sometimes, such transformed equation has no exact solution. For this reason it is solved by an approximate technique [80] or by a numerical method [26]. In this case, the perturbation method facilitates the numerical method. The amplitude and phase variables α_l, φ_l, $l = 1, 2$ change slowly with time. So, the numerical calculation requires only a few numbers of points. On the contrary, a direct attempts to solve equation (5.1), the numerical calculation requires a great number of points, since the solution dealing with some harmonic terms. Often one is interested not only in the oscillating processes itself, *i.e.*, finding $x(t,\varepsilon)$ in terms of t; but mainly in the behavior of the amplitudes and phases, which characterized the oscillating processes (see [26] for details). Sometimes the variational equations for amplitudes and phases are used to investigate the stability of a nonlinear system described by equation (5.1) [93].

For a small value of ε, the perturbation solution shows a good agreement with numerical solution. As a check, undamped solution [given by equation (5.22)] has been compared with corresponding numerical solution (computed by Runge-Kutta fourth order procedure) in Fig. 5.1, 5.2 and 5.3 when $3\omega_1 - \omega_2$ is small. The determination of a damped solution (significant) was a difficult task, especially when $n > 2$ [35]. But the new technique [92] facilitates the KBM method to obtain the damped and over-damped solutions as well as resonant (both internal and external) and non-resonant solutions. The damped solution has also been compared with the numerical solution in Fig. 5.4 and 5.5. It is noted that the damped solution [given by equation (5.36)] can be used to an over-damped system replacing ω_l, $l = 1, 2$ by $i\omega_l$ only, which is the basic principle of the unified theory [53,80,92]. The over-damped solution is shown in Fig. 5.6.

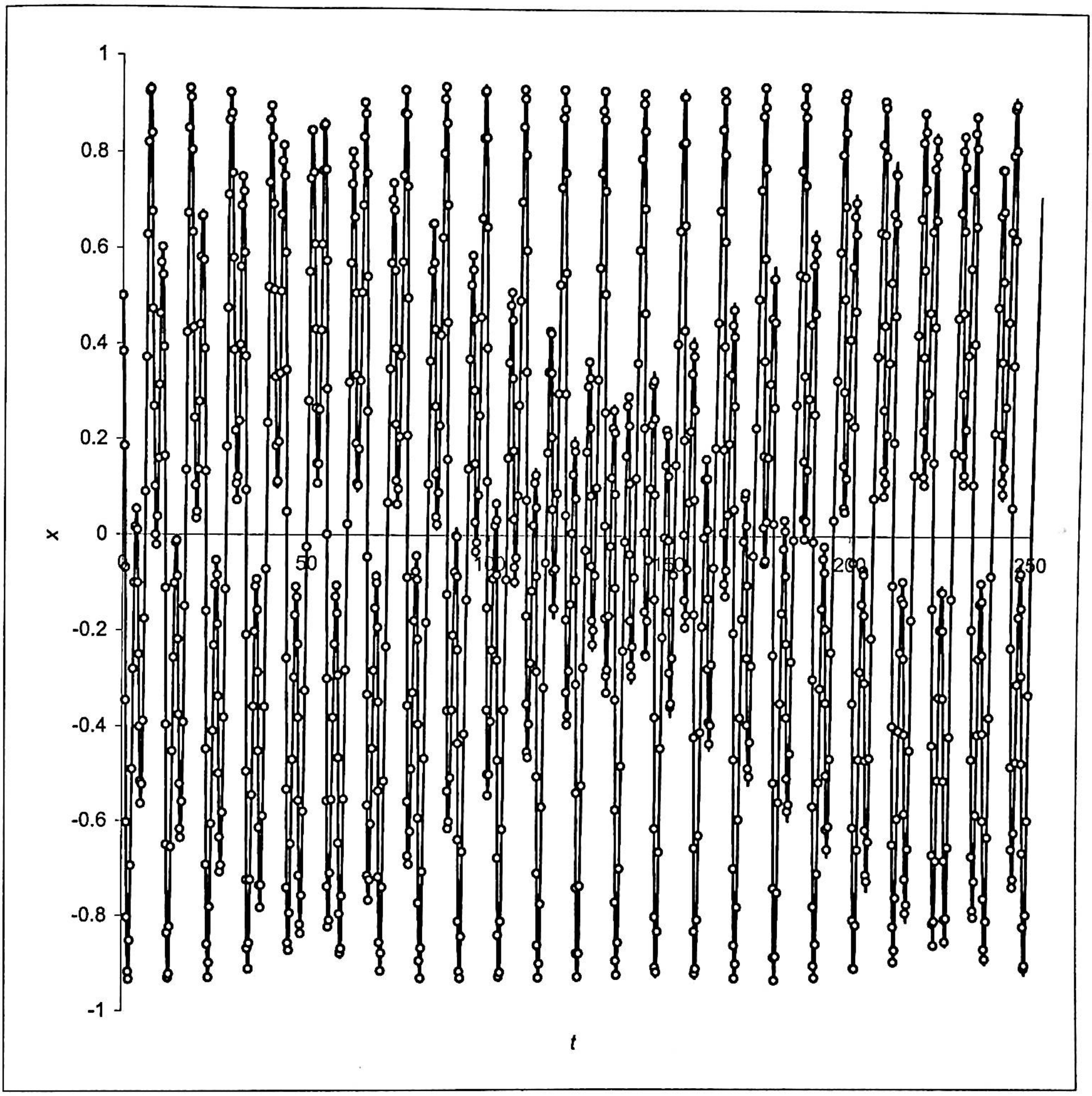

Fig. 5.1 Undamped solution Eq. (5.22) has been denoted by -o-. Here a, b, φ_1 and φ_2 have been evaluated by Eq. (5.19) with initial conditions $a_0 = b_0 = 0.5$, $\varphi_{1,0} = \varphi_{2,0} = 0$ [or, $x(0) = 1.0$, $\dot{x}(0) = 0.0$, $\ddot{x}(0) = -1.615556, \dddot{x}(0) = 0.0$]; $\omega_1 = 1/1.7, \omega_2 = 1.7$ and $\varepsilon = 0.1$. Corresponding numerical solution (calculated by Runge-Kutta fourth-order procedure) has been denoted by —.

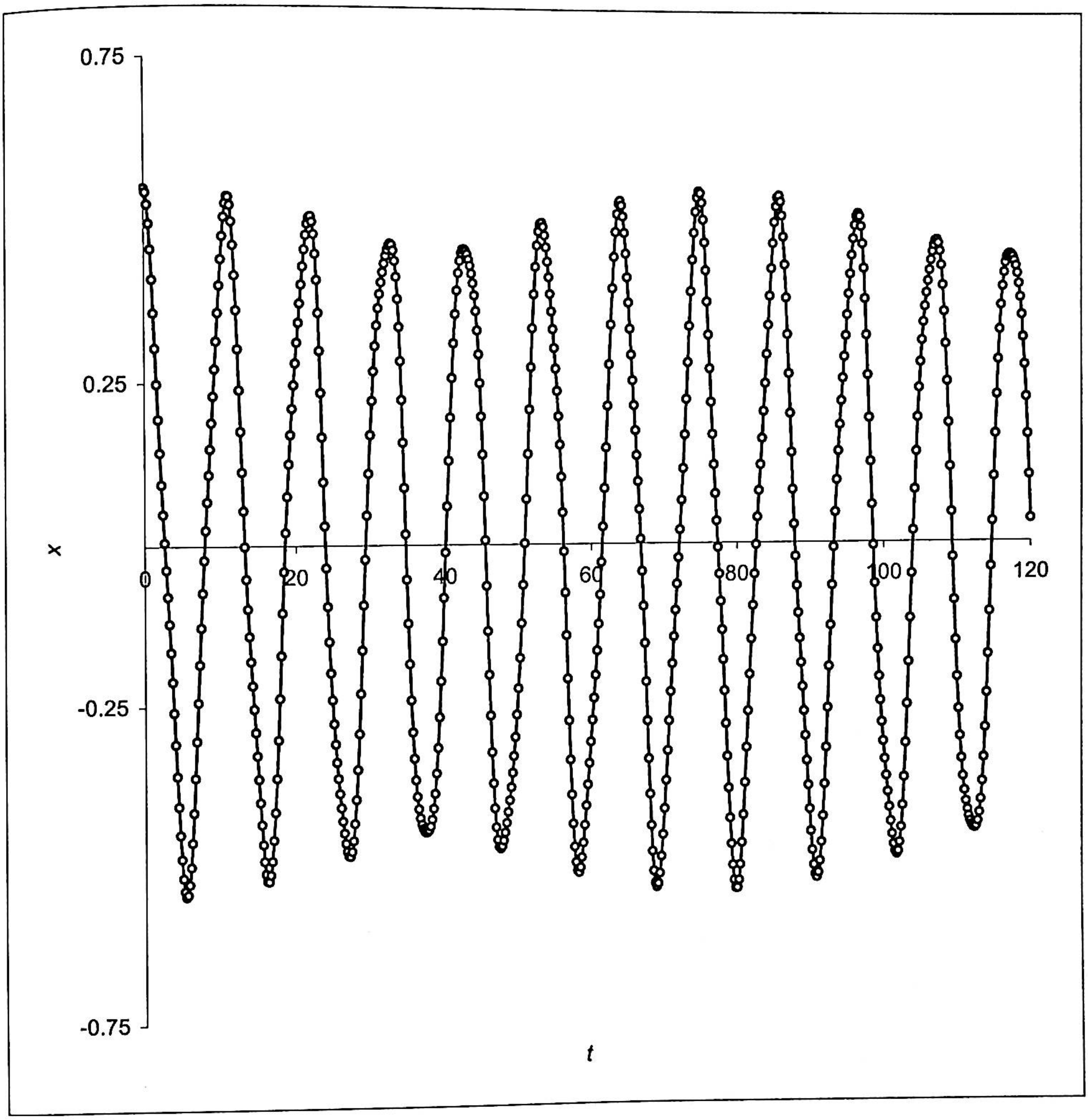

Fig. 5.2 Undamped solution Eq. (5.22) has been denoted by -o-. Here a, b, φ_1 and φ_2 have been evaluated by Eq. (5.19) with initial conditions $a_0 = 0.5$, $b_0 = 0.05$, $\varphi_{1,0} = \varphi_{2,0} = 0$, [or, $x(0) = 0.55$, $\dot{x}(0) = 0.0$, $\ddot{x}(0) = -0.315903, \dddot{x}(0) = 0.0$]; $\omega_1 = 1/1.68$, $\omega_2 = 1.68$ and $\varepsilon = 0.1$. Corresponding numerical solution has been denoted by —.

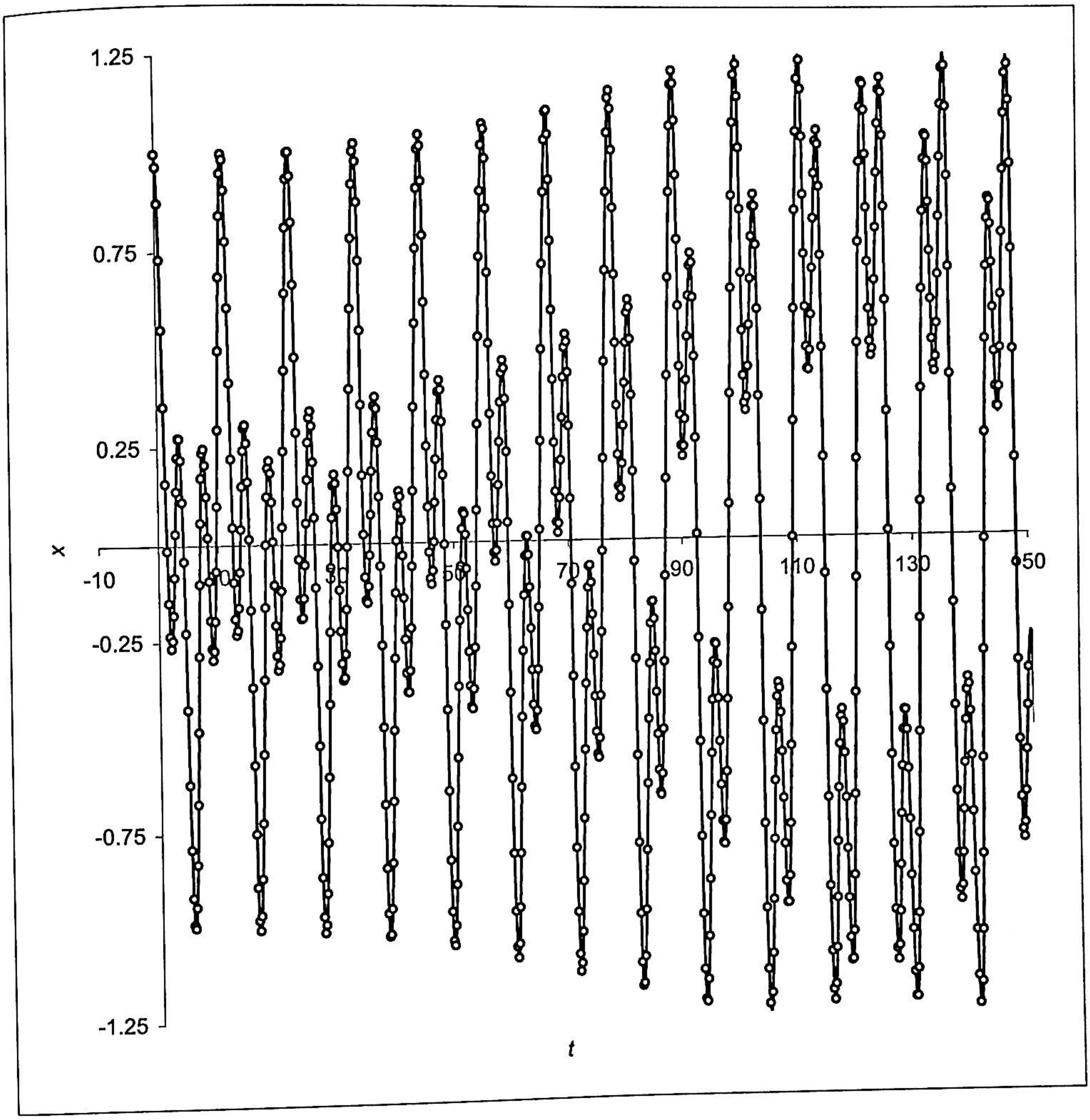

Fig. 5.3 Undamped solution Eq. (5.22) has been denoted by -o-. Here a, b, φ_1 and φ_2 have been evaluated by Eq. (5.19) with initial conditions $a_0 = 0.5$, $b_0 = 0.5$, $\varphi_{1,0} = \pi/2$, $\varphi_{2,0} = 0$ [or, $x(0) = 0.5$, $\dot{x}(0) = -0.281134$, $\ddot{x}(0) = -1.493474$, $\dddot{x}(0) = 0.085899$]; $\omega_1 = 1/1.72$, $\omega_2 = 1.72$ and $\varepsilon = 0.1$. Corresponding numerical solution has been denoted by —.

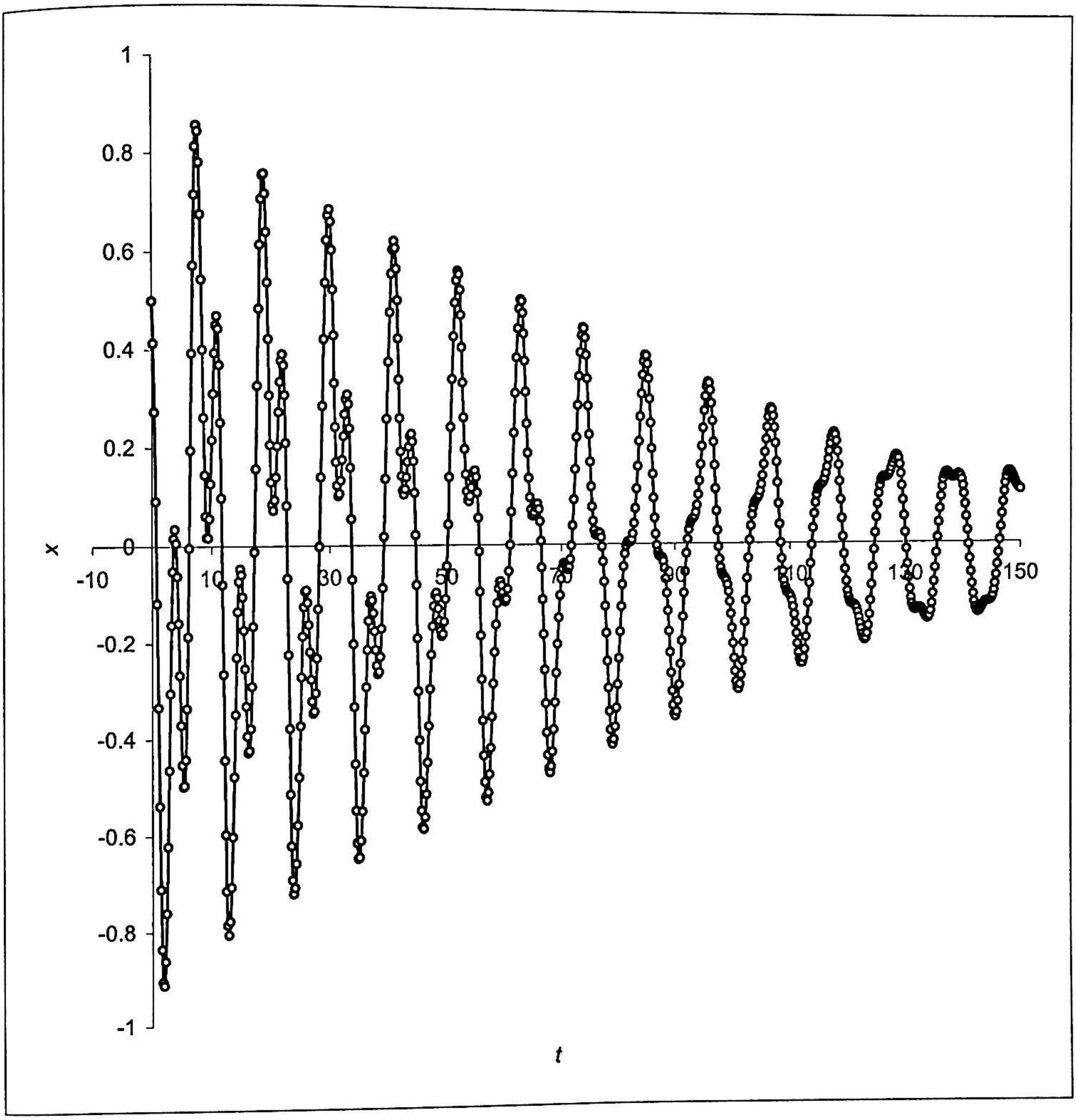

Fig. 5.4 Damped solution Eq. (5.36) has been denoted by -o-. Here a, b, φ_1 and φ_2 have been evaluated by Eq. (5.35) with initial conditions $a_0 = 0.5$, $b_0 = 0.5$, $\varphi_{1,0} = \pi/2$, $\varphi_{2,0} = 0$ [or, $x(0) = 0.5$, $\dot{x}(0) = -0.289304$, $\ddot{x}(0) = -1.461226$, $\dddot{x}(0) = 0.170034$]; $\mu_1 = 0.01$, $\mu_2 = 0.02$, $\omega_1 = 1/1.7$, $\omega_2 = 1.7$ and $\varepsilon = 0.15$. Corresponding numerical solution has been denoted by —.

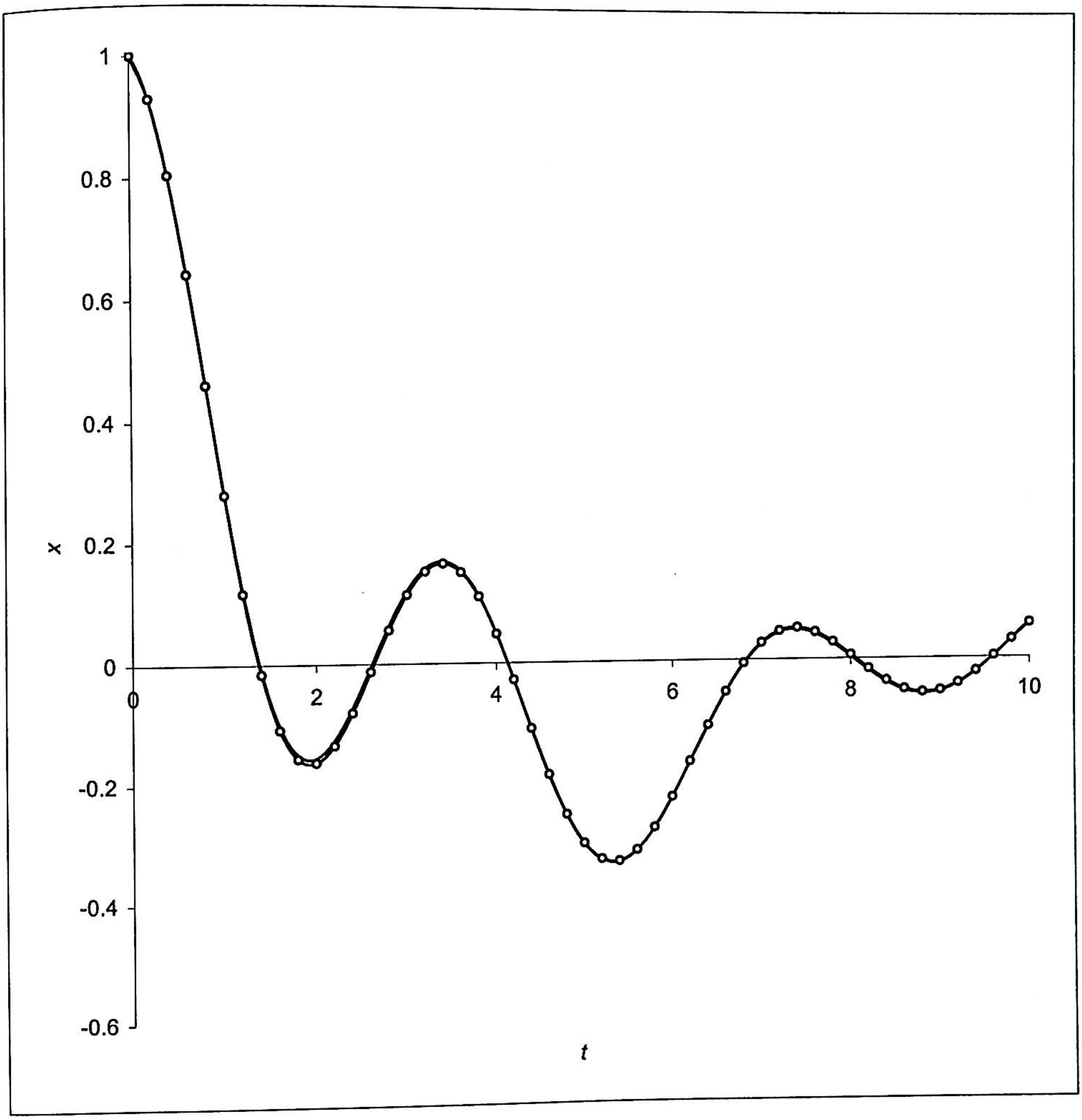

Fig. 5.5 Damped solution Eq. (5.36) has been denoted by -o-. Here a, b, φ_1 and φ_2 have been evaluated by Eq. (5.35) with initial conditions $a_0 = b_0 = 0.5$, $\varphi_{1,0} = \varphi_{2,0} = 0$ [or, $x(0) = 1.0$, $\dot{x}(0) = -0.201534$, $\ddot{x}(0) = -1.613776$, $\dddot{x}(0) = 0.956182$]; $\mu_1 = \mu_2 = 0.2$, $\omega_1 = 1/1.73$, $\omega_2 = 1.73$ and $\varepsilon = 0.25$. Corresponding numerical solution has been denoted by —.

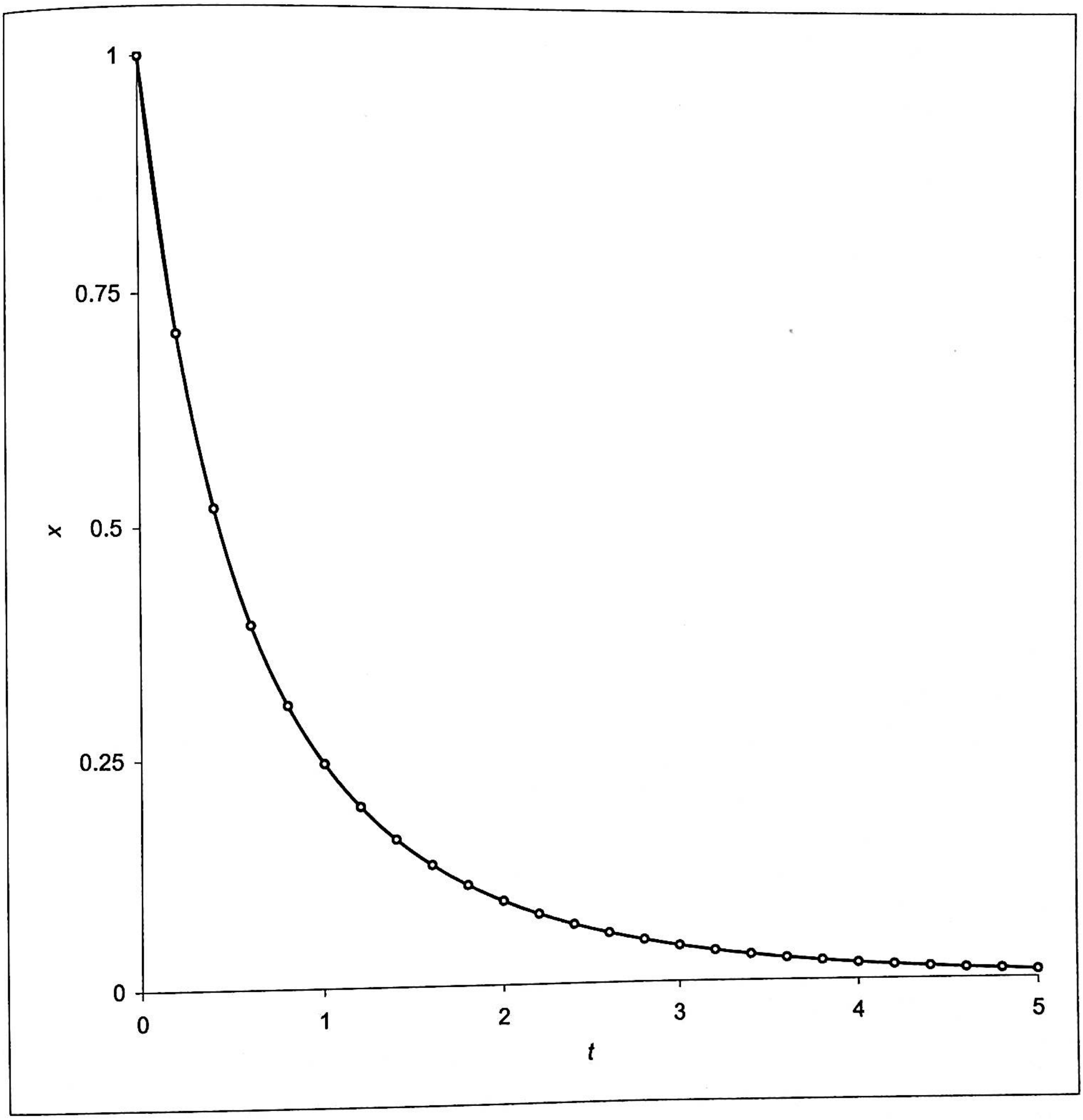

Fig. 5.6 Over-damped solution obtained from Eq. (5.36) replacing ω_1 by $i\omega_1$ and ω_2 by $i\omega_2$ respectively, has been denoted by -o-. Here a, b, φ_1 and φ_2 have been evaluated by Eq. (5.35) replacing ω_1 by $i\omega_1$ and ω_2 by $i\omega_2$ respectively, with initial conditions $a_0 = b_0 = 0.5$, $\varphi_{1,0} = 0$, $\varphi_{2,0} = 0$ [or, $x(0) = 1.0, \dot{x}(0) = -1.842748, \ddot{x}(0) = 4.537942, \dddot{x}(0) = -13.241129$]; $3\mu_1 = \mu_2 = \sqrt{3} + 3/(2\sqrt{2})$, $3\omega_1 = \omega_2 = \sqrt{3} - 3/(2\sqrt{2})$ and $\varepsilon = 0.2$. Corresponding numerical solution has been denoted by —.

5.5 Conclusion

A method is developed to tackle the case of internal resonance, which is simple, systematic and easier than Bojadziev [26] and Bojadziev and Hung [27]. The method is the generalization of the KBM method, which cover the cases, when the eigen-values of the corresponding unperturbed equation, are real, complex or imaginary. For time dependent differential systems, Bojadziev [26], Bojadziev and Hung [27] used at least two trail solutions; one is for the resonant case and the other is for the non-resonant case. But we have used only one trail solution for both the resonant and non-resonant cases, which is an improvement of this method.

Appendix 5.A

When the difference of $3\omega_1$ and ω_2 is significant, u_1 excludes only first harmonic terms. Therefore, equation (5.10) can be separated in the following way [80]

$$(D^2+\omega_2^2)(D+i\omega_1)(e^{i\omega_1 t}A_1)=3a_1{}^2a_2e^{i\omega_1 t}+6a_1a_3a_4e^{i\omega_1 t}, \tag{5.A.1}$$

$$(D^2+\omega_2^2)(D-i\omega_1)(e^{-i\omega_1 t}A_2)=3a_2{}^2a_1e^{-i\omega_1 t}+6a_2a_3a_4e^{-i\omega_1 t}, \tag{5.A.2}$$

$$(D^2+\omega_1^2)(D+i\omega_2)(e^{i\omega_2 t}A_3)=3a_3{}^2a_4e^{i\omega_2 t}+6a_1a_2a_3e^{i\omega_2 t}, \tag{5.A.3}$$

$$(D^2+\omega_1^2)(D-i\omega_2)(e^{-i\omega_2 t}A_4)=3a_4{}^2a_3e^{-i\omega_2 t}+6a_1a_2a_4e^{-i\omega_2 t}, \tag{5.A.4}$$

and

$$\begin{aligned}(D^2+\omega_1^2)(D^2+\omega_2^2)u_1 &= a_3{}^3e^{3i\omega_2 t}+a_4{}^3e^{-3i\omega_2 t}+3a_1{}^2a_3e^{i(2\omega_1+\omega_2)t}+3a_2{}^2a_4e^{-i(2\omega_1+\omega_2)t}\\ &+3a_3{}^2a_1e^{i(2\omega_2+\omega_1)t}+3a_4{}^2a_2e^{-i(2\omega_2+\omega_1)t}+3a_3{}^2a_2e^{i(2\omega_2-\omega_1)t}+3a_4{}^2a_1e^{-i(2\omega_2-\omega_1)t}\\ &+3a_2{}^2a_3e^{-i(2\omega_1-\omega_2)t}++3a_1{}^2a_4e^{i(2\omega_1-\omega_2)t}+a_1{}^3e^{3i\omega_1 t}+a_2{}^3e^{-3i\omega_1 t}.\end{aligned} \tag{5.A.5}$$

Solving equations (5.A.1)-(5.A.5), we obtain

$$\begin{aligned}A_1&=\frac{-3(a_1{}^2a_2+2a_1a_3a_4)}{2i\omega_1(\omega_1{}^2-\omega_2{}^2)},\quad A_2=\frac{3(a_2{}^2a_1+2a_2a_3a_4)}{2i\omega_1(\omega_1{}^2-\omega_2{}^2)},\\ A_3&=\frac{3(a_3{}^2a_4+2a_1a_2a_3)}{2i\omega_2(\omega_1{}^2-\omega_2{}^2)},\quad A_4=\frac{-3(a_4{}^2a_3+2a_1a_2a_4)}{2i\omega_2(\omega_1{}^2-\omega_2{}^2)},\end{aligned} \tag{5.A.6}$$

and

$$\begin{aligned}u_1&=\frac{a_3{}^3e^{3i\omega_2 t}+a_4{}^3e^{-3i\omega_2 t}}{8\omega_2^2(9\omega_2^2-\omega_1^2)}+\frac{3a_1{}^2a_3e^{i(2\omega_1+\omega_2)t}+3a_2{}^2a_4e^{-i(2\omega_1+\omega_2)t}}{[(2\omega_1+\omega_2)^2-\omega_1^2][(2\omega_1+\omega_2)^2-\omega_2^2]}\\ &+\frac{3a_3{}^2a_1e^{i(2\omega_2+\omega_1)t}+3a_4{}^2a_2e^{-i(2\omega_2+\omega_1)t}}{[(2\omega_2+\omega_1)^2-\omega_1^2][(2\omega_2+\omega_1)^2-\omega_2^2]}+\frac{3a_3{}^2a_2e^{i(2\omega_2-\omega_1)t}+3a_4{}^2a_1e^{-i(2\omega_2-\omega_1)t}}{[(2\omega_2-\omega_1)^2-\omega_1^2][(2\omega_2-\omega_1)^2-\omega_2^2]}\\ &+\frac{a_1{}^3e^{3i\omega_1 t}+a_2{}^3e^{-3i\omega_1 t}}{8\omega_1^2(9\omega_1^2-\omega_2^2)}+\frac{3a_1{}^2a_4e^{i(2\omega_1-\omega_2)t}+3a_2{}^2a_3e^{-i(2\omega_1-\omega_2)t}}{[(2\omega_1-\omega_2)^2-\omega_2^2][(2\omega_1-\omega_2)^2-\omega_1^2]}.\end{aligned} \tag{5.A.7}$$

Substituting the values of $A_1,\cdots,A_4$ from equation (5.A.6) into equation (5.4), and using the transformations $a_1=\frac{1}{2}ae^{i\varphi_1}$, $a_2=\frac{1}{2}ae^{-i\varphi_1}$, $a_1=\frac{1}{2}be^{i\varphi_2}$, $a_4=\frac{1}{2}be^{-i\varphi_2}$, we obtain

$$\begin{aligned}&\dot{a}=0,\quad \dot{b}=0,\\&\dot{\varphi}_1=\varepsilon m_1^*(a^2+2b^2),\quad \dot{\varphi}_2=\varepsilon r_1^*(2a^2+b^2),\end{aligned} \tag{5.A.8}$$

where m_1^* and r_1^* are given by equation (5.20) and u_1 becomes

$$\begin{aligned}u_1=&\frac{b^3\cos(3\omega_2t+3\varphi_2)}{32\omega_2^2(9\omega_2^2-\omega_1^2)}+\frac{3a^2b\cos(2\omega_1t+\omega_2t+2\varphi_1+\varphi_2)}{4[(2\omega_1+\omega_2)^2-\omega_1^2][(2\omega_1+\omega_2)^2-\omega_2^2]}\\&+\frac{3ab^2\cos(2\omega_2t+\omega_1t+2\varphi_2+\varphi_1)}{4[(2\omega_2+\omega_1)^2-\omega_1^2][(2\omega_2+\omega_1)^2-\omega_2^2]}+\frac{3ab^2\cos(2\omega_2t-\omega_1t+2\varphi_2-\varphi_1)}{4[(2\omega_2-\omega_1)^2-\omega_1^2][(2\omega_2-\omega_1)^2-\omega_2^2]}\\&+\frac{a^3\cos(3\omega_1t+3\varphi_1)}{32\omega_1^2(9\omega_1^2-\omega_2^2)}+\frac{3a^2b\cos(2\omega_1t-\omega_2t+2\varphi_1-\varphi_2)}{4[(2\omega_1-\omega_2)^2-\omega_2^2][(2\omega_1-\omega_2)^2-\omega_1^2]}.\end{aligned} \tag{5.A.9}$$

It is obvious that the fifth and sixth terms of u_1 become indefinite when $3\omega_1-\omega_2\to 0$, or these terms become $O(\varepsilon^{-1})$ when $3\omega_1-\omega_2=O(\varepsilon)$.

Appendix 5.B

Consider *Duffing equation* with an external force

$$\ddot{x}+\omega^2x=-\varepsilon x^3+\varepsilon E\sin\nu t. \tag{5.B.1}$$

When $\varepsilon=0$, equation (5.B.1) has two eigen-values $\lambda_1=i\omega$ and $\lambda_2=-i\omega$; $x_0=a_1e^{\lambda_1t}+a_2e^{\lambda_2t}=a_1e^{i\omega t}+a_2e^{-i\omega t}$ and

$$f^{(0)}=-(a_1^3e^{3i\omega t}+3a_1^2a_2e^{i\omega t}+3a_1a_2^2e^{-i\omega t}+a_2^3e^{-3i\omega t}).$$

We consider the situation when the difference of ω and ν is small, *i.e.*, $\omega-\nu=O(\varepsilon)$. In this case u_1 excludes the terms of $3a_1^2a_2e^{i\omega t}$ and $3a_1a_2^2e^{-i\omega t}$. Moreover, in our assumption u_1 excludes $E\sin\nu t$ when $\omega-\nu=O(\varepsilon)$ and the equations of u_1, A_1 and A_2 become

$$\left(\frac{d}{dt}-i\omega\right)\left(\frac{d}{dt}+i\omega\right)u_1=-(a_1^3e^{3i\omega t}+a_2^3e^{-3i\omega t}), \tag{5.B.2}$$

$$\left(\frac{d}{dt}+i\omega\right)\left(A_1e^{i\omega t}\right)=-3a_1^2a_2e^{i\omega t}+\frac{E}{2i}e^{i\nu t}, \tag{5.B.3}$$

and

$$\left(\frac{d}{dt}-i\omega\right)\left(A_2e^{-i\omega t}\right)=-3a_1a_2^2e^{-i\omega t}-\frac{E}{2i}e^{-i\nu t}. \tag{5.B.4}$$

Solutions of equations (5.B.3)-(5.B.4) are

$$\begin{aligned} A_1e^{i\omega t} &= -\frac{3a_1^2a_2e^{i\omega t}}{2i\omega}+\frac{Ee^{i\nu t}}{2i(i\nu+i\omega)}, \\ A_2e^{-i\omega t} &= \frac{3a_1a_2^2e^{-i\omega t}}{2i\omega}-\frac{Ee^{-i\nu t}}{2i(-i\nu-i\omega)}. \end{aligned} \tag{5.B.5}$$

or,

$$\begin{aligned} A_1 &= -\frac{3a_1^2a_2}{2i\omega}+\frac{Ee^{i(\nu-\omega)t}}{2i(i\nu+i\omega)}, \\ A_2 &= \frac{3a_1a_2^2}{2i\omega}-\frac{Ee^{-i(\nu-\omega)t}}{2i(-i\nu-i\omega)}. \end{aligned} \tag{5.B.6}$$

Substituting the values of A_1 and A_2 from equation (5.B.6) into equation (5.4), we obtain

$$\begin{aligned} \dot{a}_1 &= -\varepsilon\left(\frac{3a_1^2a_2}{2i\omega}+\frac{Ee^{i(\nu-\omega)t}}{2(\nu+\omega)}\right), \\ \dot{a}_2 &= \varepsilon\left(\frac{3a_1a_2^2}{2i\omega}-\frac{Ee^{-i(\nu-\omega)t}}{2(\nu+\omega)}\right). \end{aligned} \tag{5.B.7}$$

Under the transformations, $a_1e^{i\omega t}=\frac{1}{2}ae^{i\varphi}$ and $a_2e^{-i\omega t}=\frac{1}{2}ae^{-i\varphi}$, we shall obtain the variational equations of a and φ in the real forms. To do so, we differentiate $a_1e^{i\omega t}=\frac{1}{2}ae^{i\varphi}$

and $a_2 e^{-i\omega t} = \frac{1}{2} a e^{-i\varphi}$ with respect to t, using the relations in (5.B.7) and simplifying them, we obtain

$$\dot{a} = -\frac{\varepsilon E \cos(\varphi - \nu t)}{\nu + \omega} + O(\varepsilon^2),$$
$$\dot{\varphi} = \omega + \frac{3\varepsilon a^2}{8\omega} - \frac{\varepsilon E \sin(\varphi - \nu t)}{a(\nu + \omega)} + O(\varepsilon^2). \qquad (5.B.8)$$

Equation (5.B.8) is similar to that obtained previously by KBM [13,34] method (see [48] for details). It is noted that the new procedure [93] is similar to KBM method, but the approach is entirely different. The method is simpler than KBM original technique (see also [93] for details).

Chapter 6

Perturbation Method for Fourth Order Nonlinear Differential Equations with Large Damping

6.1 Introduction

While finding the solutions of nonlinear damped oscillatory systems by perturbation methods, in most cases the solutions are found near the undamped solutions; and solutions near the critical damping are not considered. Shamsul [77] has investigated an approximate solution of the nonlinear differential equation

$$\ddot{x} + 2k\dot{x} + \omega^2 x = -\varepsilon\, f(x, \dot{x}); \qquad 0 < k < \omega, \tag{6.1}$$

characterized by large damping effects, *i. e.*, the damping is less than critical damping. First Krylov and Bogoliubov [34] developed a technique to discuss the transient response of equation (6.1), when the damping is small, *i. e.*, $k = 0$. Then the technique was amplified and justified mathematically by Bogoliubov and Mitropolskii [13], and later extended by Popov [66] to damped oscillatory systems. It is noteworthy that, because of the importance of physical phenomena involving damping, Mendelson [46] and Bojadziev [27] rediscovered Popov's results. Murty *et al* [52] extended the KBM method to discuss the transient response of an over-damped case. Murty [53] has also presented a unified KBM method to solve a second order nonlinear differential equation which covers the over-damped, damped and under damped cases. Osiniskii [57] and Mulholland [50] extended the KBM method to third order nonlinear systems. Shamsul [80] has presented a unified KBM method to solve n-th order nonlinear systems. Recently, Shamsul [92] has modified the formula presented in [80].

Later, Shamsul [81] extended the method of second order systems [77] to solve n-th order nonlinear systems with large damping. Shamsul, Bellal and Shanta [91] also used the method presented in [77] to nonlinear systems with varying coefficients.

In case of strong damping force (especially, when the damping force is slightly smaller than the critically damping force), the solution presented in Chapter 3 do not give desired results. In this Chapter, an approximate solution is found for strong damping force based on the work of Shamsul [77, 91]. The results obtained by this method in this Chapter are better than the solutions obtained in Chapter 3. Actually, the new solution is a complement of the solution presented in Chapter 3.

6.2 The Method

Consider a weakly nonlinear system governed by the differential equation

$$x^{(4)} + k_1 \dddot{x} + k_2 \ddot{x} + k_3 \dot{x} + k_4 x = \varepsilon\, f(x, \dot{x}, \ddot{x}, \dddot{x}), \qquad (6.2)$$

where $x^{(4)}$, denotes the fourth derivative of x, over-dot is used for the first, the second and the third derivatives with respect to t, ε is a small parameter, k_j, $j = 1, 2, 3, 4$ are constants and f is the nonlinear function.

When $\varepsilon = 0$, equation (6.2) has four distinct eigen-values, say $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and the unperturbed solution becomes

$$x(t,0) = \sum_{j=1}^{4} a_{j,0} e^{\lambda_j t}, \qquad (6.3)$$

where $a_{j,0}$, $j = 1, 2, 3, 4$ are arbitrary constants.

When $\varepsilon \neq 0$, following [80], an asymptotic solution of equation (6.2) is sought in the form

$$x(t,\varepsilon)=\sum_{j=1}^{4}a_j(t)e^{\lambda_j t}+\varepsilon u_1(a_1,a_2,a_3,a_4,t)+O(\varepsilon^2), \quad (6.4)$$

where u_1 is a function of a_j, $j=1,2,3,4$ and a_j satisfies the first order differential equation

$$\dot{a}_j=\varepsilon A_j(a_1,a_2,a_3,a_4,t)+O(\varepsilon^2) \quad (6.5)$$

and A_j, $j=1,2,3,4$ are also functions of a_1, a_2, a_3, a_4 and t.

To obtain a first order solution of equation (6.2), Shamsul [80] has presented a formula of the form

$$\sum_{j=1}^{4}\left(\prod_{k=1,\,k\neq j}^{n}(D-\lambda_k)(e^{\lambda_j t}A_j)\right)+\prod_{j=1}^{4}(D-\lambda_j)u_1=f^{(0)}(a_1,a_2,a_3,a_4,t), \quad (6.6)$$

where $D=\dfrac{\partial}{\partial t}$, $f^{(0)}=f(x_0,\dot{x}_0,\ddot{x}_0,\dddot{x}_0)$ and $x_0=\sum_{j=1}^{4}a_j(t)e^{\lambda_j t}$. According to [80], equation (6.6) can be separated into five individual equations to determine u_1 and A_j, $j=1,2,3,4$ subject to the condition that, u_1 excludes all fundamental terms and A_j, $j=1,2,3,4$ are independent of phases (see [13,34] for details). It has been mentioned before that, Krylov, Bogoliubov and Mitropolskii have studied nonlinear systems with small damping effects. Popov [66], Murty *et al.* [52], Murty [53], Bojadziev [27] strictly followed Krylov, Bogoliubov and Mitropolskii assumptions that A_j, $j=1,2,3,4$ are independent of phases, even if the system is strongly damped. But in case of strong damping effects, Shamsul [77] observed that if u_1 is independent of phases and first harmonics and A_j, $j=1,2,3,4$ depend only on amplitudes, the solutions do not give desired results; and for this reason in this case u_1 is not independent of first harmonics and A_j, $j=1,2,3,4$ depend on both amplitudes and phases.

Clearly, solution (6.4) starts containing some unusual variables, a_j, $j=1,2,3,4$ rather than amplitudes and phases. Yet this form is very important. The construction of equation

(6.6) is simple and it can be brought to the usual KBM form by the transformation $a_{2l-1} = \frac{1}{2}\alpha_l e^{i\varphi_l}$, $a_{2l} = \frac{1}{2}\alpha_l e^{-i\varphi_l}$, $l = 1, 2$, where α_l and φ_l are amplitude and phase variables (see [92] for details).

Equation (6.6) can also be rewritten as

$$\sum_{l=1}^{2}\left(\prod_{k=1,\,k\neq 2l-1,\,2l}^{4}(D-\lambda_k)[(D-\lambda_{2l})(e^{\lambda_{2l-1}t}A_{2l-1})+(D-\lambda_{2l-1})(e^{\lambda_{2l}t}A_{2l})]\right)$$
$$+\prod_{j=1}^{4}(D-\lambda_j)u_1 = f^{(0)}(a_1,a_2,a_3,a_4,t). \tag{6.7}$$

Changing the variables a_j, $j = 1, 2, 3, 4$ by $a_{2l-1} = \frac{1}{2}\alpha_l e^{i\varphi_l}$, $a_{2l} = \frac{1}{2}\alpha_l e^{-i\varphi_l}$, $l = 1, 2$, together with the substitutions $\lambda_{2l-1} = -\mu_l + i\omega_l$, $\lambda_{2l} = -\mu_l - i\omega_l$, $A_{2l-1} = \frac{1}{2}(\widetilde{A}_l + i\widetilde{B}_l)$, and $A_{2l} = \frac{1}{2}(\widetilde{A}_l - i\widetilde{B}_l)$, equation (6.7) becomes

$$\sum_{l=1}^{2}\left(\prod_{k=1,\,k\neq 2l-1,\,2l}^{4}(D-\lambda_k)[e^{-\mu_l t}\{\cos\psi_l(D\widetilde{A}_l - 2\omega_l\alpha_l\widetilde{B}_l) - \sin\psi_l(2\omega_l\widetilde{A}_l + \alpha_l D\widetilde{B}_l)\}]\right)$$
$$+\prod_{j=1}^{4}(D-\lambda_j)u_1 = f^{(0)}(\alpha_1,\alpha_2,\psi_1,\psi_2,t), \qquad \psi_l = \omega_l t + \varphi_l. \tag{6.8}$$

Therefore, the transformed equations of equation (6.5) for α_l, φ_l are

$$\dot{\alpha}_l = \varepsilon\widetilde{A}_l + O(\varepsilon^2), \quad \dot{\varphi}_l = \varepsilon\widetilde{B}_l + O(\varepsilon^2). \tag{6.9}$$

In accordance with the formal KBM [13,34] method, $f^{(0)}(\alpha_1,\alpha_2,\psi_1,\psi_2,t)$ can be expanded in Fourier series as

$$f^{(0)}(\alpha_1,\alpha_2,\psi_1,\psi_2,t) = \sum_{n=0}^{\infty}(F_n\cos n\psi_l + G_n\sin n\psi_l)$$

Therefore, according to [77], equation (6.8) can be resolved in the way

$$\sum_{l=1}^{2}\left(\prod_{k=1,\,k\neq 2l-1,\,2l}^{4}(D-\lambda_k)[e^{-\mu_l t}\{\cos\psi_l(D\widetilde{A}_l-2\omega_l\alpha_l\widetilde{B}_l)-\sin\psi_l(2\omega_l\widetilde{A}_l+\alpha_l D\widetilde{B}_l)\}]\right) \tag{6.10}$$
$$=[F_1\cos\psi_l+G_1\sin\psi_l]\cos^2(\varphi_2(0)).$$

and

$$\prod_{j=1}^{4}(D-\lambda_j)u_1=F_0+[F_1\cos\psi_l+G_1\sin\psi_l]\sin^2(\varphi_2(0))$$
$$+\sum_{n=2}^{\infty}(F_n\cos n\psi_l+G_n\sin n\psi_l). \tag{6.11}$$

The particular solutions of equations (6.9)-(6.10) give the unknown functions A_1, A_2, B_1, B_2 and u_1. Thus the determination of first order solution is complete. In this case A_1, A_2, B_1, B_2 depend on both amplitudes and phases and u_1 is not independent of first harmonics.

6.3 Example

As an example of the above procedure, we consider a fourth order nonlinear differential equation

$$x^{(4)}+k_1\dddot{x}+k_2\ddot{x}+k_3\dot{x}+k_4x=\varepsilon\, x^3, \tag{6.12}$$

Here $f=x^3$.

Therefore, $x_0=a_{1,0}e^{\lambda_1 t}+a_{2,0}e^{\lambda_2 t}+a_{3,0}e^{\lambda_3 t}+a_{4,0}e^{\lambda_4 t}$.

Here, we have dealt with monofrequent oscillations (see [72] for details), that is oscillations that are predominantly near one or the other of the linear modes of motion. Therefore, $a_1=0,\ a_2=0$. Such oscillations arise naturally and are of considerable interest in certain types of non-autonomous systems and certain autonomous systems. Now using the transformation $a_{2l-1}=\frac{1}{2}\alpha_l e^{i\varphi_l}$, $a_{2l}=\frac{1}{2}\alpha_l e^{-i\varphi_l}$, together with the substitutions $\lambda_{2l-1}=-\mu_l+i\omega_l$,

$\lambda_{2l} = -\mu_l - i\omega_l$, $A_{2l-1} = \frac{1}{2}(\widetilde{A}_l + i\widetilde{B}_l)$, and $A_{2l} = \frac{1}{2}(\widetilde{A}_l - i\widetilde{B}_l)$, $l = 1, 2$, $\alpha_1 = a, \alpha_2 = b$, we obtain

$$f^{(0)} = \{be^{-\mu_2 t}\cos(\omega_2 t + \varphi_2)\}^3$$

or,

$$f^{(0)} = \frac{b^3 e^{-3\mu_2 t}}{4}\{3\cos(\omega_2 t + \varphi_2) + \cos 3(\omega_2 t + \varphi_2)\}$$

Here $F_1 = \dfrac{3}{4}b^3 e^{-3\mu_2 t}$, $F_3 = \dfrac{1}{4}b^3 e^{-3\mu_2 t}$

Therefore, in accordance with [77, 91], equations (6.10)-(6.11) respectively become

$$\begin{aligned} & e^{-\mu_l t}\{\cos(\omega_2 t + \varphi_2)(D\widetilde{A}_2 - 2\omega_2 b\widetilde{B}_2) - \sin(\omega_2 t + \varphi_2)(2\omega_2\widetilde{A}_2 + bD\widetilde{B}_2)\} \\ & = \frac{3b^3 e^{-3\mu_2 t}[\{(3\mu_1 - \mu_2)^2 - \omega_1{}^2 + \omega_2{}^2\}\cos(\omega_2 t + \varphi_2) - 2\omega_2(3\mu_1 - \mu_2)\sin(\omega_2 t + \varphi_2)]\cos^2(\varphi_2(0))}{4[(3\mu_1 - \mu_2)^2 + (\omega_1 - \omega_2)^2][(3\mu_1 - \mu_2)^2 + (\omega_1 + \omega_2)^2]} \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} & \left((D + \mu_1)^2 + \omega_1{}^2\right)\left((D + \mu_2)^2 + \omega_2{}^2\right)u_1 \\ & = \frac{b^3}{4}e^{-3\mu_2 t}\cos 3(\omega_2 t + \varphi_2) + \frac{3}{4}b^3 e^{-3\mu_2 t}\cos(\omega_2 t + \varphi_2)\sin^2(\varphi_2(0)) \end{aligned} \tag{6.14}$$

Therefore, comparing the coefficients of $\cos(\omega_2 t + \varphi_2)$ and $\sin(\omega_2 t + \varphi_2)$, we obtain

$$(D\widetilde{A}_2 - 2\omega_2 b\widetilde{B}_2) = \frac{3b^3 e^{-2\mu_2 t}\{(3\mu_1 - \mu_2)^2 + \omega_1{}^2 - \omega_2{}^2\}\cos^2(\varphi_2(0))}{4[(3\mu_1 - \mu_2)^2 + (\omega_1 - \omega_2)^2][(3\mu_1 - \mu_2)^2 + (\omega_1 + \omega_2)^2]} \tag{6.15}$$

$$(2\omega_2\widetilde{A}_2 + bD\widetilde{B}_2) = \frac{3b^3 e^{-2\mu_2 t}\{-2\omega_2(3\mu_1 - \mu_2)\}\cos^2(\varphi_2(0))}{4[(3\mu_1 - \mu_2)^2 + (\omega_1 - \omega_2)^2][(3\mu_1 - \mu_2)^2 + (\omega_1 + \omega_2)^2]} \tag{6.16}$$

The solutions of equations (6.15)-(6.16) are

$$\widetilde{A}_2 = l_1 b^3 e^{-2\mu_2 t} \cos^2(\varphi_2(0)),$$
$$\widetilde{B}_2 = m_1 b^2 e^{-2\mu_2 t} \cos^2(\varphi_2(0)), \tag{6.17}$$

where

$$l_1 = -\frac{3\,[\mu_2\{(3\mu_2-\mu_1)^2-\omega_2{}^2+\omega_1{}^2\}-2\omega_2{}^2(3\mu_2-\mu_1)]}{8(\mu_2{}^2+\omega_2{}^2)[(3\mu_2-\mu_1)^2+(\omega_2-\omega_1)^2][(3\mu_2-\mu_1)^2+(\omega_2+\omega_1)^2]},$$

$$m_1 = -\frac{3\,\omega_2[2\mu_2(3\mu_2-\mu_1)+(3\mu_2-\mu_1)^2-\omega_2{}^2+\omega_1{}^2]}{8(\mu_2{}^2+\omega_2{}^2)[(3\mu_2-\mu_1)^2+(\omega_2-\omega_1)^2][(3\mu_2-\mu_1)^2+(\omega_2+\omega_1)^2]}. \tag{6.18}$$

Substituting the values of (6.17) into equation (6.9), we obtain

$$\dot{b} = \varepsilon\, l_1 b^3 e^{-2\mu_2 t} \cos^2(\varphi_2(0)),$$
$$\dot{\varphi}_2 = \varepsilon\, m_1 b^2 e^{-2\mu_2 t} \cos^2(\varphi_2(0)). \tag{6.19}$$

Also the particular solution of (6.14) is

$$u_1 = b^3 e^{-3\mu_2 t}\{\cos 3(\omega_2 t+\varphi_2)\, g_{2,1} + \sin 3(\omega_2 t+\varphi_2)\, h_{2,1}$$
$$+ \left(\cos(\omega_2 t+\varphi_2)\, c_{1,2} + \sin(\omega_2 t+\varphi_2)\, d_{1,2}\right)\sin^2(\varphi_2(0))\cos(\varphi_2)\}, \tag{6.20}$$

where

$$g_{2,1} = \frac{\mu_2{}^2(3\mu_2-\mu_1)^2+(2\omega_2{}^2-\mu_2{}^2)(9\omega_2{}^2-\omega_1{}^2)-18\mu_2\omega_2{}^2(3\mu_2-\mu_1)-2(3\mu_2-\mu_1)^2\omega_2{}^2}{16(\mu_2{}^2+\omega_2{}^2)(\mu_2{}^2+4\omega_2{}^2)\{(3\mu_2-\mu_1)^2+(3\omega_2-\omega_1)^2\}\{(3\mu_2-\mu_1)^2+(3\omega_2+\omega_1)^2\}},$$

$$h_{2,1} = \frac{-3\omega_2[2\,\mu_2{}^2(3\mu_2-\mu_1)+\mu_2(3\mu_2-\mu_1)^2-\mu_2(21\omega_2{}^2-\omega_1{}^2)+4\mu_1\omega_2{}^2]}{16(\mu_2{}^2+\omega_2{}^2)(\mu_2{}^2+4\omega_2{}^2)\{(3\mu_2-\mu_1)^2+(3\omega_2-\omega_1)^2\}\{(3\mu_2-\mu_1)^2+(3\omega_2+\omega_1)^2\}},$$

$$c_{1,2} = \frac{3[\mu_2(3\mu_2-\mu_1)^2-2\omega_2{}^2(3\mu_2-\mu_1)+\mu_2(\omega_1{}^2-\omega_2{}^2)]}{16\mu_2(\mu_2{}^2+\omega_2{}^2)\{(3\mu_2-\mu_1)^2+(\omega_1-\omega_2)^2\}\{(3\mu_2-\mu_1)^2+(\omega_1+\omega_2)^2\}}, \tag{6.21}$$

$$d_{1,2} = \frac{-3\omega_2[(3\mu_2-\mu_1)^2+2\omega_2(3\mu_2-\mu_1)+\omega_1{}^2-\omega_2{}^2]}{16\mu_2(\mu_2{}^2+\omega_2{}^2)\{(3\mu_2-\mu_1)^2+(\omega_1-\omega_2)^2\}\{(3\mu_2-\mu_1)^2+(\omega_1+\omega_2)^2\}},$$

In general, two equations of (6.19) are solved by numerical method [26]. An approximate solution of (6.19) may also be found by assuming that b is constant in the right hand sides of the equations. However, the first equation of (6.19) has an exact solution. Therefore, the first equation of (6.19) is solved analytically and an approximate solution is found for the second equation of (6.19).

Therefore, the first order solution of the equation (6.12) is given by

$$x = b e^{-\mu_2 t} \cos(\omega_2 t + \varphi_2) + \varepsilon\, u_1. \tag{6.22}$$

6.4 Results and Discussions

In order to test the accuracy of the approximate solutions obtained by this perturbation method, we compare the approximate solutions to the numerical solutions. For such comparison, we consider $k_1 = 0.5$, $k_2 = \sqrt{0.5}$, $\omega_1 = 0.5$, $\omega_2 = \sqrt{1.5}$, $\varphi_{1,0} = 0$, $\varphi_{2,0} = 1.570796$ and $\varepsilon = 0.1$. Solutions obtained by equation (6.22), in which b and φ_2 are calculated by (6.19) with initial conditions $x(0) = -0.000108$, $\dot{x}(0) = -1.219775$, $\ddot{x}(0) = 1.709554$ and $\dddot{x}(0) = 0.044153$ [or $a_0 = 0$, $b_0 = 1.0$], and is plotted in Fig. 6.1 (denoted by -o-). A second solution of (6.12) is computed by a fourth order Runge-Kutta formula with a small time increment $\Delta t = 0.05$ and the results are plotted in Fig. 6.1 (denoted by --). From the figure it is clear that the perturbation solutions (6.22) together with (6.19) and (6.20) agree with the numerical solutions.

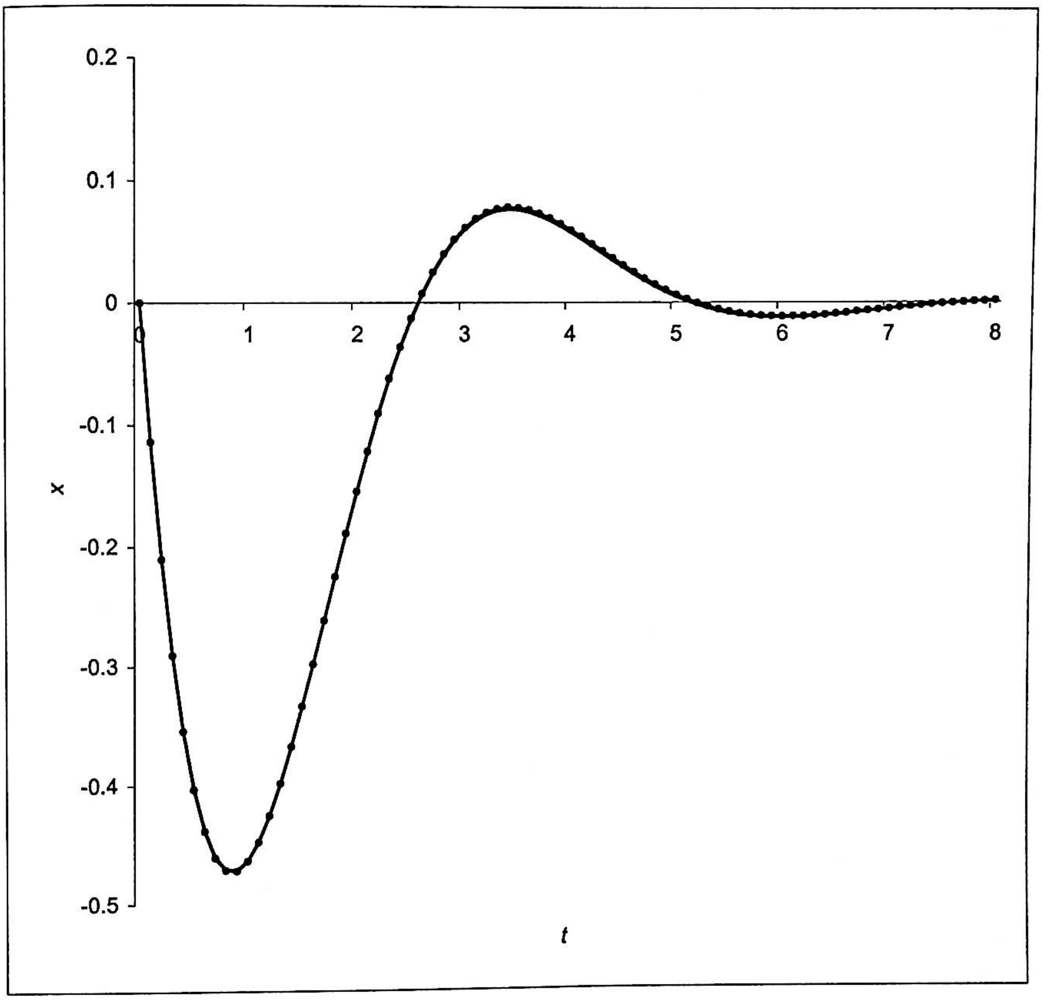

Fig. 6.1 Solution of Eq. (6.22): (i) Perturbation Solution denoted by -o- (ii) Numerical Solution denoted by --. For $k_1 = 0.5, k_2 = \sqrt{0.5}$, $\omega_1 = 0.5$, $\omega_2 = \sqrt{1.5}$, $\varepsilon = 0.1$. Initial conditions $a_0 = 0, b_0 = 1.0$, $\varphi_{1,0} = 0$ and $\varphi_{2,0} = \pi/2$ or [$x(0) = -0.000108$, $\dot{x}(0) = -1.219775$, $\ddot{x}(0) = 1.709554$, $\dddot{x}(0) = 0.044153$].

6.5 Conclusion

An asymptotic solution of a fourth order nonlinear differential equation, based on the theory of Krylov, Bogoluibov and Mitropolskii, has been found for large damping effects. The solutions obtained in Chapter 3 is useful when the damping force is small, but in the case of large damping effects the solution gives incorrect result. However, the solution found in this Chapter is useful for systems with large damping. The solutions obtained by this method (concerning this Chapter) remain valid before the moment of critical damping.

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