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Quantum Effects of Black Holes

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Certificate From The Supervisor

This is to certify that the thesis entitled "Quantum Effects of Black Holes" submitted by Kausari Sultana, who got her name registered in July/2007 M.Phil./Ph.D. batch for the award of the degree of Doctor of Philosophy in Science (Applied Mathematics) of the University of Rajshahi, is absolutely based on her own work under my supervision and that neither this thesis nor any part of it has been submitted for any degree or any other academic award anywhere before.

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Declaration

I do hereby declare that the thesis entitled "Quantum Effects of Black Holes" submitted by me to the University of Rajshahi for the award of Ph.D. degree in Science (Applied Mathematics) has not been submitted to any Institute or University for any degree or award.

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Dedicated To My Parents

Abstract

We study quantum mechanical aspect of black holes. We give a brief review of black hole radiation which is commonly called Hawking radiation. So far several different methods have been employed to investigate this radiation and all these results suggest in favor of the existence of Hawking radiation. However, there still exist several aspects of the Hawking effect which have yet to be clarified. We outline some arguments from previous works on the subject and then attempt to present the more satisfactory derivations of Hawking radiation by using semi-classical tunneling mechanism for nonrotating and rotating background spacetimes. We employ three kinds of methods for investigating the tunneling radiation: the null-geodesic method, the Hamilton-Jacobi ansatz, and the Damour-Ruffini method. All these methods lead to the same conclusion. However, the Hamilton-Jacobi ansatz is more simple and the physical picture in this method is more clear. We also discuss thermodynamic properties like entropy of different black holes. We obtain inner horizon entropy and Bekenstein-Smarr Formula as well.

In some recent derivations thermal characters of the inner horizon have been employed; however, the understanding of possible role that may play the inner horizons of black holes in black hole thermodynamics is still somewhat incomplete. Motivated by this problem we investigate Hawking

radiation of black holes by considering thermal characters of both the outer and inner horizons. We investigate Hawking radiation of electrically and magnetically charged Dirac particles (as well as scalar particles) from more general black hole spacetimes (such as Demiański-Newman and Kerr-Newman-Kasuya-Taub-NUT-Anti-de Sitter black hole).

Taking into account conservation of energy and the back-reaction of particles to the spacetime, we calculate the emission rate and find it proportional to the change of Bekenstein-Hawking entropy. The radiation spectrum deviates from the precisely thermal one and the investigation specifies a quantum-corrected radiation temperature dependent on the black hole background and the radiation particle's energy, angular momentum, and charges. It also has been found that Dirac particles are emitted at the same temperature as scalar particles from a black hole. It depicts the robustness of the semi-classical tunneling technique.

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I express my deepest gratitude to my parents for their encouragement and special attention towards my higher education. It is their encouragement and blessings that enabled me to cross the hurdles of life. I am thankful to my spouse and my kids for their endless support. They are anxiously awaiting for the completion of the thesis. Thanks are also due to my sisters, my in-laws and all the members of my family for standing beside me.

Author

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"I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me."

Sir Issac Newton[1]

[1]In D. Brewster, *Memoirs of the life, writings and discoveries of Sir Isaac Newton*, Edinburgh: Thomas Constable and Co., Hamilton, Adams and Co., London 1855 (reprint Johnson Reprint Corporation, New York and London, 1965 Vol. **2**, p. 407).

Chapter 1

Introduction

The current understanding of physics identifies all the forces in nature into three categories: strong, electro-weak and gravity. The first two forces have been successfully described by the quantum field theory up to energy scales of the order of 100 GeV. The electro-weak interaction is the unification of electromagnetic and weak forces, successfully described by the Weinberg-Salam theory. The strong interaction is described by quantum chromodynamics and the remaining one, the gravity, is described by the general theory of relativity. There have been attempts to some success in the direction of grand unified theory (GUT) which incorporates the strong interaction with the electroweak interaction. However, because we still lack a quantum description of the gravitational interaction – *quantum gravity*, there has yet to be successfully included gravity in a theory of everything (TOE).

There has been a large number of theoretical approaches on the problem of quantum gravity with some successes, but none of them have given a complete theory that works at Planck energy scales ($\sim 1.22 \times 10^{19} GeV$). The way to quantum gravity faces a lot of difficulties because the resulting theory is not renormalizable. This means, physically meaningful observables contain nonremovable infinities. It was believed for sometime that the supergravity theories might overcome the non-renormalisabilty of general relativity, but detail calculations have led to the conclusion that they also suffer from the same problem. The string theory has potentiality to solve these problems. However, we have not found any solid result in string theory yet and it depends on the future progress. It is thus yet to formulate a widely accepted consistent theory that combines the general theory of relativity with the principle of quantum theory.

There exist compelling reasons to believe that quantum gravitational effects will be important only at energy scales of the order of Planck energy. There is a domain of 17 orders of magnitude between the Planck energy and an energy scale of the order of 100 GeV. In this energy domain the gravitational field can be assumed to behave classically and the matter fields can be assumed to have a quantum nature. Describing classical gravity by the general theory of relativity, one is led to the subject of quantum field theory in curved spacetimes. This is a semi-classical theory and the gravitational field is retained in this theory as a classical background while the matter fields are quantized according to the conventional quantum field theory.

Even though we have not yet confirmed that black holes do really exist, it has been predicted that they exist in the universe as a consequence of the general theory of relativity. They are among the most remarkable predictions of the Einstein field equation, which evoke mysterious aspects of gravity. In particular, the Einstein equation suggests that the spacetime is curved by the effects of gravity. A very strong gravity can curve the spacetime to form a closed region from which nothing, not even photons, can escape. The closed region is called the black hole. Thus a black hole cannot classically allow the emission of radiation. If only a classical system is considered, it would be impossible to define a temperature for a black hole since it would be impossible for anything to be in thermal equilibrium with a black hole. This is due to that everything would go into the black hole but nothing will come out. Entry of matter, which has its own entropy, into the black hole, results in the decrease of the total entropy of the universe, and this contradicts the second law of thermodynamics. It was Bekenstein [1] who first conjectured that there was a fundamental relationship between the properties of black holes and the laws of thermodynamics and showed that the black hole possesses entropy similar to its surface area. As the black hole absorbs matter, its entropy increases and the decrease of the exterior entropy is then balanced, preserving the second law of thermodynamics. The surface gravity, which is the gravitational acceleration experienced at the surface of the black hole or any object, is related with temperature of the body in thermal equilibrium. Soon after the significant work of Bekenstein, Hawking showed that quantum mechanically black holes can emit radiations [2, 3, 4, 5, 6]. The radiation from the black hole is commonly called the *Hawking radiation*. Hawking was also able to show that this black hole radiation was purely thermal. Black holes thus have a well defined temperature and can truly be thought of as thermodynamic objects. This was an important discovery since classically nothing could escape from a black hole. Hawking radiation thus emphasizes the importance of trying to find a full quantum theory of gravity.

Hawking radiation is one of the most striking effects which are widely accepted by now. However, there are several aspects which have yet to be clarified. In particular, the entropy is interpreted as a count of the number of states in statistical mechanics, but the entropy of a black hole with a finite temperature has not been derived by counting the number of quantum states associated with the black hole. It is thought that this problem has a close link with the fact that the quantum theory of gravity has not been explicitly formulated yet, and it is not an easy task to construct a consistent quantum gravity.

All the known derivations of Hawking radiation have not reached an impeccable conclusion yet. A new physics will be found, if a complete quantum theory of gravity is formulated. The discovery of Hawking radiation also brought forth new mysteries such as the *information loss* problem, which results from the argument of whether the black hole radiation should be purely thermal or not. If the black hole radiates thermal radiation like the black body radiation, it will not contain any information with it. After the black hole evaporates, the information of what made up the black hole will be gone forever. This information loss problem has a particular concern in quantum gravity. Then a moot question arises whether information will actually be lost or the radiation should have to be modified so that it is not truly thermal. Different physicists disagree over whether or not black holes should lose information. The most famous example of this is the Thorne-Hawking-Preskill bet. Kip Thorne and Stephen Hawking made a public bet in 1997 with John Preskill that information would be lost in a black hole and Preskill bet that information must not be lost. In 2004 Hawking publicly conceded the bet but Thorne has not conceded and the issue still remains as an open problem. Furthermore, there is a dispute regarding the reaction of the radiation to the spacetime. When the black hole generates Hawking radiation, the black hole parameters (energy, charge, and angular momentum) fluctuate. This effect was not considered in the past. Hawking derived the black-hole radiation as precisely thermal spectrum only under the assumption that the spacetime is invariant.

There are several methods employed by now for deriving Hawking radiation [2–69] and calculating the black hole temperature. The original Hawking's method considered the creation of a black hole in the context of a collapse geometry. The analysis calculates the Bogoliubov coefficients between the initial and final states of incoming and outgoing radiation [2, 12]. Damour-Ruffini [7] calculated particles' emitting rate from black holes by analytically extending the outgoing wave from outside of horizon to inside. This technique is the generalization of the classical approach of barrier penetration to curved spaces endowed with future horizons. It allows one to recover most directly the spectrum of the Hawking radiation. Soon after the works of Gibbons and Hawking [4, 5], the more popular method of analytic continuation to a Euclidean section (the Wick Rotation method) emerged. Turning on the methods of finite-temperature quantum field theory, an analytic continuation $t \to i\tau$ of the black hole metric is executed. The periodicity of τ (denoted by β) is preferred to remove a conical singularity that would otherwise be present at fixed points of the $U(1)$ isometry generated by $\partial/\partial\tau$ (the event horizon in the original Lorentzian section). The black hole is then thought to be in equilibrium with a scalar field having inverse temperature β at infinity.

For calculating black hole temperature, there have been developed some other methods such as the black hole tunneling methods [15–68] and the anomaly method [69]. The tunneling method provides a dynamical model of the black hole radiation and hence it is a particularly interesting method for calculating black hole temperature. It was originally applied to a Schwarzschild black hole [15–17]. Because of the semi-classical nature of the model, it was not considered to be as powerful as it has turned out to be. In the 1990's, Kraus and Wilczek [15] proposed a semi-classical method of modeling Hawking radiation as a tunneling effect, which has garnered a lot of interest [15–68]. In this method the imaginary part of the action is calculated for the (classically forbidden) process of s-wave emission across the horizon [15–17]. The Boltzmann factor for emission at the Hawking temperature is related to this imaginary part of the action. Using the Wentzel-Kramers-Brillouin (WKB) approximation the tunneling probability for the classically forbidden trajectory of the s-wave coming from inside to outside the horizon is given by

$$\Gamma \propto \exp(-2\,\mathrm{Im}\, I), \tag{1.0.1}$$

where I is the classical action of the trajectory to leading order in $\hbar$ (here we set $\hbar = 1$). When the action is expanded in terms of the particle energy, the Hawking temperature is recovered at linear order. That is, for $2I = \beta\omega + \mathcal{O}(\omega^2)$, one finds the regular Boltzmann factor

$$\begin{aligned} \Gamma &\propto \exp[-(\beta\omega + \mathcal{O}(\omega^2))] \\ &\simeq \exp(-\beta\omega) \end{aligned} \tag{1.0.2}$$

for a particle of energy ω where β is the inverse temperature of the horizon. The higher order terms describe a self-interaction effect resulting from energy conservation [16, 19]. Two different approaches are there to calculate the imaginary part of the action for the emitted particle. Following the work of Kraus and Wilczek [15–17], Parikh and Wilczek [19] first developed the black hole tunneling method which is known as the null geodesic method. The other approach to black hole tunneling is the Hamilton-Jacobi ansatz used by Marco Angheben et al. [35] and further developed by Kerner and Mann [44, 60], which is an extension of the complex path analysis of Srinivasan et al. [26–29].

The null geodesic method studies a null s-wave emitted from the black hole. Analyzing the full action in detail on the basis of the previous work[15–17], the only part of the action that contributes an imaginary term is found to be $\int_{r_{\rm in}}^{r_{\rm out}} p_r dr$, where p_r is the momentum of the emitted null s-wave. Then by utilizing Hamilton's equation with the knowledge of the null geodesics, it is possible to calculate the imaginary part of the action. The Hamilton-Jacobi ansatz, on the other hand, considers an emitted scalar particle, ignoring its self-gravitation. The action of the particle satisfies the relativistic Hamilton-Jacobi equation. Using the symmetries of the metric, one could choose an appropriate ansatz for the form of the action. This method is derived by using the WKB approximation to the Klein-Gordon equation. Kerner and Mann [60] first extended this method to fermion particles by employing the WKB approximation to the Dirac equation. Recently, Ding [70] has further improved this method by viewing the Hawking radiation as a series of infinite small quasi-static emission process.

Another new method was proposed by Liu [71] to model black hole radiation. Using the Damour-Ruffini method [7] Liu investigated Hawking radiation of massive Klein-Gordon particles from a Reissner-Nordström black hole [71]. Extending Liu's work to charged Dirac particles' Hawking radiation from a Kerr-Newman black hole, Zhou and Liu [72] arrived at the same terminations as the previous works.

In calculating the black hole temperature, the tunneling method has a lot of strengths compared to other methods. The calculations in this method are straightforward and relatively simple. The tunneling method is robust in the sense that it can be applied to a wide variety of exotic spacetimes. It has been successfully applied to spacetimes such as Kerr and Kerr-Newman cases [37, 39, 44], black rings [42], the 3-dimensional BTZ black hole [35, 43], Vaidya black hole [50], other dynamical black holes [54], Taub-NUT spacetimes [44], Gödel spacetimes [55], and Hot-NUT-Kerr-Newman-Kasuya Spacetimes [56, 57]. The tunneling method has also been applied to horizons that are not black hole horizons, such as Rindler spacetimes [26, 44] and the Unruh temperature [9] has been retrieved. The tunneling method has also been applied to the cosmological horizons of de Sitter spacetimes [23, 31, 32, 35, 45, 61, 68]. The applications to de Sitter spacetimes demonstrate that the tunneling method has a particular advantage over the Wick rotation method. This is due to that the Wick rotation method cannot be applied when a Schwarzschild black hole is embedded in a de Sitter spacetime but the tunneling method can be applied. The tunneling method has another strength that it can be extended beyond the emission of scalar particles and can model particles that have spin [60, 64–68]. The importance of the tunneling method lies in the fact that it gives an intuitive picture of black hole radiation. The trajectory of an s-wave particle is from the inside of the black hole to the outside, a classically forbidden process. It follows from energy conservation that the radius of the black hole shrinks as a function of the energy of the outgoing particle and in this sense the particle creates its own tunneling barrier. This also yields a dynamical model of black hole radiation because the mass of the black hole decreases. However, this is a slow dynamics since the mass cannot be changing rapidly for this model.

There is still an open problem on black hole entropy [73, 74, 75]. The Nernst theorem demands that the entropy of a system must vanish as its temperature goes to zero. If this assertion is applied to black holes, one finds that the entropy of the black hole with two horizons does not vanish as its temperature approaches absolute zero [76, 77]. However, if the black hole with two horizons is considered as a thermodynamics system composed of two subsystems: the outer horizon and the inner horizon, the Nernst theorem is found to be satisfied. This is because the entropy of the black hole then contains contributions of both the outer and inner horizons [78, 79, 80]. Recently, thermodynamics properties of the inner horizon of a Kerr-Newman black hole [81] and tunneling effect of two horizons from a Reissner-Nordström black hole [82] have been investigated by Jun Ren. All these works are in agreement with Parikh's work.

One of the aims of this thesis is to investigate Hawking radiation in some interesting black hole spacetimes as a tunneling phenomena and show that Hawking radiation is covariant. The second aim of the thesis is to analyze the inner horizon radiation of black holes with two horizons and redefine the entropy of the black hole to satisfy the Nernst theorem. A chapter wise summary of the thesis is given below.

In Chapter 2, we review some properties of black holes. These properties will be useful to understand the contents of the following chapters. In Chapter 3, we review Hawking's original derivation of black hole radiation and briefly describe Unruh effect and Damour-Ruffini method of calculating black-hole evaporation. We also review the derivation of Hawking radiation by using tunneling methods (i.e. the null geodesic method and the Hamilton-Jacobi ansatz).

Chapter 4 is concerned with the investigation of Hawking radiation of charged particles via tunneling of both horizons from Reissner-Nordström-Taub-NUT (RNTN) black holes [83]. In some recent derivations thermal characters of the inner horizon have been employed [78–82, 84–87]; however, the understanding of possible role that may play the inner horizons of black holes in black hole thermodynamics is still somewhat incomplete. Motivated by this problem we investigate Hawking radiation of the RNTN black hole by considering thermal characters of both the outer and inner horizons. We solve the Klein-Gordon equation for massive particles. Using Damour-Ruffini method [7] and Liu's technique [71] we then calculate the charged particles' Hawking Radiation via tunneling of both horizons from RNTN black holes. The inner horizon admits thermal character with positive temperature and entropy proportional to its area, and it thus may contribute to the total entropy of the black hole in the context of Nernst theorem. Exploiting the thin film brick wall model [88] which is based on the brick wall model proposed by 't Hooft [89], we compute the entropy of the inner horizon of the RNTN black hole. Considering conservations of energy and charge and the back-reaction of emitting particles to the spacetime, the emission spectra are obtained for both the inner and outer horizons. The total emission rate is the product of the emission rates of the inner and outer horizons, and it yields the same conclusion as the previous works (done by null geodesic method and Hamilton-Jacobi ansatz). In the limit of vanishing NUT parameter, this chapter gives result for the RN black hole. The work of this chapter draws conclusion analogous to the previous works.

In chapter 5 we the investigate Hawking radiation by using null-geodesic method. We analyze tunneling of charged and magnetized massive particles from Taub-NUT-Reissner-Nordström-Anti-de Sitter (TNRN-AdS) black holes endowed with electric as well as magnetic charges [90]. The TNRN-AdS black hole is the NUT charged RN black hole in the AdS space. It reduces in special cases to the Taub-NUT-AdS and Taub-NUT black holes. The AdS spacetime not only is interesting in the context of brane-world scenarios based on the setup of Randall and Sundrum but also plays leading role in the familiar AdS/CFT [91] conjecture. By studying thermodynamics of the asymptotically AdS spacetime, it is possible to get some insights into the thermodynamic behavior of some strong coupling CFTs from the correspondence between the supergravity in asymptotically AdS spacetimes and CFT [92]. On the other hand, recent developments in string/M theory have greatly stimulated the study of NUT charged black hole phenomena in AdS spaces. In particular, these black hole backgrounds are interesting in the context of AdS/CFT conjecture [93, 94, 95] and supergravity. The Taub-NUT metric plays an important role in the conceptual development of general relativity. As "counter example to almost anything" [96], the Taub-NUT spacetime has peculiar character. The entropy of various Taub-NUT black holes is not proportional to the area of the event horizon and their free energy can have negative value [93, 95, 97–100]. The NUT charged AdS black hole has a boundary metric that has closed timelike curves. Quantum field theory behaves significantly different in this space. It is of interest to understand AdS-CFT correspondence in this type of spaces [101]. The presence of closed timelike curves in the NUT charged AdS black hole spacetimes can be avoided, if one takes into account the universal covering of such AdS black hole backgrounds, which is not globally hyperbolic. In view of the above considerations the TNRN-AdS black hole deserves investigation in a broader context. The study of this chapter is interesting in this regard. Our concern in this chapter is to analyze the basic property of the TNRN-AdS black hole and investigate quantum tunneling radiation. We find that the entropy of the TNRN-AdS black hole is not proportional to the event horizon area. The obtained tunneling radiation agrees with that obtained in chapter 4 by using Liu's method [71]. We also discuss the Schwarzschild-AdS, Taub-NUT-AdS, and RN-AdS black hole cases, which are special types of the TNRN-AdS black hole. The derived results can provide results for the TNRN black hole, the RN black hole, the TN black hole, and the Schwarzschild black hole. In view of the above considerations the work of this chapter is well motivated.

Since a black hole has a well defined temperature, it should radiate all types of particles like a black body at that temperature (ignoring grey-body effects). Therefore, the expected emission spectrum should contain particles of all spins. Indeed, the implications of this expectation were studied around four decades ago [10, 11].

Chapter 6 is concerned with the study of tunneling radiation and temperature of Demiański-Newman black holes [102]. We use improved Hamilton-Jacobi approach [70] by viewing the Hawking radiation as a series of infinite small quasi-static emission process and compute tunneling rate of charged and magnetized scalar as well as fermion particles from Demiański-Newman black holes. The Demiański-Newman spacetime [103] is a five-parameter stationary axisymmetric solution of the Einstein-Maxwell equations. The Demiański-Newman black hole background is interesting in that it generalizes the well-known Kerr-Newman spacetime with two intriguing parameters the gravitomagnetic and magnetic monopoles. In the stationary pure vacuum limit, the Demiański-Newman metric reduces to the combined Kerr-NUT and Taub-NUT solutions. It is interesting that the spacetimes with the NUT charge are not asymptotically flat but asymptotically locally flat [93, 100, 101] and they possess several special properties. As discussed in [44], tunneling and temperature of Taub-NUT black holes can be formally carried out and the physical interpretation is less problematic in the context of the Hamilton-Jacobi ansatz than the null-geodesic method. The Taub-NUT space has played an important role in the conceptional development of general relativity and in the construction of brane solutions in string theory and M-theory. The singularities of the NUT charged spacetime, the Misner strings [96], can be avoided by periodic time coordinate. One of the interesting properties of NUT charged spaces is the existence of closed timelike curves which violates the causality condition. The half-closed timelike geodesics in Taub area can be explored in NUT area, so the naked singularity exists. The NUT charged black holes have been of particular interest in AdS/CFT conjecture [93, 94, 95]. In AdS backgrounds, Lorentzian sector of these spacetimes boundary metric is similar with the Gödel metric [104]. In recent years the thermodynamics of various Taub-NUT spacetimes has become a subject of intense study. Entropy of these spacetimes is not just a quarter area at the horizon and their free energy can sometimes be negative [93, 97–101, 105, 106]. It was ingeniously suggested by Dirac relatively long ago that the magnetic monopole does exist in nature, but it was neglected due to the failure to detect such an object. However, in recent years, the development of gauge theories has shed new light on it. Several recent extensions of the standard model of particle physics predict existence of magnetic monopoles and it has grown interests in the possibility of magnetically charged black holes. The string theory [107] also admits the existence of such objects. The importance of the Demiański-Newman solution lies in that it gives a single constituent with the whole set of parameters which may have a physical sense in axisymmetric many-body systems of aligned sources [108, 109]. In view of the above considerations, the research on the Demiański-Newman black hole is necessary and meaningful.

We investigate scalar particles' emission by the charged Klein-Gordon equation and fermion particles' emission by the charged Dirac equation in covariant form [110]. We also calculate the change of total entropy of the system including black hole and radiating particles. The result shows that the change in total entropy is $\Delta S > 0$ (indicating the process as irreversible) but very small and can be neglected. This has some difference from Parikh's work (null geodesic method) in which $\Delta S = 0$. It also suggests that the probing of radiating particles of the black hole is connected with the change of the black hole entropy.

The study of this chapter demonstrates that the black hole emits tunneling radiation spectrum of massive and massless (scalar or fermion) particles at the same Hawking temperature in the semi-classical limit in which the WKB approximation is applicable. The result of this chapter is accordant with the results obtained in chapter 5 by the null geodesic method and in chapter 4 by Liu's method. It also demonstrates that the physical picture in Hamilton-Jacobi method is more clear. In special cases the study gives results for the Kerr-Newman black hole, the Kerr-NUT black, the Kerr black hole, the Taub-NUT black hole, the Reissner-Nordström black hole, and the Schwarzschild black hole.

In chapter 7, we investigate Hawking radiation of electrically and magnetically charged Dirac particles from the dyonic Kerr-Newman-Kasuya-Taub-NUT-Anti-de Sitter (KNKTNAdS) black hole by considering thermal characters of both the outer and inner horizons [111]. The particles are described by the Dirac equations in the curved spacetime described in terms of the Newman-Penrose formalism. We calculate the temperature of the inner horizon of the black hole by following the Liu's [71] technique based on Damour-Ruffini method [7]. We demonstrate the existence of thermal characters of the inner horizon. Like as for the Reissner-Nordström-Taub-NUT black hole case [83] discussed in chapter 4, the inner horizon of the KNKTN-AdS black hole emits positive energy particles inside the inner horizon (towards the singularity) with a positive temperature. In order to maintain a local energy balance, antiparticles with negative energy are emitted away from the singularity through the inner horizon. This is a process analogous to that takes place at the outer horizon according to the Hawking effect—at the outer horizon antiparticles go in and particles come out. The real particle remains inside the inner horizon and finally meets with the singularity. But the antiparticle enters the intermediate region between the horizons. Traveling across the intermediate region this antiparticle finally comes out from the white hole horizon, if the backscattering effects are neglected. The situation is, however, quite complicated because the vacuum states corresponding to a freely falling observer near the inner horizon of the black hole and the white hole horizon are entirely different. Since the white hole horizon emits thermal radiation [87], outside the KNKTN-AdS black hole two simultaneous radiation processes could be found—one is the normal black hole radiation and the other one is "white hole radiation," caused by the pair creation effects at the inner horizon. The white hole radiation may be thought of as absorption of energy, since it radiates only antiparticles with negative energy. Because the white hole horizon absorbs no energy classically, this feature contradicts with the classical result in a similar way as does the evaporation process at the outer horizon of black holes.

The KNKTN-AdS spacetime is stationary and the Killing vector field $(\partial/\partial t)^a$ is time-like in the regions both outside the outer horizon and inside the inner horizon. Hence, the surface gravity can be well-defined on the inner horizon. We calculate the inner horizon entropy proportional to its area by membrane model [112, 113], which is the modified form of the brick-wall model, proposed by 't Hooft [89]. So, the entropy of the KNKTNAdS black hole might include the contributions of both the outer and inner horizons. The redefined entropy then satisfies the Nernst theorem. In special cases, the work of this chapter gives results for a wide range of black holes including (i) the Kerr-Newman-Kasuya black hole, (ii) the Kerr-Newman-AdS black hole, (iii) the Kerr-Newman-Taub-NUT black hole, (iv) the Taub-NUT-Reissner-Nordström-AdS black hole, (v) the Reissner-Nordström-Taub-NUT black hole, (vi) the Reissner-Nordström black hole, (vii) the Taub-NUT-AdS black hole, and (viii) the Taub-NUT black hole. The results are in agreement with the results of previous works done by null geodesic method (chapter 5) and Hamilton-Jacobi ansatz (chapter 6).

Finally, in chapter 8, we present our conclusions and the future outlook. Some technical details of the work presented in the thesis are given in the Appendices A–G (chapter 9).

In this thesis, we use the natural system of units

$$c = G = \hbar = k_B = 1 \tag{1.0.3}$$

unless stated otherwise, where c is the speed of light in vacuum, G is the gravitational constant, $\hbar$ is the reduced Planck constant (Dirac's constant), and k_B is the Boltzmann's constant.

This thesis is mainly based on the following publications:

1. M.H. Ali and **K. Sultana**, "Charged Particles' Hawking Radiation via Tunneling of Both Horizons from Reissner-Nordström-Taub-NUT Black Holes", Int.J.Theor.Phys. **52**, (2013) 2802–2817; [**Chapter 4**]

2. M.H. Ali and **K. Sultana**, "Tunneling of Charged Massive Particles from Taub-NUT-Reissner-Nordström-AdS Black Holes", Int. J. Theor. Phys. **53**, (2014) 1441–1453; [**Chapter 5**]

3. M.H. Ali and **K. Sultana**, "Tunneling and temperature of Demiański-Newman black holes", Int. J. Theor. Phys. **DOI:**10.1007/s10773-014-2154-1 (in press); [**Chapter 6**]

4. M.H. Ali and **K. Sultana**, "Charged Dirac Particles' Hawking Radiation via Tunneling of Both Horizons and Thermodynamics Properties of Kerr-Newman-Kasuya-Taub-NUT-AdS Black Holes", Int. J. Theor. Phys. **52**, (2013) 4537–4556; [**Chapter 7**]

and the list of references.

Chapter 2

Black Holes: Basic Concepts and Properties

Black holes are one of the most fascinating objects predicted by Einstein's field equations with having mysterious aspect of gravity. They manifest regions of space with enormously strong gravitational fields from which nothing, not even light, can escape. Their rotational behaviors are predicted by the existence of rotating stars. By ejecting rotation energy, a rotating black hole gradually reduces to a nonrotating black hole. Further, an isolated black hole can be endowed with a net electric charge.

During the past four decades, research in the theory of black holes in general relativity has brought to light strong indications of a very intense and fundamental relationship between gravitation, thermodynamics, and quantum theory. The basis of this relationship is black hole thermodynamics, where it turns up that certain laws of black hole mechanics are, in fact, simply the ordinary laws of thermodynamics applied to a system containing a black hole. Indeed, the discovery of the thermodynamic behavior of black holes—achieved primarily by classical and semi-classical analyzes—has given rise to most of our present physical insights into the nature of quantum phenomena occurring in strong gravitational fields. In this chapter we would like to review some basic concepts and properties of black holes.

The contents of this chapter are as follows. In section 2.1, we discuss about the basics of relativity and Einstein's field equations. In section 2.2, we briefly describe a black hole and its spacetime. In section 2.3, we refer to the Penrose diagram and show how to describe it. In section 2.4, we discuss the process of energy extraction from a rotating black hole classically. In section 2.5, we would like to discuss analogies between black hole physics and thermodynamics. In section 2.6, we review the argument, suggested by Bekenstein, regarding black hole entropy. These introductory discussions will be useful in understanding the contents of the following chapters.

2.1 Theory of Relativity and Field Equations

In 1905, Albert Einstein determined that the laws of physics are the same for all non-accelerating observers, and that the speed of light in a vacuum is independent of the motion of all observers. This is the theory of special relativity. It introduces a new framework for all of physics and proposed new concepts of space and time, matter and energy. Minkowski space or Minkowski spacetime (named after the mathematician Hermann Minkowski) is the mathematical space setting spanned by (t, x, y, z) in which Einstein's theory of special relativity is most conveniently formulated. In this setting the three ordinary dimensions of space are combined with a single dimension of time to form a 4-dimensional manifold for representing a spacetime. Minkowski space is often contrasted with Euclidean space. A Euclidean space has only spacelike dimensions, while a Minkowski space has one additional timelike dimension along with space-like dimensions. Therefore, the symmetry group of a Euclidean space is the Euclidean group, but it is the Poincaré group for a Minkowski space. The metric of the Minkowski spacetime is described by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.1.1}$$

The spacetime interval between two events in Minkowski space is either a space-like, light-like (or "null") or time-like according as $ds^2 > 0$, $ds^2 = 0$, or $ds^2 < 0$. The Minkowski space describes physical systems over finite distances only where no appreciable gravitation does exist in the Newtonian limit.

In the case of significant gravitation, one must abandon special relativity in favor of the full theory of general relativity according to which spacetime becomes curved. However, even in such cases, Minkowski space might still be a good description in an infinitesimal region surrounding any point (excluding gravitational singularities). That is, in the presence of gravity spacetime is described by a curved 4-dimensional manifold for which the tangent space to any point is a 4-dimensional Minkowski space. In the realm of weak gravity, spacetime becomes flat and looks globally (not just locally) like Minkowski space. So, Minkowski space is often referred to as flat spacetime.

Soon after publishing the special theory of relativity, Einstein began thinking about how to incorporate gravity into his new relativistic framework. He started with a simple thought experiment regarding an observer in free fall and carried out the search for a relativistic theory of gravity.

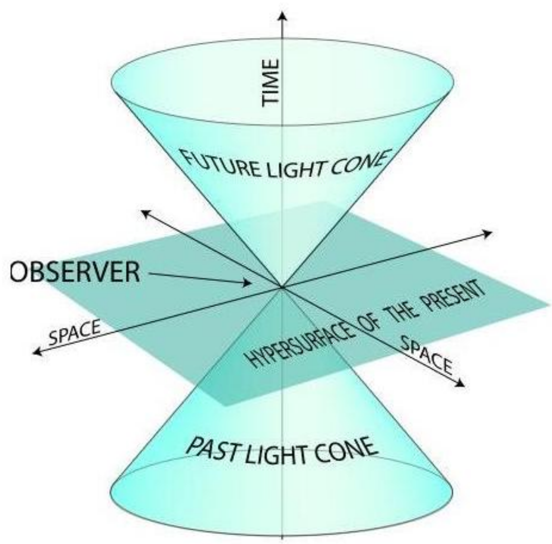

Figure 2.1: The Minkowski spacetime in 3-dimensions.

After numerous detours and false starts, he culminated the work through the presentation to the Prussian Academy of Science in November 1915 [114], which is now known as the Einstein field equation. This equation determines the influence of matter and radiation to the geometry of space and time and forms the core of Einstein's general theory of relativity. It is given by [115]

$$R_{\mu\nu} - \frac{1}{2}R\mathrm{g}_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \qquad (2.1.2)$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $\mathrm{g}_{\mu\nu}$ is the metric tensor of spacetime, G is Newton's gravitational constant, c is the speed of light, and $T_{\mu\nu}$ is the energy-momentum tensor. These quantities are defined by

$$R = R^{\mu}_{\mu} = g^{\mu\nu} R_{\mu\nu}, \tag{2.1.3}$$

$$R_{\mu\nu} = R^{\rho}{}_{\mu\nu\rho}, \tag{2.1.4}$$

$$R^{\rho}{}_{\mu\nu\sigma} = \partial_{\nu}\Gamma^{\rho}_{\mu\sigma} - \partial_{\sigma}\Gamma^{\rho}_{\mu\nu} + \Gamma^{\alpha}_{\mu\sigma}\Gamma^{\rho}_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\rho}_{\alpha\sigma}, \tag{2.1.5}$$

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_{\nu} g_{\alpha\mu} + \partial_{\mu} g_{\alpha\nu} - \partial_{\alpha} g_{\mu\nu}), \tag{2.1.6}$$

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{2.1.7}$$

where $R^{\rho}{}_{\mu\nu\sigma}$ is the Riemann-Christoffel tensor or the curvature tensor, $\Gamma^{\rho}_{\mu\nu}$ is the Christoffel symbol and ds is the metric or line element. As determined by the metric (2.1.7) the expression on the left-hand side of the equation (2.1.2) represents the curvature of spacetime and the expression on the right-hand side represents the distribution of matter fields. Thus, the Einstein equation (2.1.2), as a set of equations, is prescribing how the curvature of spacetime is related to the distribution of matter and energy in the universe.

2.2 Black Holes

A black hole is a region in spacetime in which the gravitational field is so strong that even light is caught and held in its grip. It curves space and warps time. A black hole is formed when a body of mass M contracts to a size less than the so-called gravitational radius $R_S = 2GM/c^2$. The boundary enclosing a black hole is called an "event horizon" because an outside observer is unable to observe events on the other side of it. Light

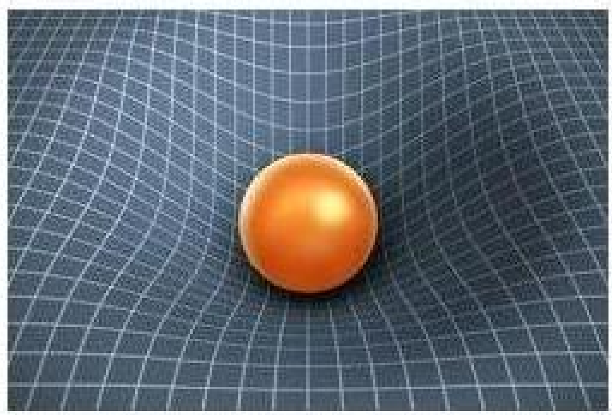

Figure 2.2: Einstein's theory of general relativity predicts that the spacetime around a massive body would be warped.

rays from inside of the event horizon cannot propagate out. A black hole may well form from a gravitational collapse of stars more massive than $8-10M_{\odot}$, where $M_{\odot}$ is the mass of the sun. The original stars, which will form the black hole, have various physical quantities and properties. After formation of the black hole by the gravitational collapse, the state of the black hole becomes a stationary state and can be characterized by three physical parameters: the mass, the angular momentum and the electrical charge. The black hole does not retain any information of the original star except these three parameters. This result is called the black hole uniqueness theorem [116–118] or the no-hair theorem [119]. The uniqueness theorem is shown in a 4-dimensional theory when the solutions of the Einstein equations satisfy the four conditions: (i) only electromagnetic field exits, (ii) asymptotically flat, (iii) stationary, and (iv) no singularity exists on and outside the event horizon. The fourth condition is based on the cosmic censorship hypothesis, proposed by Penrose [120], according to which "Naked singularities" cannot form from gravitational collapse in an asymptotically flat spacetime that is nonsingular on some initial spacelike hypersurface (Cauchy surface).

The Einstein's equation (2.1.2) is a quadratic nonlinear differential equation and is very difficult to solve for the general solution. However, there exist four exact solutions of the Einstein's equation describing black hole solutions with or without charge and angular momentum. These are the following:

- The Schwarzschild solution (1916) [121]: It is static and spherically symmetric solution with having only mass M.

- The Reissner-Nordström solution (1918) [122,123]: It is static and spherically symmetric solution depending on mass M and electric charge Q.

- The Kerr solution (1963) [124]: It is stationary, axisymmetric, and depends on mass M and angular momentum J.

- The Kerr-Newman solution (1965) [125]: It is stationary, axisymmetric, and depends on all three parameters M, J, Q.

The Schwarzschild solution is derived by solving Einstein's equations in vacuum, $R_{\mu\nu} = 0$, and it describes a vacuum spacetime around a mass M.

Its metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.2.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric of a unit 2-sphere. The most general solution, corresponding to the final state of black hole equilibrium, is the 3-parameters Kerr-Newman family and its spacetime is described by the metric, in Boyer-Lindquist coordinates,

$$\begin{aligned} ds^2 &= \mathrm{g}_{\mu\nu} dx^\mu dx^\nu \\ &= -\frac{\Delta - a^2 \sin^2\theta}{\Sigma} dt^2 + 2\frac{\Delta - (r^2 + a^2)}{\Sigma} a \sin^2\theta dt d\varphi \\ &\quad + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta d\varphi^2, \end{aligned} \tag{2.2.2}$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \tag{2.2.3}$$

$$\Sigma = r^2 + a^2 \cos^2\theta, \tag{2.2.4}$$

$a \equiv J/M$ is the angular momentum per unit mass. The metric (2.2.2) becomes the Kerr metric for $Q = 0$, the Reissner-Nordström metric for $a = 0$, and the Schwarzschild metric (2.2.1) for $Q = 0$, $a = 0$. As $r \to \infty$ the metric (2.2.2) approaches the Minkowski metric (2.1.1). Thus, the Kerr-Newman metric (2.2.2) is asymptotically flat.

The event horizons of the Kerr-Newman black hole appear at those fixed values of r for which $\mathrm{g}^{rr} = 0$. Since $\mathrm{g}^{rr} = \Delta/\Sigma$, and $\Sigma \geq 0$, this occurs if $\Delta(r) = r^2 - 2Mr + a^2 + Q^2 = 0$. There are three possibilities: $M^2 > a^2 + Q^2$, $M^2 = a^2 + Q^2$, and $M^2 < a^2 + Q^2$. The last case produces a naked singularity, while $M^2 = a^2 + Q^2$ features the extremal case of the black hole. There are two radii at which $\Delta = 0$, given by

$$r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2}. \tag{2.2.5}$$

These two radii are null surfaces and turn out to be horizons; the outer horizon at r_+ is the event horizon and the inner horizon at r_- is the Cauchy horizon of the black hole.

On the event horizon $r = r_+$, the metric (2.2.2) takes the form

$$ds^2 = \frac{(r_+^2 + a^2)^2}{\Sigma_+} \sin^2\theta d\varphi^2 + \Sigma_+ d\theta^2, \tag{2.2.6}$$

because both t and r are constant on the horizon, i.e., $dt = dr = 0$. Here, $\Delta_+ = r_+^2 - 2Mr_+ + a^2 + Q^2 = 0$ and $\Sigma_+ = \Sigma(r_+)$. Hence, the area of the black hole A is given by

$$\begin{aligned} A &= \int \sqrt{g_{\theta\theta}(r_+) g_{\varphi\varphi}(r_+)} d\theta d\varphi \\ &= 4\pi(r_+^2 + a^2) = 4\pi(2Mr_+ - Q^2). \end{aligned} \tag{2.2.7}$$

Differentiating (2.2.7), we get

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ, \tag{2.2.8}$$

where κ, Ω_H, Φ_H are respectively the surface gravity, the angular velocity and the electrical potential on the horizon, defined by

$$\kappa = \frac{4\pi(r_+ - M)}{A}, \tag{2.2.9}$$

$$\Omega_H = \frac{4\pi a}{A}, \tag{2.2.10}$$

$$\Phi_H = \frac{4\pi r_+ Q}{A}. \tag{2.2.11}$$

The relation (2.2.8) is known as the energy conservation law in black hole physics.

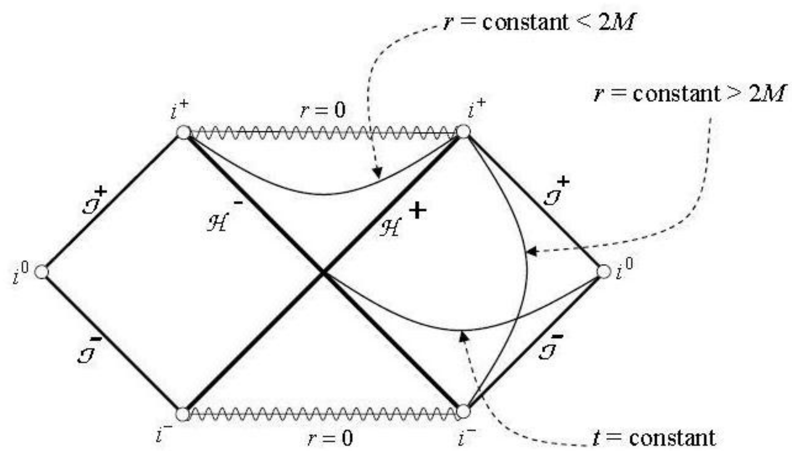

Figure 2.3: Penrose diagram for Schwarzschild spacetime.

2.3 Penrose Diagram

Penrose diagrams were employed first by the researchers Brandon Carter and Roger Penrose, acknowledging whom they are more properly (but less frequently) called Penrose-Carter diagrams (or Carter-Penrose diagrams). They are also called conformal diagrams, or simply spacetime diagrams.

Penrose diagram [126] is concerned with mapping an infinite spacetime onto a finite manifold with a boundary using a conformal transformation.

The conformal transformation rescales the metric

$$f^*\mathbf{ds} = \Lambda^{-2}\mathbf{ds}, \quad \Lambda(x) \neq 0, \tag{2.3.1}$$

preserving the causal structure (see Appendix-A). So, the sign of the norm $\mathbf{ds}(\mathbf{v}, \mathbf{v})$ for any given vector $\mathbf{v}$ is preserved under the conformal transformation. That is, space-like vectors are mapped to space-like vectors, light-like to light-like vectors, and time-like to time-like vectors. Using conformal transformations we can pull the infinities of spacetime back onto a finite and bounded region.

We consider the Schwarzschild black hole case whose metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.3.2}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2, \tag{2.3.3}$$

is a metric on a unit 2-sphere. The metric (2.3.2) has a curvature singularity at $r = 0$ and a coordinate singularity at $r = 2M$. The curvature singularity cannot be removed while the coordinate singularity can be removed by using appropriate coordinates. The global structure of the analytically extended Schwarzschild solution can be depicted in a simple way by using Penrose diagram of the r-t plane (Fig. 2.3). In this diagram null geodesics are at $\pm 45°$ to the vertical. Each point of the diagram is a 2-sphere of area $4\pi r^2$. Applying a conformal transformation, infinity has been brought to a finite distance. In the diagram infinity is represented by the two diagonal lines (really null surfaces) labelled $\mathcal{J}^+$ and $\mathcal{J}^-$, and the points i^+, i^-, and i^0. The two curvature singularities are at the lines $r = 0$. The two diagonal lines $r = 2M$ (really null surfaces) are the future and past event horizons which divide the solution up into regions from which one cannot escape to the future null infinity $\mathcal{J}^+$ and the fast null infinity $\mathcal{J}^-$. There is another infinity and asymptotically flat region on the left of the diagram.

The Penrose diagram can be drawn through some coordinate transformations (see, for example, [127]). Straight lines of constant time and space coordinates become hyperbolas, which appear to converge at points in the corners of the diagram. These points represent "conformal infinity" for space and time. The notations appearing in Fig. 2.3 are as followings:

$$i^0 = \begin{cases} t; \text{finite} \\ r \to \infty, \end{cases} \qquad i^\pm = \begin{cases} t \to \pm\infty \\ r; \text{finite}, \end{cases} \tag{2.3.4}$$

$$\mathcal{J}^- = \begin{cases} t \to -\infty \\ r \to +\infty, \end{cases} \qquad \mathcal{J}^+ = \begin{cases} t \to +\infty \\ r \to +\infty, \end{cases} \tag{2.3.5}$$

and the curvature singularities of the Schwarzschild metric at $r = 0$ are straight lines that stretch from timelike infinity in one asymptotic region to timelike infinity in the other. The i^+ and i^- (future and past infinity) are distinct from $r = 0$—there are plenty of timelike paths that do not hit the singulary. The heavy lines $\mathcal{H}^+$ and $\mathcal{H}^-$ stand for

$$\mathcal{H}^+ = \begin{cases} t \to +\infty \\ r = 2M, \end{cases} \qquad \mathcal{H}^- = \begin{cases} t \to -\infty \\ r = 2M, \end{cases} \tag{2.3.6}$$

and these are respectively the future event horizon and the past event horizon.

The geometry of the spacetime is realized through the knowledge of its causal structure, as defined by the light cones. We therefore consider radial null geodesics for which $\theta = 0 = \varphi$ and $ds^2 = 0$. From the metric (2.3.2) we then obtain

$$t = \pm r^* + \text{constant}, \tag{2.3.7}$$

where r^* is the tortoise coordinate defined by [128, 129]

$$r^* = r + 2M \ln \left| \frac{r - 2M}{2M} \right| . \tag{2.3.8}$$

The tortoise coordinate is only sensibly related to r for $r \geq 2M$. In terms of the tortoise coordinate the Schwarzschild metric (2.3.2) takes the form

$$ds^2 = \left(1 - \frac{2M}{r}\right)(-dt^2 + dr^{*2}) + r^2 d\Omega^2, \tag{2.3.9}$$

As r ranges from $2M$ to ∞, r^* ranges from $-\infty$ to ∞. We next consider coordinates that are naturally adapted to the null geodesics, that is, $d(t \pm r^*) = 0$ on radial null geodesics. We use the Eddington-Finkelstein coordinates defined by [130, 131]

$$v = t + r_*, \quad -\infty < v < \infty, \tag{2.3.10}$$

$$u = t - r_*, \quad -\infty < u < \infty, \tag{2.3.11}$$

where v =constant characterizes the infalling radial null geodesic and u =constant defines the outgoing radial null geodesic. The metric (2.3.9) then becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv du + r^2 d\Omega^2, \tag{2.3.12}$$

where r is related to v and u by

$$\frac{1}{2}(v - u) = r + 2M \ln \left(\frac{r}{2M} - 1\right) .$$

In these coordinates $r = 2M$ is "infinitely far away" (at either $v = -\infty$ or $u = +\infty$). In order to change to coordinates that pull these points into finite coordinate value, we introduce the Kruskal-Szekeres coordinates (U, V) [132, 133] defined (for $r > 2M$) by

$$\begin{cases} V = e^{\frac{v}{4M}} = \left(\frac{r}{2M} - 1\right)^{1/2} e^{(r+t)/4M}, \\ U = -e^{\frac{-u}{4M}} = -\left(\frac{r}{2M} - 1\right)^{1/2} e^{(r-t)/4M}. \end{cases} \tag{2.3.13}$$

In the (V, U, θ, φ) system the metric (2.3.12) is

$$ds^2 = -\frac{32M^3}{r} e^{-\frac{r}{2M}} dV dU + r^2 d\Omega^2, \tag{2.3.14}$$

which is completely nonsingular at $r = 2M$. Clearly, $r = 2M$ corresponds to $UV = 0$, i.e. either $U = 0$ or $V = 0$ and the singularity at $r = 0$ corresponds to $UV = 1$.

For $r < 2M$, these coordinates are given by

$$\begin{cases} V = e^{\frac{v}{4M}} = \left(\frac{r}{2M} - 1\right)^{1/2} e^{(r+t)/4M}, \\ U = e^{\frac{-u}{4M}} = \left(\frac{r}{2M} - 1\right)^{1/2} e^{(r-t)/4M}, \end{cases} \tag{2.3.15}$$

and the metric (2.3.12) is written

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} dV dU + r^2 d\Omega^2. \tag{2.3.16}$$

To bring infinities appeared in V or U into finite coordinate values (such as $\frac{\pi}{2}$ or $-\frac{\pi}{2}$) we make the following coordinate transformations

$$\begin{cases} \tilde{V} = \tan^{-1}\left(\frac{V}{4M\sqrt{2M}}\right), \\ \tilde{U} = \tan^{-1}\left(\frac{U}{4M\sqrt{2M}}\right). \end{cases} \tag{2.3.17}$$

Both $\tilde{V}$ and $\tilde{U}$ are null coordinates in the sense that $\partial/\partial\tilde{V}$ and $\partial/\partial\tilde{U}$ are null vectors. For somewhat more comfortable working in a system where one coordinate is timelike and the rest are spacelike, we use the coordinate transformations defined by

$$\begin{cases} \tilde{T} = \frac{1}{2}(\tilde{V} + \tilde{U}), \\ \tilde{R} = \frac{1}{2}(\tilde{V} - \tilde{U}). \end{cases} \tag{2.3.18}$$

We now draw i^+ and $\mathcal{J}^+$, as an illustration. The i^+ is expressed by

$$i^+ = \begin{cases} t \to +\infty \\ r; \text{finite.} \end{cases} \tag{2.3.19}$$

There are two cases: $r > 2M$ and $r < 2M$. For the first case, (2.3.10), (2.3.11) and (2.3.19) give

$$i^+ = \begin{cases} v \to +\infty \\ u \to +\infty. \end{cases} \tag{2.3.20}$$

When $r < 2M$, by inserting (2.3.19) into (2.3.15), we see that v and u agree with (2.3.20). From (2.2.13) and (2.3.20), V and U are given by

$$i^+ = \begin{cases} V \to +\infty \\ U \to 0, \end{cases} \tag{2.3.21}$$

and from (2.3.17) and (2.3.21), $\tilde{V}$ and $\tilde{U}$ become

$$i^+ = \begin{cases} \tilde{V} \to +\frac{\pi}{2} \\ \tilde{U} \to 0. \end{cases} \tag{2.3.22}$$

Using (2.3.22) in (2.3.18), $\tilde{T}$ and $\tilde{R}$ turn out to be

$$i^+ = \begin{cases} \tilde{T} \to +\frac{\pi}{4} \\ \tilde{R} \to +\frac{\pi}{4}. \end{cases} \tag{2.3.23}$$

The region i^+ as in (2.3.19) is thus represented by $(\tilde{R}, \tilde{T}) = (\frac{\pi}{4}, \frac{\pi}{4})$ in the Penrose diagram, when r takes finite values with $r \neq 2M$ (Fig. 2.4).

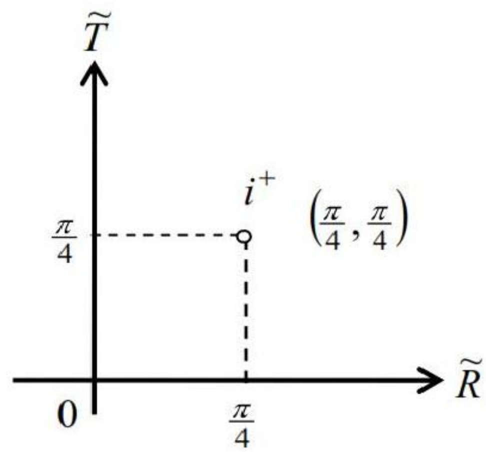

Figure 2.4: The region of i^+ in Penrose diagram.

Following the similar way as that of i^+ we draw $\mathcal{J}^+$ whose region is expressed by

$$\mathcal{J}^+ = \begin{cases} t \to +\infty \\ r \to +\infty \\ u; \text{finite.} \end{cases} \tag{2.3.24}$$

We have only to consider the case of $r > 2M$, since r is at infinity.

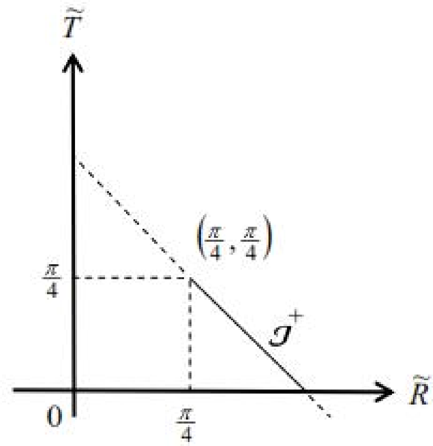

Figure 2.5: The region of $\mathcal{J}^+$ in Penrose diagram.

Inserting (2.2.24) into (2.3.10) and (2.3.11), v and u are found as follows:

$$\mathcal{J}^+ = \begin{cases} v \to +\infty \\ u; \text{finite.} \end{cases} \tag{2.3.25}$$

With (2.3.13) and (2.3.25), V and U become

$$\mathcal{J}^+ = \begin{cases} V \to +\infty \\ U; \text{finite,} \end{cases} \tag{2.3.26}$$

while using (2.3.26) in (2.3.17) give, for $\tilde{V}$ and $\tilde{U}$,

$$\mathcal{J}^+ = \begin{cases} \tilde{V} \to +\frac{\pi}{2} \\ \tilde{U}; \text{finite.} \end{cases} \tag{2.3.27}$$

Substituting (2.3.27) into (2.3.18) yield, for $\tilde{T}$ and $\tilde{R}$,

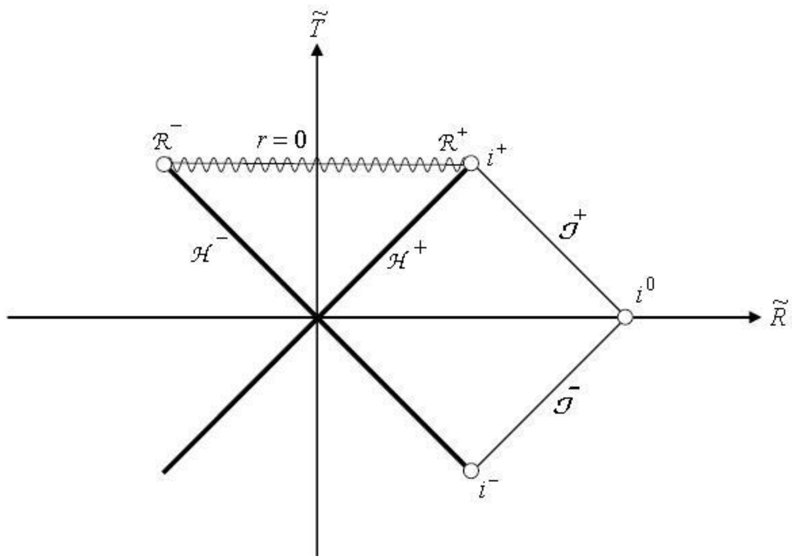

Figure 2.6: The Penrose diagram corresponding to Table 2.2.

$$\mathcal{J}^+ = \begin{cases} \tilde{T} = \frac{1}{2}(\frac{\pi}{2} + \tilde{U}) \\ \tilde{R} = \frac{1}{2}(\frac{\pi}{2} - \tilde{U}). \end{cases} \tag{2.3.28}$$

From (2.3.28) the region $\mathcal{J}^+$ as given in (2.3.24) is represented by the segment of a line

$$\tilde{T} = \frac{\pi}{2} - \tilde{R} \tag{2.3.29}$$

in the Penrose diagram (Fig. 2.5).

Following the similar procedure as above we can draw other points and segments (Tab. 2.2). In Table 2.2 below, the name of the variable is retained if it is finite and is not uniquely fixed. Figure 2.6 expresses the diagram drawn by using Tab. 2.2. The regions $\mathcal{R}^+$ and $\mathcal{R}^-$ respectively

Table 2.2: Coordinate values in each region

Region	(t,r)	(v, u)	(V,U)	$\left(\tilde{V}, \tilde{U}\right)$	$\left(\tilde{T}, \tilde{R}\right)$
i^+	$(+\infty, r)$	$(+\infty, +\infty)$	$(+\infty, 0)$	$\left(+\frac{\pi}{2}, 0\right)$	$\left(+\frac{\pi}{4}, +\frac{\pi}{4}\right)$
i^-	$(-\infty, r)$	$(-\infty, -\infty)$	$(0, -\infty)$	$\left(0, -\frac{\pi}{2}\right)$	$\left(-\frac{\pi}{4}, +\frac{\pi}{4}\right)$
i^0	$(t, +\infty)$	$(+\infty, -\infty)$	$(+\infty, -\infty)$	$\left(+\frac{\pi}{2}, -\frac{\pi}{2}\right)$	$\left(0, +\frac{\pi}{2}\right)$
$\mathcal{J}^+$	$(+\infty, +\infty)$	$(+\infty, u)$	$(+\infty, U)$	$\left(\frac{\pi}{2}, \tilde{U}\right)$	$\tilde{T} = \frac{\pi}{2} - \tilde{R}$
$\mathcal{J}^-$	$(-\infty, +\infty)$	$(v, -\infty)$	$(V, -\infty)$	$\left(\tilde{V}, -\frac{\pi}{2}\right)$	$\tilde{T} = \tilde{R} - \frac{\pi}{2}$
$\mathcal{H}^+$	$(+\infty, 2M)$	$(v, +\infty)$	$(V, 0)$	$\left(\tilde{V}, 0\right)$	$\tilde{T} = \tilde{R}$
$\mathcal{H}^-$	$(-\infty, 2M)$	$(-\infty, u)$	$(0, U)$	$\left(0, \tilde{U}\right)$	$\tilde{T} = -\tilde{R}$
$\mathcal{R}^+$	$(+\infty, 0)$	$(+\infty, +\infty)$	$(+\infty, 0)$	$\left(+\frac{\pi}{2}, 0\right)$	$\left(+\frac{\pi}{4}, +\frac{\pi}{4}\right)$
$\mathcal{R}^-$	$(-\infty, 0)$	$(-\infty, -\infty)$	$(0, +\infty)$	$\left(0, +\frac{\pi}{2}\right)$	$\left(+\frac{\pi}{4}, -\frac{\pi}{4}\right)$

represent the following regions:

$$\mathcal{R}^+ = \begin{cases} t \to +\infty \\ r = 0, \end{cases} \qquad \mathcal{R}^- = \begin{cases} t \to -\infty \\ r = 0, \end{cases} \tag{2.3.30}$$

and the $r = 0$ line combines $\mathcal{R}^+$ and $\mathcal{R}^-$. The region for $r = 0$ has finite t, but we cannot uniquely determine the point in the region $r = 0$. That is, we do not know how to draw an exact line of the region $r = 0$. There are some missing parts in Fig. 2.6 in comparison with Fig. 2.3. We can draw them by defining the other universe where time proceeds reversely by analogy with our universe. However, we skip them as they are not important in the body of the present thesis.

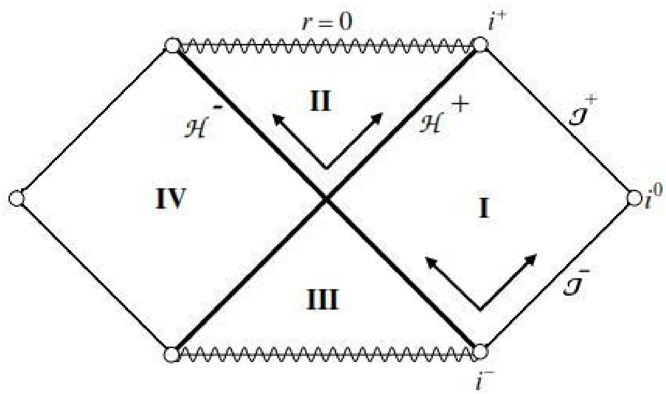

Figure 2.7: The Penrose diagram for the Schwarzschild solution.

Infinite time or radial coordinates are represented as points or lines in the Penrose diagram. The lines of 45^o to the vertical represent null geodesics. Every point in the diagram describes a 2-dimensional sphere of area $4\pi r^2$. That means, angular coordinates (θ, φ) as in (2.3.2) are attached to each point of the diagram.

There are four regions in the Penrose diagram, divided by the two diagonal lines $\mathcal{H}^+$ and $\mathcal{H}^-$ (Fig. 2.7). They respectively represent our universe (region-I), a black hole (region-II), the other universe (region-IV) that time reversely proceeds by comparison with our universe, and a white hole (region-III) which is the time reversal of a black hole and ejects matter from the horizon. The null geodesics in the region-I can arrive at $\mathcal{J}^+$ or the black hole through the horizon $\mathcal{H}^+$. However, the null geodesics in the region-II (inside the black hole) cannot arrive at our universe through the horizon $\mathcal{H}^+$.

Let us consider that a black hole is formed by the gravitational collapse

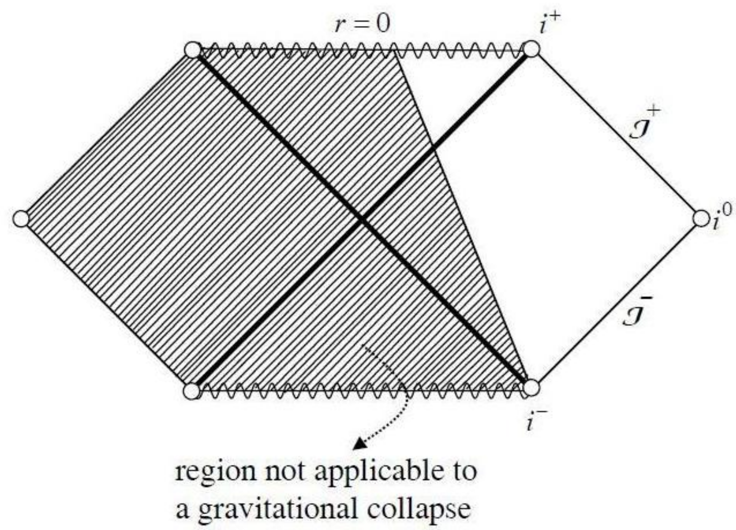

Figure 2.8: The development of the collapsing object in the Penrose diagram.

of a star with a heavy mass [2]. Its metric is that of the Schwarzschild solution only in the region outside the collapsing matter and only in the asymptotic future. We consider Hawking's exposition and for simplicity, we assume that the gravitational collapse is spherically symmetric. The collapse of this type of objects starts at the point i^- and its passing is later than light (Fig. 2.8), because the collapsing object has a mass. For exactly spherical collapse, the metric is exactly the Schwarzchild metric everywhere outside the surface of the collapsing object which is represented by a timelike geodesic in the Penrose diagram (Fig. 2.8). Inside the object the metric is completely different. The past event horizon, the past $r = 0$ singularity and the other asymptotically flat region do not exist. These are replaced by a timelike curve representing the origin of polar coordinates. Figure 2.9 depicts the appropriate Penrose diagram. The

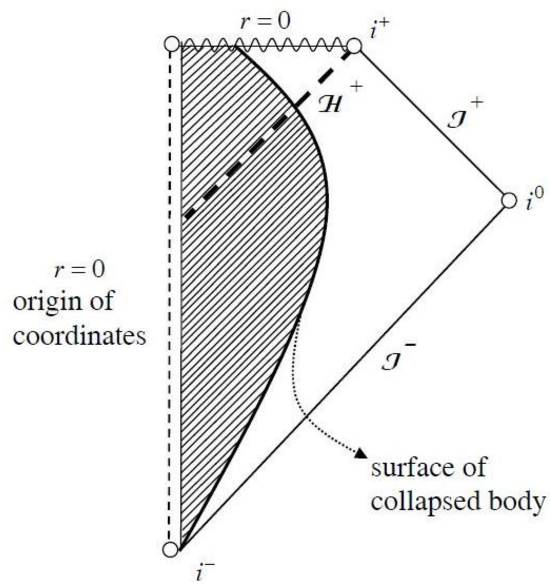

Figure 2.9: The Penrose diagram of a spherically symmetric collapsing body producing a black hole.

origin of coordinate is the vertical dotted line since the metric inside the object might be nonsingular at the origin.

2.4 Energy Extraction from Kerr Black Holes

Even though a black hole is, by definition, a region of forbidden escape for any body and light rays, situations are possible in which energy can be extracted via certain physical processes. This energy is released from the field associated with the black hole and surrounding it. However, energy extraction is possible if the black hole rotates or is charged. The process of energy extraction from a rotating black hole was first proposed by Roger Penrose [120], which is called the "Penrose process" (or "Penrose mechanism") and can be explained in the classical theory. For recent works on Penrose process we would like to mention the Refs. [134–139]. The radiance associated with the Penrose process is called the black hole superradiance.

2.4.1 Penrose Process

The Kerr black hole spacetime is stationary but not static and the asymptotic time-translation Killing vector $K = \partial_t$ is not null along null surfaces at $r_\pm$. So, the horizons at $r_\pm$ are not Killing horizons. The norm of the timelike Killing field,

$$K^\mu K_\mu = \mathrm{g}_{tt} = \frac{a^2 \sin^2\theta - \Delta}{\Sigma}, \tag{2.3.1}$$

does not vanish at the event horizon. In fact, $K^\mu K_\mu = a^2 \sin^2\theta/\Sigma \geq 0$ at r_+. Hence, the Killing vector is spacelike at the event horizon, except at the poles at $\theta = 0, \pi$ where it is null. The locus of points where $K^\mu K_\mu = 0$ defines the stationary limit surface and is described by $(r - M)^2 = M^2 - a^2 \cos^2\theta$. The part of the stationary limit surface, which lies outside the black hole, is given by $r = M + \sqrt{M^2 - a^2\cos^2\theta}$. The region

$$r_+ < r < M + \sqrt{M^2 - a^2 \cos^2\theta} \tag{2.4.2}$$

is called the ergosphere (Fig. 2.10) in which the asymptotic time translation Killing field $K^\mu = (\partial/\partial t)^\mu$ becomes spacelike. All observers in the ergosphere must rotate in the direction of rotation of the black hole (the

φ-direction); however, they can still move toward or away from the event horizon and have no trouble existing the ergosphere. The ergosphere is a place where interesting things can happen even before an observer crosses the event horizon.

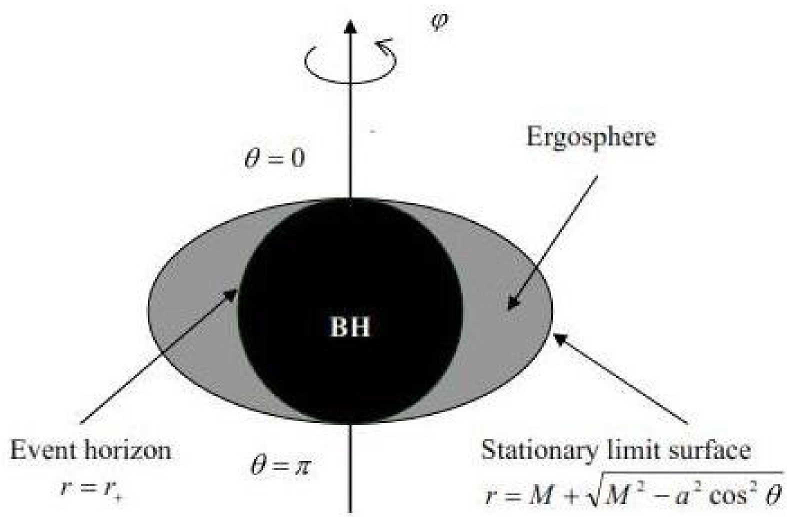

Figure 2.10: Ergosphere: the region between the event horizon and the stationary limit horizon (side view).

We consider the conserved quantities of the Kerr spacetime associated with the Killing vectors $K = \partial_t$ and $R = \partial_\varphi$. The actual energy and angular momentum of the particle with the four-momentum $p^\mu = m\frac{dx^\mu}{d\tau}$ are respectively

$$\begin{aligned} E =& -K_\mu p^\mu \\ =& m\left(1 - \frac{2Mr}{\Sigma}\right)\frac{dt}{d\tau} + \frac{2mMar}{\Sigma}\sin^2\theta\frac{d\varphi}{d\tau}, \end{aligned} \tag{2.4.3}$$

and

$$
\begin{aligned}
L &= R_\mu p^\mu \\
&= -\frac{2mMar}{\Sigma}\sin^2\theta\frac{dt}{d\tau} \\
&\quad +\frac{m(r^2+a^2)^2 - m\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\frac{d\varphi}{d\tau},
\end{aligned}
\tag{2.4.4}
$$

where m is the rest mass of the particle and $Q = 0$ in Δ, Σ. Because both K^μ and p^μ are timelike at infinty, their inner product is negative. But we want the energy to be positive, so there is the minus sign in the definition of E. However, K^μ becomes spacelike inside the ergosphere. Hence, we can imagine particles in the ergosphere for which

$$E = -K_\mu p^\mu < 0. \tag{2.4.5}$$

A particle inside the ergosphere with negative energy must either remain in the ergosphere, or be accelerated until its energy is positive if it is to escape. This realization leads to a way to extract energy from a rotating black hole. This method is known as the "Penrose process". The idea is simple. Starting from outside the ergosphere, a particle with momentum $p^{(0)\mu}$ and energy $E^{(0)} = -K_\mu p^{(0)\mu} > 0$ enters the ergosphere. The particle inside the ergosphere will not remain stationary and will decay into a pair of particles with momenta $p^{(1)\mu}$ and $p^{(2)\mu}$ (Fig. 2.11):

$$p^{(0)\mu} = p^{(1)\mu} + p^{(2)\mu}. \tag{2.4.6}$$

When contracted with the Killing vector K_μ, the result gives

$$E^{(0)} = E^{(1)} + E^{(2)}. \tag{2.4.7}$$

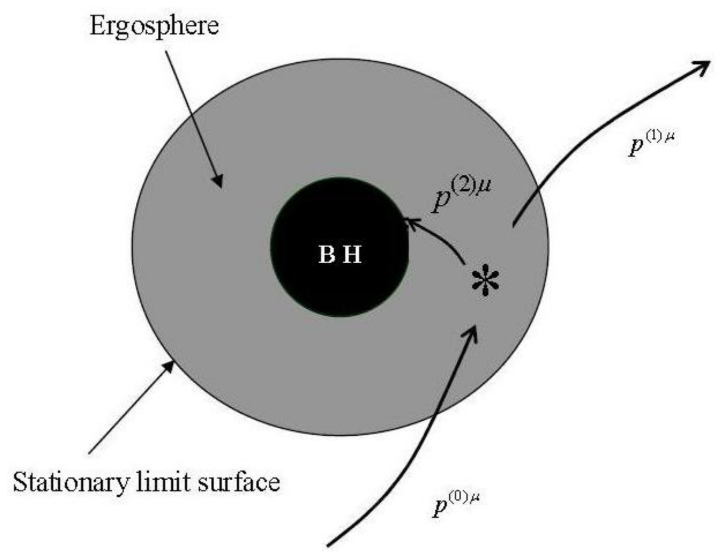

Figure 2.11: The Penrose process of energy extraction from Kerr black hole (top view).

We can consider that $E^{(2)} < 0$ as per (2.4.5). The particle $p^{(2)\mu}$ with negative energy $E^{(2)}$ falls into the horizon, while the other particle $p^{(1)\mu}$ escapes to infinity with a larger energy than that of the original infalling particle:

$$E^{(1)} > E^{(0)}. \tag{2.4.8}$$

Thus, the particle has emerged with more energy than it entered with. That is, energy can be classically extracted from a black hole. This is the Penrose process. In fact, the Penrose process extracts energy from the rotating black hole by decreasing its angular momentum; the negative energy particle carries a negative angular momentum, i.e., an angular momentum opposite to that of the black hole. By losing the total angular momentum the Kerr black hole becomes a Schwarzschild black hole. Since the ergosphere does not exist in the Schwarzschild black hole, there can not occur further energy extraction.

We now look into this process more precisely to find the limit on energy extraction from a Kerr black hole. For the Kerr spacetime, the event horizon is a Killing horizon for the Killing vector χ^μ, defined by

$$\chi^\mu = K^\mu + \Omega_H R^\mu, \tag{2.4.9}$$

where Ω_H is the angular velocity of the horizon defined in (2.2.10). It is evident that χ^μ is null at the event horizon for $K = \partial_t$ and $R = \partial_\varphi$. Since χ^μ is future directed null on the horizon and the particle with momentum $p^{(2)\mu}$ crosses the event horizon moving forward in time (i.e., $p^{(2)\mu}$ is future directed null), we have

$$0 > p^{(2)\mu}\chi_\mu = p^{(2)\mu}(K_\mu + \Omega_H R_\mu) = -E^{(2)} + \Omega_H L^{(2)} \tag{2.4.10}$$

or equivalently,

$$L^{(2)} < \frac{E^{(2)}}{\Omega_H}. \tag{2.4.11}$$

This equation depicts that a negative-energy particle entering the black hole carries negative angular momentum, i.e., it moves against the the hole's rotation. When the black hole swallows a particle, its parameters are modified by $\delta M = E^{(2)}$, $\delta J = L^{(2)}$. Then, from (2.4.11), the change in black hole parameters is governed by

$$\delta J < \frac{\delta M}{\Omega_H}. \tag{2.4.12}$$

If the black hole swallows more and more particles with future directed null $p^{(2)\mu}$, there exists the "ideal" process in which $\delta J = \frac{\delta M}{\Omega_H}$.

Although the Penrose process can extract energy from the black hole (thereby can decrease M), it cannot violate the area theorem: The area of the event horizon is nondecreasing [140]. The irreducible mass, $M_{\rm irr}$, of the black hole is defined in term of its area, $A = 4\pi(r_+^2 + a^2)$, by [141]

$$M_{\rm irr}^2 = \frac{A}{16\pi}, \tag{2.4.13}$$

which gives

$$M_{\rm irr}^2 = \frac{1}{2}\left(M^2 + \sqrt{M^4 - J^2}\right). \tag{2.4.14}$$

Inverting equation (2.4.14), we get

$$M^2 = M_{\rm irr}^2 + \frac{1}{4}\frac{J^2}{M_{\rm irr}^2} \geq M_{\rm irr}^2. \tag{2.4.15}$$

Hence, the mass of a black hole cannot be reduced below $M_{\rm irr}$ via the Penrose process. Thus, we obtain $\delta A \geq 0$ (Area Theorem).

2.4.2 Superradiance

The Penrose process demonstrates that the maximum amount of energy permitted by the area theorem can be extracted, in principle, from a rotating black hole. However, it is not a practical energy extraction method [142, 143]. It is interesting that there is a wave analog of the Penrose process, called as superradiant scattering or superradiance [144–147]. It allows energy to be extracted from a black hole in a relatively simple manner. Consider that a scalar, electromagnetic, or gravitational wave is incident upon a black hole. Then a part of the wave (the "transmitted wave") will be absorbed by the black hole and a part of the wave (the "reflected wave") will escape back to infinity. Usually the transmitted wave will carry positive energy into the black hole, and the reflected wave will have less energy than the incident wave. But for a wave of the form

$$\psi = \mathrm{Re}[\psi_0(r,\theta)e^{-i\omega t}e^{im\varphi}] \tag{2.4.16}$$

with

$$0 < \omega < m\Omega_H, \tag{2.4.17}$$

the transmitted wave will carry negative energy into the black hole. This is similar to the negative energy fragment in the Penrose process for particles. As a result, the reflected wave will return to infinity with greater amplitude and energy than the incident wave. This can be showed most easily for the case of scalar waves.

Consider the energy-momentum tensor

$$T_{\mu\nu} = \nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}\mathrm{g}_{\mu\nu}(\nabla_\lambda\psi\nabla^\lambda\psi + \mu_0^2\psi^2)$$

of a Klein-Gordon scalar field ψ: $\nabla^\mu\nabla_\mu\psi - \mu_0^2\psi = 0$, and define the energy current by

$$J_\mu = -T_{\mu\nu}K^\nu. \tag{2.4.18}$$

Since $T_{\mu\nu}$ is a symmetric tensor and is covariantly constant (i.e., a conserved quantity), we find that J_μ is also conserved:

$$\begin{aligned} \nabla^\mu J_\mu &= -(\nabla^\mu T_{\mu\nu})K^\nu - T_{\mu\nu}(\nabla^\mu K^\nu) \\ &= -\frac{1}{2}T_{\mu\nu}(\nabla^\mu K^\nu + \nabla^\nu K^\mu) = 0, \end{aligned} \tag{2.4.19}$$

where Killing fields satisfy the Killing equation

$$\nabla^\mu K^\nu + \nabla^\nu K^\mu = 0. \tag{2.4.20}$$

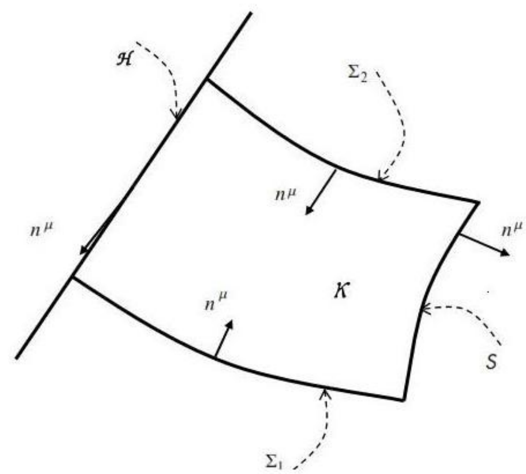

Figure 2.12: A spacetime diagram showing the region $\mathcal{K}$.

In order to know the presence or absence of the superradiance, we need to integrate $\nabla^\mu J_\mu = 0$ [equation (2.4.19)] over the region $\mathcal{K}$ of spacetime displayed in intuitive Fig. 2.12 with respect to Gauss's theorem. The precise figure for the region $\mathcal{K}$ of spacetime is shown in Figure 2.13 by Penrose diagram. The spacelike hypersurface Σ_2 is a "time translate" of Σ_1 by δt. The timelike hypersurface $\mathcal{H}$ is the event horizon at $r = r_+$ and the timelike hypersurface $\mathcal{S}$ represents a "large sphere" at infinity. By using Gauss's theorem, we obtain

$$\begin{aligned} 0 = \int_{\mathcal{K}} \sqrt{-\mathrm{g}} d^4x (\nabla_\mu J^\mu) &= \int_{\partial\mathcal{K}} d\Sigma_\mu J^\mu \\ = \int_{\Sigma_1(t)} n_\mu J^\mu d\Sigma &+ \int_{\Sigma_2(t+\partial t)} n_\mu J^\mu d\Sigma \\ + \int_{\mathcal{H}(r_+)} n_\mu J^\mu d\Sigma &+ \int_{\mathcal{S}(\infty)} n_\mu J^\mu d\Sigma, \end{aligned} \tag{2.4.21}$$

where $\partial\mathcal{K}$ is the boundary of the region $\mathcal{K}$ and $d\Sigma_\mu = n_\mu d\Sigma$ is a 3-dimensional suitable area element. The unit vector n^μ is outwardly normal to the region $\mathcal{K}$. For a wave with time dependence $e^{-i\omega t}$ the integrals over Σ_1 and Σ_2 cancel by time translation symmetry. The integral of $J_\mu n^\mu$ over $\mathcal{S}$ represents the net energy flow (i.e., the outgoing minus incoming enrgy) out of $\mathcal{K}$ to infinity during the time δt. On the other hand, the integral of $J_\mu n^\mu$ over $\mathcal{H}$ represents the net energy flow into into the black hole. Thus, we get from (2.4.21),

$$\int_{\mathcal{S}(\infty)} n_\mu J^\mu d\Sigma = -\int_{\mathcal{H}(r_+)} n_\mu J^\mu d\Sigma. \tag{2.4.22}$$

For the positive (negative) value of the quantity on the right-hand side in (2.4.22), the outgoing energy flow is larger (smaller) than the incident one and the superradiance is present (absent).

In order to calculate the right-hand side in (2.4.22), we write the vector normal to the event horizon in terms of the Killing field χ^μ (defined in (2.4.9)) as $n^\mu = -\chi^\mu$. The appearance of this relation might be surprising, since the Killing field is tangent to the horizon. In fact, this result is known on the concept that the vector which is normal to the horizon is tangent to itself on the horizon (the null hypersurface). We present a proof in Appendix-B. Hence, we obtain

$$\begin{aligned}\int_{\mathcal{H}(r_+)} n_\mu J^\mu d\Sigma &= -\int_{\mathcal{H}(r_+)} \chi_\mu (-T^\mu{}_\nu K^\nu) d\Sigma \\ &= \int_{\mathcal{H}(r_+)} \chi^\mu T_{\mu\nu} K^\nu d\Sigma,\end{aligned} \tag{2.4.23}$$

where we have used the definition (2.4.18). For a massless scalar field

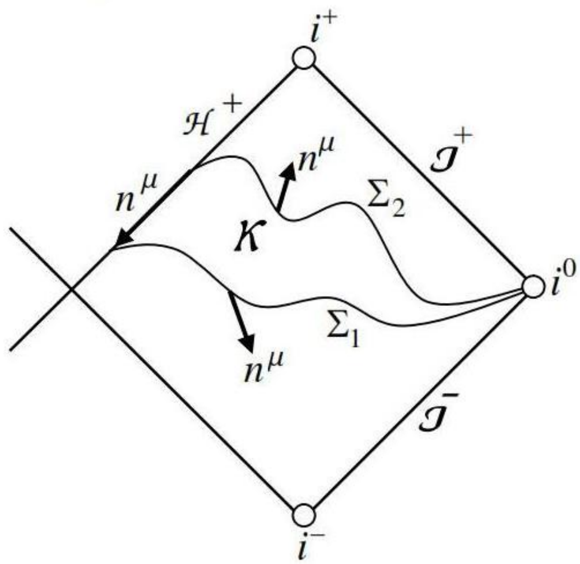

Figure 2.13: The region $\mathcal{K}$ of spacetime in Penrose diagram.

without interactions, the action in curved spacetime is given by

$$S = \int \sqrt{-\mathrm{g}} d^4x[\mathcal{L}] = \int \sqrt{-\mathrm{g}} d^4x \left[\frac{1}{2}\nabla_\mu\psi\nabla^\mu\psi\right], \tag{2.4.24}$$

where $\mathcal{L}$ is the Lagrangian density. The energy-momentum tensor of this scalar field is given by

$$\begin{aligned} T_{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\nabla^\mu\psi)}\nabla_\nu\psi - \mathrm{g}_{\mu\nu}\mathcal{L} \\ &= \frac{1}{2}(\nabla_\mu\psi)(\nabla_\mu\psi) - \frac{1}{2}\mathrm{g}_{\mu\nu}(\nabla_\lambda\psi)(\nabla^\lambda\psi), \end{aligned} \tag{2.4.25}$$

With (2.4.25) we obtain from (2.4.23)

$$\begin{aligned}
&\int_{\mathcal{H}(r_+)} n_\mu J^\mu d\Sigma \\
&\qquad = \int_{\mathcal{H}(r_+)} d\Sigma \left[\frac{1}{2}(\chi^\mu \nabla_\mu \psi)\,(K^\mu \nabla_\mu \psi) \right. \\
&\qquad\quad \left. -\frac{1}{2}\chi^\mu K_\mu (\nabla_\lambda \psi)(\nabla^\lambda \psi) \right] \\
&\qquad = \int_{\mathcal{H}(r_+)} d\Sigma \left[\frac{1}{2}(\chi^\mu \nabla_\mu \psi)(K^\mu \nabla_\mu \psi) \right], \qquad (2.4.26)
\end{aligned}$$

since $\chi^\mu K_\mu = 0$ on the horizon. Asymptotically, we have

$$\chi^\mu \nabla_\mu = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \varphi},$$

$$K^\mu \nabla_\mu = \frac{\partial}{\partial t}. \qquad (2.4.27)$$

Using (2.4.16) and (2.4.27), we obtain, asymptotically,

$$\frac{1}{2}(\chi^\mu \nabla_\mu \psi)(K^\mu \nabla_\mu \psi) = \frac{1}{2}\omega(\omega - m\Omega_H)\widetilde{\psi}^2, \qquad (2.4.28)$$

where $\widetilde{\psi} = \psi_0(r,\theta)\sin(\omega t - m\varphi)$. Inserting (2.4.28) into (2.4.26), we find

$$\int_{\mathcal{H}(r_+)} n_\mu J^\mu d\Sigma = \frac{1}{2}\omega(\omega - m\Omega_H)\int_{\mathcal{H}(r_+)} d\Sigma \widetilde{\psi}^2. \qquad (2.4.29)$$

On the event horizon, $d\Sigma = dAdv$ where A is the surface area of the event horizon and the retarded time v is an affine parameter on the horizon. Then, since the value of the integration in (2.4.29) with respect to v generally diverges, we need to evaluate the energy flow per unit time. The time averaged flux on the event horizon is given by

$$\begin{aligned} \int_{\mathcal{S}(\infty)} n_\mu J^\mu dA &= -\int_{\mathcal{H}(r_+)} n_\mu J^\mu dA \\ &= -\frac{1}{2}\omega(\omega - m\Omega_H)\left|\widetilde{\psi}_0\right|^2, \end{aligned} \tag{2.4.30}$$

where

$$|\widetilde{\psi}_0|^2 = \int_{\mathcal{H}(r_+)} dA \widetilde{\psi}^2$$

and the integral value is positive for ω in the range $0 < \omega < m\Omega_H$, given in (2.4.17). Thus the outgoing energy flow is larger than the incident one and the superradiance is present for the scalar field. One can apply the above method for fermion fields. However, in that case the right-hand side of (2.4.22) always vanishes and as a result, the superradiance is found absent in the fermionic case [148, 149].

2.5 Black Hole Physics and Thermodynamics

Black hole physics studies the properties of black holes in the context of generalized thermodynamics. The analogy between the laws of black hole physics and the laws of thermodynamics brought forth the laws of black hole thermodynamics. It is said that J.M. Greif worked as a pioneer with making use of thermodynamic methods in black hole physics. He studied the possibility of defining the entropy of a black hole, but failed to make a concrete proposal [150]. This was because of lacking many of the recent results in black hole physics. Subsequently, Bekenstein, Bardeen, Carter and Hawking, and others examined properties of black holes and clarified analogies between black hole physics and thermodynamics [1, 151].

Consider that two Schwarzschild black holes with masses M_1 and M_2 merge and form a black hole with a mass $M = M_1 + M_2$. If their areas are respectively $A_1 = 16\pi M_1^2$, $A_2 = 16\pi M_2^2$, and $A_{1+2} = 16\pi(M_1 + M_2)^2$, we obtain an inequality for black hole areas given by $A_1 + A_2 \leq A_{1+2}$. In fact, the area theorem of black hole [140] states that in any physically allowed process, the total area of all black holes in the universe cannot decrease,

$$\delta A \geq 0. \tag{2.5.1}$$

This law carries a strong resemblance to the second law of thermodynamics, according to which the total entropy S of all matter in the universe cannot decrease in any physically allowed process, i.e.

$$\delta S \geq 0. \tag{2.5.2}$$

This similarity might seem to be of a very superficial nature. The area theorem is a mathematically rigorous consequence of general relativity. However, the second law of thermodynamics is believed not to be a rigorous consequence of the laws of nature. It is rather a law which holds with irresistible possibility for systems with a large number of degrees of freedom. Nevertheless, this formal analog for black holes of the second law of thermodynamics extends to the other laws of thermodynamics as well and the relationship of the laws of black hole physics with the laws of thermodynamics is of a fundamental nature [152].

In thermodynamics, an increasing of entropy develops a part of energy that is no longer converted into work. The same thing happens in black hole physics as well. We have shown in section 2.4 that a part of energy can be extracted by the Penrose process from a rotating black hole such as a Kerr black hole. But all of its energy cannot be extracted. The Kerr black hole gradually decreases the angular momentum by the Penrose process and becomes a Schwarzschild black hole. By the Hawking's area theorem, the mass of this black hole is then larger than a mass of a Schwarzschild black hole obtained by setting $a = 0$ in the original Kerr black hole. This mass is called an irreducible mass of the black hole and in the case of a Kerr-Newman black hole, it is given by

$$M_{\mathbf{ir}} = \sqrt{\frac{A}{16\pi}} = \frac{1}{\sqrt{2}}\left(M^2 + \sqrt{M^4 - J^2 - M^2Q^2} - \frac{Q^2}{2}\right)^{1/2}.$$

The relation (2.4.12) leads to $\delta M_{\mathbf{ir}} > 0$, i.e. the irreducible mass can never be reduced by Penrose process. The $M_{\mathbf{ir}}$ is regarded as an inactive energy because it cannot be converted to work. The black hole area A increases as the irreducible mass $M_{\mathbf{ir}}$ increases. Thus the increase of A corresponds to a degradation of the black hole energy in the thermodynamic sense.

The energy conservation law in black hole physics is given by (2.2.8):

$$dM = \frac{\kappa}{8\pi}dA + \Omega_H dJ + \Phi_H dQ. \tag{2.5.3}$$

Equations like (2.5.3) first induced people to think about a correspondence between black holes and thermodynamics. The first law of thermodynamics is

$$d\mathcal{E} = TdS - pdV, \tag{2.5.4}$$

where $\mathcal{E}$ is the energy of the system, T is the temperature, S is the entropy, p is the pressure, and V is the volume, so the pdV term represents work done to the system. It is natural to think of the terms $\Omega_H dJ$ and

$\Phi_H dQ$ in (2.5.3) as work done on the black hole by rotation and electromagnetism. We next compare the remaining first term in each relation, i.e., $\frac{\kappa}{8\pi}dA$ and TdS. Then the analogy begins to get shape if we take into consideration of identifying the thermodynamic quantities energy, entropy, and temperature with the black-hole mass, area, and surface gravity:

$$\left.\begin{aligned} \mathcal{E} &\longleftrightarrow M, \\ S &\longleftrightarrow A/4, \\ T &\longleftrightarrow \kappa/2\pi, \end{aligned}\right\} \tag{2.5.5}$$

using units in which $G = \hbar = c = k_B = 1$. The above correspondence is essentially perfect in the context of classical general relativity, with each law of thermodynamics corresponding to a law of black hole mechanics.

We recall properties of both the surface gravity of a black hole and temperature. By definition, a surface gravity of a black hole is the strength of the gravitational field on the event horizon. Consider that a system in thermal equilibrium have settled to a stationary state, analogous to a stationary black hole. According to the zeroth law of thermodynamics, the temperature is constant throughout the system in thermal equilibrium. The analogous statement for black holes is that stationary black holes have constant surface gravity on the event horizon, as found in (2.2.9). This is true, at least under the same reasonable assumptions under which the event horizon is a Killing horizon. Thus we have found that the first law (2.5.4) is equivalent to (2.5.3). The second law, which states that entropy never decreases, is simply the statement that the area of the black hole horizon never decreases.

Finally, the third law states that it is impossible to achieve absolute zero temperature ($T = 0$) in any physical process, or that the entropy

Table 2.5: Black hole and Thermodynamics

Law	Thermodynamics	Black hole physics
Zeroth	T is constant throughout body in thermal equilibrium	κ is constant over horizon of stationary black hole
First	$d\mathcal{E} = TdS - pdV$	$dM = \frac{\kappa}{8\pi}dA + \Omega_H dJ + \Phi_H dQ$
Second	$\delta S \geq 0$ in any process	$dA \geq 0$ in any process
Third	Impossible to achieve $T = 0$ by a physical process	Impossible to achieve $\kappa = 0$ by a physical process

must go to zero ($S \to 0$) as the temperature goes to zero ($T \to 0$). This is also called Nernst's theorem. This doesn't quite work for black holes; e.g., $\kappa = 0$ corresponds to extremal black holes, which don't necessarily have a vanishing area. As $\kappa \to 0$, the area A may remain finite. Actually, the thermodynamic third law doesn't work either, in the sense that there are ordinary physical systems that violate it. Even though the third law applies to some situations, it is not genuinely fundamental. Table 2.5 displays the close mathematical correspondence between the laws of black hole physics and the ordinary laws of thermodynamics.

The correspondence in (2.5.5) is a little refutable in the sense that by equating TdS with $\kappa dA/8\pi$ we do not know how to separately normalize S/A or T/κ. Moreover, $\mathcal{E}$ and M are not merely analogs in the formulas but present the same physical quantity: total energy. Because a black hole is a perfect absorber but doesn't emit anything, the thermodynamic temperature of a black hole in the classical general relativity is absolute zero. It thus appears that κ could not physically represent a temperature. Nevertheless, in 1973 Bekenstein [1] first claimed that $TdS = \frac{\kappa}{8\pi}dA$, so that the temperature of the black hole was proportional to the surface gravity and that the entropy was proportional to the area. Hawking [3] later showed this and calculated the temperature of a black hole to be explicitly

$T = \kappa/2\pi$. This describes that the relationship between laws of black hole physics and thermodynamics may be more than an analogy. The black hole laws of Table 2.5 may be just the ordinary laws of thermodynamics applied to a black hole. The relation $T = \kappa/2\pi$ also leads to interpret $A/4$ as an actual entropy of the black hole. Then we get a generalized second law, proposed by Bekenstein [1, 153], that the combined entropy of matter and black holes never decreases:

$$\delta\left(S + \frac{A}{4}\right) \geq 0. \tag{2.5.6}$$

However, we usually like to relate the entropy of a system with the logarithm of the number of accessible quantum states. So, some tension occurs between this concept and the no-hair theorem, according to which there is in fact only one possible state for a black hole of fixed charge, mass, and spin. Probably this behavior seems to be an indication of a profound feature of the interaction between quantum mechanics and gravitation.

2.6 Black Hole Entropy

In this section, we briefly describe Bekenstein's derivation of black hole entropy. We explain the entropy of a particle with the least information in information theory. When a particle falls into a black hole, the entropy of the hole is increased. We calculate the black hole entropy.

2.6.1 Entropy in Information Theory

Entropy is the degradation of the matter and energy in the universe to an ultimate state of inert uniformity. In terms of the Boltzmann's formula the entropy is defined by

$$S = k_B \ln W, \tag{2.6.1}$$

where W (stands for "Wahrscheinlichkeit"–the German word for probability) is the number of microstates corresponding to a given macrostate and k_B is the Boltzmann's constant.

Following information theory [154–158] and Brillouin's classic work linking it to thermodynamics [159], Bekenstein proposed for S an information theoretic implication. The thermal entropy of an ideal gas certainly decreases due to the isothermal compression. After the isothermal compression, one has better information about the position of the molecules, since they become more localized. As a matter of fact, the increase in information is formalized by the relation

$$\Delta I = -\Delta S, \tag{2.6.2}$$

where ΔS is the decrease in entropy. This equation is the basis for Brillouin's identification of information with negative entropy [159]. Thus the entropy measures lack of information about the actual internal configuration of the system.

Let p_n be the probability of occurrence of an internal configuration labelled by the positive integer n. Then the entropy associated with the system is given by Boltzmann's formula (with the Boltzmann constant $k_B = 1$)

$$S = -\sum_n p_n \ln p_n. \tag{2.6.3}$$

Evidently, this entropy is dimensionless. It means that we choose to measure temperature in units of energy. Then Boltzmann's constant is dimensionless. Availability of a new information about the system imposes some constraints on the probabilities p_n. For example, the probabilities of a dice are respectively $\frac{1}{6}$ from 1 to 6. Then the entropy is $\ln 6$. If we get new information that "There are even numbers (or even numbers are given)," the probability of getting odd numbers becomes zero, i.e., $p_1 = p_3 = p_5 = 0$. The probability of getting even numbers is $\ln \frac{1}{3}$ and so the entropy is $\ln 3$. Thus, the entropy locally decreases as new information is available, as is depicted by Brillouin's identification (2.6.2).

Let the conventional unit of information be the "bit," which we may define as the information available when the answer to a yes-or-no question is precisely known (zero entropy). The unit is, of course, dimensionless. Corresponding to (2.6.2), a bit is also numerically equal to the maximum entropy that can be associated with a yes-or-no question, i.e., the entropy when no information whatever is available about the answer. The entropy in the yes-or-no question is written, from (2.6.3), as

$$\begin{aligned} S &= -p_{\text{yes}} \ln p_{\text{yes}} - p_{\text{no}} \ln p_{\text{no}} \\ &= -p_{\text{yes}} \ln p_{\text{yes}} - (1 - p_{\text{yes}}) \ln(1 - p_{\text{yes}}). \end{aligned} \tag{2.6.4}$$

The entropy is maximized when $p_{\text{yes}} = p_{\text{no}} = \frac{1}{2}$. So, one bit is equal to $\ln 2$ of information.

Let us now return to the black hole case and consider that a particle falls into a black hole. As the particle disappears some information is lost with it. An amount of information of the particle would depend on the knowledge of the internal states of the particle. The minimum amount of information lost for the particle is that contained in the answer to the question "does the particle exist or not?" Before the particle falls into the black hole, the answer is known to be "yes". But after the particle falls into the black hole, we have no information whatever about the answer.

This is because one knows nothing about the physical conditions inside the black hole. Thus one cannot determine the probability of the particle continuing to exist or being destroyed. One must, then, accept the loss of one bit of information at the very least. This implies that the entropy is increased by

$$\Delta\mathcal{S} = \ln 2, \tag{2.6.5}$$

before and after the particle with the tiniest information falls into the black hole.

2.6.2 Minimum Increase of Black Hole Area

We estimate the minimum possible increase in the Kerr-Newman black-hole area which must result when the black hole captures a spherical particle of rest mass μ and proper radius b. The "rationalized area" of a black hole α, used by Bekenstein, is given by

$$\alpha = \frac{A}{4\pi}. \tag{2.6.6}$$

where A is the black hole area as in (2.2.7). Then the first law of black hole physics (2.2.8) is written as

$$dM = \Theta_H d\alpha + \Omega_H dJ + \Phi_H dQ. \tag{2.6.7}$$

where Θ_H is defined by

$$\Theta_H = \frac{r_+ - M}{2\alpha}. \tag{2.6.8}$$

The particle may fall into the black hole by following different ways, all of which bring the increase of the black hole area. For inserting the particle into the black hole we consider the method which results in the smallest increase of the black hole area. This method has already been considered by Christodoulou to introduce the concept of irreducible mass [141, 160]. The essence of this method is that if a freely falling point particle is caught by a Kerr-Newman black hole, then the irreducible mass as well as the area of the black hole is left unchanged. Bekenstein generalized Christodoulou's method to a particle with a proper radius and showed that the increased area of the black hole is no longer precisely zero if the particle falls into the black hole. We assume that the freely falling particle is neutral. Then, the particle follows a geodesic of the Kerr-Newman metric (2.2.2) when falling freely. The horizon is located at $r = r_+$ defined by (2.2.5).

First integrals for geodesic motion in the background of Kerr-Newman black hole have been derived by Carter [161]. As a starting point of the analysis, Christodoulou applied the first integral (derivation is in Appendix-C)

$$E^2[r^4 + a^2(r^2 + 2Mr - Q^2)] - 2E(2Mr - Q^2)ap_\varphi$$

$$-(r^2 - 2Mr + Q^2)p_\varphi^2 - (\mu^2 r^2 + q)\Delta = (p_r\Delta)^2, \quad (2.6.9)$$

where $E = -p_t$ is the conserved energy, p_φ is the conserved component of angular momentum in the direction of the axis of symmetry, q is Carter's fourth constant of the motion, μ is the rest mass of the particle and p_r is its covariant radial momentum.

When (2.6.9) is solved for E, following Christodoulou, the result gives

$$E = \zeta a p_\varphi + \left[\left(\zeta^2 a^2 + \frac{r^2 - \xi\zeta}{\xi} \right) p_\varphi^2 + \frac{(\mu^2 r^2 + q)\Delta + (p_r\Delta)^2}{\xi} \right]^{1/2}, \quad (2.6.10)$$

where

$$\xi = r^4 + a^2(r^2 + 2Mr - Q^2),$$

$$\zeta = \frac{(2Mr - Q^2)}{\xi}. \tag{2.6.11}$$

At the event horizon $\Delta = 0$, so that we find, using (2.2.3),

$$\xi \mid_{r=r_+} = \xi_+ = (r_+^2 + a^2)^2,$$

$$\zeta \mid_{r=r_+} = \zeta_+ = \frac{1}{r_+^2 + a^2}, \tag{2.6.12}$$

and $\eta_+ a = \Omega_H$, where Ω_H is defined by (2.2.10). The coefficient of p_φ^2 in (2.6.10) vanishes at the horizon:

$$\zeta_+^2 a^2 + \frac{r_+^2 - 2Mr_+ + Q^2}{\xi_+} = \frac{\Delta(r_+)}{(r_+^2 + a^2)^2} = 0, \tag{2.6.13}$$

and the coefficient of $\mu^2 r^2 + q$ also vanishes. However, since $p_r = \mathrm{g}_{rr} p^r$,

$$p_r \Delta = (r^2 + a^2 \cos^2\theta) p^r, \tag{2.6.14}$$

which does not vanish at the horizon in general. If the orbit of the particle intersects the horizon, we have from (2.6.10) that

$$E = \Omega_H p_\varphi + \frac{\mid p_r \Delta \mid_+}{\sqrt{\xi_+}}. \tag{2.6.15}$$

As a result of the capture, the black hole's mass increases by E and its component of angular momentum in the direction of the symmetry axis increases by p_φ. Hence, corresponding to (2.6.7) the black hole's rationalized area α will increase by

$$\frac{\mid p_r\Delta \mid_+}{\Theta_H\sqrt{\xi_+}}.$$

Christodoulou pointed that this increase vanishes if the particle is captured from a turning point in its orbit in which case $\mid p_r\Delta \mid_+ = 0$. Then, the relation (2.6.15) becomes

$$E = \Omega_H p_\varphi. \tag{2.6.16}$$

The above analysis indicates the possibility that a black hole can capture a point particle without increasing its area.

Following Bekenstein's extension, we now show that this result is changed when the particle has a nonzero proper radius b. The relation (2.6.10) always describes the motion of the center of mass of the particle at the moment of capture. To generalize Christodoulou's result to the present case, it should be clear that one should evaluate (2.6.10) not at $r = r_+$, but at $r = r_+ + \delta$, where δ is determined by

$$\int_{r_+}^{r_+ + \delta} \sqrt{\mathrm{g}_{rr}}dr = b. \tag{2.6.17}$$

Here, $r = r_+ + \delta$ is a point a proper distance b outside the horizon. Using g_{rr} as in (2.2.2) we find

$$b = 2\sqrt{\frac{\delta(r_+^2 + a^2\cos^2\theta)}{r_+ - r_-}}, \tag{2.6.18}$$

where

$$r_+ - r_- \gg \delta,$$

i.e., black hole is not nearly extreme. Expanding the argument of the square root in (2.6.10) in powers of δ, replacing δ by its value given in

(2.6.18), and keeping only terms to $\mathcal{O}(b)$ we obtain

$$E = \Omega_H p_\varphi + \frac{1}{2} b \left(\frac{r_+ - r_-}{r_+^2 + a^2} \right) \cdot \frac{1}{\sqrt{r_+^2 + a^2 \cos^2 \theta}} \times \sqrt{\left(\frac{r_+^2 - a^2}{r_+^2 + a^2} \right) p_\varphi^2 + \mu^2 r_+^2 + q}. \tag{2.6.19}$$

This relation (2.6.19) is the generalization to $\mathcal{O}(b)$ of Christodoulou's condition (2.6.16). Carter's kinetic constant q, appeared in the derivation of (2.6.9)(see Appendix-C), is given by

$$q = \cos^2 \theta \left[a^2 (\mu^2 - E^2) + \frac{p_\varphi^2}{\sin^2 \theta} \right] + p_\theta^2. \tag{2.6.20}$$

For the reality of the θ momentum p_θ, it follows that

$$q \geq \cos^2 \theta \left[a^2 (\mu^2 - E^2) + \frac{p_\varphi^2}{\sin^2 \theta} \right], \tag{2.6.21}$$

the equality holds when $p_\theta = 0$. With replacing E in (2.6.21) by $\Omega_H p_\varphi$ [as in (2.6.16)], we obtain

$$q \geq \cos^2 \theta \left[a^2 \mu^2 + p_\varphi^2 \left(\frac{1}{\sin^2 \theta} - a^2 \Omega_H^2 \right) \right]. \tag{2.6.22}$$

One can find $a^2 \Omega_H^2 \leq \frac{1}{4}$ for a Kerr-Newman black hole and $1/\sin^2 \theta \geq 1$. Since the coefficient of p_φ^2 is positive, we can take for the constant q the value

$$q \geq a^2 \mu^2 \cos^2 \theta, \tag{2.6.23}$$

when $p_\varphi = 0$. Substituting (2.6.23) into (2.6.19), we obtain

$$E \geq \Omega_H p_\varphi + \frac{1}{2}\mu b \left(\frac{r_+ - r_-}{r_+^2 + a^2} \right), \tag{2.6.24}$$

which is correct to $\mathcal{O}(b)$. The equality sign in (2.6.24) corresponds to the case $p_\varphi = p_\theta = p^r = 0$ at the point of capture. The increase in the rationalized area of the black hole, computed by means of (2.6.7), (2.6.8) and (2.6.24), is

$$\Delta\alpha \geq 2\mu b. \tag{2.6.25}$$

This gives the fundamental lower bound on the increase in the area of the black hole:

$$(\Delta\alpha)_{\min} = 2\mu b, \tag{2.6.26}$$

which is independent of M, Q and J. By making b smaller, $(\Delta\alpha)_{\min}$ can be made smaller. However, it must be remembered that b can be no smaller than the particle's Compton wavelength $\frac{\hbar}{\mu}$, or the Schwarzschild radius 2μ. For the Compton wavelength is larger than the Schwarzschild radius $\frac{\hbar}{\mu} \geq 2\mu$, viz., the mass of the particle satisfies $\mu \leq \sqrt{\frac{\hbar}{2}}$, one can make b smaller to $\frac{\hbar}{\mu}$. On the contrary, if the Schwarzschild radius is larger than the Compton wavelength $\frac{\hbar}{\mu} < 2\mu$, viz., the mass of the particle satisfies $\mu > \sqrt{\frac{\hbar}{2}}$, one can make b smaller to 2μ. The relation (2.6.26) is then given by $2\hbar$, when $b = \frac{\hbar}{\mu}$, and given by $4\mu^2$, when $b \simeq 2\mu$. When $4\mu^2 > 2\hbar$, one can find a lower bound of the rationalized area of a Kerr-Newman black hole as

$$(\Delta\alpha)_{\min} = 2\hbar, \tag{2.6.27}$$

as the black hole captures the particle.

2.6.3 Information Loss and Black-Hole Entropy

In section 2.5, we already have observed that a black hole area is similar to the entropy in thermodynamics. Even though there are clear analogies between them, we do not know how to associate the black-hole area to its entropy. In this subsection, we would like to present the treatment followed by Bekenstein [1].

According to the no-hair theorem [119], a black hole in equilibrium (Kerr-Newman black hole) can be completely described (insofar as an exterior observer is concerned) by just three parameters: mass, charge, and angular momentum. Black holes in equilibrium having the same set of these three parameters may still have different "internal configurations." For example, a black hole may have been formed by the collapse of a normal star, or a neutron star, or by the collapse of a geon[1]. These various alternatives may be considered as different possible internal configurations of one and the same black hole described by their (common) mass, charge, and angular momentum. It is then natural to introduce the concept of black-hole entropy as the measure of the inaccessibility of information (to an exterior observer) as to which specific internal configuration of the black hole is really recognized in a given type.

The black-hole entropy we are speaking of is not the thermal entropy inside the black hole. Indeed, our black-hole entropy refers not to one particular black hole, but to the equivalence class of all black holes which have the same mass, charge, and angular momentum. The discussion of section 2.5 predisposes us to choose black-hole area to take as a measure of this black-hole entropy. However, in order to be more general, Bekenstein

[1]The word "geon" is the abbreviation for the phrase "gravitational-electromagnetic entity;" more in Ref.: J.A. Wheeler, "Geons," Phys. Rev. **97**, (1955) 511–536.

assumed that the entropy of a black hole, S_{BH}, is some monotonically increasing function of its rationalized area as in (2.6.6):

$$S_{BH} = f(\alpha). \tag{2.6.28}$$

Due to the gradual loss of information, the entropy of an evolving thermodynamic system increases. This result is a consequence of the washing out of the most of the initial conditions. The effects of the initial conditions are also washed out (the black hole loses its hair) as a black hole approaches equilibrium; only mass, charge, and angular momentum are left as determinants of the black hole at late times. It would be thus expected that the loss of information about initial peculiarities of the black hole will be reflected in a gradual increase in S_{BH}. Indeed the relation (2.6.28) predicts just this. As the black hole evolves S_{BH} increases monotonically by Hawking's theorem.

One possible choice for f in (2.6.28) is

$$f(\alpha) \propto \sqrt{\alpha}, \tag{2.6.29}$$

which is untenable on some reasons. We consider two black holes which are at a distant from each other so that they interact weakly. Then we can take the total black hole entropy to be the sum of S_{BH} of each black hole. Let the black holes move closer together and finally merge, and form a black hole which settles down to equilibrium. During the process no information about the black hole interior can become available. On the contrary, much information is lost as the final black hole loses its hair. So, we expect that the final black-hole entropy exceeds the initial one. By the assumption (2.6.29), this suggests that the irreducible mass $M_{\mathrm{ir}} = \sqrt{\frac{A}{16\pi}}$ of the final black hole exceeds the sum of irreducible masses of the initial black holes. We suppose that all three black holes are Schwarzschild ($M = M_{\rm ir}$). It then predicts that the final black-hole mass exceeds the initial one. But this is nonsense because the total black-hole mass only decreases due to gravitational radiation losses. We thus see that the choice in (2.6.29) is untenable.

The next simplest choice for f is

$$f(\alpha) = \gamma\alpha, \tag{2.6.30}$$

with γ a constant. If we repeat the above argument for this new f, the result leads to the conclusion that the final black-hole area must exceed the total initial black-hole area. But this is true from Hawking's theorem. Thus the choice (2.6.30) leads to no contradiction. So, we adopt (2.6.30) for the moment.

From comparison of (2.6.29) and (2.6.30), the units of γ is found as [length]^2. However, no constant with such units exists in classical general relativity. If we turn to quantum physics in desperation, we observe only one truly universal constant with the correct units: $\hbar^{-1}$, that is, the reciprocal of the Planck length squared. Thus Bekenstein represented (2.6.28) as

$$S_{BH} = \frac{\eta\alpha}{\hbar}, \tag{2.6.31}$$

where η is a dimensionless constant, expected to be of order unity. Bekenstein also proposed this expression earlier from a different point of view [162]. It is well known [163] that $\hbar$ also appears in the formulas for the entropy of many thermodynamic systems that are conventionally regarded as classical, for example, the Boltzmann ideal gas, and the Sackur-Tetrode equation [164]. This is a manifestation of the fact that entropy is, in a sense, a count of states of the system, and the underlying states of any system are always quantum in nature. It is therefore not totally surprising that $\hbar$ appears in (2.6.31).

To determine the value of η, Bekenstein considered that a particle falls into a Kerr-Newman black hole. As it disappears some information is lost with it. In subsection 2.6.1, we presented that the loss of one bit of information before and after the particle with the least information falls into a black hole, i.e., the increased entropy is

$$\Delta S = \ln 2.$$

In subsection 2.6.2, we presented that when a spherical particle with a radius as large as the Compton wavelength falls into a black hole, the minimum increase of the black hole area is given by (2.6.26). From (2.6.26), we find the increase of black hole entropy as follows:

$$(\Delta S_{BH})_{\min} = 2\hbar \frac{df(\alpha)}{d\alpha}. \tag{2.6.32}$$

As Bekenstein conjectured this entropy agrees with the loss of one bit of information (2.6.5). We thus have

$$2\hbar \frac{df(\alpha)}{d\alpha} = \ln 2. \tag{2.6.33}$$

In the left-hand side of (2.6.33), the limit as in (2.6.27) can be found only for a particle with dimension given by its Compton wavelength. Only such an "elementary particle" may be considered as having no internal structure. One can thus consider that the loss of information associated with the loss of such a particle should be minimum. When (2.6.33) is integrated over α, one finds

$$f(\alpha) = \left(\frac{1}{2}\ln 2\right)\frac{\alpha}{\hbar}. \tag{2.6.34}$$

From (2.6.30), we have the black hole entropy

$$S_{BH} = \left(\frac{1}{2}\ln 2\right)\frac{\alpha}{\hbar}, \tag{2.6.35}$$

which is of the same form as (2.6.31).

Bekenstein showed the dependence of the black hole entropy S_{BH} on the black hole area α from the above discussion, and expressed the black hole entropy, using some conjectures, in conventional units by

$$S_{BH} = \left(\frac{1}{2}\ln 2\right)\frac{k_B c^3}{4\pi\hbar G}A, \tag{2.6.36}$$

The relation

$$\eta = \frac{1}{2}\ln 2$$

is obtained from the assumption that the smallest possible radius of a particle is precisely equal to its Compton-wavelength whereas the actual radius is not so sharply defined. Moreover, an amount of information of such a particle might be more than $\ln 2$. This is because the particle has information for the mass and the radius. According to the current understanding, the black hole entropy is given by

$$S_{BH} = \frac{1}{4}\frac{k_B c^3}{\hbar G}A. \tag{2.6.37}$$

We notice that the value of η in (2.6.37) is slightly different from that in (2.6.36). Nevertheless, Bekenstein mentioned in his paper [153] that it would be somewhat pretentious to attempt to estimate the precise value of the constant $\frac{\eta}{\hbar}$ without a full understanding of the quantum reality which underlies a "classical" black hole. He, surprisingly, already suggested that the derivation of black hole radiation needs the consideration of quantum theory.

Bekenstein defined as well a characteristic temperature for a Kerr-Newman black hole by the relation

$$\frac{1}{T_{BH}} = \left(\frac{\partial S_{BH}}{\partial M}\right)_{J,Q}, \tag{2.6.38}$$

which is the analog of the thermodynamic relation

$$\frac{1}{T} = \left(\frac{\partial S}{\partial \mathcal{E}}\right)_{V}. \tag{2.6.39}$$

By using both (2.6.7) and (2.6.35) in (2.6.38), we can obtain

$$T_{BH} = \frac{2\hbar}{\ln 2}\Theta_H. \tag{2.6.40}$$

But Bekenstein did not regard T_{BH} as the temperature of the black hole. If a black hole has a temperature, some radiation from the black hole may emerge. This conflicts with the classical definition of black holes. A black hole, by definition, can only absorb matter but cannot radiate matter. For this reason Bekenstein did not suggest that a black hole has a temperature.

Chapter 3

Radiation from Black Holes

In this chapter, we would like to review several previous works on deriving radiation from black holes. The chapter is arranged as follows. In section 3.1, we explain radiation from black holes using Penrose diagram in the Schwarzschild black hole spacetime. In section 3.2, we review Hawking's original derivation of black hole radiation. In section 3.3, we discuss Unruh radiation briefly. In section 3.4, we describe the Damour-Ruffini method of deriving Hawking radiation. In section 3.5, we review the null geodesic method used by Parikh and Wilczek [19] that followed from the work of Kraus and Wilczek [15, 16, 17]. In section 3.6, we review an alternate method for calculating black hole tunneling that makes use of the Hamilton-Jacobi equation as an ansatz.

3.1 Hawking Radiation

A black hole cannot radiate but absorb matter in the context of the classical theory. As proposed by Bekenstein, a black hole has an entropy from the point of view of information theory. However, there was no suggestion that a black hole has a temperature. So, the complete correspondence between black hole physics and thermodynamics could not be obtained. Nevertheless, Hawking showed that a black hole can radiate its energy by taking quantum effects into account [2]. Moreover, it was found that a black hole behaves like a black-body with a certain temperature. The radiation from the black hole is commonly called Hawking radiation.

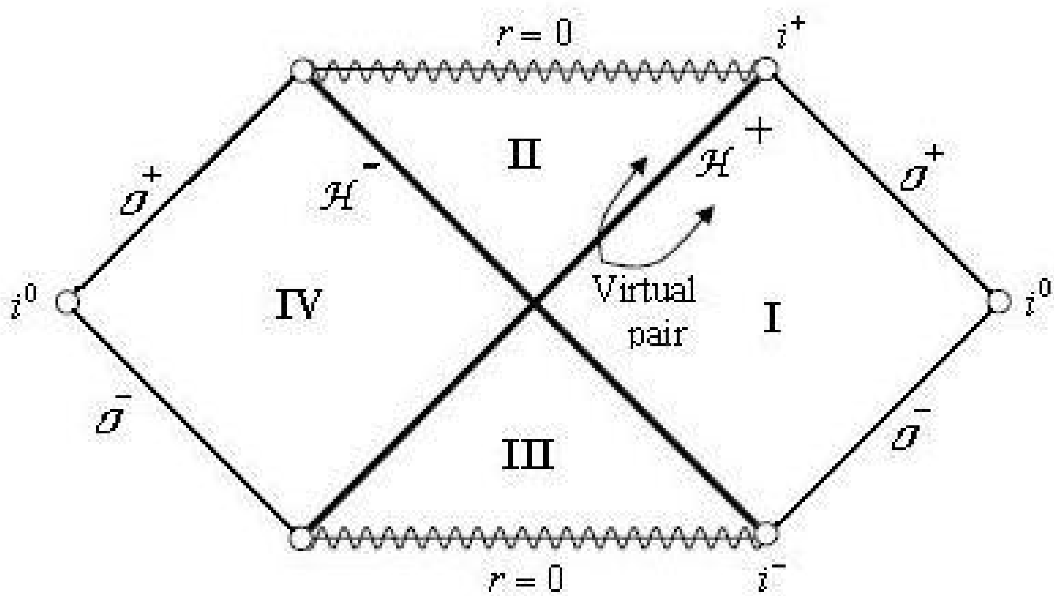

Figure 3.1: Penrose diagram of the maximally extended Schwarzschild spacetime.

One way to understand the origin of the radiation is as follows. According to quantum field theory, it is possible to consider spontaneous particle-antiparticle pair production near the event horizon of a black hole. Usually, such a pair annihilates itself very rapidly. However, there may happen that one of them—particle or antiparticle—is swallowed by the black hole before the annihilation such that the other one is free to escape away from the black hole. We illustrate this event in Figure 3.1, the Penrose diagram of the maximally extended Schwarzschild black hole spacetime. It can be demonstrated that as a net effect more antiparticles than particles fall through the horizon towards the singularity. As a result, an observer outside the black hole, i.e., at the region I of the Figure 3.1, observes a particle flux which seems to come out from the black hole. In Figure 3.1 the regions I and IV represent spacetime surrounding the regions II (black hole) and III (white hole), while the regions I and III are causally separated. Consider that a particle-antiparticle pair is spontaneously created near the event horizon $\mathcal{H}^+$ of the black hole in region I. It is then possible that either a particle or an antiparticle is swallowed by the black hole such that the other one is free to escape to the future null infinity at $\mathcal{J}^+$.

3.2 Hawking's Original Derivation

Hawking [2] showed by applying quantum field theory in black hole physics that black holes radiate matter. Let us consider a free massless scalar field, for simplicity, which in Minkowski space satisfies the Klein-Gordon equation

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\Phi = 0, \tag{3.2.1}$$

where Φ is a massless Hermitian scalar field, $\eta_{\mu\nu}$ is the Minkowski metric (2.1.1) and $\partial_\mu = \frac{\partial}{\partial x^\mu}$ is the partial derivative. The ordinary derivative of Φ is also written as $\Phi_{,\mu}$. We decompose the field into positive and negative frequency components

$$\Phi = \sum_i \left(f_i \boldsymbol{a}_i + f_i^* \boldsymbol{a}_i^\dagger \right), \tag{3.2.2}$$

where $\{f_i\}$ form a complete orthonormal family of complex valued solutions of the wave equation $\eta^{\mu\nu}\partial_\mu\partial_\nu f_i = 0$ which contain only positive frequencies with respect to the usual Minkowski time coordinate. The operators $\boldsymbol{a}_i$ and $\boldsymbol{a}_i^\dagger$ are respectively the annihilation and creation operators for particles in the i-th state. The vacuum state $\mid 0\rangle$ is defined by

$$\boldsymbol{a}_i|0\rangle = 0, \quad \text{for all} \quad i, \tag{3.2.3}$$

i.e., it is the state from which one cannot annihilate any particle. The orthonormal condition is defined by

$$\rho_M(f_i, f_j^*) = \frac{i}{2}\int_V \left(f_i \partial_t f_j^* - f_j^* \partial_t f_i \right) d^3x = \delta_{ij}, \tag{3.2.4}$$

V being a suitable closed space.

Let us extend the quantum field theory from Minkowski spacetime to curved spacetime produced by the intense gravity of a black hole. Physical laws must hold in any coordinate system. The partial derivative contained in these laws must be replaced by the covariant derivative in the curved spacetime, represented by $\nabla_\mu \Phi = \Phi_{;\mu}$. Of course, the covariant derivative of a scalar field Φ is given by $\nabla_\mu \Phi = \partial_\mu \Phi$, while the covariant derivative of a vector field A_ν is given by

$$\nabla_\mu A_\nu = \partial_\mu A_\nu + \Gamma^\alpha_{\nu\mu} A_\alpha,$$

where $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol defined in (2.1.6).

The Klein-Gordon equation (3.2.1) is thus represented in curved spacetime by

$$\mathrm{g}^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = 0. \tag{3.2.5}$$

One cannot decompose the field in curved spacetime into its positive and negative frequency parts because positive and negative frequencies have no invariant meaning in curved spacetime. One can however require that the $\{f_i\}$ and the $\{f_i^*\}$ together form a complete basis for solutions of the wave equations with

$$\rho(f_i, f_j^*) = \frac{i}{2}\int_\Sigma \left(f_i\nabla_\mu f_j^* - f_j^*\nabla_\mu f_i\right) d\Sigma^\mu = \delta_{ij}, \tag{3.2.6}$$

where $d\Sigma$ stands for an area element and Σ is called a Cauchy surface which represents a suitable surface.

Consider a black hole formed by gravitational collapse. In the case of exactly spherical collapse, the appropriate Penrose diagram is shown in Fig. 2.9. The Schwarzschild metric is asymptotically flat (the Minkowski metric) in the past null infinity $\mathscr{J}^-$, since $r \to \infty$. Then the field operator Φ, which satisfies the Klein-Gordon equation (3.2.5), can be expanded as

$$\Phi = \sum_i \left(f_i\boldsymbol{a}_i + f_i^*\boldsymbol{a}_i^\dagger\right), \tag{3.2.7}$$

where $\{f_i\}$ is a family of solutions of the wave equation $\mathrm{g}^{\mu\nu}\nabla_\mu\nabla_\nu f_i = 0$, satisfying the orthonormality conditions (3.2.6), and the surface Σ is $\mathscr{J}^-$. This family of solutions form a complete family on past null infinity $\mathscr{J}^-$ and contains only positive frequencies with respect to the canonical affine parameter on $\mathscr{J}^-$. Naturally, the operators $\boldsymbol{a}_i$ and $\boldsymbol{a}_i^\dagger$ are interpreted as the annihilation and creation operators for ingoing particles i.e. for particles at past null infinity $\mathcal{J}^-$. The vacuum state at $\mathcal{J}^-$ is defined by

$$\boldsymbol{a}_i|0_-\rangle = 0. \tag{3.2.8}$$

The massless field operator Φ can also be determined in the region outside the event horizon by their data on the event horizon and on future null infinity $\mathcal{J}^+$. Thus we can also express Φ in the form

$$\Phi = \sum_i \left(p_i \boldsymbol{b}_i + p_i^* \boldsymbol{b}_i^\dagger + q_i \boldsymbol{c}_i + q_i^* \boldsymbol{c}_i^\dagger \right), \tag{3.2.9}$$

where $\{p_i\}$ are solutions of the wave equation which can escape to $\mathcal{J}^+$ and $\{q_i\}$ are solutions of the wave equation which cannot escape to $\mathcal{J}^+$ since they are absorbed by the future event horizon $\mathcal{H}^+$, i.e., $\{p_i\}$ are zero at $\mathcal{H}^+$ and $\{q_i\}$ are zero at $\mathcal{J}^+$. With the positive frequency condition on $\{p_i\}$, the operators $\boldsymbol{b}_i$ and $\boldsymbol{b}_i^\dagger$ can be regarded as the annihilation and creation operators for outgoing particles, i.e. for particles on $\mathcal{J}^+$, and the operators $\boldsymbol{c}_i$ and $\boldsymbol{c}_i^\dagger$ respectively stand for the annihilation and creation operators at $\mathcal{H}^+$. The vacua at $\mathcal{J}^+$ and $\mathcal{H}^+$ are thus defined by

$$\boldsymbol{b}_i|0_+\rangle = 0, \tag{3.2.10}$$

$$\boldsymbol{c}_i|0_{\mathcal{H}+}\rangle = 0. \tag{3.2.11}$$

It is not clear whether one should impose some positive frequency condition on $\{q_i\}$. However, the choice of the $\{q_i\}$ does not affect the calculation of the emission of particle to $\mathcal{J}^+$ since the $\{q_i\}$ are zero at $\mathcal{J}^+$. We would like to consider particles which start from $\mathcal{J}^-$, pass through the collapsing body and can escape to $\mathcal{J}^+$. We require that $\{p_i\}$ and $\{p_i^*\}$ are a complete orthonormal family satisfying

$$\rho'(p_i, p_j^*) = \frac{i}{2}\int_{\Sigma'} \left(p_i \nabla_\mu p_j^* - p_j^* \nabla_\mu p_i\right) d\Sigma'^\mu = \delta_{ij}. \qquad (3.2.12)$$

The relation (3.2.12) is satisfied even if one uses Σ which appeared in

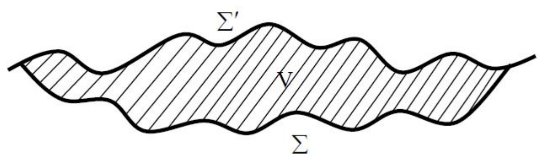

Figure 3.2: The volume V is enclosed by surfaces Σ and Σ'

(3.2.6) instead of Σ'. If the stable surface Σ' differs from Σ, Σ' can smoothly intersect with Σ at certain points since Σ' is not parallel to Σ. If V (Fig. 3.2) is the 4-dimensional volume enclosed by these two surfaces, we obtain by Gauss theorem

$$\rho(p_i, p_j^*) - \rho'(p_i, p_j^*) = \int_V d^4x\sqrt{-\mathrm{g}}\ \nabla^\mu \left(p_i \nabla_\mu p_j^* - p_j^* \nabla_\mu p_i\right), \qquad (3.2.13)$$

where $\mathrm{g} = \det(\mathrm{g}_{\mu\nu})$ and $\sqrt{-\mathrm{g}}$ stands for the Jacobian with respect to the transformation from d^4x to $d\Sigma$. Since

$$\begin{aligned}\nabla^\mu \left(p_i \nabla_\mu p_j^* - p_j^* \nabla_\mu p_i\right) &= \nabla^\mu p_i \nabla_\mu p_j^* + p_i \nabla^\mu \nabla_\mu p_j^* \\ &\quad - \nabla^\mu p_j^* \nabla_\mu p_i - p_j^* \nabla^\mu \nabla_\mu p_i \\ &= p_i \nabla^\mu \nabla_\mu p_j^* - p_j^* \nabla^\mu \nabla_\mu p_i,\end{aligned}$$

by using the Klein-Gordon equation (3.2.5), we find from (3.2.13),

$$\nabla^{\mu}\left(p_i\nabla_{\mu}p_j^* - p_j^*\nabla_{\mu}p_i\right) = 0, \tag{3.2.14}$$

and also it follows that

$$\rho(p_i, p_j^*) = \rho'(p_i, p_j^*). \tag{3.2.15}$$

Thus, $\rho(p_i, p_j^*)$ does not depend on Σ. It means that if the Gauss theorem is satisfied, we can freely choose the surface Σ in (3.2.6). The above discussion is also valid for a scalar field with a mass [165].

A collapsing body will appear in the transitional time between $\{f_i\}$ and $\{p_i\}$. Since we do not know the metric inside this region, we do not know the corresponding solutions. By the analogy of the tunneling effect, $\{p_i\}$ (which appear at $\mathcal{J}^+$) can be expressed as the linear combinations of the $\{f_i\}$ and $\{f_i^*\}$:

$$p_i = \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f_j^*\right), \tag{3.2.16}$$

where α_{ij} and β_{ij} are proportionality coefficients. Substituting (3.2.16) into (3.2.9), we get

$$\Phi = \sum_i \left\{ \sum_j \left(\boldsymbol{b}_i\alpha_{ij} + \boldsymbol{b}_j^{\dagger}\beta_{ij}^*\right) f_i + \sum_j \left(\boldsymbol{b}_i\beta_{ij} + \boldsymbol{b}_j^{\dagger}\alpha_{ij}^*\right) f_i^* \right\}, \tag{3.2.17}$$

since $\{q_i\} = 0$ at $\mathcal{J}^+$. Comparing (3.2.17) with (3.2.7), we obtain

$$\boldsymbol{a}_i = \sum_j \left(\boldsymbol{b}_i\alpha_{ij} + \boldsymbol{b}_j^{\dagger}\beta_{ij}^*\right), \tag{3.2.18}$$

$$\boldsymbol{a}_i^{\dagger} = \sum_j \left(\boldsymbol{b}_i\beta_{ij} + \boldsymbol{b}_j^{\dagger}\alpha_{ij}^*\right). \tag{3.2.19}$$

The inverse transformations with respect to $\boldsymbol{b}_j$ and $\boldsymbol{b}_j^\dagger$ are also obtained as

$$\boldsymbol{b}_i = \sum_j \left(\alpha_{ij}^* \boldsymbol{a}_j - \beta_{ij}^* \boldsymbol{a}_j^\dagger \right), \tag{3.2.20}$$

$$\boldsymbol{b}_i^\dagger = \sum_j \left(\alpha_{ij} \boldsymbol{a}_j^\dagger - \beta_{ij} \boldsymbol{a}_j \right). \tag{3.2.21}$$

These are called the Bogoliubov transformations. Details of this calculation are in Appendix-D.

We have already defined the initial vacuum as in (3.2.8). Operating the annihilation operator $\boldsymbol{b}_i$ on the initial vacuum state $|0_-\rangle$, we obtain

$$\begin{aligned} \boldsymbol{b}_i|0_-\rangle &= \sum_j \left(\alpha_{ij}^* \boldsymbol{a}_j - \beta_{ij}^* \boldsymbol{a}_j^\dagger \right) |0_-\rangle \\ &= \sum_j -\beta_{ij}^* \boldsymbol{a}_j^\dagger |0_-\rangle \neq 0. \end{aligned} \tag{3.2.22}$$

Since the coefficients β_{ij} will not be zero in general, the initial vacuum state will not appear to be a vacuum state to an observer at $\mathcal{J}^+$. Thus particles are created by the gravitational field and emitted to infinity.

We now determine the number of particles created at $\mathcal{J}^+$ from the initial vacuum $|0_-\rangle$. The expectation value of the number operator $N_i \equiv \boldsymbol{b}_i^\dagger \boldsymbol{b}_i$ for the i-th outgoing mode is

$$N_i = \langle 0_- | \boldsymbol{b}_i^\dagger \boldsymbol{b}_i | 0_- \rangle = \sum_{j,k} \langle 0_- | \beta_{ik} \beta_{ij}^* \boldsymbol{a}_k \boldsymbol{a}_j^\dagger | 0_- \rangle. \tag{3.2.23}$$

With the commutation relation of the creation-annihilation operators, given by

$$\left[\boldsymbol{a}_i, \boldsymbol{a}_j^\dagger \right] = \delta_{ij}, \tag{3.2.24}$$

the relation (3.2.23) becomes

$$N_i = \sum_{j,k} \beta_{ik}\beta_{ij}^*\delta_{jk} = \sum_j |\beta_{ij}|^2. \tag{3.2.25}$$

This is the number of particles which propagate to infinity among the particle pairs created by the vacuum. To find the value, we need to calculate the coefficients β_{ij}.

Solving the Klein-Gordon equation (3.2.5), we obtain (see Appendix-E)

$$f_{\omega' lm} = \frac{F_{\omega'}(r)}{r\sqrt{2\pi\omega'}} e^{i\omega' v} Y_{lm}(\theta, \varphi), \tag{3.2.26}$$

$$p_{\omega lm} = \frac{P_{\omega}(r)}{r\sqrt{2\pi\omega}} e^{i\omega u} Y_{lm}(\theta, \varphi), \tag{3.2.27}$$

where $Y_{lm}(\theta, \varphi)$ is the spherical harmonics and $f_{\omega' lm}$ stands for f_i. The frequencies ω and ω' are eigenvalues given by

$$i\partial_t f_{\omega' lm} = \omega' f_{\omega' lm}, \tag{3.2.28}$$

$$i\partial_t p_{\omega lm} = \omega f_{\omega lm}. \tag{3.2.29}$$

The advanced time v is an affine parameter at $\mathcal{J}^-$, while the retarded time u is an affine parameter at $\mathcal{J}^+$. They are defined as in (2.3.10) and (2.3.11). The solutions $f_{\omega' lm}$ and $p_{\omega lm}$ are obtained by approximating the Klein-Gordon equation at $r \to \infty$. The integration constants $F_{\omega'}(r)$ and $P_\omega(r)$ contain a tiny effect depending on r.

By taking a continuous limit in (3.2.16), (3.2.20) and (3.2.25), we obtain

$$p_\omega = \int_0^\infty (\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f_{\omega'}^*)\, d\omega', \tag{3.2.30}$$

$$b_\omega = \int_0^\infty \left(\alpha_{\omega\omega'}\boldsymbol{a}_{\omega'} + \beta_{\omega\omega'}\boldsymbol{a}^\dagger_{\omega'}\right) d\omega', \tag{3.2.31}$$

and

$$N_\omega = \int_0^\infty |\beta_{\omega\omega'}|^2 d\omega', \tag{3.2.32}$$

where we have dropped indices l and m since the wave functions with different indices l and m are not connected to each other in a spherically symmetric system. The coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ can be evaluated by performing the Fourier transform in (3.2.30). Substituting (3.2.26) into (3.2.30) and then multiplying the both sides by $\int_{-\infty}^{\infty} e^{-i\omega'' v}dv$, we find

$$\int_{-\infty}^{\infty} dve^{(-i\omega' v)}p_\omega = 2\pi \int_0^\infty d\omega' \Bigg[\alpha_{\omega\omega'}\frac{F_{\omega'}(r)}{r\sqrt{2\pi\omega'}}\delta(\omega' - \omega'') \\ -\beta_{\omega\omega'}\frac{F_{\omega'}(r)}{r\sqrt{2\pi\omega'}}\delta(\omega' + \omega'')\Bigg], \tag{3.2.33}$$

The second term on the right-hand side vanishes since $(\omega' + \omega'') \neq 0$. We thus obtain

$$\alpha_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}} \int_{-\infty}^{\infty} dve^{-i\omega' v}p_\omega. \tag{3.2.34}$$

As for $\beta_{\omega\omega'}$, we similarly obtain

$$\beta_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}} \int_{-\infty}^{\infty} dve^{i\omega' v}p_\omega. \tag{3.2.35}$$

Both (3.2.34) and (3.2.35) contain u and v. The relation of between u and v can be derived from the following connection condition. We consider the wave function p_ω which reaches $\mathcal{J}^+$. When viewed backwards, the wave function is found to propagate into two groups. The first group $p_\omega^{(1)}$, which will be scattered by the Schwarzschild field outside the collapsing body,

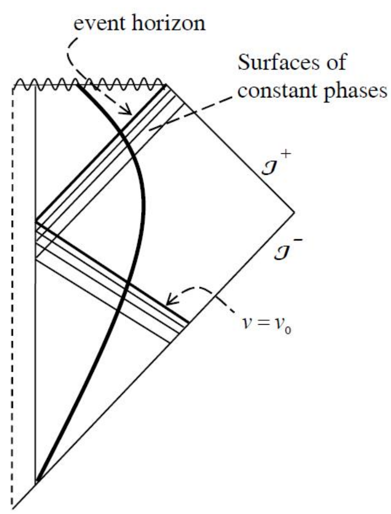

Figure 3.3: The solution p_ω has an infinite number of cycles near the event horizon and near the surface $v = v_o$.

will end up on $\mathcal{J}^-$ with the same frequency ω. The second group $p_\omega^{(2)}$ will enter the collapsing body where it will be partly scattered and partly reflected through the center, eventually emerging to $\mathcal{J}^-$. The group $p_\omega^{(2)}$ is the part that produces the interesting effect. Since the retarded time coordinate u is infinite at the horizon, the surfaces of constant phase of the solution $p_\omega^{(2)}$ will pile up near the horizon (Fig. 3.3). An observer on the collapsing body would see the wave to have a very large blue-shift. Since its effective frequency was very high, the wave would propagate by geometric optics through the centre of the body and out on $\mathcal{J}^-$. The part $p_\omega^{(2)}$ would have an infinite number of cycles on $\mathcal{J}^-$ just before the advanced time $v = v_o$ where v_o is the latest time that a null geodesic could leave $\mathcal{J}^-$, pass through the centre of the body and escape to $\mathcal{J}^+$ before being trapped by the event horizon. We can estimate the form of $p_\omega^{(2)}$ on $\mathcal{J}^-$ near $v = v_o$ in the following way. Consider that x is a point on the event horizon outside the collapsing body. Let l^μ be a null vector tangent to the horizon at x and n^μ be a future-directed null vector at x which is directed radially inwards. They are normalized so that

$$l^\mu n_\mu = -1. \tag{3.2.36}$$

For a very small constant $\varepsilon > 0$, the vector $-\varepsilon n^\mu$ will connect the point x on the event horizon with a nearby null surface of constant retarded time u and hence with a surface of constant phase of the solution $p_\omega^{(2)}$. The vectors l^μ and n^μ transport parallelly along the null geodesic γ through x which generates the horizon. The vector $-\varepsilon n^\mu$ always connects the event horizon with the same surface of constant phase of $p_\omega^{(2)}$. To find the relation between ε and the phase of $p_\omega^{(2)}$, we imagine in Fig. 2.8 of chapter 2 that the collapsing body did not exist but one analytically continued the empty space Schwarzchild solution back to cover the whole Penrose diagram. Then the pair (l^μ, n^μ) can be transported back along to the point where future and past event horizons intersected so that the vector $-\varepsilon n^\mu$ would lie along the past event horizon. Let λ be the affine parameter along the past event horizon. The parameter λ is such that at the point of intersection of the two horizons,

$$\lambda = 0 \quad \text{and} \quad \frac{dx^\mu}{d\lambda} = n^\mu.$$

The affine pardameter λ is related to the retarded time u on the past horizon by

$$\lambda = -Ce^{\kappa u}, \tag{3.2.37}$$

where C is a constant and κ is the surface gravity of the black hole defined by

$$\nabla_\nu K^\mu K^\nu = -\kappa K^\mu$$

with K^μ the time translation Killing vector. For a Schwarzchild black hole, the surface gravity is given by

$$\kappa = \frac{1}{4M}. \tag{3.2.38}$$

From (3.2.37) it follows that the vector $-\varepsilon n^\mu$ connects the future event horizon with the surface of constant phase

$$-\frac{\omega}{\kappa}(\ln\varepsilon - \ln C)$$

of the solution $p^{(2)}_\omega$. This result also applies to the real spacetime (including the collapsing body) in the region outside the body. The solution $p^{(2)}_\omega$, near the event horizon, will obey the geometric optics approximation as it passes through the body because its effective frequency will be very high. Thus, even if we extend the null geodesic γ back past the end-point of the event horizon and out onto $\mathcal{J}^-$ at $v = v_o$ and parallelly transports n^μ along γ, the vector $-\varepsilon n^\mu$ will still connect γ to a surface of constant phase of the solution $p^{(2)}_\omega$. Since the vector n^μ on $\mathcal{J}^-$ is parallel to the Killing vector K^μ which is tangent to the null geodesic generators of $\mathcal{J}^-$, the vector n^μ is given by

$$n_\mu = DK^\mu, \tag{3.2.39}$$

where D is a constant. One finds that $p_\omega^{(2)}$ is zero for $v > v_o$ because the particle is captured by the black hole and cannot escape to $\mathcal{J}^+$. Thus, for $v_o - v$ small and positive, the phase of $p_\omega^{(2)}$ on $\mathcal{J}^-$ will be

$$-\frac{\omega}{\kappa}\left[\ln(v_o - v) - \ln D - \ln C\right]. \tag{3.2.40}$$

Then on $\mathcal{J}^-$ the wave function $p_\omega^{(2)} \sim 0$ for $v > v_o$, while

$$p_\omega^{(2)} \sim \frac{P_\omega^-}{r\sqrt{2\pi\omega}} \exp\left[-i\frac{\omega}{\kappa}\ln\left(\frac{v_0 - v}{CD}\right)\right] \quad \text{for } v < v_o, \tag{3.2.41}$$

where $P_\omega^- \equiv P_\omega(2M)$ is the value of the radial function for P_ω on the past event horizon in the analytically continued Schwarzchild solution. The expression for $p_\omega^{(2)}$ in (3.2.41) is valid only for $v_o - v$ small and positive. The amplitude at earlier advanced times will be different and the frequency with respect to v will approach the original frequency ω.

Performing integrations of both (3.2.34) and (3.2.35), we obtain (see Appendix-F)

$$\alpha_{\omega\omega'}^{(2)} \approx \frac{1}{2\pi} P_\omega^- (CD)^{i\omega/\kappa} e^{-i\omega' v_o} \left(\sqrt{\frac{\omega'}{\omega}}\right) \times\Gamma\left(1 - \frac{i\omega}{\kappa}\right)(-i\omega')^{-1+i\omega/\kappa}, \tag{3.2.42}$$

$$\beta_{\omega\omega'}^{(2)} \approx -i\alpha_{\omega(-\omega')}^{(2)}. \tag{3.2.43}$$

By expressing $\beta_{\omega\omega'}^{(2)}$ in terms of $\alpha_{\omega\omega'}^{(2)}$ from both (3.2.42) and (3.2.43), we obtain

$$\beta_{\omega\omega'}^{(2)} = e^{2i\omega' v_0} e^{(i\omega/\kappa - 1)\ln(-1)} \alpha_{\omega\omega'}^{(2)}. \tag{3.2.44}$$

The factor $(-i\omega')^{-1+i\omega/\kappa}$ has a logarithmic singularity at $\omega' = 0$. We analytically continue $\alpha^{(2)}_{\omega\omega'}$ anticlockwise round this singularity and obtain

$$|\beta^{(2)}_{\omega\omega'}| = e^{-\pi\omega/\kappa}\,|\alpha^{(2)}_{\omega\omega'}|, \tag{3.2.45}$$

using $\ln(-1) = i\pi$. This relation is valid for the large values of ω'.

The total number of particles created at $\mathcal{J}^+$ in the frequency range ω to $\omega + d\omega$ has the averaged value

$$d\omega \int_0^\infty d\omega' |\beta_{\omega\omega'}|^2.$$

This integral diverges because $|\beta_{\omega\omega'}|$ goes like $(\omega')^{-\frac{1}{2}}$ at large ω'. It is considered that this infinite total number of created particles corresponds to a finite steady rate of emission continuing for an infinite time. To evaluate the finite rate of emission, Hawking defined wave packets p_{jn} by

$$p^{(2)}_{jn} = \varepsilon^{-\frac{1}{2}} \int_{j\varepsilon}^{(j+1)\varepsilon} e^{-2\pi i n\omega/\varepsilon} p^{(2)}_{\omega} d\omega, \tag{3.2.46}$$

where j and n are integers, $j \geq 0$, $\varepsilon > 0$. For small ε these wave packets would have frequency $j\varepsilon$ and would be peaked around retarded time $u = 2\pi n\varepsilon^{-1}$ with width ε^{-1}. We can expand $\{p_{jn}\}$ in terms of $\{f_\omega\}$

$$p^{(2)}_{jn} = \int_0^\infty \left(\alpha^{(2)}_{jn\omega'} f_{\omega'} + \beta^{(2)}_{jn\omega'} f^*_{\omega'}\right) d\omega'. \tag{3.2.47}$$

Comparing (3.2.47) with the relation (3.2.46) which is obtained by using (3.2.27), we find that the proportionality coefficient $\alpha_{jn\omega'}$ is given by

$$\alpha^{(2)}_{jn\omega'} = \frac{1}{\sqrt{\varepsilon}} \int_{j\varepsilon}^{(j+1)\varepsilon} e^{-2\pi i n\omega\varepsilon^{-1}} \alpha^{(2)}_{\omega\omega'} d\omega. \tag{3.2.48}$$

By substituting (3.2.42) into (3.2.48) for $j \gg \varepsilon$ and $n \gg \varepsilon$, we obtain

$$\begin{aligned} |\alpha^{(2)}_{jn\omega'}| &= \left| \frac{P^-_\omega}{2\pi\sqrt{\omega}} \Gamma\left(1 - \frac{i\omega}{\kappa}\right) \frac{1}{\sqrt{\varepsilon\omega'}} \right| \\ &\times \left| \int_{j\varepsilon}^{(j+1)\varepsilon} \exp\left[i\omega'' \left(-\frac{2\pi n}{\varepsilon} + \frac{\log\omega'}{\kappa} \right) \right] d\omega'' \right| \\ &= \left| \frac{P^-_\omega}{\pi\sqrt{\omega}} \Gamma\left(1 - \frac{i\omega}{\kappa}\right) \frac{\sin\frac{1}{2}\varepsilon z}{z\sqrt{\varepsilon\omega'}} \right|, \end{aligned} \tag{3.2.49}$$

where $\omega = j\varepsilon$ and $z = \frac{1}{\kappa}\ln\omega' - \frac{2\pi n}{\varepsilon}$. The relation (3.2.45) remains unchanged in these transformations

$$|\beta^{(2)}_{jn\omega'}| = e^{-\pi\omega/\kappa}\, |\alpha^{(2)}_{jn\omega'}|, \tag{3.2.50}$$

and the proportionality coefficient $|\beta_{jn\omega'}|$ thus behaves as $\sqrt{\frac{\varepsilon}{\omega'}}$. The logarithmic divergence of the integral can be controlled by an effect of ε. Then, in the wave-packet mode p_{jn}, the expectation value of the number of particles created and emitted to infinity $\mathcal{J}^-$ is given by

$$N_{jn} = \int_0^\infty |\beta^{(2)}_{jn\omega'}|^2 d\omega'. \tag{3.2.51}$$

To evaluate this we consider the wave-packet p_{jn} propagating backwards from $\mathcal{J}^+$. Until now, we have disregarded the change in the amplitude of the wave function. Nevertheless, a fraction of the particles would actually be scattered at the horizon. Consequently, a fraction of the wave packet with

$$\rho(f_{jn}, f^*_{jn}) = 1$$

as in (3.2.6) will be scattered by the static Schwarzschild field and the others will enter the collapsing body. The wave packets which reach $\mathcal{J}^+$

would satisfy

$$\rho(p_{jn}, p^*_{jn}) = \Gamma_{jn} < 1$$

where Γ_{jn} is called the gray body factor. Then, the orthonormal condition (3.2.12) would become

$$\Gamma_{jn} = \int_0^\infty \left(|\alpha^{(2)}_{jn\omega'}|^2 - |\beta^{(2)}_{jn\omega'}|^2 \right) d\omega'. \tag{3.2.52}$$

Using (3.2.50) in (3.2.52), we obtain

$$\int_0^\infty |\beta^{(2)}_{jn\omega'}|^2 d\omega' = \frac{\Gamma_{jn}}{\exp(\frac{2\pi\omega}{\kappa}) - 1}. \tag{3.2.53}$$

From (3.2.51) and (3.2.53), we find

$$N_{jn} = \frac{\Gamma_{jn}}{\exp(\frac{2\pi\omega}{\kappa}) - 1}, \tag{3.2.54}$$

giving the total number of particles created in the mode $p^{(2)}_{jn}$. Ignoring the gray body factor, the total number of particles N is given by

$$N = \frac{1}{\exp(\frac{2\pi\omega}{\kappa}) - 1}. \tag{3.2.55}$$

As given by the Bose-Einstein statistics in thermodynamics, the total number of particles for the black body radiation is

$$N = \frac{1}{\exp(\frac{\omega}{T}) - 1}, \tag{3.2.56}$$

where ω is the frequency of the particle and T is temperature of the system. Thus a black hole which has Hawking temperature T_{BH}, defined by

$$T_{BH} = \frac{\kappa}{2\pi}, \tag{3.2.57}$$

behaves as a black body and the black hole continuously emits radiation. Here κ is the surface gravity of the black hole. It shows that the temperature of the black hole is proportional to its surface gravity, as already conjectured by the corresponding relationship between black hole physics and thermodynamics. The black hole entropy S_{BH} is also found from a thermodynamic consideration:

$$dS_{BH} = \frac{1}{T_{BH}} dM. \tag{3.2.58}$$

On integration (3.2.58) gives

$$S_{BH} = \frac{A}{4}, \tag{3.2.59}$$

where A is the black hole area. It shows that the black hole entropy is proportional to its area. Since a black hole can radiate matter, it has temperature and entropy.

From the above analysis Hawking suggested that a black hole can evaporate and the temperature of a Schwarzschild black hole is given by

$$T_{BH} = \frac{1}{8\pi M}, \tag{3.2.60}$$

which is obtained with $\kappa = \frac{1}{4M}$ in (3.2.57). This shows that the temperature of the black hole is inversely proportional to its mass. We thus find that the temperature of the black hole is higher as its mass is smaller and the temperature is lower as the mass is larger. It is known that the temperature for a black hole with the solar mass is much lower than the temperature of the cosmic microwave background (CMB) radiation. Black holes of this size are absorbing radiation faster than they are emitting it and so they are increasing their masses. There might be tiny black holes in the early universe [166, 167]. If the temperature of a tiny black hole is higher than the temperature of the CMB radiation, such tiny black holes would then be radiation-dominated. As this tiny black hole radiates matter, its mass reduces but the temperature increases. As a result, it increasingly radiates matter. Thus, it would be expected that the black hole will evaporate at some point.

Hawking radiation can also be shown to occur in the cases of other black holes. For a Kerr-Newman black hole, the relation (3.2.55) is extended to

$$N = \frac{1}{\exp[\frac{2\pi}{\kappa}(\omega - m\Omega_H - e\Phi_H)] - 1}. \tag{3.2.61}$$

Here, m is a magnetic quantum number of the emitted matter field, e is the charge of the matter field, Ω_H is the angular velocity of the black hole, Φ_H is the electrical potential of the black hole and κ is given by not $\frac{1}{4M}$ but by $\frac{4\pi(r_+ - M)}{A}$ as in (2.2.9).

Because the black holes actually have temperature and entropy, the first law of the black hole physics is written as

$$dM = T_{BH}dS_{BH} + \Omega_H dJ + \Phi_H dQ, \tag{3.2.62}$$

and the second law is given by

$$\Delta S_{BH} + \Delta S_C = \Delta(S_{BH} + S_C) \geq 0, \tag{3.2.63}$$

where S_C is the entropy of the matter outside the black hole. It was depicted that black holes can radiate by using quantum effects. As was demonstrated in section 2.4, a part of energy can be extracted from a rotating black hole by the Penrose process. However, this cannot break the classical Hawking's black hole area theorem (2.5.1). On the contrary, Hawking radiation decreases the black hole area, and the classical Hawking's black hole area theorem is violated [168]. Thus the second law as given in (3.2.63) needs to be generalized. This consideration was already carried out by Bekenstein [1, 153] before Hawking's original paper [2]. The generalized second law always holds in any physical process.

3.3 Unruh Effect

The Unruh effect is the flat space analog of Hawking's effect. An observer who is accelerating with respect to the conventional zero-temperature Minkowski vacuum state will observe a thermal spectrum of particles, with a temperature that depends linearly on the magnitude of the acceleration. The Unruh effect has played a crucial role in our understanding that the particle content of a field theory is observer dependent. This effect is important as a way to understand the phenomenon of particle emission from black holes and cosmological horizons.

As derived in (3.2.57), an observer outside a Schwarzschild black hole experiences a bath of thermal Hawking radiation of temperature [2, 3]

$$T = \frac{g}{2\pi}, \tag{3.3.1}$$

because

$$\kappa = \frac{1}{4M} = \frac{M}{r_s^2} = g$$

with $r_s = 2M$ the Schwarzschild radius. Here g is the local acceleration due to gravity. In some manner, the background gravitational field interacts with the quantum fluctuations of the electromagnetic field. It then results that energy can be transferred to the observer as if he(she) were in an oven filled with black-body radiation. Naturally, the effect is strong only when the background field is strong. An utmost example is that if the temperature $T \sim 1$ MeV or more, virtual electron-positron pairs emerge from the vacuum into real particles.

Shortly after Hawking's discovery it was shown by Unruh [9, 169] that an accelerated observer in a gravity-free environment experiences the same physics (locally) as an observer at rest in a gravitational field. Thus, in zero gravity, an accelerated observer should find him(her)self in a thermal bath of radiation characterized by temperature

$$T = \frac{a}{2\pi}, \tag{3.3.2}$$

where a is the acceleration in the observer's instantaneous rest frame. There are many papers on the subject (see, for instance, [170–179]). A simple derivation of the temperature Eq.(3.3.2) is given in Appendix-G. The Unruh effect is radically changing the notion of the vacuum and the debunking the idea that "particles" are fundamental entities in quantum field theory.

The existence of Unruh radiation clarifies aspects of the equivalence between radiation in uniform acceleration and in a uniform gravitational field. The results of Hawking-Unruh radiation indicate profound consequences for the unification of quantum field theory and general relativity and initiated intense debates over unresolved questions that are still actively investigated today. In particular, if black holes are not really "black," then question arises regarding the ultimate fate of black holes. Do they end up with naked singularities, or such occurrences will be prevented by a long-sought fusion of quantum mechanics and general relativity into a coherent theory of quantum gravity? Furthermore, consider that a quantum mechanical pure state is dropped into a black hole and there results a pure thermal (uncorrelated) radiation. Then how does one explain the apparent non-unitary evolution of a pure state to a mixed state?

3.4 Damour-Ruffini Method

In this section we briefly describe the method of calculating black-hole evaporation, developed by Damour and Ruffini [7] through a generalization of the classical approach of barrier penetration to curved spaces endowed with future horizons. This method allows one to recover most directly the Hawking's result described in the preceding section.

It had been shown that the total mass-energy of a black hole can have three components: the irreducible mass, the Coulomb energy, and the rotational energy [140, 141, 160]. The Coulomb and the rotational energy could be extractable in principle by a set of classical gedanken experiments [180–182]. However, Hawking [2, 3] suggested that by vacuum polarization processes the irreducible mass of a black hole could be radiated away. Damour and Ruffini [7] developed a treatment of barrier penetration [183–186], giving a clear understanding of this phenomenon. They considered (i) a Kerr-Newman geometry endowed with a vacuum future horizon, (ii) a massive charged scalar field Φ fulfilling the covariant Klein-Gordon equation in that backgroun geometry, and (iii) assumed analyticity properties of the wave function Φ in the complexified manifold. There exist explicit asymptotic expressions for the field Φ near the horizon and at spatial infinity. Physically, inside the horizon, a spacelike Killing vector ξ_t exists, which allows a classical particle as "seen" from infinity to reach a negative-energy state. This phenomenon allows, in the quantum description, an antiparticle to reach positive-energy states. These states are classically confined in the black hole, but quantum mechanically can be tunneled out by a wave function “over” the horizon which gives rise to the creation of a pair: one particle (positive energy) going out and one antiparticle (negative energy) falling back towards the singularity. Obviously, this approach only requires the existence of a future horizon and is totally independent of any dynamical details of the process leading to the formation of this horizon.

The most general black hole is the Kerr-Newman black hole

$$\begin{aligned} ds^2 = \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 - \frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\varphi)^2 \\ + \frac{\sin^2\theta}{\Sigma}[(r^2+a^2)d\varphi - adt]^2, \end{aligned} \tag{3.4.1}$$

where

$$\Delta = r^2 - 2Mr + a^2 + e^2 = (r - r_+)(r - r_-),$$

$$r_\pm = M \pm \sqrt{M^2 - a^2 - e^2}, \quad \Sigma = r^2 + a^2\cos^2\theta.$$

The parameters M, e, a are respectively the mass, charge, and specific angular momentum of the black hole. The future event horizon is denoted by r_+. Introducing the tortoise coordinate r_*,

$$dr_* = \frac{r^2 + a^2}{\Delta}dr, \tag{3.4.2}$$

we have, when $r \sim r_+$ ($r > r_+$),

$$r_* \sim \frac{1}{2\kappa}\ln(r - r_+) \tag{3.4.3}$$

with

$$\kappa = \frac{1}{2} \cdot \frac{r_+ - r_-}{r_+^2 + a^2}. \tag{3.4.4}$$

For the case of a Schwarzschild black hole ($a = 0 = e$), the scalar function Φ satisfying the covariant Klein-Gordon equation can be given by (see Appendix-E)

$$\Phi_\omega = \frac{E_\omega(r_*, t)}{r\sqrt{2\pi|\omega|}} Y_l^m(\theta, \varphi). \tag{3.4.5}$$

Here, Y_l^m is the usual spherical harmonics, while E_ω is monochromatic in time. We take $\omega > 0$, i.e., a flux of particles at infinity, and treat the flux of antiparticles as usual by charge conjugation. There exist two linearly independent solutions just outside the horizon r_+:

$$E_\omega^{\text{in}} = e^{-i\omega(t+r_*)} = e^{-i\omega v}, \tag{3.4.6}$$

and

$$\begin{aligned} E_\omega^{\text{out}} &= e^{-i\omega(t-r_*)} = e^{2i\omega r_*} e^{-i\omega v} \\ &= (r - 2M)^{i4M\omega} e^{-i\omega v}, \end{aligned} \tag{3.4.7}$$

where $v(= t + r_*)$ is the advanced Eddington-Finkelstein coordinate, in which the metric is well behaved and analytic over the whole coordinate range $0 < r < +\infty$, $-\infty < v < +\infty$, including r_+.

Equation (3.4.6) corresponds to a wave purely ingoing on r_+ and can be extended inside $r < 2M$. On the other hand, equation (3.4.7) represents an outgoing wave which has an infinite number of oscillations as $r \to 2M$ and hence cannot be directly extended to the region inside r_+. In the following we use the well-known result of flat-space relativistic wave theories and generalize it to analytic curved spaces. The wave function $\Phi(x)$ describing a particle state (positive frequencies) can be analytically extended to complex points of the form $z = x + iy$ if y is in the past cone. Likewise, for an antiparticle state (negative frequencies) y has to lie in the future cone.

In Finkelstein coordinates, the vector $\frac{\partial}{\partial r}$ is everywhere null and past-directed. So, the prescription $r \to r - i0$ will give the unique continuation of (3.4.6) describing an antiparticle state,

$$\bar{P}_\omega = \bar{N}_\omega \Phi_\omega^{\text{out}}(r - 2M - i0). \tag{3.4.8}$$

Introducing the Heaviside step function Y,

$$Y(r) = \begin{cases} 1, & r \geq 0, \\ 0, & r < 0, \end{cases} \tag{3.4.9}$$

we can write

$$\begin{aligned} \bar{P}_\omega = \bar{N}_\omega \big[& Y(r - 2M)\Phi_\omega^{\text{out}}(r - 2M) \\ & + e^{4\pi M\omega} Y(2M - r)\Phi_\omega^{\text{out}}(2M - r) \big], \end{aligned} \tag{3.4.10}$$

where $\bar{N}_\omega$ is a normalization factor such that

$$\langle \bar{P}_{\omega_1}, \bar{P}_{\omega_2} \rangle = -\delta(\omega_1 - \omega_2)\delta_{l_1 l_2}\delta_{m_1 m_2}. \tag{3.4.11}$$

Since Φ_ω was already normalized, it follows that

$$|\bar{N}_\omega|^2 = \frac{1}{e^{8\pi M\omega} - 1} \tag{3.4.12}$$

In equation (3.4.10) the wave $\bar{P}_\omega$ has two parts, one is outgoing from the horizon and another is falling on the singularity (displayed in Fig.

3.4). The probability flux carried away by this outgoing wave is simply $|\bar{N}_\omega|^2/2\pi$ per unit of time. Only a fraction Γ of this flux will be transmitted to infinity, where Γ is the transmission coefficient of the potential and centrifugal barrier, and a fraction will be partially back-scattered into the hole. Using (3.4.12) the outgoing flux of particles at infinity is given as

$$\frac{\Gamma}{2\pi(e^{8\pi M\omega} - 1)} \tag{3.4.13}$$

per unit of time and per unit range of frequency. This is Hawking's result [2, 3], obtained in the preceding section.

Figure 3.4 also displays an antiparticle wave of strength

$$|\bar{N}_\omega|^2 e^{8\pi M\omega} = 1 + |\bar{N}_\omega|^2$$

with positive energy flux outgoing in the past from the singularity, which can always be interpreted as a negative-energy flux of antiparticles $|\bar{N}_\omega|^2$ ingoing in the future toward the singularity.

Let us now consider a scalar field is in the Kerr-Newman geometry. Using the analogs of the Eddington-Finkelstein coordinates and a corresponding gauge transformation for the electromagnetic field, it can be found that the normalized ingoing wave $\Phi_\omega^{\rm in}$ is regular at r_+ but $\Phi_\omega^{\rm out}$ contains a factor $(r - r_+)^{i(\omega-\omega_o)/\kappa}$, where κ is given by (3.4.4) and

$$\omega_o = m\Omega + \epsilon V, \tag{3.4.14}$$

m being the usual azimuthal quantum number of the particle, ϵ its charge, and Ω and V being respectively the angular velocity and the electric potential of the black hole. Because the vector $\partial/\partial r$ in these coordinates are still null and past-directed, we can describe an antiparticle by the same

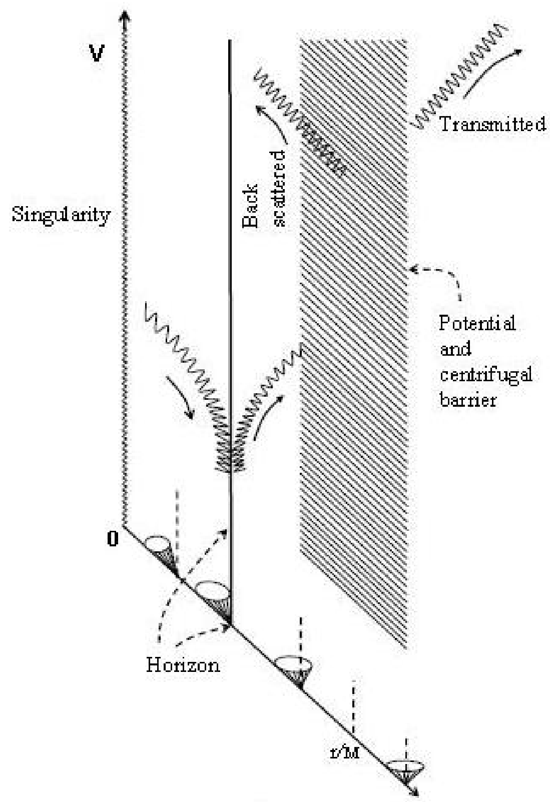

Figure 3.4: Splitting of the antiparticle state $\bar{P}_\omega$ into two components in usual Eddington-Finkkestein coordinates.

prescription as before, $r \to r - i0$. This yields the splitting similar to demonstrated by (3.4.10):

$$\bar{P}_\omega = \bar{N}_\omega \big[Y(r - r_+) \Phi_\omega^{\text{out}}(r - r_+) \\ + e^{\pi(\omega - \omega_o)/\kappa} Y(r_+ - r) \Phi_\omega^{\text{out}}(r_+ - r) \big]. \qquad (3.4.15)$$

In the present situation, however, there occur two different situations for an energy of the wave, $\omega > \omega_o$ or $\omega < \omega_o$.

For the case $\omega > \omega_o$, the norm of $\Phi_\omega^{\rm out}$ is positive and its flux is $+(2\pi)^{-1}$. So, one finds

$$|\bar{N}_\omega|^2 = \frac{1}{e^{2\pi(\omega-\omega_o)/\kappa} - 1}$$

and as usual only a positive fraction Γ of this flux is transmitted to infinity through the combined potential and centrifugal barriers.

For $\omega > \omega_o$, the norm of $\Phi_\omega^{\rm out}$ is negative and its flux is negative as well, $-(2\pi)^{-1}$ (antiparticles). So, one gets

$$|\bar{N}_\omega|^2 = \frac{1}{1 - e^{2\pi(\omega-\omega_o)/\kappa}}.$$

We are now in the condition of level crossing [183–186] between the horizon and spatial infinity so that a negative fraction Γ of this flux will be transmitted to infinity.

Thus one observes in both cases a positive flux at infinity (particles) given by

$$\frac{\Gamma}{2\pi[e^{2\pi(\omega-\omega_o)/\kappa} - 1]} \tag{3.4.16}$$

per unit of time and per unit range of frequency.

The drastic difference between those two regimes appears clearly if one considers for the black-hole's effective temperature the limit $\kappa/2\pi \to 0$. In the case of the Schwarzschild black hole, the particle creation rate then goes to zero. In the Kerr-Newman black hole case the rate goes also to zero if $\omega > \omega_o$, but it tends to $-\Gamma/2\pi$ in the range $\mu < \omega < \omega_o$ [183–186].

3.5 Tunneling Method

In this section we briefly review two basic approaches to model black hole radiation as a quantum tunneling process. These are the null geodesic method [19] and the Hamilton-Jacobi ansatz [35]. The calculation will demonstrate the tunneling of uncharged scalar particles from general non-rotating black holes [44].

3.5.1 Null Geodesic Method

We consider the null geodesic method used by Parikh and Wilczek [19] that followed from the work of Kraus and Wilczek [15, 16, 187]. In this approach Hawking radiation is regarded as a quantum tunneling process. The tunneling barrier is created by the outgoing particle itself, whose trajectory is from the inside of the black hole to the outside, a classically forbidden process. For this tunneling process, the probability of tunneling is proportional to the exponential of (negative) two times the imaginary part of the classical action in the WKB limit. The radius of the black hole shrinks, on account of energy conservation, as a function of the energy of the outgoing particle. In response to the motion of the particle, the horizon shrinks and in this sense the particle creates its own tunneling barrier.

The Schrödinger equation in the WKB approximation gives a wave function of the form

$$\Phi \propto \exp(iI/\hbar),$$

where I is solved along the classically forbidden trajectory and as a result, I will be complex. Then $\Phi\Phi^*$ is a semi-classical tunneling probability for the emitted particle and it can be written in the form:

$$\Gamma \propto \exp(-2\,\mathrm{Im}\,I), \tag{3.5.1}$$

where $\hbar$ has been set equal to unity. The Hamilton-Jacobi ansatz (discussed in the next subsection) also uses this as a starting point of its calculation. However, the Hamilton-Jacobi method applies the WKB approximation to the Klein-Gordon equation instead of the Schrödinger equation. These two methods thus end up differing in how the action is calculated.

In the null geodesic method the only part of the action contributing an imaginary term to the final tunneling probability is $\int_{r_{\text{in}}}^{r_{\text{out}}} p_r dr$, where p_r is the (radial) momentum of the emitted null s-wave. Other contributions to the action I are in general terms of the form $-\int E dt$, $\int p_\varphi d\varphi$, and $\int p_\theta d\theta$ (known from Hamilton's principle) and are ignored because they do not contribute to the final tunneling rate. For a stationary spacetime, the energy integral simply corresponds to $-Et$ which is entirely real and does not contribute to the tunneling probability (3.5.1). The angular terms are also real and hence do not contribute. It is also possible to simply ignore any effects of the angular terms by assuming that the emitted s-wave is only moving radially. In this case the angular terms are automatically zero. Indeed, Kraus and Wilczek solved the most general action for the full system of the shell and the background completely [15, 16, 187] which provides a more explicit proof that only $\int_{r_{\text{in}}}^{r_{\text{out}}} p_r dr$ contributes to the tunneling rate as claimed.

The spacetime of a general non-rotating black hole is described by the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2, \tag{3.5.2}$$

which covers a broad range of black hole metrics. Both $f(r)$ and $g(r)$ in (3.5.2) vanish at the black hole horizon r_o (i.e. $f(r_o) = g(r_o) = 0$). We assume that the black hole is non-extremal, that is, the two functions $f(r)$ and $g(r)$ only have first order zeros at the horizon. In other words, the first derivatives of these functions exist at the horizon and are non-vanishing (i.e. $f'(r_o) \neq 0$, $g'(r_o) \neq 0$).

For the null geodesic method the metric must be converted into Painlevé form [188] so that there will no longer be a singularity at the horizon. This is easily attained through the transformation:

$$t \to t - \int \sqrt{\frac{1-g(r)}{f(r)g(r)}} dr. \tag{3.5.3}$$

In Painlevé form the metric (3.5.2) becomes

$$ds^2 = -f(r)dt^2 + 2\sqrt{f(r)}\sqrt{\left(\frac{1}{g(r)} - 1\right)} drdt + dr^2 + r^2 d\Omega^2. \tag{3.5.4}$$

The Painlevé form of the metric is a prerequisite for the null geodesic calculation. This coordinate system also has a number of interesting features in addition to removing the singularity at the horizon. The metric in this coordinates has the properties that at any fixed time the spatial geometry is flat and at any fixed radius the boundary geometry is the same as that of the unaltered metric (3.5.2). For the metric (3.5.4) the radial null geodesics are given by

$$\dot{r} = \sqrt{\frac{f(r)}{g(r)}} \left[\pm 1 - \sqrt{1-g(r)}\right], \tag{3.5.5}$$

where the $+(-)$ sign corresponds to outgoing(ingoing) null geodesics.

In the spherically symmetric case, the emitted particle (corresponding to the plus sign in (3.4.5)) is taken to be in an outgoing s-wave mode. Since f' and g' are both non-zero at the horizon, $\frac{f(r)}{g(r)}$ is well defined there. So, $\dot{r} = 0$ at the horizon. The imaginary part of the action for an outgoing

s-wave from $r_{\rm in}$ to $r_{\rm out}$ is given by

$$I = \int_{r_{\rm in}}^{r_{\rm out}} p_r dr = \int_{r_{\rm in}}^{r_{\rm out}} \int_0^{p_r} dp'_r dr, \tag{3.5.6}$$

where $r_{\rm in}$ and $r_{\rm out}$ are the respective initial and final radii of the black hole. The trajectory between these two radii is the barrier through which the particle must tunnel.

We assume that the total energy of the spacetime was originally M and that the emitted s-wave has energy $\omega' \ll M$. Utilizing conservation of energy to this approximation, the s-wave moves in a background spacetime of energy $M \to M - \omega'$. We now evaluate the integral and in this regard, we use Hamilton's equation $\dot{r} = \frac{dH}{dp_r}|_r$ to switch the integration variable from momentum to energy ($dp_r = \frac{dH}{\dot{r}}$). This gives

$$I = \int_{r_{\rm in}}^{r_{\rm out}} \int_M^{M-\omega} \frac{dr}{\dot{r}} dH = \int_0^{\omega} \int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{\dot{r}} (-d\omega'), \tag{3.5.7}$$

where $dH = -d\omega'$ as total energy $H = M - \omega'$ with M constant. The $\dot{r}$ is implicitly a function of $M - \omega'$. In particular, this function is known for the Schwarzschild case and then the integral in (3.5.7) can be solved exactly in terms of ω [19]. For a Schwarzschild black hole $f(r) = g(r) = \left(1 - \frac{2M}{r}\right)$ and the radial geodesic with the black hole mass $M - \omega'$ (i.e. when background spacetime is reduced in mass by ω') is given by

$$\dot{r} = \left(1 - \sqrt{\frac{2(M-\omega')}{r}}\right). \tag{3.5.8}$$

Then

$$I = \int_0^{\omega} \int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{1 - \sqrt{2(M-\omega')/r}} (-d\omega'), \tag{3.5.9}$$

and hence

$$\begin{aligned}\operatorname{Im} I &= \operatorname{Im} \int_0^{\omega} +4\pi i(M-\omega')d\omega' \\ &= +4\pi\omega\left(M-\frac{\omega}{2}\right).\end{aligned} \tag{3.5.10}$$

The sign is positive since $r_{\text{in}} > r_{\text{out}}$, because the black hole horizon before emission is located at $r_{\text{in}} = 2M$ and the black hole horizon after emission is at $r_{\text{out}} = 2(M-\omega)$. This was established by Parikh and Wilczek in their paper [19] by changing the order of integration. Inserting this into the expression for the semi-classical emission rate (3.5.1), we have

$$\Gamma \sim \exp(-8\pi\omega\left(M-\frac{\omega}{2}\right) = \exp(+\Delta S_{BH}), \tag{3.5.11}$$

where ΔS_{BH} is the change in Bekenstein-Hawking entropy S_{BH} of the black hole. Considering the lowest order of ω, we find that the expression reduces to $\exp(-8\pi M\omega)$ which is the same as the Boltzmann factor (i.e. $\exp[-\frac{E}{k_B T}]$) for a particle of energy ω at the Hawking Temperature $T_H = \frac{1}{8\pi M}$ (with $k_B = 1$). The ω^2 correction arises from the physics of energy conservation and it (roughly speaking) self-consistently raises the effective temperature of the hole as it radiates. The exact result must be correct which can be seen on physical grounds by considering the limit in which the emitted particle carries away the entire mass and charge of the black hole (corresponding to the transmutation of the black hole into an outgoing shell). There can be only one such outgoing state. Further, there are $\exp(S_{BH})$ states in total. Then statistical mechanics asserts that the probability of finding a shell containing all of the mass of the black hole is proportional to $\exp(-S_{BH})$, as above. However, there has been some question to the validity of the higher order terms of the tunneling rate. This is because it has been claimed that the semi-classical tunneling probability is not invariant under canonical transformations in general [46]. When only the lowest order of ω is used, the resulting Boltzmann factor is invariant under such canonical transformations.

We now return to the general expression for the action (3.5.7). Performing a series expansion in ω, we find

$$\begin{aligned} I &= \int_0^\omega \int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{\dot{r}(r, M-\omega')}(-d\omega') \\ &= -\omega \int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{\dot{r}(r,M)} + \mathcal{O}(\omega^2) \\ &\simeq \omega \int_{r_{\rm out}}^{r_{\rm in}} \frac{dr}{\dot{r}(r,M)}. \end{aligned} \tag{3.5.12}$$

This integral needs to be estimated to proceed any further. Since the black hole decreases in mass as the s-wave is emitted, $r_{\rm in} > r_{\rm out}$ and consequently, the radius of the event horizon decreases. The limits on the integral indicate that, over the course of the classically forbidden trajectory, the outgoing particle starts from $r_{\rm in} = r_o(M) - \varepsilon$, just inside the initial position of the horizon, and crosses the contracting horizon to materialize at $r_{\rm out} = r_o(M-\omega) + \varepsilon$, just outside the final position of the horizon. Here $r_o(M)$ is the location of the event horizon of the original background spacetime before the emission of particles. We use the notation r_o for $r_o(M)$ in the following. With this generalization, no explicit knowledge of the total energy or mass is required because r_o is simply the radius of the event horizon before any particle is emitted.

A pole occurs at the horizon where $\dot{r} = 0$. Having in mind that $f'(r_o)$ and $g'(r_o)$ are both non-zero at the horizon for a non-extremal black hole, we find that $1/\dot{r}$ only has a simple pole at the horizon with a residue of

$\frac{2}{\sqrt{f'(r_o)g'(r_o)}}$. Therefore the imaginary part of the action becomes

$$\operatorname{Im} I = \frac{2\pi\omega}{\sqrt{f'(r_o)g'(r_o)}} + \mathcal{O}(\omega^2). \tag{3.5.13}$$

So, the tunneling probability is

$$\Gamma \sim \exp(-2 \operatorname{Im} I) = \exp[-\beta\omega + \mathcal{O}(\omega^2)] \tag{3.5.14}$$

and the resulting Hawking temperature $T_H = \beta^{-1}$ is

$$T_H = \frac{\sqrt{f'(r_o)g'(r_o)}}{4\pi}. \tag{3.5.15}$$

Obviously, for Schwarzschild black hole the correct result of $T_H = \frac{1}{8\pi M}$ follows once again. For the Reissner-Nordström black hole, $f = g = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$ and its non-extremal case is when $M^2 > Q^2$. Equation (3.5.15) yields a temperature of

$$T_H = \frac{1}{2\pi} \frac{\sqrt{M^2 - Q^2}}{(M + \sqrt{M^2 - Q^2})^2}, \tag{3.5.16}$$

exactly the same as obtained in [19]. When the horizons do not have a simple pole, that is, for extremal black holes, the situations need to be handled separately [44].

3.5.2 Hamilton-Jacobi Ansatz

We now review an alternate method for calculating black hole tunneling radiation that makes use of the Hamilton-Jacobi equation as an ansatz [35]. This method ignores the effects of the particle self-gravitation and is developed by Padmanabhan and his collaborators [26, 27, 28]. In general the method uses the WKB approximation to solve a wave equation. Kerner and Mann further developed [44] and extended this method to model fermion particles [60, 67]. The simplest case to model is scalar particles, and then we need to apply the WKB approximation to the Klein-Gordon equation. The result, to the lowest order of WKB approximation, gives a differential equation. This equation can be solved by plugging in a suitable ansatz, which is chosen by using the symmetries of the space-time to assume separability. After inserting a suitable ansatz, the resulting equation can be integrated along the classically forbidden trajectory, which starts inside the horizon and finishes at the outside observer (usually at infinity). Because this trajectory is classically forbidden, the equation must have a simple pole located at the horizon. Then it is necessary to apply the method of complex path analysis and deflect the path around the pole. Since we are only concerned with calculating the semi-classical tunneling probability, we need to multiply the resulting wave equation by its complex conjugate. Therefore the portion of the trajectory that starts outside the black hole and continues to the observer will not contribute to the final tunneling probability and can be safely ignored. Thus the only part of the wave equation that contributes to the tunneling probability is the contour around the black hole horizon. A visual representation of the deformation of the contour is displayed in Figure 3.5.

We consider a general (non-extremal) black hole metric of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + C(r)h_{ij}dx^i dx^j. \tag{3.5.17}$$

The Klein-Gordon equation for a scalar field Φ is

$$\mathrm{g}^{\mu\nu}\partial_\mu\partial_\nu\Phi - \frac{m^2}{\hbar^2}\Phi = 0. \tag{3.5.18}$$

Using the WKB approximation by assuming an ansatz of the form

$$\Phi(t, r, x^i) = \exp\left[\frac{i}{\hbar} I(t, r, x^i) + I_1(t, r, x^i) + \mathcal{O}(\hbar)\right] \tag{3.5.19}$$

and inserting this into the Klein-Gordon equation, the Hamilton-Jacobi equation to the lowest order in $\hbar$ is obtained as

$$-\left[\mathrm{g}^{\mu\nu}\partial_\mu I \partial_\nu I + m^2\right] + \mathcal{O}(\hbar) = 0. \tag{3.5.20}$$

Also

$$\Gamma \propto |\Phi|^2 = \exp\left(-\frac{2\,\mathrm{Im}\, I}{\hbar}\right). \tag{3.5.21}$$

For the Hamilton-Jacobi ansatz it is common [35] to skip these early steps. So, we simply start a calculation by assuming that the classically forbidden trajectory from inside to outside the horizon is given by

$$\Gamma \propto \exp(-2\,\mathrm{Im}\, I), \tag{3.5.22}$$

setting $\hbar = 1$. The classical action I satisfies the relativistic Hamilton-Jacobi equation

$$\mathrm{g}^{\mu\nu}\partial_\mu I \partial_\nu I + m^2 = 0, \tag{3.5.23}$$

which for the black hole metric is explicitly

$$-\frac{(\partial_t I)^2}{f(r)} + g(r)(\partial_r I)^2 + \frac{h^{ij}}{C(r)}\partial_i I \partial_j I + m^2 = 0. \tag{3.5.24}$$

Its solution can be put in the form

$$I = -Et + W(r) + J(x^i) + K, \tag{3.5.25}$$

where

$$\partial_t I = -E, \quad \partial_r I = W'(r), \quad \partial_i I = J_i, \tag{3.5.26}$$

and K and J_i's are constants (K can be complex). Since ∂_t is the timelike killing vector for this coordinate system, E is detected as the energy of the particle by an observer at infinity. This is because the norm of the timelike killing vector ∂_t at infinity is (minus) unity. Solving for $W(r)$ we obtain

$$W_{\pm}(r) = \pm \int \frac{dr}{\sqrt{f(r)g(r)}} \sqrt{E^2 - f(r)\left(m^2 + \frac{h^{ij} J_i J_j}{C(r)}\right)}, \tag{3.5.27}$$

since the equation was quadratic in terms of $W(r)$. The solution W_+ corresponds to scalar particles moving away from the black hole (i.e. outgoing), while the other solution W_- corresponds to particles moving toward the black hole (i.e. incoming). The action can only get imaginary parts due to the pole at the horizon or from the imaginary part of K. The probabilities of crossing the horizon in each direction are proportional to

$$\mathrm{Prob}[out] \propto \exp[-2\hbar^{-1} \operatorname{Im} I] = \exp[-2\hbar^{-1}(\operatorname{Im} W_+ + \operatorname{Im} K)], \tag{3.5.28}$$

$$\mathrm{Prob}[in] \propto \exp[-2\hbar^{-1} \operatorname{Im} I] = \exp[-2\hbar^{-1}(\operatorname{Im} W_- + \operatorname{Im} K)]. \tag{3.5.29}$$

If the probability is normalized, any incoming particles crossing the horizon have a 100% chance of entering the black hole. For this it is necessary to set $\operatorname{Im} K = -\operatorname{Im} W_-$ and since $W_+ = -W_-$ this implies that the probability of a particle tunneling from inside to outside the horizon is given by

$$\Gamma \propto \exp[-4 \operatorname{Im} W_+], \tag{3.5.30}$$

setting $\hbar = 1$. One may start with an ansatz for the action that does not contain the constant K. In that case it is necessary to take a ratio of (3.5.28) and (3.5.29) to have the correct tunneling rate (3.5.30).

We now integrate (3.5.27) for W_+ around the pole at the horizon. In order to get the correct result, it is important to parameterize in terms of the proper spatial distance [35]. For the null-geodesic method the Painlevé coordinate r was the proper spatial distance. In this case the proper spatial distance between any two points at some fixed t is given by

$$d\sigma^2 = \frac{dr^2}{g(r)} + C(r)h_{ij}dx^i dx^j. \tag{3.5.31}$$

Like the null geodesic method we are only concerned with radial rays. So, the only proper spatial distance we are concerned with is radial

$$d\sigma^2 = \frac{dr^2}{g(r)}. \tag{3.5.32}$$

Using the near horizon approximation,

$$f(r) = f'(r_o)(r - r_o) + \cdots$$

$$g(r) = g'(r_o)(r - r_o) + \cdots \tag{3.5.33}$$

we find the proper radial distance as

$$\sigma = \int \frac{d(r)}{\sqrt{g(r)}} \simeq 2\frac{\sqrt{r - r_o}}{\sqrt{g'(r_o)}}. \tag{3.5.34}$$

Hence, for particles emitted radially, we obtain

$$W_+(\xi) = \frac{1}{\sqrt{g'(r_o)f'(r_o)}} \int \frac{d\xi}{\xi} \sqrt{E^2 - \xi^2 g'(r_o)f'(r_o)\left(m^2 + \frac{h^{ij}J_iJ_j}{C(r_o)}\right)}, \tag{3.5.35}$$

where we have conveniently set $\xi = \sigma/2$. Performing the integration

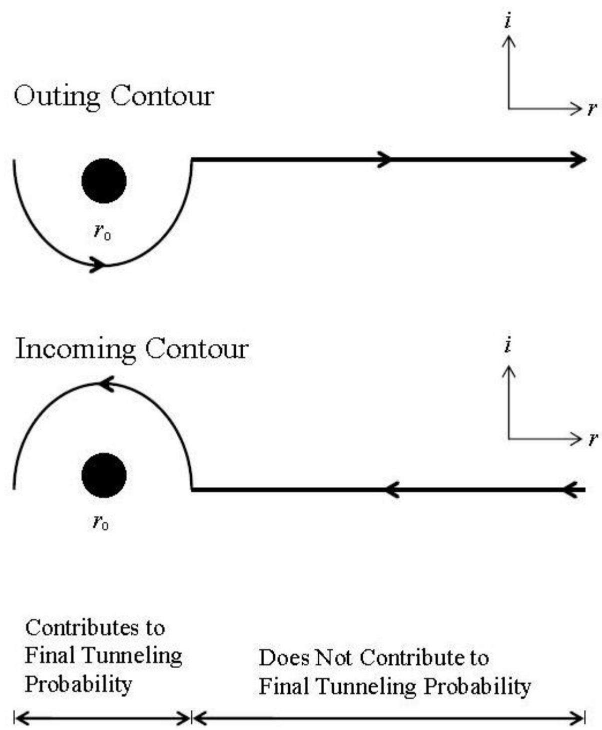

Figure 3.5: Diagram of contours between black hole and observer for outgoing and incoming trajectories

around the pole at the horizon and dropping the + subscript from W, we obtain

$$W = \frac{\pi i E}{\sqrt{f'(r_o)g'(r_o)}}. \tag{3.5.36}$$

This results in a tunneling probability given by

$$\Gamma = \exp\left[-\frac{4\pi}{\sqrt{f'(r_o)g'(r_o)}}E\right], \tag{3.5.37}$$

and yields the usual Hawking temperature

$$T_H = \frac{\sqrt{g'(r_o)f'(r_o)}}{4\pi}. \tag{3.5.38}$$

However, the correct Hawking temperature can be derived by parameterizing the outgoing probability in terms of the proper radial distance and ignoring the incoming probability [35, 44].

Chapter 4

Charged Particles' Hawking Radiation via Tunneling of both Horizons from Reissner-Nordström-Taub-NUT Black Holes

In some recent derivations thermal characters of the inner horizon have been employed; however, the understanding of possible role that may play the inner horizons of black holes in black hole thermodynamics is still somewhat incomplete. Motivated by this problem we investigate Hawking radiation of the Reissner-Nordström-Taub-NUT (RNTN) black hole by considering thermal characters of both the outer and inner horizons [83]. The work is presented in this chapter. We apply Damour-Ruffini method and the thin film brick wall model to calculate the temperature and the entropy of the inner horizon of the RNTN black hole. The inner horizon admits thermal character with positive temperature and entropy proportional to its area, and it thus may contribute to the total entropy of the black hole in the context of Nernst theorem. Considering conservations of energy and charge and the back-reaction of emitting particles to the spacetime, the emission spectra are obtained for both the inner and outer horizons. The total emission rate is the product of the emission rates of the inner and outer horizons, and it deviates from the purely thermal spectrum and can bring some information out. Thus, the result can be treated as an explanation to the information loss paradox.

The chapter is structured as follows. In the section 4.1, we present an introduction concerning the work of this chapter. In section 4.2, we calculate the temperature of inner horizon and point out that there exists a quantum effect, "Hawking absorption," at the inner horizon. In section 4.3, we redefine the entropy of the black hole and show that the redefined entropy satisfies the Nernst theorem. We derive the Bekenstein-Smarr formula using the inner horizon parameters, which shows that the first law of black hole thermodynamics is also tenable at the inner horizon. In section 4.4, considering conservation of energy and charge and taking into account the particles' back-reaction, we investigate tunneling effect including the inner horizon of the RNTN black hole. The result shows that the total tunneling rate is in agreement with the Parikhs standard result and there is no loss of information. Finally, in section 4.5, we present our concluding remarks.

4.1 Introduction

The signifying discovery of Stephen Hawking (reviewed in chapter 3) that quantum mechanically black holes radiate thermal radiation with a spectrum similar to that of a black body. This initiated a great development in the research of black hole thermodynamics. Since Hawking radiation is an exact thermal spectrum [5], there arise two obvious disputes: the first is information loss, which states that the black-hole radiation does not take any information of an inner matter of the black hole. Thus all information including unitary property will be lost by a vaporized black hole. It means that the pure quantum state will decay to a mixed state. The second dispute is regarding the reaction of the radiation to the spacetime. When the black hole produces Hawking radiation, the state parameters (energy and charge) describing the black hole will fluctuate. This effect was not considered in the past. Hawking derived the black-hole radiation as purely thermal spectrum only under the condition that the spacetime is invariable.

Till now there have been proposed at least three kinds of methods to resolve the two problems, two of which are the semi-classical approach proposed by Parikh and Wilczek (reviewed in subsection 3.5.1), and the Hamilton-Jacobi method (reviewed in subsection 3.5.2 of chapter 3). In addition, using the Damour-Ruffini method (reviewed in section 3.4 of chapter 3), Liu has proposed a new method [71] to investigate Hawking radiation of massive Klein-Gorden particles from a Reissner-Nordström black hole. It leads to the same terminations as the previous works, when conservation of energy and the particles' back-reaction are taken into consideration.

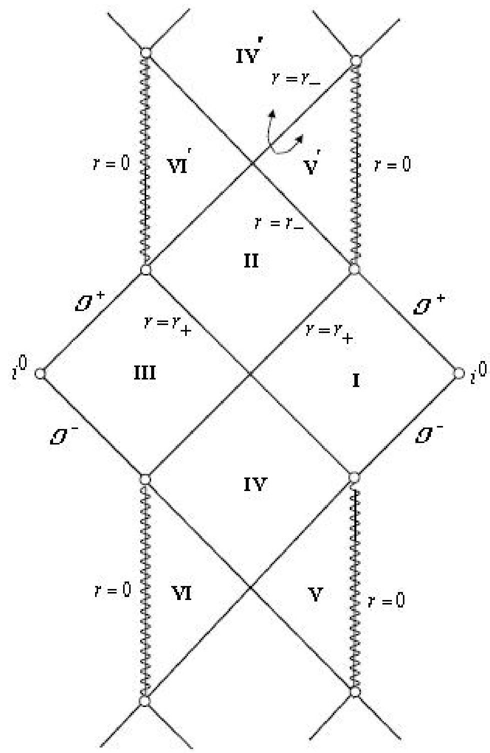

Figure 4.1: Maximally extended Reissner-Nordström spacetime.

As illustrated in Fig. 3.1 (in chapter 3) for the Schwarzschild black hole case, an observer outside the hole observes a particle flux which seems to come out from the black hole. Like the event horizon of a Schwarzschild black hole, the outer horizon of a Reissner-Nordström black hole radiates. However, it is interesting to see what kind of processes is predicted by the virtual pair production mechanism if one looks at the inner horizon of the Reissner-Nordström black hole. Think of a maximally extended

Reissner-Nordström spacetime (see Fig. 4.1). It is obvious that the causal relationship between the regions V′ and IV′ is analogous to that between the regions I and II, respectively. Hence, as shown in the Fig. 4.1, a virtual particle-antiparticle pair which grows very close to the inner horizon $r = r_-$ in the region V′ can avoid annihilation if either the particle or the antiparticle falls into the region IV′ and the other one remains in V′. Thus the pair production mechanism indicates that the inner horizon does radiate and this radiation is directed inwards, i.e., towards the singularity. Nevertheless, this does not provide any information about the radiation itself. It remains unclear whether the inner horizon radiates particles or antiparticles.

Using analytic continuation of the Klein-Gordon field, Wu and Cai [84, 85, 86] have performed explicit calculation considering the radiation of the inner horizons. Their analysis gives for the inner horizon the negative temperature which seems to contradict the general attitude towards the black hole thermodynamics [189, 190, 191, 192] and the very foundations of thermodynamics itself. So, the true nature of the inner horizon radiation is still somewhat unclear.

In Ref. [87] Peltola and Mäkelä have found creation of virtual particle-antiparticle pairs at the inner horizon of a maximally extended Reissner-Nordström spacetime such that real particles with positive energy and temperature are emitted towards the singularity from the inner horizon and antiparticles with negative energy are radiated away from the singularity through the inner horizon. If the backscattering effects are neglected, these antiparticles emitted away from the singularity by the inner horizon will go through the intermediate region between the horizons of the maximally extended Reissner-Nordström spacetime, and finally they will come out of the white hole—at least when the black hole is almost extreme. Thus, there is found a new effect for maximally extended Reissner-Nordström spacetimes which is called “white hole radiation”. The energy spectrum of the antiparticles leads to a positive temperature for the white hole horizon. So, in addition to the radiation effects of black hole horizons, the white hole horizon radiates. The quantum effect at the outer horizon causes the black hole radiation, whereas the white hole radiation is caused by the quantum effects at the inner horizon of the Reissner-Nordström black hole.

There is still no complete knowledge of understanding the possible role that plays the black hole's inner horizon in the black hole thermodynamics. As suggested by Bekenstein, the entropy of a black hole is proportional to the area of its event horizon surface [1, 2]. The temperature of the black hole is described by the surface gravity of the event horizon [151]. There is an open problem on black hole entropy [73, 74, 75]. The Nernst formulation of the third law of ordinary thermodynamics (often referred to as the Nernst theorem) demands that the entropy of a system must vanish as its temperature goes to zero. This assertion is commonly believed to be a fundamental law of thermodynamics. But the entropy of a black hole is non-vanishing as its temperature goes to absolute zero [76, 77].

As studied in [78] the inner horizon can have thermal character and the thermodynamics system of the black hole then is composed of two subsystems: the outer horizon and the inner horizon. The work of this chapter is to show that the tunneling effect of the inner horizon might have to be taken into account as there exists thermal characters of the inner horizon. Recently, Ren investigated thermodynamics properties of the inner horizon of a Kerr-Newman black hole [81] and tunneling effect of two horizons from a Reissner-Nordström black hole [82]. The result is in agreement with Parikh's work and shows no loss of information.

In this chapter we calculate, following Liu's method [71] which is based on the Damour-Ruffini method (reviewed in section 3.4 of chapter 3), the temperature of the inner horizon of the Reissner-Nordström-Taub-NUT black hole which is the Reissner-Nordström black hole generalized with the NUT parameter and prove the existence of thermal characters of the inner horizon. Like the RN black hole [82, 87] the radiation emitted by the inner horizon of the RNTN black hole is directed towards the singularity $r = 0$ and the observer at rest with respect to the inner horizon must be situated inside the two-sphere $r = r_-$. Hence, the roles of the ingoing and the outcoming modes interchange. The inner horizon emits particles inside the inner horizon with a positive temperature. When real particles with energy ω are emitted towards the singularity from the inner horizon, it is necessary to maintain a local energy balance that antiparticles with energy $-\omega$ are emitted away from the singularity through the inner horizon. The process is analogous to the one which takes place at the outer horizon according to the Hawking effect—at the outer horizon antiparticles go in and particles come out. This is true at the inner horizon as well. The real particle remains inside the inner horizon and finally meets with the singularity, while the antiparticle enters the intermediate region between the horizons. One may speculate on the possibility that it travels across the intermediate region and finally comes out from the white hole horizon, if the backscattering effects are neglected. However, the situation is quite complicated because the vacuum states corresponding to a freely falling observer near the inner horizon of the black hole and the white hole horizon are completely different. The analysis in [87] predicts that not only does the black hole horizon emit thermal radiation with a black body spectrum but thermal radiation is emitted by the white hole horizon as well. Thus outside the black hole there exists two simultaneous radiation processes: the normal black hole radiation, and the “white hole radiation” which is caused by the pair creation effects at the inner horizon. The white hole radiation contains only antiparticles with negative energy and this may be understood as an absorption of energy by the white hole horizon. However, this feature contradicts with the classical results in a similar way as does the evaporation process at black hole horizons.

The RNTN spacetime is stationary and the Killing vector field $(\partial/\partial t)^a$ is time-like in both the regions outside the outer horizon and inside the inner horizon. Thus the surface gravity on the inner horizon can be well defined. Further, the entropy of the inner horizon is also proportional to its area. We calculate the inner horizon entropy by applying thin film brick wall model [88] which is based on the brick wall model proposed by ’t Hooft [89]. The entropy obtained for the inner horizon is also proportional to its area and the cut-off factor is 90β, which is same as in the calculation of the entropy of the outer horizon. The entropy of the RNTN black hole should include the contributions of both the outer and inner horizons. The redefined entropy vanishes as the temperature of the RNTN black hole approaches zero and thus the Nernst theorem is satisfied.

The RNTN black hole has the metric [193]

$$ds^2 = -\Delta(dt + \Omega d\varphi)^2 + \Delta^{-1}dr^2$$

$$+(r^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2), \qquad (4.1.1)$$

where

$$\Delta = \frac{r^2 - 2Mr + Q^2 - n^2}{r^2 + n^2} = \frac{(r - r_+)(r - r_-)}{r^2 + n^2},$$
$$\Omega = 2n\cos\theta + \Omega', \quad Q^2 = Q_{\rm el}^2 + Q_{\rm mag}^2,$$
$$Q_{\rm el} = Q\frac{r^2 - n^2}{r^2 + n^2}, \quad Q_{\rm mag} = \frac{2Qnr}{r^2 + n^2}. \tag{4.1.2}$$

The horizons of the black hole are located at

$$r_\pm = M \pm \sqrt{M^2 + n^2 - Q^2}. \tag{4.1.3}$$

The electric and magnetic potentials associated with this metric are respectively given by

$$V_{\rm el} = -\frac{Qr}{r^2 + n^2} \quad \text{and} \quad V_{\rm mag} = \frac{Qn}{r^2 + n^2}, \tag{4.1.4}$$

while the associated electromagnetic field is

$$F = Q\frac{r^2 - n^2}{(r^2 + n^2)^2}(dt + \Omega d\varphi) \wedge dr + \frac{2Qnr}{r^2 + n^2}\sin\theta d\theta \wedge d\varphi. \tag{4.1.5}$$

Here, M is the mass, $Q_{\rm el}$ the electric charge, $Q_{\rm mag}$ the magnetic charge, and n the NUT (gravitational monopole) charge of the black hole. The NUT charge plays the role of a magnetic mass by inducing a topology in the Euclidean section at infinity that is a Hopf fibration of a circle over a 2-sphere. In a recent work [194], the NUT parameter has been interpreted as generating a "rotational effect". The constant Ω' is set equal to $-2n\,(2n)$ to make the half-axis $\theta = 0\,(\theta = \pi)$ explicitly regular, leaving the other half-axis—the Misner string—singular, since $d\varphi$ is not a well-behaved one-form at $\theta = o, \pi$. Because changing Ω' from $-2n$ to $2n$ can be reproduced by changing the time coordinate from t to $t' = t - 4n\varphi$, both half-axes can be made regular. This is because of closed timelike curves since it requires that both t and t' should be periodic with period $8\pi n$. The spacetime of the metric (4.1.1) is asymptotically flat and its axial-symmetry is not caused by the rotation of the black hole.

The RNTN metric (4.1.1) is Kerr-Newman-like in regard to that it has a crossed spacetime metric component $\mathrm{g}_{t\varphi}$ which generates gravimagnetic effects. In the Kerr-Newman metric the cross term breaks spherical symmetry and produces an ergosphere and frame dragging. On the contrary, the cross term in (4.1.1) does not generate ergosphere, but it produces an effect analogous to the dragging of inertial frames. It is interesting that the metric (4.1.1) represents (i) the Taub-NUT black hole for $Q = 0$; (ii) the magnetically charged RN black hole [195] for $n = 0$, $Q_{\rm el} = 0$, $0 < Q_{\rm mag} < M$; (iii) the generic RN black hole for $n = 0$, $Q_{\rm mag} = 0$, $0 < Q_{\rm el} < M$; (iv) the extremal RN black hole for $Q = M$, $n = 0$; and (v) the Schwarzschild black hole for $Q = 0$, $n = 0$. The extremely charged RN black hole represent an extreme limit in the context of the cosmic censorship hypothesis. Because the body with charge equal to higher than the extremal value is undressed by the event horizon and produces a naked singularity [152, 196]. If the horizon is sufficiently small, the magnetically charged RN solution develops a classical instability in the context of spontaneously broken gauge theories, which has significant implications for the evolution of a magnetically charged black hole [195]. It leads, in particular, to the possibility of evaporating a black hole completely, leaving in its place a nonsingular magnetic monopole. The magnetic monopole hypothesis was propounded by Dirac [197] relatively long ago. The ingenious suggestion by Dirac that magnetic monopole does exist in nature, but it was neglected due to the failure to detect such objects. In recent years, the development of gauge theories has shed new light on it and the string theory [107] also admits the existence of such objects. The Taub-NUT black hole plays an important role in the conceptional development of general relativity and in the construction of brane solutions in string theory and M-theory [198, 199, 200]. As “a counter example to almost anything” [96], the Taub-NUT spacetime has been of particular interest in recent years. It plays the role in furthering our understanding of the AdS/CFT correspondence [44, 93, 94, 95]. The existence of the closed time-like geodesics violates the causality condition. The half-closed time-like geodesics in Taub area can be explored in NUT area, so the naked singularity exists. Meanwhile, its angular velocity is zero and no super-radiation occurs at the event horizon. Hawking radiation from the Taub-NUT black hole has been investigated in [201] and the result is in accordance with Parikh and Wilczek's opinion.

4.2 Temperature of Inner Horizon

The metric (4.1.1) admits a timelike Killing vector field ξ^μ. Hence, from the definition of the surface gravity [152]

$$\kappa^2 = -\frac{1}{2}(\nabla^a \chi^b)(\nabla_a \chi_b)$$

we obtain [152, 196]

$$\kappa = -\frac{1}{2}\left(\sqrt{\frac{g^{rr}}{-g_{tt}}}\frac{dg_{tt}}{dr}\right)_{\text{Horizon}}, \tag{4.2.1}$$

which gives the surface gravity of the outer horizon,

$$\kappa_+ = \frac{r_+ - r_-}{2(r_+^2 + n^2)}. \tag{4.2.2}$$

Since $(\frac{\partial}{\partial t})^a$ is a time-like Killing vector field in the region $r < r_-$, (4.2.1) is suitable to the surface gravity of the inner horizon κ_- and we obtain

$$\kappa_- = -\frac{r_+ - r_-}{2(r_-^2 + n^2)}. \tag{4.2.3}$$

The surface gravity of the inner horizon κ_- is negative, since it is directed to the singularity, not to the horizon, opposite to κ_+ which is directed to the outer horizon. The outer horizon of the RNTN black hole is a future horizon for the observer outside the hole $r > r_+$, while the inner horizon is a "past horizon" for the observer inside the hole $r < r_-$. This means that the inner horizon is a horizon of a white hole for the observer in the region $r < r_-$. Because the physical process near the white holes is a time reversal of the physical process near the black holes, we can expect "Hawking absorption" for the white hole as one expects Hawking radiation for the black hole.

Based on the metric (4.1.1), we have

$$\begin{aligned} &\mathrm{g}^{00} = -(r^2+n^2)\Delta^{-1}(\mathrm{g}+\Delta\Omega^2)/\mathrm{g}, \quad \mathrm{g}^{11} = \Delta, \\ &\mathrm{g}^{22} = (r^2+n^2)^{-1}, \quad \mathrm{g}^{33} = -(r^2+n^2)/\mathrm{g}, \\ &\mathrm{g}^{03} = (r^2+n^2)\Omega/\mathrm{g}, \quad \mathrm{g} = -(r^2+n^2)^2\sin^2\theta. \end{aligned} \tag{4.2.4}$$

Substituting (4.2.4) into the Klein-Gordon equation

$$\frac{1}{\sqrt{-\mathrm{g}}}(\partial_\mu - \mathrm{i}qA_\mu)\left[\sqrt{-\mathrm{g}}\mathrm{g}^{\mu\nu}(\partial_\nu - \mathrm{i}qA_\nu)\right]\Phi = \mu_0^2\Phi, \tag{4.2.5}$$

where

$$A_\mu = (-\frac{Qr}{r^2+n^2}, 0, 0, -\frac{\Omega Qr}{r^2+n^2})$$

is the four-potential of the electromagnetic field, and q and μ_0 are charge and mass of the KG particle, we obtain

$$\begin{aligned}&\left\{\mathrm{g}^{00}(r^2+n^2)\frac{\partial^2}{\partial t^2} - \frac{2\Omega}{\sin^2\theta}\frac{\partial}{\partial t}\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial r}\left[(r^2+n^2)\mathrm{g}^{11}\frac{\partial}{\partial r}\right]\right.\\ &\qquad\left. + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} - 2\mathrm{i}qQr\frac{r^2+n^2}{\Delta}\frac{\partial}{\partial t}\right\}\Phi\\ &= \left[(r^2+n^2)\mu_0^2 - \frac{1}{\Delta}q^2Q^2r^2\right]\Phi.\end{aligned} \tag{4.2.6}$$

Letting $\Phi = \exp(-\mathrm{i}\omega t)Y_{lm}(\theta,\varphi)R(r)$, we find

$$\begin{aligned}&\frac{d}{dr}\left[(r^2+n^2)\mathrm{g}^{11}\frac{d}{dr}\right]R(r)\\ &\qquad= \left[l(l+1) + (r^2+n^2)\mu_0^2 - \frac{K^2}{\Delta}\right]R(r),\end{aligned} \tag{4.2.7}$$

$$\begin{aligned}&\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{2\mathrm{i}\omega\Omega}{\sin^2\theta}\frac{\partial}{\partial\varphi}\right\}Y_{lm}(\theta,\varphi)\\ &\qquad= \left[l(l+1) - \frac{\omega^2\Omega^2}{\sin^2\theta}\right]Y_{lm}(\theta,\varphi),\end{aligned} \tag{4.2.8}$$

where $K = (r^2+n^2)\omega - qQr$.

Introducing the tortoise coordinate transformation

$$d\hat{r} = \frac{r^2+n^2}{(r-r_+)(r-r_-)}dr, \tag{4.2.9}$$

$$\hat{r} = r + \frac{1}{2\kappa_+}\ln\left(\frac{|r-r_+|}{r_+}\right) - \frac{1}{2|\kappa_-|}\ln\left(\frac{|r-r_-|}{r_-}\right), \tag{4.2.10}$$

we have

$$\frac{d}{dr} = \frac{r^2+n^2}{(r-r_+)(r-r_+)}\frac{d}{d\hat{r}}, \tag{4.2.11}$$

$$\frac{d^2}{dr^2} = \frac{2r(r-r_+)(r-r_-)-2(r^2+n^2)(r-M)}{(r-r_+)^2(r-r_-)^2}\frac{d}{d\hat{r}} + \left(\frac{r^2+n^2}{(r-r_+)(r-r_+)}\right)^2\frac{d^2}{d\hat{r}^2}. \tag{4.2.12}$$

With these results (4.2.7) transforms into

$$\frac{d^2}{d\hat{r}^2}R(r) + \frac{2r\Delta}{(r^2+n^2)}\frac{d}{d\hat{r}}R(r) = \frac{\Delta}{(r^2+n^2)}\left[l(l+1)+(r^2+n^2)\mu_0^2-\frac{K^2}{\Delta}\right]R(r). \tag{4.2.13}$$

Near the horizon, (4.2.13) reduces to

$$\frac{d^2R(r)}{d\hat{r}^2} + (\omega-\omega_0)^2R(r) = 0, \tag{4.2.14}$$

where

$$\omega_0 = \frac{qQr_\pm}{r_\pm^2+n^2}.$$

This is the standard form of wave equation on the horizons. Using this equation Hawking radiation near the outer horizon can be derived. In this chapter, we are interested in investigating the case near the inner horizon ($r < r_-$). The solution of (4.2.14) is

$$R = \exp[\pm \mathrm{i}(\omega-\omega_0)\hat{r}], \tag{4.2.15}$$

Considering the time factor, the solutions near the inner horizon r_- are given by

$$\Psi = \exp[-\mathrm{i}\omega t \pm \mathrm{i}\omega\tilde{r}], \tag{4.2.16}$$

where

$$\tilde{r} = \frac{\omega - \omega_0}{\omega}\hat{r}.$$

Thus on the inner horizon surface, we have the outgoing and ingoing waves, respectively, given by

$$\Psi_{\rm out}=\exp[-\mathrm{i}\omega(t-\tilde{r})] = \exp[-\mathrm{i}\omega u], \tag{4.2.17}$$

$$\begin{aligned}\Psi_{\rm in}=&\exp[-\mathrm{i}\omega(t+\tilde{r})]\\ =&\exp[-\mathrm{i}\omega u - 2\mathrm{i}(\omega-\omega_0)\hat{r}],\end{aligned} \tag{4.2.18}$$

where $u = t - \tilde{r}$ is the retarded Eddington-Finkelstein coordinate. Since $r \to r_-$ corresponds to $\tilde{r} \to -\infty$, and $\tilde{r} \to 0$ as $r \to 0$, (4.2.17) is just the outgoing wave emitted by the inner horizon, while (4.2.18) represents the ingoing wave to the inner horizon. Obviously, as $r \to r_-$,

$$\hat{r} \sim \frac{1}{2\kappa_-}\ln(r_- - r). \tag{4.2.19}$$

Hence, the ingoing wave is written as

$$\Psi_{\rm in}(r < r_-) = e^{-\mathrm{i}\omega u}(r_- - r)^{-\mathrm{i}(\omega-\omega_0)/\kappa_-}, \tag{4.2.20}$$

which is singular at $r = r_-$. We take this singularity as the center of a circle with radius $|r-r_-|$. By analytical continuation rotating $-\pi$ along the upper-half in the complex r-plane, into the "one-way membrane" region between the inner and outer horizons, we have

$$(r_- - r) \to |r_- - r|e^{-\mathrm{i}\pi} = (r - r_-)e^{-\mathrm{i}\pi}. \tag{4.2.21}$$

Thus, Ψ_{in} in the region $r_- < r < r_+$ can be written as

$$\Psi_{\text{in}}(r > r_-) = \Psi'_{\text{in}} e^{-\pi(\omega-\omega_0)/\kappa_-}, \tag{4.2.22}$$

where

$$\begin{aligned}\Psi'_{\text{in}} =& e^{-\mathrm{i}\omega u}(r-r_-)^{-\mathrm{i}(\omega-\omega_0)/\kappa_-} \\ =& e^{-\mathrm{i}\omega u} e^{-2\mathrm{i}(\omega-\omega_0)\hat{r}}.\end{aligned} \tag{4.2.23}$$

Using the works of Damour and Ruffini [7] and Sannan [202], it is possible to calculate the emission rate at the inner horizon. The total ingoing wave function can be written in the form

$$\Psi = N_\omega[Y(r_- - r)\Psi_{\text{in}}(r < r_-) + Y(r - r_-)\Psi'_{\text{in}}(r > r_-)], \tag{4.2.24}$$

where

$$Y(r) = \begin{cases} 1, & r \geq 0, \\ 0, & r < 0. \end{cases} \tag{4.2.25}$$

The normalization condition

$$(\Psi, \Psi) = N_\omega^2(1 \pm e^{(\omega-\omega_0)/T_-}) = \pm 1 \tag{4.2.26}$$

shows that the inner horizon absorbs thermal radiation from the region $r < r_-$. The thermal spectrum and temperature of this radiation are, respectively, given by

$$N_\omega^2 = \frac{\Gamma_-}{1-\Gamma_-} = \frac{1}{e^{(\omega-\omega_0)/T_-} \pm 1}, \tag{4.2.27}$$

$$T_- = \frac{-\kappa_-}{2\pi}. \tag{4.2.28}$$

The temperature of the inner horizon is positive and is in agreement with the findings of Refs. [189, 190, 191, 192]. Thus there exists some thermal radiation from the region $r < r_-$ to the inner horizon with temperature T_-. This thermal radiation is absorbed by the inner horizon and the corresponding quantum effect is named "Hawking absorption"[203, 204]. Similar as the outer horizon of the black hole is in thermal equilibrium with the thermal radiation outside the black hole, the inner horizon is in thermal equilibrium with the thermal radiation inside the inner horizon. The inner horizon absorbs thermal radiation at temperature T_-, and at the same time it emits thermal radiation at temperature T_-. Thus, the inner horizon is a thermal system with temperature T_-. The radiations of the outer horizon and the inner horizon are separate and simultaneously ongoing processes in the spacetime, and an observer situated at the exterior region of the black hole observes the both types of radiation. Then the most remarkable result is that, in contrast to common beliefs, the inner horizon is not a passive spectator but an active participant in the radiation processes [205, 206] of the RNTN black hole. We can explain Hawking radiation as follows. The inner horizon absorbs the positive energy particles created near the singularity. Transiting the "one-way membrane" region $r_- < r < r_+$, these particles arrive at the outer horizon. Being scattered by the outer horizon, they travel to infinity as Hawking radiation.

In (4.2.27) Γ_- symbolizes the tunneling rate at the inner horizon and is given by

$$\Gamma_- = \left|\frac{\Psi_{\text{in}}(r > r_-)}{\Psi_{\text{in}}(r < r_-)}\right|^2 = e^{-2\pi(\omega-\omega_0)/\kappa_-}. \tag{4.2.29}$$

The resulting temperature (4.2.28) is in agreement with the statistical Hawking temperature [75] computed as usual by dividing the surface gravity by 2π. We suppose that the area theorem is also applicable on the inner horizon as the same as the outer horizon. The area of outer horizon A_+ and the area of inner horizon A_- are given by

$$A_\pm = \pm \int \sqrt{\mathrm{g}} d\theta d\varphi = \pm 4\pi (r_\pm^2 + n^2), \tag{4.2.30}$$

where $\mathrm{g} = (r_\pm^2 + n^2)^2 \sin^2\theta$ is the determinant of the 2-dimensional metric on the inner and outer horizons. Since the inner horizon is like the horizon of a white hole, A_- is defined as minus.

4.3 Inner Horizon Entropy and Bekenstein–Smarr Formula

We calculate the entropy of the inner horizon by using the thin film brick wall model [88] which is based on the brick wall model proposed by 't Hooft [89]. This model treats the entropy as being associated with the field in the considered small region, where the local thermal equilibrium and the statistical laws are valid [207]. Hence, the field outside the horizon is supposed to be non-zero only in the thin film bordered by $r_+ + \varepsilon$ and $r_+ + \varepsilon + \delta$. Here, ε, δ are positive infinitesimal parameters with ε the ultraviolet cut-off distance and δ the thickness of the thin film. One can work out the entropy of the outer horizon by using this model. Since a time-like Killing vector field exists in the region $r < r_-$, the field in the thin film

$$(r_- - \varepsilon) \to (r_- - \varepsilon - \delta)$$

can be regarded as non-zero when we calculate the entropy of the inner horizon.

Rewriting $R(r)$ as $R(r) = e^{\mathrm{i}S(r)}$, we obtain from (4.2.7) the following equation by WKB approximation

$$\begin{aligned} k_r^2 &= \left(\frac{\partial S(r)}{\partial r}\right)^2 \\ &= \frac{1}{(r-r_+)(r-r_-)} \\ &\quad \times \left[\frac{(r^2+n^2)^2\tilde{\omega}^2}{(r-r_+)(r-r_-)} - \mu_0^2(r^2+n^2) - l(l+1)\right], \end{aligned} \tag{4.3.1}$$

where k_r is wave number and

$$\tilde{\omega} = (\omega - qV_{\mathrm{el}}).$$

The constraint of semi-classical quantum condition applied on k_r is

$$n\pi = \int_{r_- - \varepsilon}^{r_- - \varepsilon - \delta} k_r dr,$$

where n is a non-negative integer. The free energy F in the theory of canonical ensemble is given by

$$\beta F = \sum_{\tilde{\omega}} \ln(1 - e^{-\beta\tilde{\omega}}).$$

Considering the states of energy as continuous and transforming summation into integration, we obtain

$$\sum \rightarrow \int_0^\infty d\tilde{\omega} \mathrm{g}(\tilde{\omega}),$$

where $\mathrm{g}(\tilde{\omega})$ is the density of states, i.e.

$$\mathrm{g}(\tilde{\omega}) = \frac{d\Gamma(\tilde{\omega})}{d\tilde{\omega}},$$

$\Gamma(\tilde{\omega})$ is the number of the microscopic states, that is,

$$\Gamma(\tilde{\omega}) = \sum_{\lambda} n_r(\tilde{\omega}, \lambda) = \int d\lambda \frac{1}{\pi} \int k_r(\tilde{\omega}, \lambda) dr,$$

where the separation constant λ is the angular quantum number which corresponds to l in the spherical spacetime case and is given by $\lambda = l(l+1)$.

The expression for the free energy can be calculated as follows:

$$\begin{aligned} F =& \frac{1}{\beta} \int_0^{+\infty} d\tilde{\omega} \mathrm{g}(\tilde{\omega}) \ln(1 - e^{-\beta\tilde{\omega}}) \\ =& - \int_0^{+\infty} d\tilde{\omega} \frac{\Gamma(\tilde{\omega})}{e^{\beta\tilde{\omega}} - 1} \\ =& \frac{-1}{\pi} \int_0^{+\infty} d\tilde{\omega} \int_r dr \int_l (2l+1) dl \frac{k_r(r, \tilde{\omega}, l)}{e^{\beta\tilde{\omega}} - 1}, \end{aligned} \tag{4.3.2}$$

where the upper limit of the integration with respect to l is taken so that k_r^2 remains positive, and the lower limit is zero. Using the expression for k_r from (4.3.1) in (4.3.2) and then integrating on l, we obtain

$$\begin{aligned} F = \frac{-2}{3\pi} \int_0^{+\infty} \frac{d\tilde{\omega}}{e^{\beta\tilde{\omega}} - 1} \int_r dr \frac{(r^2 + n^2)^3}{(r - r_+)^2 (r - r_-)^2} \\ \times \left[\left\{ \tilde{\omega}^2 - \frac{(r - r_+)(r - r_-)}{(r^2 + n^2)} \mu_0^2 \right\} \right]^{\frac{3}{2}}. \end{aligned} \tag{4.3.3}$$

The integration with respect to r is quite difficult. But the thin film brick-wall model imposes us to take only the free energy of a thin layer near horizon of a black hole. So, the integration with respect to r must be limited in the region

$$r_- - \varepsilon - \delta \leq r \leq r_- - \varepsilon.$$

This choice sets the coefficient of μ_0^2 to zero, and the integration of (4.3.3) with respect to $\tilde{\omega}$ results $\pi^4/(15\beta^4)$. Thus, (4.3.3) reduces to

$$F = \frac{-2\pi^3}{45\beta^4}\frac{(r_-^2+n^2)^3}{(r_+-r_-)^2}\int_{r_--\varepsilon}^{r_--\varepsilon-\delta}\frac{dr}{(r-r_-)^2}$$

$$\cong \frac{2\pi^3}{45\beta^4}\frac{(r_-^2+n^2)^3}{(r_+-r_-)^2}\frac{\delta}{\varepsilon(\varepsilon+\delta)} \tag{4.3.4}$$

for an observer in $r < r_-$. With the temperature of the inner horizon

$$T_- = \frac{r_+-r_-}{4\pi(r_-^2+n^2)} = \frac{1}{\beta},$$

we obtain the entropy

$$S_- = \beta^2\frac{\partial F}{\partial\beta}$$

$$= -\frac{\pi(r_-^2+n^2)}{90\beta}\frac{\delta}{\varepsilon(\varepsilon+\delta)}. \tag{4.3.5}$$

If we select an appropriate cut-off distance ε and thickness of thin film δ to satisfy

$$\frac{\delta}{\varepsilon(\varepsilon+\delta)} = 90\beta,$$

the entropy of the inner horizon is

$$S_- = \frac{1}{4}A_- = -\pi(r_-^2+n^2). \tag{4.3.6}$$

Because the inner horizon is a horizon of a white hole, the entropy contributed by the inner horizon is chosen negative for the observer outside the black hole. The entropy is also proportional to the area of the inner horizon and cut-off is 90β which is same as that in the calculation of the entropy of the outer horizon.

The entropy and the temperature of the inner horizon satisfy the familiar formula $\frac{1}{T} = \frac{dS}{dm}$:

$$T_- = \left(\frac{dS_-}{dM + V_{\mathrm{el}-} dQ} \right)^{-1} = -\frac{\kappa_-}{2\pi}$$

$$= \frac{r_+ - r_-}{4\pi(r_-^2 + n^2)}. \tag{4.3.7}$$

It implies that the temperature or the entropy of the inner horizon is negative and another is positive. In several papers before, the entropy was positive and the temperature was negative. But, there is no clear explanation why the temperature of the inner horizon is negative. Indeed, our understanding of the essence of the black hole entropy is still incomplete. However, the negative entropy of the inner horizon can make possible the entropy of the black hole with two horizons to satisfy the Nernst theorem.

We regard the total entropy of the black hole as the sum of the contributions of the outer and inner horizons [78]:

$$S_{BH} = S_+ + S_-$$

$$= \pi(r_+^2 + n^2) - \pi(r_-^2 + n^2) = \pi(r_+^2 - r_-^2). \tag{4.3.8}$$

Evidently, the redefined entropy of the black hole $S_{BH} \to 0$ as its temperature

$$T = \frac{r_+ - r_-}{4\pi(r_+^2 + n^2)}$$

goes to absolute zero, and consequently, the Nernst theorem is satisfied.

We now obtain the Bekenstein-Smarr formula using the parameters of the inner horizon. From (4.2.2), (4.2.3) and (4.2.30), we obtain

$$QV_\pm - 2n^2\kappa_\pm = r_\mp = M \mp \sqrt{M^2 - Q^2 + n^2}, \tag{4.3.9}$$

$$\frac{1}{4\pi}\kappa_\pm A_\pm = \sqrt{M^2 - Q^2 + n^2}, \tag{4.3.10}$$

where

$$V_\pm = \frac{Qr_\pm}{r_\pm^2 + n^2}.$$

From (4.3.9) and (4.3.10), we obtain the Bekenstein-Smarr integral formulae using the parameters of the horizons [78]

$$M = \pm\frac{1}{4\pi}\kappa_\pm A_\pm + V_\pm Q - 2n^2\kappa_\pm. \tag{4.3.11}$$

Differentiating (4.3.11), we have

$$\delta M = \pm\frac{1}{4\pi}\kappa_\pm \delta A_\pm \pm \left(\frac{1}{4\pi}A_\pm \mp 2n^2\right)\delta\kappa_\pm$$

$$+V_\pm \delta Q + Q\delta V_\pm. \tag{4.3.12}$$

Using

$$\frac{1}{4\pi}A_\pm \delta\kappa_\pm = \frac{M\delta M}{\sqrt{M^2 - Q^2 + n^2}} - \frac{Q\delta Q}{\sqrt{M^2 - Q^2 + n^2}}$$
$$-\frac{\sqrt{M^2 - Q^2 + n^2}}{A_\pm}\delta A_\pm, \tag{4.3.13}$$

$$\frac{1}{4\pi}A_\pm \delta V_\pm = \frac{Qr_\pm}{\sqrt{M^2 - Q^2 + n^2}}\delta M + \frac{Mr_\pm - 2Q^2 + n^2}{\sqrt{M^2 - Q^2 + n^2}}\delta Q$$
$$\mp\frac{Qr_\pm}{A_\pm}\delta A_\pm, \tag{4.3.14}$$

in (4.3.12), we obtain the differential equation of Bekenstein-Smarr formulae adopting the parameters of the outer horizon

$$\delta M = C_+ \frac{1}{8\pi} \kappa_+ \delta A_+ + C'_+ V_+ \delta Q, \tag{4.3.15}$$

where

$$C_+ = \left(1 - \frac{n^2 r_-}{(n^2 - Q^2) r_+}\right),$$
$$C'_+ = \left(1 + \frac{n^2}{r_+^2}\right), \tag{4.3.16}$$

and adopting the parameters of the inner horizon

$$\delta M = -C_- \frac{1}{8\pi} \kappa_- \delta A_- + C'_- V_- \delta Q, \tag{4.3.17}$$

where

$$C_- = \left(1 - \frac{n^2 r_+}{(n^2 - Q^2) r_-}\right),$$
$$C'_- = \left(1 + \frac{n^2}{r_-^2}\right). \tag{4.3.18}$$

In the limit $n = 0$, (4.3.15) and (4.3.17) reduce to the Reissner-Nordström black hole case. When the temperature and the entropy of the inner horizon calculated in (4.2.28) and (4.3.6) are substituted in (4.3.17), with defining $\tilde{T}_- = C_- T_-$ and $\tilde{V}_- = C_- V_-$, the result gives

$$\delta M = \tilde{T}_- \delta S_- + \tilde{V}_- \delta Q. \tag{4.3.19}$$

Thus, the first law of black hole thermodynamics is justified.

4.4 Back-reaction of Radiation

We consider that the emitting particles have back-reaction on the space-time. When a particle with energy ω_i and charge q_i tunnels out of the inner horizon and then out of the black hole, the mass M and charge Q of the black hole should be replaced by $(M - \omega_i)$ and $(Q - q_i)$, in consideration of energy conservation and charge conservation. Then,

$$\Gamma_{-i} = e^{-2\pi(\omega_i - \omega_{oi})/\kappa_{-i}}, \tag{4.4.1}$$

where

$$\omega_{0i} = -q_i V_{\mathrm{el}-i} = \frac{q_i(Q - q_i)r_{-i}}{r_{-i}^2 + n^2},$$

$$r_{\pm i} = (M - \omega_i) \pm \sqrt{(M - \omega_i)^2 - (Q - q_i)^2 + n^2},$$

$$\begin{aligned} \kappa_{-i} &= -\frac{(r_{+i} - r_{-i})}{2(r_{-i}^2 + n^2)} \\ &= -\frac{\sqrt{(M - \omega_i)^2 - (Q - q_i)^2 + n^2}}{((M - \omega_i) - \sqrt{(M - \omega_i)^2 - (Q - q_i)^2 + n^2})^2 + n^2}. \end{aligned} \tag{4.4.2}$$

Considering emission of many particles and thinking that they radiate one by one, we have

$$\Gamma_- = \prod_i \Gamma_{-i} = e^{-2\pi(\omega_i - \omega_{oi})/\kappa_{-i}}. \tag{4.4.3}$$

If the emission is regarded as a continuous procession, the sum in (4.4.3) should be replaced by integration

$$\Gamma_- = e^{-2\pi \int (d\omega' + V'_{\mathrm{el}} dq')/\kappa'_-} = e^{-2\pi\Theta_-}, \tag{4.4.4}$$

where

$$\Theta_{-} = -\int_{(0,0)}^{(\omega,q)} \left\{ \frac{((M-\omega')-\sqrt{(M-\omega')^2-(Q-q')^2+n^2})^2+n^2}{\sqrt{(M-\omega')^2-(Q-q')^2+n^2}} d\omega' \right. \\ \left. - \frac{(Q-q')((M-\omega')-\sqrt{(M-\omega')^2-(Q-q')^2+n^2})}{\sqrt{(M-\omega')^2-(Q-q')^2+n^2}} dq' \right\}. \tag{4.4.5}$$

To calculate it, we make use of the inner horizon entropy S_{-} derived in (4.3.6) and obtain

$$\begin{aligned} \Delta S_{-} &= \pi(r_{-}^2 - r_{-}'^2) \\ &= \pi[(M-\sqrt{M^2-Q^2+n^2})^2 \\ &\quad -((M-\omega)-\sqrt{(M-\omega)^2-(Q-q)^2+n^2})^2], \end{aligned} \tag{4.4.6}$$

where

$$\Delta S_{-} = S_{-}(M-\omega, Q-q) - S_{-}(M,Q)$$

is the difference between the entropies of the inner horizon before and after the emission. Equation (4.4.5) can be calculated out as follow:

$$\begin{aligned} \Theta_{-} &\approx -\frac{1}{2\pi}\int_{(0,0)}^{(\omega,q)} \left\{ \frac{\partial(\Delta S_{-})}{\partial\omega'} d\omega' - \frac{\partial(\Delta S_{-})}{\partial q'} dq' \right\} \\ &= -\frac{1}{2\pi}\int d(\Delta S_{-}) = -\frac{1}{2\pi}\Delta S_{-}, \end{aligned} \tag{4.4.7}$$

hence, the emitting rate of the inner horizon Γ_{-} is given by

$$\Gamma_{-} = e^{\Delta S_{-}}. \tag{4.4.8}$$

Applying the same method, the emitting rate of the outer horizon is obtained

$$\Gamma_+ = e^{\Delta S_+}. \tag{4.4.9}$$

Thus, the total tunneling rate is

$$\Gamma = \Gamma_+ \cdot \Gamma_- = e^{\Delta S_{BH}}, \tag{4.4.10}$$

where $S_{BH} = (S_+ + S_-)$ is the Bekenstein-Hawking entropy of the black hole, defined in (4.3.8). This result is in agreement with Parikh's work. Evidently, the derived emission spectrum actually deviates from pure thermality, and this result is consistent with an underlying unitary theory. To compare with the thermal spectrum, we expand Γ in ω and q. Thus, the total tunneling rate is

$$\Gamma = e^{-\beta(\omega-\omega_0)+O(\omega,q)^2}, \tag{4.4.11}$$

where the leading-order term is the Boltzman factor. The higher-order terms of ω and q generate a deviation from a purely thermal spectrum. Further, considering the modification idea of the surface gravity and temperature due to one-loop back-reaction effects [58, 208, 209] according to which

$$\frac{\kappa(M)}{\kappa_0(M)} = 1 + \frac{\alpha}{M^2} = \frac{T(M)}{T_0(M)}$$

for a Schwarzschild black hole, (4.4.11) can be put in the form,

$$\Gamma = e^{-\beta'(\omega-\omega_0)}, \quad \beta' = \beta\left[1 - \frac{O(\omega,q)}{\beta(\omega-\omega_0)}\right], \tag{4.4.12}$$

where β' can be treated as an inverse quantum-corrected temperature.

In quantum mechanics, the emitting rate is obtained by

$$\Gamma(i \to f) = |A_{if}|^2 \delta_p, \tag{4.4.13}$$

where $|A_{if}|^2$ is the square of the amplitude for the tunneling action. The phase space factor δ_p is derived by averaging the number n_i of microstates of the initial state and the number n_f of microstates of the final state, that is, $\delta_p = n_f/n_i$. Since $S_j \sim \ln n_j$, i.e., $n_j \sim e^{S_j} \quad (j = i, f)$, then

$$\Gamma = \frac{e^{S_f}}{e^{S_i}} = e^{S_f - S_i} = e^{\Delta S}. \tag{4.4.14}$$

Obviously, equation (4.4.14) is consistent with our result. Thus, equation (4.4.10) satisfies the underlying unitary theory in quantum mechanics and thereby provides a might explanation to the black hole information puzzle.

4.5 Concluding Remarks

The main concern of this study is to investigate the thermal character of the inner horizon, redefine the entropy to satisfy the Nernst theorem, and to derive Hawking radiation via tunneling effect of both inner and outer horizons from a Reissner-Nordström-Taub-NUT black hole. The study is interesting in the context of black hole physics. We find the inner horizon temperature as positive by Damour-Ruffini method, and the entropy, by thin film brick wall model, proportional to the area of the inner horizon. Since the inner horizon temperature is positive, there is no interpretative problem concerning the thermodynamical properties of the radiation of the inner horizon. In addition to the radiation effects of black hole horizons, also the white hole horizon radiates. The black hole radiation is caused by the quantum effects at the outer horizon, whereas the white hole radiation is caused by the quantum effects at the inner horizon of the RNTN black hole.

The inner horizon emits particles inside the inner horizon with a positive temperature has a most important consequence. When real particles with energy ω are emitted towards the singularity from the inner horizon, it is necessary for a local energy balance that antiparticles with energy $-\omega$ are emitted away from the singularity through the inner horizon. The process is analogous to the one which, according to the Hawking effect, takes place at the outer horizon of the RNTN black hole– at the outer horizon antiparticles go in and particles come out. Thus, in contrast to common beliefs, the inner horizon is not a passive spectator but an active participant in the radiation processes of the black hole. The radiations of the outer horizon (i.e., black hole horizon) and the inner horizon (i.e., white hole horizon) are separate and simultaneously ongoing processes in the spacetime, and an observer situated at the exterior region of the black hole observes the both types of radiation. The emission of antiparticles out of the white hole, in turn, may be understood as an absorption of energy by the white hole horizon. As no energy may be absorbed classically by the white hole horizon, this feature contradicts with the classical results in the same way as does the evaporation process at black hole horizons.

As (4.3.7) suggests, the temperature or the entropy of the inner horizon is negative and another is positive. The positive temperature implies that the inner horizon entropy is negative. It is not clear why the inner horizon entropy is negative. In fact, our understanding of the essence of the black hole entropy is still incomplete. However, the negative entropy of the inner horizon can make possible the entropy of the black hole with two horizons to satisfy the Nernst theorem. In our study the emission process satisfies the law of black hole thermodynamics,

$$\frac{dM + V_{\mathrm{el}} dQ}{T} = dS,$$

which is only reliable for the reversible process; for an irreversible process,

$$dS > \frac{dM + V_{\mathrm{el}} dQ}{T}.$$

The emission process in this analyze is thus an reversible one. In this picture, by the process of entropy flux, the two horizons and the outside spacetime approach an thermal equilibrium. As the black hole radiates, its entropy decreases but the total entropy of the system remains constant, and the information is preserved. However, the existence of the negative heat capacity makes an evaporating black hole a highly unstable system, and the thermal equilibrium between the black hole and the outside becomes unstable (there will exist difference in temperature). The process is then irreversible and the underlying unitary theory is not satisfied, i.e., information does not conserve during the evaporation. Further, our study is still a semi-classical analysis in which the radiation is treated as point particles. This type of approximation can only be valid in the low energy regime. To properly address the information loss problem, a better understanding of physics at the Planck scale is a necessary prerequisite, particularly that of the last stages of the endpoint of Hawking evaporation.

The study of this chapter provides in special cases the results for the two interesting black holes: (i) the Reissner-Nordström black hole result for $n = 0$, as obtained in [82], and (ii) the Taub-NUT black hole result for $Q = 0$. Further, the procedure of this chapter could be applied to any black hole with two horizons to obtain Hawking radiation via tunneling phenomenon of both horizons. The radiation effect of the inner horizon has much importance because it supports the idea that all horizons of spacetime emit radiation. In fact, the Hawking radiation relates the theory of general relativity with quantum field theory and statistical thermodynamics. It is generally believed that a deeper investigation of Hawking radiation would facilitate to set up a satisfactory quantum theory of gravity. So, the Hawking radiation demands intense efforts to investigate in a broader context. In this regard, the work of this chapter is interesting.

Chapter 5

Tunneling of Charged Massive Particles from Taub-NUT-Reissner-Nordström-AdS Black Holes

In this chapter we apply the null-geodesic method to investigate tunneling radiation of charged and magnetized massive particles from Taub-NUT-Reissner-Nordström black holes endowed with electric as well as magnetic charges in Anti-de Sitter (AdS) spaces [90]. The geodesics of charged massive particle tunneling from the black hole is not lightlike, but can be determined by the phase velocity. We find that the tunneling rate is related to the difference of Bekenstein-Hawking entropies of the black hole before and after the emission of particles. The entropy differs from just a quarter area at the horizon of black holes with NUT parameter. The emission spectrum is not precisely thermal anymore and the deviation from the precisely thermal spectrum can bring some information out, which can be treated as an explanation to the information loss paradox. The result can also be treated as a quantum-corrected radiation temperature, which is dependent on the black hole background and the radiation particle's energy and charges.

The organization of this chapter is as follows. In the section 7.1, we present an introduction relating the work of this chapter. In section 7.2, we express the TNRN-AdS black hole spacetime in Painlevé coordinate system, and obtain the radial geodesic equation of a charged and magnetized massive particle. In section 7.3, we use null geodesics method to analyze tunneling radiation and present results for the Schwarzschild-AdS, TN-AdS, RN-AdS, and TNRN-AdS black holes. Finally, we give our concluding remarks in section 7.4.

5.1 Introduction

Hawking's discovery, reviewed in chapter 3, that the collapsing black hole, at late times, radiates particles in all modes of the quantum field, with characteristic thermal spectrum as a strict black-body spectrum. This finding has positive implication in understanding and investigating star evolution. It promotes our knowledge of the black-hole thermodynamics.

When a negative energy antiparticle is absorbed, the black hole mass decreases, while temperature and charge potential increase. Then the black hole can spontaneously transfer an amount of heat and charge to the positive energy particle. As a result, the black hole temperature and charge potential further increase and further transfer. Acquiring enough energy the positive energy particle can escape away to infinity as Hawking radiation. This effect results the black hole to shrink. Black hole's thermal radiation spectrum subsequently raises two obvious disputes: the first is information loss. Since the black-body spectrum can provide only a temperature parameter, the black-hole radiation as an exact black-body spectrum will not take any information of the black hole inner matter. Hence, all information including unitary property will be lost by a vaporized black hole. This implies that the pure quantum state will decay to a mixed state, violating the established unitary principle in quantum mechanics. The second is regarding the reaction of the radiation to the spacetime. When the black hole generates Hawking radiation, the black holes state parameters (energy and charge) will fluctuate. But this effect was not considered in the past. The original derivation of Hawking radiation is under the condition that the spacetime is invariable and it leads to the precisely thermal black-hole radiation spectrum. To address these two problems, tunneling phenomenon of Hawking radiation is considered as a more effective technique.

In 2000, Parikh and Wilczek proposed a semi-classical method of modeling Hawking radiation as a tunneling effect. A review work on this method is presented in subsection 3.5.1 of chapter 3. Recently, Qi [210] investigated by the null-geodesic method the tunneling radiation of the massive charged particle from the Reissner-Nordström-NUT black hole. In this chapter we employ the null-geodesic method to analyze tunneling radiation of charged and magnetized massive particles from Taub-NUT-Reissner-Nordström-AdS (TNRN-AdS) black holes endowed with electric as well as magnetic charges. The TNRN-AdS black hole is the NUT charged RN black hole in the AdS space. It reduces in special cases to the Taub-NUT-AdS and Taub-NUT black holes. The AdS spacetime not only is interesting in the context of brane-world scenarios based on the setup of Randall and Sundrum but also plays leading role in the familiar AdS/CFT [91] conjecture. By studying thermodynamics of the asymptotically AdS spacetime, it is possible to get some insights into the thermodynamic behavior of some strong coupling CFTs from the correspondence between the supergravity in asymptotically AdS spacetimes and CFT [92]. On the other hand, recent developments in string/M theory have greatly stimulated the study of NUT charged black hole phenomena in AdS spaces. In particular, these black hole backgrounds are interesting in the context of AdS/CFT conjecture [93, 94, 95] and supergravity. The Taub-NUT metric plays an important role in the conceptual development of general relativity. As "counter example to almost anything" [96], the Taub-NUT spacetime has peculiar character. The entropy of various Taub-NUT black holes is not proportional to the area of the event horizon and their free energy can have negative value [93, 95, 97, 98, 99, 100]. The NUT charged AdS black hole has a boundary metric that has closed timelike curves. Quantum field theory behaves significantly different in this space. It is of interest to understand AdS-CFT correspondence in this type of spaces [101]. The presence of closed timelike curves in the NUT charged AdS black hole spacetimes can be avoided, if one takes into account the universal covering of such AdS black hole backgrounds, which is not globally hyperbolic. In view of the above considerations the TNRN-AdS black hole deserves investigation in a broader context. The study of this chapter is interesting in this regard.

Our concern in this chapter is to analyze the basic property of the TNRN-AdS black hole and investigate quantum tunneling radiation. We find that the entropy of the TNRN-AdS black hole is not proportional to the event horizon area. The obtained result demonstrates that the radiation spectrum is not strictly thermal and is consistent with underlying unitary quantum theory. We also discuss the Schwarzschild-AdS, Taub-NUT-AdS, and Reissner-Nordström-AdS black hole cases, which are special types of the TNRN-AdS black hole.

5.2 TNRN-AdS Spacetimes in Painlevé coordinate and Radial Geodesics

The TNRN-AdS black hole spacetime can be expressed by the metric

$$ds^2 = -\Delta(dt + \Omega d\varphi)^2 + \Delta^{-1}dr^2$$

$$+(r^2 + n^2)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{5.2.1}$$

where

$$\Delta = \frac{r^2 - 2Mr + Q^2 + \Phi^2 - n^2}{r^2 + n^2} + \frac{r^2 + 5n^2}{\ell^2},$$

$$\Omega = 2n\cos\theta + \Omega', \tag{5.2.2}$$

Beside the negative cosmological parameter $\Lambda = -3/\ell^2$, the metric (5.2.1) possesses four parameters: the mass parameter M, the NUT parameter n, the electric charge parameter Q, and the magnetic monopole parameter Φ. The NUT parameter induces a topology at infinity in the Euclidean section that is a Hopf fibration of a circle over a 2-sphere and behaves like a "magnetic mass." As interpreted by Aliev [194] the NUT parameter generates a "rotational effect." Relatively long ago, Dirac predicted the existence of the magnetic monopole theoretically, but it was neglected on account of the failure to detect such object in the following years. In recent years, however, the development of gauge theories [211, 212] has shed new light on it and the string theory [107] also admits the existence of this object. It is thus very interesting and necessary to deal with the background with magnetic charge. Since $d\varphi$ is not a well-behaved one-form at $\theta = o, \pi$, the constant Ω' is set equal to $-2n\,(2n)$ to make the half-axis $\theta = 0\,(\theta = \pi)$ explicitly regular, leaving the other half-axis—the Misner string—singular. Changing Ω' from $-2n$ to $2n$ can be reproduced by changing the time coordinate from t to $t' = t - 4n\varphi$; hence, both half-axes can be made regular. This is due to closed timelike curves since it requires that both t and t' should be periodic with period $8\pi n$.

The TNRN-AdS metric (5.2.1) represents (i) the Schwarzschild-AdS black hole for $n = Q = \Phi = 0$; (ii) the TN-AdS black hole for $Q = \Phi = 0$ and (iii) the RN-AdS black hole for $n = 0$. The electric potential A_μ and the magnetic-like potential B_μ associated with the metric (5.2.1) can be written as

$$A_\mu = -\frac{Qr}{r^2 + n^2}(dt + \Omega d\varphi),$$

$$B_\mu = -\frac{\Phi r}{r^2 + n^2}(dt + \Omega d\varphi). \tag{5.2.3}$$

The event horizon r_+ and inter horizon r_- of the black hole are located at the real roots of $\Delta = 0$, which are given, following Ref. [213], as below

$$r_{\pm} = \frac{1}{2}(\alpha \pm \beta), \tag{5.2.4}$$

where

$$\alpha = \sqrt{u - \ell^2 - 6n^2}, \quad \beta = \sqrt{-u - \ell^2 - 6n^2 + \frac{4M\ell^2}{\alpha}},$$

$$u = \frac{\ell^2 + 6n^2}{3} + \frac{\ell^{4/3}(\mathcal{M}_+^2 - \mathcal{M}_-^2)^{2/3}}{(2N^2 - \mathcal{M}_+^2 - \mathcal{M}_-^2)^{1/3}} + \ell^{4/3}(2N^2 - \mathcal{M}_+^2 - \mathcal{M}_-^2)^{1/3},$$

$$N^2 = M^2 + \sqrt{(M^2 - \mathcal{M}_+^2)(M^2 - \mathcal{M}_-^2)}. \tag{5.2.5}$$

The two critical mass parameters $\mathcal{M}_{\pm}$ are given by

$$\mathcal{M}_{\pm} = \frac{\ell}{3\sqrt{6}}\sqrt{\zeta(3\vartheta - \zeta^2) \pm \eta^3} \tag{5.2.6}$$

where

$$\eta = (\zeta^2 + \vartheta)^{1/2}, \quad \zeta = \left(1 + \frac{6n^2}{\ell^2}\right),$$

$$\vartheta = \frac{12}{\ell^2}\left\{(Q^2 + \Phi^2 - n^2) + \frac{5n^4}{\ell^2}\right\}. \tag{5.2.7}$$

Expanding the expressions in (5.2.5) in powers of $1/\ell$ with $M/\ell \ll 1$, we obtain

$$r_{\pm} = r_{o\pm} - \frac{r_{o\pm}^2}{2\ell^2}\frac{2Mr_{o\pm} - Q^2 - \Phi^2 + n^2}{r_{o\pm} - M} + \mathcal{O}\left(\frac{1}{\ell^4}\right), \tag{5.2.8}$$

where

$$r_{o\pm} = M \pm \sqrt{M^2 - Q^2 - \Phi^2 + n^2}. \tag{5.2.9}$$

It shows that the event horizon lies in the range $r_- < r_+ < r_{o+}$. In the limit $\ell \to \infty$, (5.2.8) gives the horizons of the TNRN black hole. The metric (5.2.1) describes a naked singularity for $M < \mathcal{M}_+$, and a black hole for $M \geq \mathcal{M}_+$, where the equality corresponds to an extreme black hole of radius

$$r_{\rm ebh} = \frac{\ell}{\sqrt{6}} \left(\eta - \zeta\right)^{1/2}.$$

If we set $d\varphi = \dot{\varphi} dt$ with

$$\dot{\varphi} = \frac{d\varphi}{dt} = -\frac{\mathrm{g}_{03}}{\mathrm{g}_{33}} = \frac{\Delta\Omega}{(r^2 + n^2)\sin^2\theta - \Delta\Omega^2}, \tag{5.2.10}$$

the metric (5.2.1) and electrical potential as well as magnetic potential can be rewritten as follows:

$$\begin{aligned} ds^2 &= -\frac{\Delta(r^2 + n^2)\sin^2\theta}{(r^2 + n^2)\sin^2\theta - \Delta\Omega^2} dt^2 + \Delta^{-1} dr^2 + (r^2 + n^2) d\theta^2 \\ &= \hat{\mathrm{g}}_{00} dt^2 + \mathrm{g}_{11} dr^2 + \mathrm{g}_{22} d\theta^2, \end{aligned} \tag{5.2.11}$$

$$\begin{aligned} A_t' &= -\frac{Qr}{r^2 + n^2} \cdot \frac{(r^2 + n^2)\sin^2\theta}{(r^2 + n^2)\sin^2\theta - \Delta\Omega^2}, \\ B_t' &= -\frac{\Phi r}{r^2 + n^2} \cdot \frac{(r^2 + n^2)\sin^2\theta}{(r^2 + n^2)\sin^2\theta - \Delta\Omega^2}. \end{aligned} \tag{5.2.12}$$

In order to remove the coordinate singularity at the horizon r_+, we make general Painlevé coordinate transformation [188]

$$dt_d = dt + F(r, \theta) dr + G(r, \theta) d\theta, \tag{5.2.13}$$

and obtain

$$
\begin{aligned}
ds^2 = \hat{\mathrm{g}}_{00}dt^2 \pm 2\sqrt{\hat{\mathrm{g}}_{00}(1-\mathrm{g}_{11})}dtdr + dr^2 \\
+ \left[\hat{\mathrm{g}}_{00}G(r,\theta)^2 + \mathrm{g}_{22}\right] d\theta^2 + 2\hat{\mathrm{g}}_{00}G(r,\theta)dtd\theta \\
+2\sqrt{\hat{\mathrm{g}}_{00}(1-\mathrm{g}_{11})}G(r,\theta)drd\theta,
\end{aligned}
\tag{5.2.14}
$$

where we have set

$$\mathrm{g}_{11} + \hat{\mathrm{g}}_{00}\{F(r,\theta)\}^2 = 1$$

and $G(r,\theta)$ is given by

$$G(r,\theta) = \int \frac{\partial F(r,\theta)}{\partial \theta} dr + C(\theta)$$

with $C(\theta)$ an arbitrary analytic function of θ. The $+(-)$ sign in (5.2.14) denotes the spacetime metric of the charged massive outgoing (ingoing) particles at the horizon. Since the charged particle's world-line is subject to Lorentz forces, it does not follow radial light-like geodesic when it tunnels across the horizon. We consider the outgoing charged particle as a massive charged shell which corresponds to de Broglie "s-wave" with phase velocity v_p and group velocity v_{g} satisfying the relationship

$$
\begin{aligned}
v_p =& \frac{dr}{dt}, \\
v_{\mathrm{g}} =& 2v_p = \frac{dr_c}{dt},
\end{aligned}
\tag{5.2.15}
$$

where r_c denotes the location of the tunneling particle. There occur simultaneously two events in different places during the tunneling process:

one is the particle tunneling into the barrier, and the other is the particle tunneling out the barrier. Since the tunneling across the barrier is an instantaneous process and the metric (5.2.14) satisfies Landau's theory of the coordinate clock synchronization, the difference of coordinate times of these two events is

$$dt = -\frac{\mathrm{g}_{01}}{\mathrm{g}_{00}} dr_c, \quad (d\theta = 0). \tag{5.2.16}$$

So the phase velocity (the radial geodesics) is

$$\begin{aligned} \dot{r} = v_p = \frac{1}{2}\frac{dr_c}{dt} &= -\frac{1}{2}\frac{\hat{\mathrm{g}}_{00}}{\sqrt{\hat{\mathrm{g}}_{00}(1-\mathrm{g}_{11})}} \\ &= \frac{\Delta}{2}\left(\frac{(r^2+n^2)\sin^2\theta}{(1-\Delta)[(r^2+n^2)\sin^2\theta - \Delta\Omega^2]}\right)^{\frac{1}{2}}. \end{aligned} \tag{5.2.17}$$

In the subsequent section, we discuss the tunneling radiation characteristics of a particle with electric and magnetic charges at the event horizon of the TNRN-AdS black hole.

5.3 Tunneling Radiation of Charged and Magnetized Massive Particles

We consider energy conservation, charge conservation, and magnetic conservation, when a particle with energy ω, charge q, and magnet ϕ tunnels out of the event horizon. Then, after emission of the particle, the mass, charge, and magnet parameters of the black hole will be replaced by $M-\omega$, $Q-q$, and $\Phi-\phi$, respectively. Applying the WKB approximation, the tunneling rate is given by

$$\Gamma \sim e^{-2\,\mathrm{Im}\,S}, \quad \mathrm{Im}\,S = \mathrm{Im}\int_{t_i}^{t_f} Ldt, \tag{5.3.1}$$

where L is the Lagrangian function of the matter-gravity system. When a particle with electric and magnetic charge tunnels out, the effect of the electromagnetic field outside the black hole should be taken into account. So the matter-gravity system consists of the black hole and the electromagnetic field outside the black hole. Since the Lagrangian function of the electromagnetic field corresponding to the generalized coordinates described by (5.2.12) is

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

we can find that the generalized coordinate is an ignorable coordinate. To eliminate the freedoms, the imaginary part of the action should be written as

$$\begin{aligned}\mathrm{Im}\,S &= \mathrm{Im}\int_{t_i}^{t_f}\left(L - P_{A_t'}\dot{A}_t' - P_{B_t'}\dot{B}_t'\right)dt \\ &= \mathrm{Im}\int_{r_i}^{r_f}\int_{(0,0,0)}^{(P_r,P_{A_t'},P_{B_t'})}\frac{dr}{\dot{r}}\left[\dot{r}dP_r' - \dot{A}_t'dP_{A_t'} - \dot{B}_t'dP_{B_t'}\right],\end{aligned} \tag{5.3.2}$$

where P_r, $P_{A_t'}$ and $P_{B_t'}$ are the canonical momentum conjugate to r, A_t' and B_t', respectively. The r_i and r_f represent the locations of the event horizon before and after the particle with electric and magnetic charge emission, and they are often regarded as the two turning points of the tunneling potential hill. The distance between them depends on the energy, charge and magnet of the outgoing particle. According to Hamilton's canonical equation of motion, we have

$$\dot{r} = \left.\frac{dH}{dp_r}\right|_{(r;A'_t,P_{A'_t};B'_t,P_{B'_t})},$$

$$dH|_{(r;A'_t,p_{A'_t};B'_t,P_{B'_t})} = d(M-\omega),$$

$$\dot{A}'_t = \left.\frac{dH}{dP_{A'_t}}\right|_{(A'_t;B'_t,P_{B'_t};r,P_r)},$$

$$dH|_{(A'_t;B'_t,P_{B'_t};r,P_r)} = \frac{r\sin^2\theta(Q-q)d(Q-q)}{(r^2+n^2)\sin^2\theta - \Delta'\Omega^2},$$

$$\dot{B}'_t = \left.\frac{dH}{dP_{B'_t}}\right|_{(B'_t;r,P_r;A'_t,P_{A'_t})},$$

$$dH|_{(B'_t;r,P_r;A'_t,P_{A'_t})} = \frac{r\sin^2\theta(\Phi-\phi)d(\Phi-\phi)}{(r^2+n^2)\sin^2\theta - \Delta'\Omega^2}, \tag{5.3.3}$$

Substituting the above formula into (5.3.2), we have

$$\begin{aligned}
\operatorname{Im} S = {} & \operatorname{Im}\int_{r_i}^{r_f}\int_{(M,Q,\Phi)}^{(M-\omega,Q-q,\Phi-\phi)} [dH|_{(r;A'_t,p_{A'_t};B'_t,P_{B'_t})} \\
& -dH|_{(A'_t;B'_t,P_{B'_t};r,P_r)} - dH|_{(B'_t;r,P_r;A'_t,P_{A'_t})}]\frac{dr}{\dot{r}} \\
= {} & \operatorname{Im}\int_{r_i}^{r_f}\int_{(M,Q,\Phi)}^{(M-\omega,Q-q,\Phi-\phi)} \frac{dr}{\dot{r}'}\Bigg[d(M-\omega') \\
& -\frac{r\sin^2\theta(Q-q')d(Q-q')}{(r^2+n^2)\sin^2\theta-\Delta'\Omega^2} - \frac{r\sin^2\theta(\Phi-\phi')d(\Phi-\phi')}{(r^2+n^2)\sin^2\theta-\Delta'\Omega^2}\Bigg],
\end{aligned} \tag{5.3.4}$$

where

$$\Delta' = \frac{r^2 - 2(M-\omega')r + (Q-q')^2 + (\Phi-\phi')^2 - n^2}{r^2+n^2} + \frac{r^2+5n^2}{\ell^2},$$

$$\dot{r}' = \frac{\Delta'}{2}\left(\frac{(r^2+n^2)\sin^2\theta}{(1-\Delta')[(r^2+n^2)\sin^2\theta - \Delta'\Omega^2]}\right)^{\frac{1}{2}}. \qquad (5.3.5)$$

The integral can be evaluated by deforming the contour around the single pole at $r = r'_+$. Performing the r integral, we obtain

$$\begin{aligned}\operatorname{Im} S = &-\frac{1}{2}\int_{(M,Q,\Phi)}^{(M-\omega,Q-q,\Phi-\phi)} \frac{4\pi}{\Delta_{,r}(r'_+)}\Bigg[d(M-\omega') \\ &-\frac{(Q-q')r'_+}{r'^2_+ + n^2}d(Q-q') - \frac{(\Phi-\phi')r'_+}{r'^2_+ + n^2}d(\Phi-\phi')\Bigg] \\ = &\ \frac{1}{2}\int_{(0,0,0)}^{(\omega,q,\phi)} \frac{4\pi}{\Delta_{,r}(r'_+)} \\ &\times\Bigg[d\omega' - \frac{(Q-q')r'_+}{r'^2_+ + n^2}dq' - \frac{(\Phi-\phi')r'_+}{r'^2_+ + n^2}d\phi'\Bigg], \qquad (5.3.6)\end{aligned}$$

where

$$\Delta_{,r}(r_+) = \partial_r \Delta|_{r=r_+}$$

and near the horizon

$$\Delta = (r - r_+)\Delta_{,r}(r_+).$$

5.3.1 Schwarzschild-AdS Black Hole Case

In this case $Q = \Phi = n = 0$, and

$$\Delta = \frac{r - 2M}{r} + \frac{r^2}{\ell^2},$$

$$r_+ = 2M(1 - \frac{4M^2}{\ell^2}),$$

so (5.3.6) reduces to

$$\begin{aligned}
\operatorname{Im} S &= -\frac{1}{2}\int_M^{M-\omega} \frac{4\pi}{\Delta_{,r}(r'_+)} d(M-\omega') \\
&= -4\pi\ell^2 \int_M^{M-\omega} \frac{(M-\omega')\{1 - \frac{4}{\ell^2}(M-\omega')^2\}}{12(M-\omega')^2\{1-\frac{4}{\ell^2}(M-\omega')^2\}^2 + \ell^2} d(M-\omega') \\
&= -2\pi \int_M^{M-\omega} \left[2(M-\omega') - \frac{32}{\ell^2}(M-\omega')^3\right] d(M-\omega') \\
&= -\frac{1}{2}[S_{BH}(M-\omega') - S_{BH}(M)] = -\frac{1}{2}\Delta S_{BH},
\end{aligned} \tag{5.3.7}$$

where the Bekenstein-Hawking entropy of the black hole is

$$S_{BH} = 4\pi M^2 \left(1 - \frac{8M^2}{\ell^2}\right)$$

and ΔS_{BH} the difference of the entropies before and after the emission of the particle. So, the tunneling probability is

$$\Gamma \sim e^{-2\operatorname{Im} S} = e^{\Delta S_{BH}}. \tag{5.3.8}$$

This result is exactly the same as obtained in [214] by the Hamilton-Jacobi ansatz. The Hawking temperature

$$T_H = \frac{\Delta_{,r}(r_+)}{4\pi} = \frac{1}{8\pi M}\left(1 + \frac{16}{\ell^2}M^2\right) \tag{5.3.9}$$

reduces to the Schwarzschild black hole temperature $T_H = \frac{1}{8\pi M}$ in the limit $\ell \to \infty$.

5.3.2 Taub-NUT-AdS Black Hole Case

In this case $Q = \Phi = 0$, and

$$\Delta = \frac{r^2 - 2Mr - n^2}{r^2 + n^2} + \frac{r^2 + 5n^2}{\ell^2},$$

$$r_+ = r_{o+} - \frac{r_{o+}^4}{2\ell^2(r_{o+} - M)},$$

so (5.3.6) gives

$$\begin{aligned}
\operatorname{Im} S &= -\frac{1}{2}\int_M^{M-\omega} \frac{4\pi}{\Delta_{,r}(r'_+)} d(M-\omega') \\
&= -\pi \int_M^{M-\omega} \frac{r'^2_{o+} + n^2}{r'_{o+} - (M-\omega')} \\
&\quad \times \left[1 - \frac{2r'_{o+}}{\ell^2}\frac{r'^2_{o+} + 3n^2}{r'_{o+} - (M-\omega')}\right] d(M-\omega') \\
&= -\frac{\pi}{2}\Bigg[r'^2_{o+} - n^2(1 - \ln r'^2_{o+}) \\
&\quad -\frac{2}{\ell^2}\left\{r'^2_{o+}(r'^2_{o+} + 6n^2) - 7n^4\right\}\Bigg]_M^{M-\omega} \\
&= -\frac{1}{2}[S_{BH}(M-\omega) - S_{BH}(M)] = -\frac{1}{2}\Delta S_{BH},
\end{aligned} \tag{5.3.10}$$

where

$$r'_{o+} = (M-\omega) + \sqrt{(M-\omega)^2 + n^2}$$

and ΔS_{BH} is the difference of the Bekenstein-Hawking entropies of the black hole before and after the emission of the particle. Therefore, the tunneling probability is

$$\Gamma \sim e^{-2\,\mathrm{Im}\,S} = e^{\Delta S_{BH}}. \tag{5.3.11}$$

The Bekenstein-Hawking entropy of the Taub-NUT-AdS black hole is defined by

$$S_{BH} = \frac{A}{4} + n^2\pi \ln\left(\frac{A}{4} - n^2\pi\right) - \frac{1}{8\pi\ell^2}(A^2 + 16n^2 A), \tag{5.3.12}$$

where

$$A = 4\pi(r_+^2 + n^2)$$

is the event horizon area of the black hole. Evidently, the entropy (5.3.12) is not just a quarter area at the horizon. In the limit $\ell \to \infty$, the tunneling rate (5.3.11) agrees with that obtained for the Taub-NUT black hole by Chen et al. [201] using the Hamilton-Jacobi ansatz and Zhao and Li [215] by Damour-Ruffini method. In those works the entropy was considered a quarter area at the horizon area, but we find it for the Taub-NUT black hole as

$$S_{BH} = \frac{A}{4} + n^2\pi \ln\left(\frac{A}{4} - n^2\pi\right).$$

Hence, our result is more interesting. For the Hawking temperature of the Taub-NUT-AdS black hole, we find

$$T_H = \frac{\Delta_{,r}(r_+)}{4\pi} = \frac{r_{o+} - M}{2\pi(r_{o+}^2 + n^2)}\left(1 + \frac{2r_{o+}}{\ell^2}\frac{r_{o+}^2 + 3n^2}{r_{o+} - M}\right), \tag{5.3.13}$$

which reduces to the Schwarzschild-AdS black hole temperature (5.3.9) in the limit $n = 0$.

5.3.3 Reissner-Nordström-AdS Black Hole Case

For this case, $n = 0$ and

$$\Delta = \frac{r^2 - 2Mr + Q^2 + \Phi^2}{r^2} + \frac{r^2}{\ell^2},$$

$$r_+ = r_{o+} - \frac{r_{o+}^4}{2\ell^2(r_{o+} - M)},$$

and then (5.3.6) yields

$$\begin{aligned}
\operatorname{Im} S &= -\frac{1}{2}\int_{(M,Q,\Phi)}^{(M-\omega',Q-q',\Phi-\phi')} \frac{4\pi}{\Delta_{,r}(r'_+)}\Bigg[d(M-\omega') \\
&\quad -\frac{(Q-q')}{r'_+}d(Q-q') - \frac{(\Phi-\phi')}{r'_+}d(\Phi-\phi')\Bigg] \\
&= -\pi\int_{(M,Q,\Phi)}^{(M-\omega',Q-q',\Phi-\phi')} \frac{r'^2_{o+}}{r'_{o+}-(M-\omega')} \\
&\quad \times\left\{1 - \frac{r'^3_{o+}}{\ell^2[r'_{o+}-(M-\omega')]}\left(1 + \frac{3r'_{o+} - 4(M-\omega')}{2[r'_{o+}-(M-\omega')]}\right)\right\} \\
&\quad \times\Bigg[d(M-\omega') - \left(1 + \frac{r'^3_{o+}}{2\ell^2[r'_{o+}-(M-\omega)]}\right) \\
&\quad \times\frac{(Q-q')}{r'_{o+}}d(Q-q') - \left(1 + \frac{r'^3_{o+}}{2\ell^2[r'_{o+}-(M-\omega)]}\right) \\
&\quad \times\frac{(\Phi-\phi')}{r'_{o+}}d(\Phi-\phi')\Bigg], \qquad (5.3.14)
\end{aligned}$$

where

$$r'_{o+} = (M-\omega) + \sqrt{(M-\omega)^2 - (Q-q)^2 - (\Phi-\phi)^2}.$$

In order to calculate more simply, we make use of the entropy $S = A/4 = \pi r_+^2$ and obtain the difference between the entropies of the horizon before and after the emission:

$$\begin{aligned}\Delta S &= \pi[r'^2_+ - r^2_+] \\ &= \pi\left[r'^2_{o+}\left(1 - \frac{1}{\ell^2}\frac{r'^3_{o+}}{r'_{o+} - (M-\omega)}\right) - r^2_{o+}\left(1 - \frac{1}{\ell^2}\frac{r^3_{o+}}{r_{o+} - M}\right)\right].\end{aligned} \tag{5.3.15}$$

Then, we obtain

$$\begin{aligned}\frac{\partial(\Delta S)}{\partial(M-\omega)} &= \frac{2\pi r'^2_{o+}}{r'_{o+} - (M-\omega)}\left[1 - \frac{r'^3_{o+}}{\ell^2[r'_{o+} - (M-\omega)]}\right. \\ &\quad \left.\times\left\{1 + \frac{3r'_{o+} - 4(M-\omega)}{2[r'_{o+} - (M-\omega)]}\right\}\right], \\ \frac{\partial(\Delta S)}{\partial(Q-q)} &= \frac{-2\pi(Q-q)r'_{o+}}{r'_{o+} - (M-\omega)}\left[1 - \frac{r'^3_{o+}}{\ell^2[r'_{o+} - (M-\omega)]}\right. \\ &\quad \left.\times\left\{1 + \frac{2r'_{o+} - 3(M-\omega)}{2[r'_{o+} - (M-\omega)]}\right\}\right], \\ \frac{\partial(\Delta S)}{\partial(\Phi-\phi)} &= \frac{-2\pi(\Phi-\phi)r'_{o+}}{r'_{o+} - (M-\omega)}\left[1 - \frac{r'^3_{o+}}{\ell^2[r'_{o+} - (M-\omega)]}\right. \\ &\quad \left.\times\left\{1 + \frac{2r'_{o+} - 3(M-\omega)}{2[r'_{o+} - (M-\omega)]}\right\}\right].\end{aligned} \tag{5.3.16}$$

With (5.3.16), (5.3.14) gives

$$\begin{aligned}\operatorname{Im} S = & -\frac{1}{2}\int_{(M,Q,\Phi)}^{(M-\omega',Q-q',\Phi-\phi')} \left[\frac{\partial(\Delta S)}{\partial(M-\omega')}d(M-\omega')\right. \\ & \left. +\frac{\partial(\Delta S)}{\partial(Q-q')}d(Q-q') + \frac{\partial(\Delta S)}{\partial(\Phi-\phi')}d(\Phi-\phi')\right] \\ = & -\frac{1}{2}\int d(\Delta S) = -\frac{1}{2}\Delta S. \end{aligned} \tag{5.3.17}$$

Hence, the tunneling rate is

$$\Gamma \sim e^{-2\operatorname{Im} S} = e^{\Delta S_{BH}}. \tag{5.3.18}$$

Replacing ℓ^2 by $-\ell^2$ yields the result for the RN-dS black hole, which agrees with the result obtained in [216] using Damour-Ruffini method. In the limit $\ell^2 \to \infty$ the result goes for the RN black hole that agrees with the result obtained in Refs. [52, 70] by Hamilton-Jacobi ansatz and Ref. [217] by Damour-Ruffini method. Different from those works our result contains contribution from the magnetic charge parameter. The Hawking temperature of the RN-AdS black hole

$$\begin{aligned} T_H = & \frac{\Delta_{,r}(r_+)}{4\pi} \\ = & \frac{r_{o+}-M}{2\pi r_{o+}^2}\left[1+\frac{r_{o+}^3(5r_{o+}-6M)}{2\ell^2(r_{o+}-M)^2}\right] \end{aligned} \tag{5.3.19}$$

reduces to the Schwarzschild-AdS black hole temperature (5.3.9) in the limit $Q = \Phi = 0$.

5.3.4 TNRN-AdS black hole case

In the TNRN-AdS black hole case, (5.3.6) gives

$$
\begin{aligned}
\operatorname{Im} S = & -\frac{1}{2}\int_{(M,Q,\Phi)}^{(M-\omega',Q-q',\Phi-\phi')} \frac{4\pi}{\Delta_{,r}(r'_{+})}\Bigg[d(M-\omega') \\
& -\frac{(Q-q')r'_{+}}{r'^{2}_{+}+n^{2}}d(Q-q') - \frac{(\Phi-\phi')r'_{+}}{r'^{2}_{+}+n^{2}}d(\Phi-\phi')\Bigg] \\
= & -\pi\int_{(M,Q,\Phi)}^{(M-\omega',Q-q',\Phi-\phi')} \frac{r'^{2}_{o+}+n^{2}}{r'_{o+}-(M-\omega')} \\
& \times\left(1-\frac{2r'_{o+}}{\ell^{2}}\frac{r'^{2}_{o+}+3n^{2}}{r'_{o+}-(M-\omega')}\right)\Bigg[d(M-\omega') \\
& -\left(1-\frac{r'^{3}_{o+}(r'^{2}_{o+}+n^{2}-2)}{2\ell^{2}(r'^{2}_{o+}+n^{2})[r'_{o+}-(M-\omega')]}\right) \\
& \times\frac{(Q-q')r'_{o+}}{r'^{2}_{o+}+n^{2}}d(Q-q') \\
& -\left(1-\frac{r'^{3}_{o+}(r'^{2}_{o+}+n^{2}-2)}{2\ell^{2}(r'^{2}_{o+}+n^{2})[r'_{o+}-(M-\omega')]}\right) \\
& \times\frac{(\Phi-\phi')r'_{o+}}{r'^{2}_{o+}+n^{2}}d(\Phi-\phi')\Bigg] \\
= & -\frac{1}{2}[S_{BH}(M-\omega)-S_{BH}(M)] = -\frac{1}{2}\Delta S_{BH}, \qquad (5.3.20)
\end{aligned}
$$

where

$$
r'_{o+} = (M-\omega)+\sqrt{(M-\omega)^{2}-(Q-q)^{2}-(\Phi-\phi)^{2}+n^{2}}.
$$

The Bekenstein-Hawking entropy of the TNRN-AdS black hole has been defined by

$$S_{BH} = 3\frac{A}{4} + n^2\pi \ln\left(\frac{A}{4} - n^2\pi\right) - \frac{\pi}{\ell^2}F + \text{const.}, \tag{5.3.21}$$

where

$$\begin{aligned} F = &\frac{9}{2}\left(\frac{A}{4\pi} - n^2\right)^2 + \frac{10}{3}M\left(\frac{A}{4\pi} - n^2\right)^{3/2} \\ &+(5M^2 + 4(Q^2 + \Phi^2) + 24n^2 - 2)\left(\frac{A}{4\pi} - n^2\right) \\ &+2M(5M^2 + 12n^2 - 2)\left(\frac{A}{4\pi} - n^2\right)^{1/2} \\ &+\frac{4Mn^3}{M^2 + n^2}\tan^{-1}\left(\frac{A}{4n^2\pi} - 1\right)^{1/2} + \frac{2n^4}{M^2 + n^2}\ln\frac{A}{4\pi} + \frac{2M^2}{M^2 + n^2} \\ &\times(5M^4 + 17M^2n^2 - 2M^2 + 12n^4) \\ &\times \ln\left\{\left(\frac{A}{4\pi} - n^2\right)^{1/2} - M\right\} + 4(Q^2 + \Phi^2 + 2n^2) \\ &\times \ln\left\{2\left(\frac{A}{4\pi} - n^2\right) - 2M\left(\frac{A}{4\pi} - n^2\right)^{1/2}\right\}, \end{aligned} \tag{5.3.22}$$

$A = 4\pi(r_+^2 + n^2)$ is the event horizon area of the black hole. Evidently, it is not proportional to the event horizon area of the black hole. In fact, if we bear in mind that $T' = \frac{\Delta_{,r}(r'_+)}{4\pi}$, we easily get

$$\frac{1}{T'}(dM' - A'_o dQ' - B'_o d\Phi') = dS'. \tag{5.3.23}$$

It implies, the result in (5.3.20) is a natural consequence of the first law of black hole thermodynamics. Thus, the emission rate of the tunneling particle is

$$\Gamma \sim e^{-2\,\mathrm{Im}\,S} = e^{\Delta S_{BH}}, \tag{5.3.24}$$

where ΔS_{BH} is the difference of the Bekenstein-Hawking entropies of the black hole before and after the emission of the particle. This is consistent with the result obtained in Ref. [210] for the RN-NUT black hole with considering the entropy a quarter area at the horizon. In our work, however, we have taken the entropy defined in (5.3.21), which gives the TNRN black hole entropy in the limit $\ell \to \infty$. Our result is therefore more interesting. For the Hawking temperature of the TNRN-AdS black hole, we find

$$T_H = \beta^{-1} = \frac{r_{o+} - M}{2\pi(r_{o+}^2 + n^2)}\left(1 + \frac{2r'_{o+}}{\ell^2}\frac{r'^2_{o+} + 3n^2}{r'_{o+} - (M - \omega')}\right), \tag{5.3.25}$$

which reduces to the Schwarzschild-AdS black hole temperature (5.3.9) in the limit $Q = \Phi = n = 0$.

Evidently, when the emission rate Γ in (5.3.24) is expanded in ω, q, ϕ, it gives

$$\Gamma = \exp[-\beta(\omega - A'_+ q - B'_+ \phi) + \mathcal{O}(\omega, q, \phi)^2], \tag{5.3.26}$$

where the leading-order term gives the Boltzman factor and the higher-order terms in ω, q, ϕ generate a deviation from a precisely thermal spectrum. Following the modified surface gravity and temperature due to one-loop back-reaction effects [58, 208, 209], (5.3.26) can be put in the form

$$\Gamma = \exp[-\beta'(\omega - A'_{+}q - B'_{+}\phi)] \tag{5.3.27}$$

with

$$\beta' = \beta\left[1 - \frac{\mathcal{O}(\omega, q, \phi)^2}{\beta(\omega - A'_{+}q - B'_{+}\phi)}\right],$$

where β' can be treated as an inverse quantum-corrected temperature.

In quantum mechanics, the number of microstates of the initial and final states are the exponent of the initial and final entropies, which results the emitting rate as $\Gamma \sim e^{S_f/S_i} = e^{\Delta S}$. Manifestly, this is consistent with our result. Thus, satisfying the underlying unitary theory our result yields a might explanation to the black hole information puzzle.

5.4 Concluding Remarks

Our concern in this study is the Hawking radiation of charged and magnetized massive particles via tunneling effect from a Taub-NUT-Reissner-Nordström-AdS black hole endowed with electric as well as magnetic charges. We use the null geodesic method and find the emission rate with treating the background spacetime as dynamical. Taking into account the particle's self-gravitation and the conservation of energy, electric charge and magnetic charge, we obtain that the emission rate is connected with the change in Bekenstein-Hawking entropy and depended on the emitted particle's energy, electric charge and magnetic charge. The result shows that the Hawking thermal radiation actually deviates from perfect thermality and is consistent with an underlying unitary theory. We discuss as well the cases for the Schwarzschild-AdS, Taub-NUT-AdS, and Reissner-Nordström-AdS black holes. The result is fully in accordance with the previous literature. We derive the expected Hawking temperature and find, in contrast to a common black hole, that the entropy is not just a quarter area at the horizon of NUT charged black holes, which is consistent with the finding of Refs. [93, 95, 97, 98, 99, 100]. The result can also be treated as a quantum-corrected radiation temperature and it depends not only on the black hole background but also on the radiation particle's energy and charges. The result of this chapter agrees with that of chapter 4 obtained by Liu's method [71] which is based on the Damour-Ruffini method (section 3.4 of chapter 3).

Chapter 6

Tunneling and Temperature of Demiański-Newman Black Holes

In this chapter we present the work of Ref. [102] in which we investigate Hawking radiation of charged and magnetized (scalar/fermion) particles from Demiański-Newman black holes by using Hamilton-Jacobi ansatz. Taking into account conservation of energy and the back-reaction of particles to the spacetime, we calculate the emission rate and find it proportional to the change of Bekenstein-Hawking entropy. The radiation spectrum deviates from the precisely thermal one and is accordant with that obtained by the null geodesic method, but its physical picture is more clear. The investigation specifies a quantum-corrected radiation temperature dependent on the black hole background and the radiation particle's energy, angular momentum, and charges.

This chapter is arranged as follows. The proceeding section is an introduction to the work of this chapter. In section 6.2 we study tunneling radiation of electrically charged scalar magnetic particles from the Demiański-Newman black holes. Utilizing WKB approximation and Hamilton-Jacobi ansatz, we derive the tunneling rate of the radiant particle. In section 6.3 we investigate charged and magnetic fermions tunneling from the Demiański-Newman black hole. We also construct an exact form of the action for massless and massive Dirac particles. Finally, section 6.4 is for conclusion.

6.1 Introduction

Black holes, the most significant prediction of Einstein's field equations, are very subtle and mysterious objects in the universe. They always play the significant role in physics and astronomy due to the people's views on the space and time, matters and gravity. Classically, they are perfect absorbers and do not emit any type of radiations. Entry of matter, which has its own entropy, into the black hole, results in the decrease of the total entropy of the universe, and this contradicts the second law of thermodynamics. Bekenstein [1] first conjectured the relation between the properties of black holes and the laws of thermodynamics and showed that the black hole possesses entropy similar to its surface area. As the black hole absorbs matter, its entropy increases and the decrease of the exterior entropy is then balanced, preserving the second law of thermodynamics. The surface gravity, which is the gravitational acceleration experienced at the surface of the black hole or any object, is related with temperature of the body in thermal equilibrium. Soon after the significant work of Bekenstein, Hawking showed that quantum mechanically black holes emit precisely thermal radiations (chapter 3). The origin of this radiation can be understood by considering spontaneous creation of particle-antiparticle pairs at or near the event horizon of the black hole by vacuum fluctuations.

Usually, such a pair annihilates itself very rapidly; but it is possible that one of them—particle or antiparticle—enters into the black hole before the annihilation so that the other one is free to escape away from the black hole. In particular, when a negative energy antiparticle is absorbed, the black hole mass decreases, and it results in increase of the black hole temperature, charge potentials and angular velocity. Then the black hole can spontaneously transfer an amount of heat, charges and angular momentum to the positive energy particle. As a result, the black hole temperature, charge potentials and angular velocity further increase and further transfer. Acquiring enough energy the positive energy particle can escape away to infinity. If as a net effect more antiparticles than particles are pulled into the black hole, an observer outside the black hole observes a particle flux which appears to come out from the black hole as Hawking radiation. This induces the black hole to shrink.

The Hawking radiation phenomenon reveals a significant correlation among thermodynamics, quantum mechanics, and gravity. The study of black hole radiation thus has been a subject of intensive and extensive research. However, the precisely thermal Hawking radiation leads to two obvious disputes: the "information loss" and violation of the underlying quantum "unitary theory" [218, 219, 220, 221]. Indeed, a moot question arises concerning the reaction of the radiation to the spacetime. When the black hole generates Hawking radiation, the black hole parameters (energy, charge, and angular momentum) fluctuate. This effect was not considered in the past. Hawking derived the black-hole radiation as precisely thermal spectrum only under the assumption that the spacetime is invariant. There have been many attempts to resolve these problems by applying semi-classical tunneling methods, such as the Parikh-Wilczek (or null-geodesic) method reviewed in chapter 3 (subsection 3.5.1).

There has been proposed another method by Srinivasan et al. [26, 27, 28], following Landau's [222] complex paths method, in which Hawking radiation is derived as tunneling across the singularity with the wave functions as semi-classical approximation modes

$$e^{\frac{i}{\hbar}I(r,t)},$$

where I is the classical action which can be expanded in powers of $\hbar/i$. This is called the Hamilton-Jacobi method reviewed in subsection 3.5.2 of chapter 3 for the simplest case (to model scalar particles). The action I satisfies the relativistic Hamilton-Jacobi equation with the solution

$$I_{\pm} = -\omega t \pm R(r) + J(x^i)$$

to the the lowest order, where the upper (lower) sign denotes outgoing (incoming) particles. A pole occurs at the horizon point r_+ in I and the probabilities of the particles are given by $\Gamma_{\pm} \sim e^{-2\,\mathrm{Im}\,I_{\pm}}$. Angheben et al. [35] and Kerner and Mann [44, 60] further developed this complex-path method by using the boundary conditions for incoming particles which fall behind the horizon along classically permitted trajectories, i.e.

$$I = -\omega t + R(r) + J(x^i) + K,$$

where K is a complex normalizing constant. The total probability is then

$$\Gamma = \Gamma_{\mathrm{out}} \sim e^{-2(\mathrm{Im}\,I_+ - \mathrm{Im}\,I_-)}.$$

In this method, however, the back-reaction effect of the emitted particle was not taken into account. In consideration of the self-gravitational interaction and unfixed background spacetime, one can find

$$2(\text{Im}\, I_+ - \text{Im}\, I_-) = \beta\omega + \mathcal{O}(\omega^2),$$

which yields at linear order the regular Boltzmann factor $\Gamma \simeq e^{-\beta\omega}$, where ω is the particle energy and β is the inverse Hawking temperature. It also leads to a quantum-corrected inverse temperature [72]

$$\beta' = \beta \left(1 + \frac{\mathcal{O}(\omega^2)}{\beta\omega}\right).$$

One can draw from this method the same conclusion as the null geodesic method. There have been studies of Hawking radiation by the Hamilton-Jacobi method in Refs. [52, 223, 224, 225, 226, 227, 228].

Recently, Ding [70] investigated Hawking radiation of charged (rotating) black holes by dividing the emission time into a series of infinite small pieces. In each of small segments the process can be treated as a quasi-static one with the background spacetime as fixed. There exists equilibrium temperature in each piece and the Hamilton-Jacobi method can be applied there. In different piece the instantaneous event horizon is different. If I_i be the action in the i-th tiny time piece after the particle tunneled across the instantaneous horizon and $\Delta I_i = I_i - I_{i-1}$, the last action is found as $I = \sum \Delta I_i \sim \int dI$. Interest in the Hamilton-Jacobi method is due to the covariant treatment of the horizon singularity through the use of the proper spatial distance. On the contrary, the null geodesic method strongly relies on a very specific choice of (regular-across-horizon) coordinates and turns upside down the relationship between Hawking radiation and back-reaction [229]. These unpleasant features can be dealt with the Hamilton-Jacobi method successfully. As indicated in Ref. [230], the null geodesic method is only suitable for the reversible process, but the factual emission process is irreversible; so there is possible to lose information. However, the Hamilton-Jacobi method can be suitable for the irreversible process and there are very few information lost in the emitting process.

In this chapter, we apply Ding's [70] approach to investigate the Hawking radiation of electrically charged magnetic scalar and Dirac particles from the Demiański-Newman black hole [103], which is a five-parameter stationary axisymmetric solution of the Einstein-Maxwell equations. The Demiański-Newman black hole background is interesting in that it generalizes the well-known Kerr-Newman spacetime with two intriguing parameters the gravitomagnetic and magnetic monopoles. In the stationary pure vacuum limit, the Demiański-Newman metric reduces to the combined Kerr-NUT and Taub-NUT solutions. It is interesting that the spacetimes with the NUT charge are not asymptotically flat but asymptotically locally flat [93, 100, 101] and they possess several special properties. As discussed in [44], tunneling and temperature of Taub-NUT black holes can be formally carried out and the physical interpretation is less problematic in the context of the Hamilton-Jacobi ansatz than the null-geodesic method. The Taub-NUT space has played an important role in the conceptional development of general relativity and in the construction of brane solutions in string theory and M-theory. The singularities of the NUT charged spacetime, the Misner strings [96], can be avoided by periodic time coordinate. One of the interesting properties of NUT charged spaces is the existence of closed timelike curves which violates the causality condition. The half-closed timelike geodesics in Taub area can be explored in NUT area, so the naked singularity exists. The NUT charged black holes have been of particular interest in AdS/CFT conjecture [93, 94, 95]. In AdS backgrounds, Lorentzian sector of these spacetimes' boundary metric is similar with the Gödel metric [104]. In recent years the thermodynamics of various Taub-NUT spacetimes has become a subject of intense study. Entropy of these spacetimes is not just a quarter area at the horizon and their free energy can sometimes be negative [93, 97, 98, 99, 100, 101, 105, 106]. It was ingeniously suggested by Dirac relatively long ago that the magnetic monopole does exist in nature, but it was neglected due to the failure to detect such an object. However, in recent years, the development of gauge theories has shed new light on it. Several recent extensions of the standard model of particle physics predict existence of magnetic monopoles and it has grown interests in the possibility of magnetically charged black holes. The string theory [107] also admits the existence of such objects. The importance of the Demiański-Newman solution lies in that it gives a single constituent with the whole set of parameters which may have a physical sense in axisymmetric many-body systems of aligned sources [108, 109]. In view of the above considerations, the research on the Demiański-Newman black hole is necessary and meaningful.

6.2 Tunneling of Electrically Charged Magnetic Scalar Particles

The metric of the Demiański-Newman black hole in Boyer-Lindquist coordinates is [103]

$$ds^2 = \frac{\Sigma}{N}\left[N\left(\frac{1}{F}dr^2 + d\theta^2\right) + \Delta\sin^2\theta d\varphi^2\right] - \frac{N}{\Sigma}\left[dt + \left(2n(1-\cos\theta) + \frac{aW\sin^2\theta}{N}\right)d\varphi\right]^2 \quad (6.2.1)$$

and the associated electromagnetic potential is

$$A = (A_t, 0, 0, A_\varphi),$$

where

$$\begin{aligned}
A_t &= -\frac{Qr + P(n - a\cos\theta)}{\Sigma}, \\
A_\varphi &= \frac{a\sin^2\theta + 2n(\cos\theta - 1)}{\Sigma} \\
&\quad \times [Qr + P(n - a\cos\theta)] + (1 - \cos\theta)P, \\
F &= r^2 - 2Mr - n^2 + a^2 + Q^2 + P^2, \\
\Sigma &= r^2 + (a\cos\theta - n)^2, \\
W &= \Sigma + a^2\sin^2\theta - F, \\
N &= F - a^2\sin^2\theta. \qquad (6.2.2)
\end{aligned}$$

In the absence of the NUT (magnetic mass) parameter n, the Demiański-Newman metric is asymptotically flat and represents a dyonic Kerr-Newman black hole spacetime with M, a, Q, P respectively being the mass, the specific angular momentum, the electric charge and the magnetic monopole parameters. The Demiański-Newman metric represents the Kerr-NUT solution for $Q = P = 0$ and the Taub-NUT solution for $Q = P = a = 0$.

The metric determinant is

$$\mathrm{g} = \det.\mathrm{g}_{\mu\nu} = -\Sigma^2 \sin^2\theta$$

and the non-vanishing elements of the inverse metric are

$$\mathrm{g}^{00} = -\frac{\Sigma}{\mathrm{N}} + \frac{\mathrm{NU}^2}{\Sigma \mathrm{F} \sin^2\theta}, \quad \mathrm{g}^{11} = \frac{\mathrm{F}}{\Sigma}, \quad \mathrm{g}^{22} = \frac{1}{\Sigma},$$

$$\mathrm{g}^{33} = \frac{\mathrm{N}}{\Sigma \mathrm{F} \sin^2\theta}, \quad \mathrm{g}^{03} = -\frac{\mathrm{NU}}{\Sigma \mathrm{F} \sin^2\theta},$$

$$U = \left\{ 2n(1-\cos\theta) + \frac{aW\sin^2\theta}{N} \right\}. \tag{6.2.3}$$

The energy of a particle changes sign as it crosses the black hole horizon. So, a pair created just inside or just outside the horizon can materialize with zero total energy, after one member of the pair has tunneled to the opposite side. When the black hole absorbs a negative energy virtual particle, its mass decreases but its temperature, electric potential, magnetic potential and angular velocity increase. The zero energy particle can gain enough energy from absorbing black hole's amount of heat Q^h and escape away to infinity. We consider the black hole and its radiation as an isolated systems. When a particle with energy ω, charge q, magnet p and angular momentum j tunnels out of the event horizon, the mass, electric charge, magnetic charge and angular momentum of the black hole should be replaced by $M-\omega$, $Q-q$, $P-p$, and $J-j$ if conservation of energy, charge, magnet and angular momentum are taken into account. Let us divide the emission process of the particle into many infinite small segments. In the first segment, the black hole temperature rises from

$$T(M,Q,P,J) \quad \text{to} \quad T'(M-\omega_1, Q-q_1', P-p_1', J-j_1')$$

and the horizon shrinks from

$$r_+(M,Q,P,a) \quad \text{to} \quad r_+'(M-\omega_1, Q-q_1', P-p_1', J-j_1').$$

This instantaneously results in increasing the black hole's electric potential

$$V_{Q+}(M,Q,J) \quad \text{to} \quad V_{Q+}'(M-\omega_1, Q-q_1', J-j_1'),$$

magnetic potential

$$V_{P+}(M,P,J) \quad \text{to} \quad V_{P+}'(M-\omega_1,, P-p_1', J-j_1')$$

and angular velocity

$$\Omega_+(M,Q,P,J) \quad \text{to} \quad \Omega_+'(M-\omega_1, Q-q_1', P-p_1', J-j_1').$$

There then occurs a spontaneous transfer of heat, electric charge, magnetic charge and angular momentum from the black hole to the particle. This process will lead to another further increase and then further transfer of the black hole temperature, electric potential, magnetic potential and angular velocity to the particle. Thus, in the i-th segment, the particle obtains energy

$$\Delta\omega_i = \omega_i - \omega_{i-1} \ll \omega,$$

charge

$$\Delta q_i' = q_i' - q_{i-1}',$$

magnet

$$\Delta p_i' = p_i' - p_{i-1}',$$

and angular momentum

$$\Delta j_i' = j_i' - j_{i-1}',$$

where

$$\Delta\omega_i = -T'(M-\omega_i)\Delta S_i' + V_{Q+}'\Delta q_i' + V_{P+}'\Delta p_i' + \Omega_+'\Delta j_i'.$$

Treating these tiny segments as many quasi-static processes, we make use of the Hamilton-Jacobi ansatz. We divide tunneling time t, rotating angle φ into infinite small pieces t_i, φ_i. A particle of instantaneous energy ω_i, charge q_i', magnet p_i' and angular momentum j_i' will effectively view the metric tensor

$$\mathrm{g}_{\mu\nu}(r(M,Q,P,J)) \quad \text{as} \quad \mathrm{g}_{\mu\nu}(r(M-\omega_i, Q-q_i', P-p_i', J-j_i')).$$

The charged Klein-Gordon equation is

$$\begin{aligned} &\frac{1}{\sqrt{-\mathrm{g}}}\left(\partial_\mu - \frac{i\phi_i'}{\hbar}A_\mu\right) \\ &\quad\times\left[\sqrt{-\mathrm{g}}\,\mathrm{g}^{\mu\nu}\left(\partial_\nu - \frac{i\phi_i'}{\hbar}A_\nu\right)\Psi\right] - \frac{m^2}{\hbar^2}\Psi = 0, \end{aligned} \tag{6.2.4}$$

where

$$\begin{aligned} \mathrm{g}^{\mu\nu} &= \mathrm{g}^{\mu\nu}(r(M-\omega_i, Q-q_i', P-p_i', J-j_i')), \\ A_\mu &= A_\mu(r(M-\omega_i, Q-q_i', P-p_i', J-j_i')), \\ \phi_i' &= \phi_i'(p_i', q_i') \end{aligned}$$

and m is the mass of the particle. In order to apply the WKB approximation, we assume the wave function

$$\Psi(t_i, r, \theta, \varphi_i) = \exp\left[\frac{i}{\hbar} I_i(t_i, r, \theta, \varphi_i) + I_1'(t_i, r, \theta, \varphi_i) + \mathcal{O}(\hbar)\right]. \qquad (6.2.5)$$

Then from (6.2.4), to leading order in $\hbar$, we get the following relativistic Hamilton-Jacobi equation

$$\begin{aligned}
&\frac{F}{\tilde{\Sigma}}\left(\frac{\partial I_i}{\partial r}\right)^2 + \frac{1}{\tilde{\Sigma}}\left(\frac{\partial I_i}{\partial \theta}\right)^2 + \frac{F - \tilde{a}^2\sin^2\theta}{\tilde{\Sigma}F\sin^2\theta}\left(\frac{\partial I_i}{\partial \varphi_i} - \phi_i' A_{\varphi_i}\right)^2 \\
&\quad - \Bigg[\frac{\tilde{\Sigma}}{F - \tilde{a}^2\sin^2\theta} - \frac{F - \tilde{a}^2\sin^2\theta}{\tilde{\Sigma}F\sin^2\theta} \\
&\quad \times \left\{2n(1-\cos\theta) - \tilde{a}\sin^2\theta\left(1 - \frac{\tilde{\Sigma}}{F - \tilde{a}^2\sin^2\theta}\right)\right\}^2\Bigg] \\
&\quad \times \left(\frac{\partial I_i}{\partial t_i} - \phi_i' A_{t_i}\right)^2 - 2\frac{F - \tilde{a}^2\sin^2\theta}{\tilde{\Sigma}F\sin^2\theta} \\
&\quad \times \left\{2n(1-\cos\theta) - \tilde{a}\sin^2\theta\left(1 - \frac{\tilde{\Sigma}}{F - \tilde{a}^2\sin^2\theta}\right)\right\} \\
&\quad \times \left(\frac{\partial I_i}{\partial t_i} - \phi_i' A_{t_i}\right)\left(\frac{\partial I_i}{\partial \varphi_i} - \phi_i' A_{\varphi_i}\right) + m^2 = 0, \qquad (6.2.6)
\end{aligned}$$

where

$$F = r^2 - 2(M - \omega_i)r - n^2 + \tilde{a}_i^2 + (Q - q_i')^2 + (P - p_i')^2,$$

$$A_{t_i} = -\frac{(Q - q_i')r + (P - p_i')(n - \tilde{a}_i\cos\theta)}{\tilde{\Sigma}},$$

$$A_{\varphi_i} = \frac{\tilde{a}_i \sin^2\theta + 2n(\cos\theta - 1)}{\tilde{\Sigma}}$$
$$\times[(Q - q_i')r + (P - p_i')(n - \tilde{a}_i\cos\theta)] + (1 - \cos\theta)(P - p_i'),$$

$$\tilde{a}_i = \frac{J - j_i'}{M - \omega_i}, \quad \tilde{\Sigma} = r^2 + (\tilde{a}_i\cos\theta - n)^2. \tag{6.2.7}$$

Its solution can be put in the form

$$I_i = -\omega_i t_i + R_i(r) + j_i'\varphi_i + \Theta_i(\theta) + K_i, \tag{6.2.8}$$

where K_i is a complex constant normalizing the action function. The Hamilton-Jacobi equation (6.2.6) gives

$$F^2\left(\frac{dR_i(r)}{dr}\right)^2 - \tilde{a}_i^2\Bigg[\tilde{j}_i' + \Bigg\{2n(1 - \cos\theta)$$
$$-\tilde{a}_i\sin^2\theta\left(1 - \frac{\tilde{\Sigma}}{F - \tilde{a}_i^2\sin^2\theta}\right)\Bigg\}\tilde{\omega}_i\Bigg]^2 + F\Lambda = 0, \tag{6.2.9}$$

where

$$\Lambda = \tilde{\Sigma}m^2 - \frac{\tilde{\Sigma}^2}{F - \tilde{a}_i^2\sin^2\theta}\tilde{\omega}_i^2 + \left(\frac{d\Theta_i}{d\theta}\right)^2 + \frac{1}{\sin^2\theta}$$
$$\times\Bigg[\tilde{j}_i' + \Bigg\{2n(1 - \cos\theta) - \tilde{a}_i\sin^2\theta$$
$$\times\left(1 - \frac{\tilde{\Sigma}}{F - \tilde{a}_i^2\sin^2\theta}\right)\Bigg\}\tilde{\omega}_i\Bigg]^2,$$

$$\tilde{\omega}_i = \omega_i + \phi_i' A_{t_i}, \quad \tilde{j}'_i = j_i' - \phi_i' A_{\varphi_i}. \tag{6.2.10}$$

Solving for R_i yields

$$R_{\pm i}(r) = \pm \int dr \frac{1}{F}\left(\tilde{a}_i^2\left[\tilde{j}_i' + \left\{2n(1-\cos\theta) - \tilde{a}_i\sin^2\theta \right.\right.\right.$$
$$\left.\left.\left.\times\left(1 - \frac{\tilde{\Sigma}}{F - \tilde{a}^2\sin^2\theta}\right)\right\}\tilde{\omega}_i\right]^2 - F\lambda\right)^{1/2}. \qquad (6.2.11)$$

Performing the integration in (6.2.11) around the pole, the imaginary parts of the action function are found as

$$\begin{aligned}
\operatorname{Im} I_{\pm i} &= \pi\frac{r'^2_+ + (n-\tilde{a}_i)^2}{2(r'_+ - M + \omega_i)}\left(\tilde{\omega}_{i+} - \frac{\tilde{j}'_{i+}\tilde{a}_i}{r'^2_+ + (n-\tilde{a}_i)^2}\right) + \operatorname{Im} K_i,\\
&= \pi\frac{r'^2_+ + (n-\tilde{a}_i)^2}{2(r'_+ - M + \omega_i)}\left(\omega_i - \frac{q_i'(Q-q_i')r'_+}{r'^2_+ + (n-\tilde{a}_i)^2}\right.\\
&\quad\left. - \frac{(n-\tilde{a}_i)p_i'(P-p_i')}{r'^2_+ + (n-\tilde{a}_i)^2} - \frac{j_i'\tilde{a}_i}{r'^2_+ + (n-\tilde{a}_i)^2}\right) + \operatorname{Im} K_i\\
&= \pi\frac{r'^2_+ + (n-\tilde{a}_i)^2}{2(r'_+ - M + \omega_i)}\\
&\quad\times\left(\omega_i - q_i'V'_{Q+} - p_i'V'_{P+} - j_i'\Omega'_+\right) + \operatorname{Im} K_i, \qquad (6.2.12)
\end{aligned}$$

where

$$r'_+ = (M-\omega_i) + \sqrt{(M-\omega_i)^2 - \tilde{a}_i^2 + n^2 - (Q-q_i')^2 - (P-p_i')^2}$$

is the instantaneous event horizon and Ω'_+ is its instantaneous angular velocity. This is same as the case of the massless particle ($m = 0$), because the extra contributions with mass vanish at the horizon. The V'_{Q+} and V'_{P+} are respectively the instantaneous electric and magnetic potentials on the event horizon.

For the particle tunneled across the i-th instantaneous horizon, the imaginary part of the action is (6.2.12), and the change between the i-th and $(i-1)$-th instantaneous imaginary parts of the action is

$$\Delta\,\mathrm{Im}\,I_{\pm i} = \pi\frac{r'^2_+ + (n-\tilde{a}_i)^2}{2(r'_+ - M + \omega_i)}(\Delta\omega_i - \Delta j'_i\Omega'_+ \\ -\Delta q'_i V'_{Q+} - \Delta p'_i V'_{P+}) + \Delta\,\mathrm{Im}\,K_i. \qquad (6.2.13)$$

As the energy, charges and angular momentum of the particle gradually approaches to ω, q, p, j, the imaginary part of its action becomes

$$\mathrm{Im}\,I = \sum \Delta\,\mathrm{Im}\,I_{\pm i} \\ = \int_{(0,0,0,0)}^{(\omega,q,p,j)} \pi\frac{r'^2_+ + (n-\tilde{a})^2}{2(r'_+ - M + \omega)} \\ \times(d\omega - V'_{Q+}dq' - V'_{P+}dp' - \Omega'_+ dj') + \mathrm{Im}\,K. \qquad (6.2.14)$$

We calculate the value of this integration by using the entropy of the event horizon. Spacetimes with nonzero NUT charge generically do not respect the usual relationship between area and entropy. However, there exists also a different viewpoint (for example, [231]) that the entropy of a Taub-NUT spacetime is still equal to one quarter of the event horizon area. Considering the difference ΔS of the entropy $S = A/4 = \pi[r_+^2 + (n-a)^2]$

of the event horizon before and after the emission, the imaginary parts of the action of tunneled particle are obtained from (6.2.14) as follows:

$$\begin{aligned}
\operatorname{Im} I_{\pm} &\simeq \pm\frac{\pi}{2}\int_{(0,0,0,0)}^{(\omega,q,p,j)}\left[\frac{\partial(\Delta S)}{\partial\omega}d\omega\right.\\
&\quad\left.+\frac{\partial(\Delta S)}{\partial q'}dq'+\frac{\partial(\Delta S)}{\partial p'}dp'+\frac{\partial(\Delta S)}{\partial j'}dj'\right]+\operatorname{Im} K\\
&=\pm\frac{\pi}{2}\left[2\omega\left(M-\frac{\omega}{2}\right)-Qq\right.\\
&\quad+\frac{1}{2}q^2-Pp+\frac{1}{2}p^2+n\tilde{a}-(M-\omega)\\
&\quad\times\sqrt{(M-\omega)^2-\tilde{a}^2+n^2-(Q-q)^2-(P-p)^2}\\
&\quad\left.+M\sqrt{M^2-a^2+n^2-Q^2-P^2}-na\right]+\operatorname{Im} K, \quad (6.2.15)
\end{aligned}$$

where

$$\tilde{a}=\frac{J-j}{M-\omega}.$$

The total emission probability (same for massive and massless particles) is then

$$\Gamma\sim e^{-2(\operatorname{Im} I_{+}-\operatorname{Im} I_{-})}=e^{\Delta S_{BH}}, \quad (6.2.16)$$

and the Hawking temperature is

$$T=\frac{1}{2\pi}\frac{r_{+}-M}{r_{+}^2+(n-a)^2}, \quad (6.2.17)$$

where ΔS_{BH} is the difference of the Bekenstein-Hawking entropies of the black hole. Indeed, (6.2.16) is a natural result if one considers the first law of black hole thermodynamics:

$$\frac{1}{T'}(d\omega-V'_{Q+}dq'-V'_{P+}dp'-\Omega'_{+}dj')=dS'.$$

Manifestly, the derived radiation spectrum deviates from precisely thermal one and is in agreement with the Parikh's work.

We now examine the total entropy change of the system comprising the black hole and radiating particles. We consider the many infinite small segments of the particle emission process. In the first segment, particle's energy increases from 0 to ϵ_1 with

$$\epsilon_1 = \omega_1 - V'_{Q+}q'_1 - V'_{P+}p'_1 - \Omega'_+ j'_1$$

and it results in decreasing the black hole entropy by

$$\Delta S'_1 = -\frac{Q^h_1}{T'(M-\omega_1)}.$$

The increase in particle entropy is

$$\Delta S''_1 = \frac{Q^h_1}{T(M)}$$

and as a result, the entropy of the system increases by

$$\begin{aligned}\Delta S_1 &= \Delta S''_1 + \Delta S'_1 \\ &= \frac{Q^h_1}{T(M)} - \frac{Q^h_1}{T'(M-\omega_1)} > 0,\end{aligned} \tag{6.2.18}$$

showing that the radiating process is irreversible, and ϵ_1 is

$$\Delta\epsilon_1 = \epsilon_1 - 0 = -T'(M-\omega_1)\Delta S'_1.$$

In the i-th segment ($i \geq 2$), particle energy increases from ϵ_{i-1} to ϵ_i with

$$\epsilon_i = \omega_i - V'_{Q+}q'_i - V'_{P+}p'_i - \Omega'_+ j'_i$$

by absorbing heat

$$\begin{aligned}\epsilon_i =& \epsilon_i - \epsilon_{i-1} \\ =& -T'(M-\omega_i)\Delta S'_i.\end{aligned}$$

So, the increase of the system entropy in the i-th segment is

$$\begin{aligned}\Delta S_i &= \Delta S''_i + \Delta S'_i \\ &= \frac{\Delta\epsilon_i}{T'(M-\omega_{i-1})} - \frac{\Delta\epsilon_i}{T'(M-\omega_i)}.\end{aligned} \tag{6.2.19}$$

As the energy of the particle approaches to ω, the black hole temperature tends to $T'(M-\omega)$ and the total increase of the system entropy becomes

$$\begin{aligned}\Delta S &= \sum \Delta S_i \\ &= \frac{\Delta\epsilon}{T(M)} - \frac{\Delta\epsilon}{T'(M-\omega)} < \frac{\Delta\epsilon}{T(M)}\end{aligned} \tag{6.2.20}$$

with

$$\Delta\epsilon_1 = \Delta\epsilon_2 = \cdots = \Delta\epsilon \equiv \Delta\omega - V'_{Q+}\Delta q - V'_{P+}\Delta p - \Omega'_+\Delta j.$$

Since $\Delta\epsilon \ll \epsilon$, the total increase of the system entropy is very small (but nonzero) and can be ignored. This agrees with Refs. [232, 233] but has some difference from [19] in which $\Delta S = 0$. This implies that the radiation process is an irreversible one and the probing of radiating particles is related to the entropy change of the black hole.

6.3 Tunneling of Fermions with Electric and Magnetic Charges

We apply the Hamilton-Jacobi ansatz in Dirac field described by the charged Dirac equation in covariant form [234]

$$i\gamma^a e_a^\mu \left(D_\mu - \frac{i\phi_i'}{\hbar} A_\mu \right) \Psi - \frac{m}{\hbar}\Psi = 0, \tag{6.3.1}$$

where e_a^μ is the vierbein (tetrad) and D_μ is the covariant derivative for fermionic fields, defined by

$$D_\mu = \partial_\mu + \Omega_\mu, \tag{6.3.2}$$

where

$$\Omega_\mu = -\frac{i}{4}\omega_{ab\mu}\sigma^{ab}, \quad \sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$$

with

$$\omega_{ab\mu} = e_a^\nu e_{b\nu;\mu}$$

the spin connection components. The Dirac matrices $\gamma^\mu = e_a^\mu \gamma^a$ satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\mathrm{g}^{\mu\nu} \times 1.$$

We choose a representation in which the γ^a's are the following chiral γ's for Minkowski space

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \tag{6.3.3}$$

where σ's are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.3.4}$$

When a particle of instantaneous energy ω_i, charges $\phi'(q_i', p_i')$ and angular momentum j_i' tunnels out of the black hole, one can find from the metric (6.2.1) the nonzero vierbein elements as follows:

$$\begin{aligned}
e_0^{t_i} &= \sqrt{\frac{\tilde{\Sigma}}{F - \tilde{a}_i^2 \sin^2\theta}}, \\
e_1^{r} &= \sqrt{\frac{F}{\tilde{\Sigma}}}, \\
e_2^{\theta} &= \frac{1}{\sqrt{\tilde{\Sigma}}}, \\
e_3^{\varphi_i} &= \frac{1}{\sin\theta}\sqrt{\frac{F - \tilde{a}_i^2 \sin^2\theta}{\tilde{\Sigma} F}}, \\
e_3^{t_i} &= -\sqrt{\frac{F - \tilde{a}_i^2 \sin^2\theta}{\tilde{\Sigma} F \sin^2\theta}} \\
&\quad \times \left\{ 2n(1-\cos\theta) - \tilde{a}_i \sin^2\theta \left(1 - \frac{\tilde{\Sigma}}{F - \tilde{a}_i^2 \sin^2\theta} \right) \right\},
\end{aligned} \tag{6.3.5}$$

where

$$\begin{aligned}
F &= r^2 - 2(M - \omega_i)r - n^2 + \tilde{a}_i^2 + (Q - q_i')^2 + (P - p_i')^2, \\
\tilde{\Sigma} &= r^2 + (\tilde{a}_i \cos\theta - n)^2, \\
\tilde{a}_i &= \frac{J - j_i'}{M - \omega_i}.
\end{aligned}$$

We denote $\xi_{\uparrow/\downarrow}$ for the eigenvectors of σ^3. The spinor wave function Ψ has two spin states: spin-up (in +ve r-direction) and spin-down (in −ve r-direction). For the spin-up and spin-down particle's solution, we assume

$$\Psi_{i\uparrow} = \begin{pmatrix} A(t_i, r, \theta, \varphi_i)\xi_\uparrow \\ \\ B(t_i, r, \theta, \varphi_i)\xi_\uparrow \end{pmatrix} e^{[(i/\hbar)I_{i\uparrow}(t_i,r,\theta,\varphi_i)]}$$

$$= \begin{pmatrix} A(t_i, r, \theta, \varphi_i) \\ 0 \\ B(t_i, r, \theta, \varphi_i) \\ 0 \end{pmatrix} e^{[(i/\hbar)I_{i\uparrow}(t_i,r,\theta,\varphi_i)]}, \tag{6.3.6}$$

and

$$\Psi_{i\downarrow} = \begin{pmatrix} C(t_i, r, \theta, \varphi_i)\xi_\downarrow \\ \\ D(t_i, r, \theta, \varphi_i)\xi_\downarrow \end{pmatrix} e^{[(i/\hbar)I_{i\downarrow}(t_i,r,\theta,\varphi_i)]}$$

$$= \begin{pmatrix} 0 \\ C(t_i, r, \theta, \varphi_i) \\ 0 \\ D(t_i, r, \theta, \varphi_i) \end{pmatrix} e^{[(i/\hbar)I_{i\downarrow}(t_i,r,\theta,\varphi_i)]}. \tag{6.3.7}$$

We analyze only the spin-up case because the spin-down case proceeds in a manner fully analogous to the spin-up case. The equations for the spin-down case are of the same form as the spin-up case except than some changes of sign. Inserting the ansatz (6.3.6) into the Dirac Equation

(6.3.1), we obtain, to leading order in $\hbar$, the follow equations:

$$Ae_0^{t_i}(\partial_{t_i}I_{i\uparrow} - \phi_i' A_{t_i}) + Be_1^r\partial_r I_{i\uparrow} + mA = 0, \tag{6.3.8}$$

$$iBe_3^{t_i}(\partial_{t_i}I_{i\uparrow} - \phi_i' A_{t_i}) + Be_2^\theta\partial_\theta I_{i\uparrow}$$
$$+iBe_3^{\varphi_i}(\partial_{\varphi_i}I_{i\uparrow} - \phi_i' A_{\varphi_i}) = 0, \tag{6.3.9}$$

$$-Be_0^{t_i}(\partial_{t_i}I_{i\uparrow} - \phi_i' A_{t_i}) - Ae_1^r\partial_r I_{i\uparrow} + mB = 0, \tag{6.3.10}$$

$$-iAe_3^{t_i}(\partial_{t_i}I_{i\uparrow} - \phi_i' A_{t_i}) - Ae_2^\theta\partial_\theta I_{i\uparrow}$$
$$-iAe_3^{\varphi_i}(\partial_{\varphi_i}I_{i\uparrow} - \phi_i' A_{\varphi_i}) = 0, \tag{6.3.11}$$

where the components Ω_μ and derivatives of A, B have been neglected, because they are all of order $\mathcal{O}(\hbar)$. Equations (6.3.8) and (6.3.10) couple (or decouple) according as $m \neq 0$ (or $m = 0$). We employ the ansatz (6.2.8):

$$I_{i\uparrow} = -\omega_i t_i + R_i(r) + j_i'\varphi_i + \Theta_i(\theta) + K_i,$$

and obtain

$$-A\sqrt{\frac{\tilde{\Sigma}}{F - \tilde{a}_i^2\sin^2\theta}}(\omega_i + \phi_i' A_{t_i}) + B\sqrt{\frac{F}{\tilde{\Sigma}}}R_i'(r) + mA = 0, \tag{6.3.12}$$

$$iB\sqrt{\frac{F - \tilde{a}_i^2\sin^2\theta}{\tilde{\Sigma}F\sin^2\theta}}$$
$$\times\left\{2n(1-\cos\theta) - \tilde{a}_i\sin^2\theta\left(1 - \frac{\tilde{\Sigma}}{F - \tilde{a}_i^2\sin^2\theta}\right)\right\}(\omega_i + \phi_i' A_{t_i})$$
$$+B\frac{1}{\sqrt{\tilde{\Sigma}}}\Theta_i'(\theta) + iB\sqrt{\frac{F - \tilde{a}_i^2\sin^2\theta}{\tilde{\Sigma}F\sin^2\theta}}(j_i' - \phi_i' A_{\varphi_i}) = 0, \tag{6.3.13}$$

$$B\sqrt{\frac{\tilde{\Sigma}}{F - \tilde{a}_i^2\sin^2\theta}}(\omega_i + \phi_i' A_{t_i}) - A\sqrt{\frac{F}{\tilde{\Sigma}}}R_i'(r) + mB = 0, \tag{6.3.14}$$

$$
\begin{aligned}
&-iA\sqrt{\frac{F-\tilde{a}_i^2\sin^2\theta}{\tilde{\Sigma}F\sin^2\theta}} \\
&\quad\times\left\{2n(1-\cos\theta)-\tilde{a}_i\sin^2\theta\left(1-\frac{\tilde{\Sigma}}{F-\tilde{a}_i^2\sin^2\theta}\right)\right\}(\omega_i+\phi_i'A_{t_i}) \\
&-A\frac{1}{\sqrt{\tilde{\Sigma}}}\Theta_i'(\theta)-iA\sqrt{\frac{F-\tilde{a}_i^2\sin^2\theta}{\tilde{\Sigma}F\sin^2\theta}}(j_i'-\phi_i'A_{\varphi_i})=0, \qquad (6.3.15)
\end{aligned}
$$

where only the positive frequency contributions are considered without loss of generality. These four equations lead to

$$
\frac{F}{\tilde{\Sigma}}R_i'^2(r)+m^2-\frac{\tilde{\Sigma}}{F-\tilde{a}_i^2\sin^2\theta}\tilde{\omega}_i^2=0, \qquad (6.3.16)
$$

$$
\begin{aligned}
&\frac{1}{\tilde{\Sigma}}\Theta_i'^2(\theta)+\frac{F-\tilde{a}_i^2\sin^2\theta}{\tilde{\Sigma}F\sin^2\theta}\left[\tilde{j}'_i+\left\{2n(1-\cos\theta)\right.\right. \\
&\qquad\left.\left.-\tilde{a}_i\sin^2\theta\left(1-\frac{\tilde{\Sigma}}{F-\tilde{a}_i^2\sin^2\theta}\right)\right\}\tilde{\omega}_i\right]^2=0, \qquad (6.3.17)
\end{aligned}
$$

from which we derive

$$
\begin{aligned}
&F^2R_i'^2(r)+F\left(\tilde{\Sigma}m^2-\frac{\tilde{\Sigma}^2}{F-\tilde{a}_i^2\sin^2\theta}\tilde{\omega}_i^2+\Theta_i'^2(\theta)\right) \\
&+\frac{F-\tilde{a}_i^2\sin^2\theta}{\sin^2\theta}\left[\tilde{j}_i'+\left\{2n(1-\cos\theta)\right.\right. \\
&\left.\left.-\tilde{a}_i\sin^2\theta\left(1-\frac{\tilde{\Sigma}}{F-\tilde{a}_i^2\sin^2\theta}\right)\right\}\tilde{\omega}_i\right]^2=0.
\end{aligned}
$$

This is exactly the same as the equation (6.2.9) and hence, the Hawking radiation is retrieved again. Since the extra contributions with mass vanish at the horizon, the result of integrating around the pole for R_i in the massive case is the same as the massless case. We thus find the interesting result that the black hole radiates different spin weight of particles, massive or massless, at the same expression (6.2.17) for temperature. In the limit $q = p = 0$, the study provides result for the fermion tunneling radiation of neutral particles with mass (neutrinos) from the Demiański-Newman black hole.

We now obtain the explicit expression for the action $I_{i\uparrow}$ in the spin-up case by solving (6.3.12)–(6.3.15) near the horizon at

$$r'_+ = (M - \omega_i) + \{(M - \omega_i)^2 - \tilde{a}_i^2 + n^2 - (Q - q'_i)^2 - (P - p'_i)^2\}^{1/2}.$$

For outgoing particles, (6.3.12) gives on integration, with $iA = B$,

$$\begin{aligned} R_i(r) &= R_{+i}(r) \\ &= -\int \frac{mA\sqrt{\tilde{\Sigma}(r'_+)}}{B\sqrt{(r - r'_+)(r'_+ - M + \omega_i)}} dr - \frac{2\tilde{\Sigma}(r'_+)\sqrt{r - r'_+}}{\tilde{a}_i \sin\theta \sqrt{r'_+ - M + \omega_i}} \\ &\quad \times \left[\omega_i - \frac{q'_i(Q - q'_i)r'_+}{\tilde{\Sigma}(r'_+)} - \frac{p'_i(P - p'_i)(n - \tilde{a}_i \cos\theta)}{\tilde{\Sigma}(r'_+)}\right], \end{aligned} \tag{6.3.18}$$

while for the incoming particles, (6.3.14) yields, with $iB = A$,

$$\begin{aligned} R_i(r) &= R_{-i}(r) \\ &= \int \frac{mB\sqrt{\tilde{\Sigma}(r'_+)}}{A\sqrt{(r - r'_+)(r'_+ - (M + \omega_i)}} dr - \frac{2\tilde{\Sigma}(r'_+)\sqrt{r - r'_+}}{\tilde{a}_i \sin\theta \sqrt{r'_+ - M + \omega_i}} \\ &\quad \times \left[\omega_i - \frac{q'_i(Q - q'_i)r'_+}{\tilde{\Sigma}(r'_+)} - \frac{p'_i(P - p'_i)(n - \tilde{a}_i \cos\theta)}{\tilde{\Sigma}(r'_+)}\right]. \end{aligned} \tag{6.3.19}$$

Equations (6.3.13) and (6.3.15) imply that

$$\Theta_i(\theta) = \frac{-i[r'^2_+ + (n-\tilde{a}_i)^2]}{\tilde{a}_i\sqrt{2-\nu+\nu\cos 2\theta}}\Bigg(\omega_i - \frac{q'_i(Q-q'_i)r'_+}{r'^2_+ + (n-\tilde{a}_i)^2}$$
$$-\frac{(n-\tilde{a}_i)p'_i(P-p'_i)}{r'^2_+ + (n-\tilde{a}_i)^2} - \frac{j'_i\tilde{a}_i}{r'^2_+ + (n-\tilde{a}_i)^2}\Bigg)$$
$$\times\Bigg[\sqrt{2}\sqrt{\csc^2\theta - \nu}\Bigg\{-\tanh^{-1}\left(\frac{\sqrt{2}\cos\theta}{\sqrt{2-\nu+\nu\cos 2\theta}}\right)$$
$$+\sqrt{\nu}\ln[\sqrt{2\nu}\cos\theta + \sqrt{2-\nu+\nu\cos 2\theta}]\Bigg\}\sin\theta\Bigg], \qquad (6.3.20)$$

where

$$\nu = \frac{\tilde{a}_i^2}{2(r-r'_+)(r'_+ - M + \omega_i)}.$$

Equations (6.3.18) and (6.3.20) with the ansatz (6.2.8) then evaluate the action in the i-th segment for the outgoing massive Dirac particles and it reduces to the massless Dirac particles' action for $m = 0$. Likewise, one can determine the action in the i-th segment for the ingoing Dirac particle either massive or massless.

6.4 Concluding Remarks

The study of Hawking radiation as a process of quantum tunneling provides physical insight into the classically forbidden phenomenon. Our concern in this chapter is to investigate by the tunneling method the radiation spectrum of electrically charged magnetic (scalar/fermion) particles from dually charged Demiański-Newman black holes. The tunneling method involves calculating the imaginary part of the action for the (classically forbidden) process of s-wave emission across the horizon, which in turn is related to the Boltzmann factor for emission at the Hawking temperature. We use the Hamilton-Jacobi ansatz and take into account self-gravitation interaction and unfixed background spacetime. The result derived in (6.2.16) shows that the spectrum is not accurately thermal. For a comparison with the purely thermal spectrum, we expand ΔS_{BH} by Taylor series in ω, q, p, j and obtain

$$\Gamma \sim e^{-\beta(\omega-\omega_o)+\mathcal{O}(\omega,q,p,j)^2} = e^{-\beta'(\omega-\omega_o)}, \tag{6.4.1}$$

where

$$\beta' = \left[1 - \frac{\mathcal{O}(\omega,q,p,j)^2}{\beta(\omega-\omega_o)}\right] \tag{6.4.2}$$

can be treated as an inverse quantum-corrected temperature. Evidently, the leading-order term in (6.4.1) gives the Boltzman factor, and the higher-order terms of ω, q, p, j are a deviation from a purely thermal spectrum. The quantum-corrected inverse Hawking temperature β' in (6.4.2) depends not only on the black hole background but also on the radiation particle's energy, charges and angular momentum. We also observe that the black hole temperature $T(M,Q,P,J)$ increases after emission of a particle to

$$T'(M-\omega, Q-q, P-p, J-j)$$

given by

$$T'(M-\omega) = \frac{1}{2\pi}\left[\frac{r_+ - M}{r_+^2 + (n-a)^2} + f(\omega,q,p,j)\right]$$

$$= T(M) + T(\epsilon), \tag{6.4.3}$$

where

$$f(\omega,q,p,j) = \frac{2a\{Ma+n(r_+-M)\}}{[r_+^2+(n-a)^2]^2} \times\left[\left(\frac{\omega}{M}-\frac{j}{J}\right)+\frac{r_+(r_+\omega-Qq-Pp)}{a\{Ma+n(r_+-M)\}}\right] -\frac{M\omega-Qq-Pp+a^2(\frac{\omega}{M}-\frac{j}{J})}{(r_+-M)[r_+^2+(n-a)^2]}+\mathcal{O}(\omega,q,p,j)^2. \tag{6.4.4}$$

In particular,

$$T'(M-\omega)\approx\frac{1}{8\pi M}\left(1+\frac{\omega}{M}\right)$$

for the Schwarzschild case. This causes the black hole to emit further. Our study shows that the black hole emits tunneling radiation spectrum of massive and massless (scalar or fermion) particles at the same temperature (6.2.17) in the semi-classical limit in which the WKB approximation is applicable. However, when dealing with the Hawking radiation of fermions tunneling, there is a subtle technical issue in selecting an appropriate ansatz for the Dirac field consistent with the choice of matrices γ^μ, and failure to make such a choice results to a breakdown in the method. We also calculate the change of total entropy of the system including black hole and radiating particles. The result shows that the change in total entropy is $\Delta S > 0$ (indicating the process as irreversible) but very small and can be neglected. This has some difference from Parikh's work [19] in which $\Delta S = 0$. It also suggests that the probing of radiating particles of the black hole is connected with the change of the black hole entropy.

The result of this work using Hamilton-Jacobi method is in agreement with that obtained by the null geodesic method. However, the physical picture in Hamilton-Jacobi method is more clear. There are some differences between the two methods. Although the null geodesic method strongly relies on a very specific choice of (regular-across-horizon) coordinates, the Hamilton-Jacobi method can directly be applied to rotating black holes without converting the metric to the corotating frame. Moreover, the factual emission process is irreversible and the null geodesic method is only suitable for the reversible process. The Hamilton-Jacobi method, on the contrary, can be suitable for the irreversible process as well and there is very few information lost in the emitting process. Further, to conserve the symmetry of the spacetime in null geodesic method, the particle should be an ellipsoid shell during the tunneling process. It implies that a should be chosen as a constant. However, this assumption needs not be considered in the Hamilton-Jacobi method and a can be substituted with $\tilde{a} = \frac{J-j}{M-\omega}$.

In fact, being a semi-classical one the tunneling radiation is treated as point particles. The validity of such an approximation can only exist in the low energy regime. To properly address the black hole radiation, a better understanding of physics at the Planck scale is a necessary prerequisite, especially that of the last stages or the endpoint of Hawking evaporation. However, our study might be reliable semi-classically. The study gives results for (i) the Kerr-Newman black hole when $n = P = 0$ [230], (ii) the Kerr-NUT black hole when $Q = P = 0$, (iii) the Kerr black hole when $n = Q = P = 0$, (iv) the Taub-NUT black hole when $a = Q = P = 0$, (v) the Reissner-Nordström black hole when $n = a = 0$ and (vi) the Schwarzschild black hole when $n = a = Q = P = 0$. Setting $q = p = 0$ in our work one can obtain the tunneling radiation spectrum of neutrinos from the Demiański-Newman black hole. The result of this chapter is accordant with that of chapters 4 and 5 obtained respectively by using

Damour-Ruffini method and null-geodesic method.

Black holes are playing a major role in relativistic astrophysics by providing mechanisms to fuel the most powerful engines in the cosmos. Indeed, the black hole is an excellent system to combine the quantization of a matter field with a curved background spacetime. It demands considerable efforts to studying the quantum thermal properties of black holes. Along this line, the study of this chapter is interesting and well justified. There will be some interest in further research to perform tunneling calculations to higher order in WKB (in both the scalar field and fermionic cases) in order to calculate grey body effects and work in this area is in progress.

Chapter 7

Charged Dirac Particles' Hawking Radiation via Tunneling of Both Horizons and Thermodynamics Properties of Kerr-Newman-Kasuya-Taub-NUT-AdS Black Holes

This chapter present the work of Ref. [111] in which we investigate Hawking radiation of electrically and magnetically charged Dirac particles from a dyonic Kerr-Newman-Kasuya-Taub-NUT-Anti-de Sitter (KNKTN-AdS) black hole by considering thermal characters of both the outer and inner horizons. The work of this chapter is a generalization of the work of chapter 4 in which our study concerns analysis of charged scalar particles in the background of the Reissner-Nordström-Taub-NUT black hole. We apply Damour-Ruffini method and membrane method to calculate the temperature and the entropy of the inner horizon of the KNKTN-AdS black hole. The inner horizon admits thermal character with positive temperature and entropy proportional to its area. The inner horizon entropy contributes to the total entropy of the black hole in the context of Nernst theorem. Considering conservation of energy, charges, angular momentum, and the back-reaction of emitting particles to the spacetime, we obtain the emission spectra for both the inner and outer horizons. The total emission rate is obtained as the product of the emission rates of the inner and outer horizons. It deviates from the purely thermal spectrum with the leading term exactly the Boltzman factor and can bring some information out. The result thus can be treated as an explanation to the information loss paradox.

The work of this chapter is organized as follows. section 7.1 is an introduction to the work. In section 7.2, we obtain the Decoupled Dirac equations of a charged particle's dynamics using Newman-Penrose formalism for the KNKTN-AdS black hole and derive the radial outgoing wave equation. In section 7.3, we solve the radial wave equation and calculate the temperature and tunneling rate of the inner horizon for the KNKTN-AdS black hole. In section 7.4, we calculate the statistical entropy of the inner and outer horizons, redefine the entropy of the black hole and show that the redefined entropy satisfies the Nernst theorem. We obtain a new Bekenstein-Smarr (BS) formula considering contributions of both the inner and outer horizons. The result shows that the first law of black hole thermodynamics is also tenable at the inner horizon. In section 7.5, considering conservation of energy and charges and taking into account the particles' back-reaction, we investigate tunneling rates of the inner and outer horizons of the KNKTN-AdS black hole. The result demonstrates that the total tunneling rate is in agreement with the Parikh's standard result and there is no loss of information. Finally, we give our concluding remarks in section 7.6.

7.1 Introduction

Quantum phenomena in gravity theories, discussed in chapter 3, predict a picture of black holes that emit radiations and can evaporate [2, 3, 4]. The origin of the radiation can be understood by considering spontaneous creation of particle-antiparticle pairs near the event horizon of a black hole. Usually, such a pair annihilates itself very rapidly; but it is possible that one of them—particle or antiparticle—is swallowed by the hole before the annihilation so that the other one is free to escape away from the hole. If as a net effect more antiparticles than particles fall through the horizon towards the singularity, an observer outside the hole observes a particle flux which appears to come out from the black hole. The familiar concept is that the outer horizon of a black hole with two horizons radiates in a similar fashion like that of the Schwarzschild event horizon. The effect of the inner horizon in the radiation process is yet not clear. There have been various derivations of the Hawking radiation with different physical assumptions [235]. Very little is known about the radiation of the inner horizon of the black hole with two horizons. It is interesting to investigate the phenomena predicted by the virtual pair production mechanism at the inner horizon of the black hole.

As the pair creation mechanism indicates, the inner horizon does radiate in the inward direction, i.e., towards the singularity. However, this provides no information about the radiation itself. In particular, it remains unclear whether the inner horizon radiates particles or antiparticles. With the analytic continuation of the Klein-Gordon field, Wu and Cai [84, 85, 86] have carried out explicit calculation considering the inner horizon radiation. Their investigation found that the temperature of the inner horizon is negative. This appears to contradict not only the general attitude towards the black hole thermodynamics [189, 190, 191, 192] but the very foundations of thermodynamics itself as well. Therefore, the true nature of the radiation of the inner horizon is still fairly indistinct. Contrary to general believe that only the outer horizon emits Hawking radiation, if one is able to show that both of the horizons radiate, then the result would support the idea that all horizons of spacetime emit radiation. This idea then may provide effective indications in the search for quantum gravity.

Peltola and Mäkelä [87] have found, in contrast to Wu and Cai, that the effective temperature for particles radiating from the inner horizon (of a maximally extended Reissner-Nordström (RN) spacetime) towards the singularity is not negative but positive. Their analysis indicates that real particles with positive energy and temperature are emitted towards the singularity from the inner horizon. It is therefore necessary to maintain the local energy balance that antiparticles with negative energy are radiated in the direction away from the singularity through the inner horizon. These antiparticles, if the backscattering effects are neglected, go through the intermediate region between the horizons, and indeed come out of the white hole—at least when the black hole is nearly extreme. This effect for maximally extended RN spacetimes is called “white hole radiation”. Correspondingly as the black hole radiation is a consequence of the quantum effects at the outer horizon, the white hole radiation is a consequence of the quantum effects at the inner horizon of the black hole. These two type radiations are separate and simultaneously ongoing processes in spacetimes containing a black hole with two horizons like the RN black hole. An observer from the exterior region of the RN black hole discovers the both types of radiation. These results are qualitatively the same for the more realistic Kerr black hole solution, because the causal structures of the RN and the Kerr spacetimes are very similar. The main result of Peltola and Mäkelä [87] is the existence of the white hole radiation, and it appears that the same result holds for the KN black holes as well. Nevertheless, there is still no complete knowledge of understanding the possible role that plays the black hole's inner horizon in the black hole thermodynamics.

Hawking radiation of black hole as an exact thermal spectrum [4] raises two obvious disputes: the information loss paradox and the violation of the underlying quantum unitary theory [218, 219, 220, 221]. Several methods have been proposed to resolve these two problems, one of which is the semi-classical approach proposed by Parikh and Wilczek [16, 17, 19, 24, 25] (reviewed in subsection 3.5.1 of chapter 3 and used in chapter 5). In this method the emission rate is calculated by treating Hawking radiation as a tunneling process and using WKB approximation. The outgoing particles themselves create the barrier. Considering self-gravitation of particles, a corrected spectrum is obtained. This method was extended to more general circumstances [38, 39, 236, 237, 238] and all of them supported the conclusion that the black hole Hawking radiation spectrum is not exactly thermal. As a result, some information can be taken out of the black hole. This leads to a possible explanation for information loss paradox and the loss of quantum unitary theory. Another method was proposed by Angheben et al. to investigate Hawking radiation [33, 35, 44] in which the classical action I of emitting particles satisfy the relativistic Hamilton-Jacobi equation (reviewed in subsection 3.5.2 of chapter 3 and used in chapter 6). The same conclusion as the first method can be drawn from this method. In calculating the particles' emitting rate, Damour-Ruffini [7] analytically extended the outgoing wave from outside of horizon to inside (reviewed in section 3.4 of chapter 3). Using the Damour-Ruffini method Liu has proposed a new method [71] to investigate Hawking radiation of massive Klein-Gordon particles from a RN black hole. When conservation of energy and the particles' back-reaction are considered, the same terminations can be obtained as the previous works. Recently, extending Liu's work to charged Dirac particles' Hawking radiation from a KN black hole, Zhou and Liu [72] found that the emission spectrum is not accurately thermal.

There is still an open problem on black hole entropy [73, 74, 75]. Bekenstein suggested that the entropy of a black hole is proportional to its event horizon surface [1, 2], while the surface gravity of the event horizon describes the temperature of the black hole [151]. The Nernst theorem demands that the entropy of a system must vanish as its temperature goes to zero. If this assertion is applied to black holes, one finds that the entropy of the black hole with two horizons, like the Kerr black hole, does not vanish as its temperature approaches absolute zero [76, 77]. However, if the black hole with two horizons is considered as a thermodynamics system composed of two subsystems: the outer horizon and inner horizon, the Nernst theorem is found to be satisfied. This is because the entropy of the black hole then contains contributions of both the outer and inner horizons [78].

Recently, thermodynamics properties of the inner horizon of a KN black hole [81] and tunneling effect of two horizons from a RN black hole [82] have been investigated by Jun Ren. Our previous work in Ref. [83] is a study of charged particles' Hawking radiation via tunneling of both horizons from the Reissner-Nordström-Taub-NUT (RNTN) black holes. All these works are in agreement with Parikh's work and show no loss of information. In this chapter we calculate, following Zhou and Liu [72], the temperature of the inner horizon of the dyonic KNKTN-AdS black hole, which is a rotating RNTN black hole in AdS space and prove the existence of thermal characters of the inner horizon. Like as in the RNTN black hole case [83], the inner horizon of the KNKTN-AdS black hole emits positive energy particles inside the inner horizon (towards the singularity) with a positive temperature. In order to maintain a local energy balance, antiparticles with negative energy are emitted away from the singularity through the inner horizon. This is a process analogous to that takes place at the outer horizon according to the Hawking effect—at the outer horizon antiparticles go in and particles come out. The real particle remains inside the inner horizon and finally meets with the singularity. But the antiparticle enters the intermediate region between the horizons. Traveling across the intermediate region this antiparticle finally comes out from the white hole horizon, if the backscattering effects are neglected. The situation is, however, quite complicated because the vacuum states corresponding to a freely falling observer near the inner horizon of the black hole and the white hole horizon are entirely different. Since the white hole horizon emits thermal radiation [87], outside the KNKTN-AdS black hole two simultaneous radiation processes could be found—one is the normal black hole radiation and the other one is "white hole radiation," caused by the pair creation effects at the inner horizon. The white hole radiation may be thought of as absorption of energy, since it radiates only antiparticles with negative energy. Because the white hole horizon absorbs no energy classically, this feature contradicts with the classical result in a similar way as does the evaporation process at the outer horizon of black holes.

The KNKTN-AdS spacetime is stationary and the Killing vector field $(\partial/\partial t)^a$ is time-like in the regions both outside the outer horizon and inside the inner horizon. Hence, the surface gravity can be well-defined on the inner horizon. We calculate the inner horizon entropy proportional to its area by membrane model [112, 113], which is the modified form of the brick-wall model, proposed by 't Hooft [89]. So, the entropy of the KNKTN-AdS black hole might include the contributions of both the outer and inner horizons. The redefined entropy then satisfies the Nernst theorem.

The KNKTN-AdS black hole metric in Boyer-Lindquist coordinates has the form

$$ds^2 = -\frac{\Delta_r}{\Sigma}\left(dt - \frac{h}{\Xi}d\varphi\right)^2 + \frac{\Sigma}{\Delta_r}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\Sigma}\left(adt - \frac{\xi^2}{\Xi}d\varphi\right)^2, \tag{7.1.1}$$

where

$$\Sigma = \tilde{\Sigma}\tilde{\Sigma}^*, \quad \tilde{\Sigma}^* = r - i(n + a\cos\theta),$$

$$\Delta_r = \xi^2\left[1 + \frac{1}{\ell^2}(r^2 + 5n^2)\right] - 2(Mr + n^2) + z^2,$$

$$
\begin{aligned}
z^2 =& Q_\mathrm{e}^2 + Q_\mathrm{m}^2, \quad \Delta_\theta = 1 - \frac{a^2}{\ell^2}\cos^2\theta, \\
\xi^2 =& r^2 + a^2 + n^2, \quad \Xi = 1 - \frac{a^2}{\ell^2}, \\
h =& a\sin^2\theta - 2n\cos\theta.
\end{aligned} \tag{7.1.2}
$$

Beside the negative cosmological constant $\Lambda = -3/\ell^2$, the metric (7.1.1) possesses five parameters. These are the mass M, the specific angular momentum a $(= J/M)$, the NUT (magnetic mass) parameter n, the electric charge Q_e, and the magnetic monopole parameter Q_m. The electrical and magnetic potentials can be written as

$$
A_\mu = (A_t, 0, 0, A_\varphi), \quad B_\mu = (B_t, 0, 0, B_\varphi), \tag{7.1.3}
$$

respectively, where

$$
\begin{aligned}
A_t =& -\frac{Q_\mathrm{e} r}{\Sigma}, \quad A_\varphi = \frac{Q_\mathrm{e} r h}{\Xi\Sigma}, \\
B_t =& -\frac{Q_\mathrm{m} r}{\Sigma}, \quad B_\varphi = \frac{Q_\mathrm{m} r h}{\Xi\Sigma}.
\end{aligned} \tag{7.1.4}
$$

The dragged angular velocity of the black hole

$$
\Omega \equiv -\frac{\mathrm{g}_{t\varphi}}{\mathrm{g}_{\varphi\varphi}},
$$

instead of vanishing at spatial infinity, has the expression

$$
\Omega_\infty = -\frac{a}{\ell^2},
$$

implying that the KNKTN-AdS metric (7.1.1) is rotating at spatial infinity. At the event horizon, $r \to r_+$ $(\Delta_r = 0)$, the angular velocity Ω tends to its constant value

$$\Omega_+ = \frac{a\Xi}{\xi^2}. \tag{7.1.5}$$

Obviously, with respect to a frame that is static at infinity, one can also define the angular velocity of the black hole

$$\tilde{\Omega}_+ \equiv \Omega_+ - \Omega_\infty = \frac{a}{\xi^2}\left(1 + \frac{r_+^2}{\ell^2}\right). \tag{7.1.6}$$

This coincides with the angular velocity of the rotating Einstein universe at infinity [91] and provides the relevant basis for a CFT dual of the bulk KNKTN-AdS black hole. This angular velocity turns out to be the most important characteristic of the KNKTN-AdS black hole in the sense that it enters their consistent thermodynamics [92].

The horizon equation:

$$\Delta_r \equiv \frac{1}{\ell^2}(r - r_+)(r - r_-)(r - r_1)(r - r_1^*) = 0, \tag{7.1.7}$$

yields two real and a pair of complex conjugate roots. The largest of the real roots r_+ gives the radius of the black holes outer event horizon and the other real root r_- represents the radius of the inner Cauchy horizon. The real solutions of (7.1.7) can be written as follows:

$$r_\pm = \frac{1}{2}\left(\alpha \pm \sqrt{\alpha^2 - 2u + \frac{4M\ell^2}{\alpha}}\right), \tag{7.1.8}$$

where the real root u of the resolvent cubic equation and α are given by

$$\begin{aligned} u = {} & \frac{\ell^2 + a^2 + 6n^2}{3} + \frac{\ell^{4/3}(\mathcal{M}_+^2 - \mathcal{M}_-^2)^{2/3}}{(2N^2 - \mathcal{M}_+^2 - \mathcal{M}_-^2)^{1/3}} \\ & + \ell^{4/3}(2N^2 - \mathcal{M}_+^2 - \mathcal{M}_-^2)^{1/3}, \end{aligned}$$

$$N^2 = M^2 + \sqrt{(M^2 - \mathcal{M}_+^2)(M^2 - \mathcal{M}_-^2)},$$

$$\alpha = \sqrt{u - \ell^2 - a^2 - 6n^2}. \tag{7.1.9}$$

Here, $\mathcal{M}_\pm$ are the two critical mass parameters defined by

$$\mathcal{M}_\pm = \frac{\ell}{3\sqrt{6}} \sqrt{\zeta(3\eta - \zeta^2) \pm (\zeta^2 + \eta)^{3/2}}, \tag{7.1.10}$$

$$\zeta = \left(1 + \frac{6n^2 + a^2}{\ell^2}\right),$$

$$\eta = \frac{12}{\ell^2}\left\{(a^2 + z^2 - n^2) + \frac{5n^2}{\ell^2}(a^2 + n^2)\right\}. \tag{7.1.11}$$

Expanding the expressions in (7.1.8) in powers of $1/\ell$ with $M/\ell \ll 1$, we obtain

$$r_\pm = r_{0\pm} - \frac{r_{0\pm}^2}{2\ell^2}\frac{2Mr_{0\pm} - z^2 + n^2}{r_{0\pm} - M} + \mathcal{O}\left(\frac{1}{\ell^4}\right), \tag{7.1.12}$$

where

$$r_{0\pm} = M \pm \sqrt{M^2 - a^2 - z^2 + n^2}. \tag{7.1.13}$$

Evidently, $r_- < r_+ < r_{0+}$ and the two horizons $r_\pm$ represent the horizons of the KN-AdS black hole [213] for $n = 0$ and $Q_\mathrm{m} = 0$. As the cosmological constant vanishes (i.e., $\ell \to \infty$),

$$\mathcal{M}_+^2 \to a^2 + z^2 - n^2$$

and $\mathcal{M}_-^2 \to -\infty$. So, only the mass parameter $\mathcal{M}_+$ has a definite physical meaning. The metric (7.1.1) describes a black hole for $a^2 < \ell^2$, $M \geq \mathcal{M}_+$, but a naked singularity for $M < \mathcal{M}_+$. The case $M = \mathcal{M}_+$ yields an extreme black hole with the radius, $r_{\rm ebh} = r_+ = r_-$, given by

$$r_{\rm ebh} = \ell\sqrt{\frac{\zeta}{6}}\left(\sqrt{1+\frac{\eta}{\zeta^2}}-1\right)^{1/2}. \tag{7.1.14}$$

In recent years, several extensions of the standard model of particle physics predict existence of magnetic monopoles and it has grown interests in the possibility of dyonic black holes. Furthermore, recent developments in string/M-theory have greatly stimulated the study of black hole solutions in AdS space. Especially, asymptotically AdS black hole backgrounds are interesting for the familiar relevance of the AdS backgrounds in the AdS/CFT conjecture and in supergravity. There exists correspondence between a weakly coupled gravity system in an AdS background and a strongly coupled conformal field theory (CFT) living on its boundary [239, 240, 241]. The AdS black hole has peculiar thermodynamical properties, according to which the canonical ensemble is well-defined [242]. The presence of closed time-like curves can be avoided, if one takes into account the universal covering of such an AdS black hole background, which is not globally hyperbolic. The singularities of the metric of NUT charged spacetime, which are called Misner strings [243], can be avoided by periodic time coordinate. In the Euclidean section this induces a periodicity proportional to the NUT charge that needs to be matched with the usual periodicity requirement following the elimination of conical singularities in the (r, t) section. Hence, the NUT charge and the rotation parameter must be analytically continued. Some works have done in this regard [93, 95, 99, 100, 244, 245]. However, it is not clear that the vanishing of the metric function at the horizon in a spacetime involving rotation yields the same physics as its non-Wick rotated version.

7.2 Dirac Equations in KNKTN-AdS Spacetime

Dirac equations of a charged particle's dynamics in the curved spacetime are described in terms of the Newman-Penrose formalism as follows [246]:

$$
\begin{aligned}
&(D + \epsilon - \rho + iq\vec{A}\cdot\vec{l})F_1 \\
&\qquad +(\delta^* + \pi - \alpha + iq\vec{A}\cdot\vec{m^*})F_2 = \frac{i\mu_q}{\sqrt{2}}G_1, \\
&(\Delta + \mu - \gamma + iq\vec{A}\cdot\vec{n})F_2 \\
&\qquad +(\delta + \beta - \tau + iq\vec{A}\cdot\vec{m})F_1 = \frac{i\mu_q}{\sqrt{2}}G_2, \\
&(\Delta + \mu^* - \gamma^* + iq\vec{A}\cdot\vec{n})G_1 \\
&\qquad +(\delta^* + \beta^* - \tau^* + iq\vec{A}\cdot\vec{m^*})G_2 = \frac{i\mu_q}{\sqrt{2}}F_1, \\
&(D + \epsilon^* - \rho^* + iq\vec{A}\cdot\vec{l})G_2 \\
&\qquad -(\delta + \pi^* - \alpha^* + iq\vec{A}\cdot\vec{m})G_1 = \frac{i\mu_q}{\sqrt{2}}F_2.
\end{aligned}
\tag{7.2.1}
$$

where μ_q, q are rest mass and charge of the particle with

$$q^2 = q_{\rm e}^2 + q_{\rm m}^2,$$

respectively. The functions F_1, F_2, G_1, and G_2 are the four components of the wave functions, D, Δ, δ, δ^* are usual differential operators, α, β, γ, ϵ,

ρ, π, μ, τ are spin coefficients, and α^*, β^*, etc, are the complex conjugates of α, β, etc. The null-vectors of the Newman-Penrose formalism are

$$\begin{aligned}
l_\mu =& \frac{1}{\Delta_r}\left[\Delta_r, -\Sigma, 0, \frac{-h\Delta_r}{\Xi}\right], \\
n_\mu =& \frac{1}{2\Sigma}\left[\Delta_r, \Sigma, 0, \frac{-h\Delta_r}{\Xi}\right], \\
m_\mu =& \frac{1}{\tilde{\Sigma}\sqrt{2\Delta_\theta}}\left[i\Delta_\theta a\sin\theta, 0, -\Sigma, \frac{-i\Delta_\theta\xi^2\sin\theta}{\Xi}\right], \\
m^*_\mu =& \frac{1}{\tilde{\Sigma}^*\sqrt{2\Delta_\theta}}\left[-i\Delta_\theta a\sin\theta, 0, -\Sigma, \frac{i\Delta_\theta\xi^2\sin\theta}{\Xi}\right], \qquad (7.2.2)
\end{aligned}$$

and their contravariant forms are

$$\begin{aligned}
l^\mu &= \frac{1}{\Delta_r}[\xi^2, \Delta_r, 0, a\Xi], \\
m^\mu &= \frac{1}{\tilde{\Sigma}\sqrt{2\Delta_\theta}}\left[\frac{ih}{\sin\theta}, 0, \Delta_\theta, \frac{i\Xi}{\sin\theta}\right], \\
n^\mu &= \frac{1}{2\Sigma}[\xi^2, -\Delta_r, 0, a\Xi], \\
m^{*\mu} &= \frac{1}{\tilde{\Sigma}^*\sqrt{2\Delta_\theta}}\left[\frac{-i\,h}{\sin\theta}, 0, \Delta_\theta, \frac{-i\Xi}{\sin\theta}\right]. \qquad (7.2.3)
\end{aligned}$$

The electrical potential and magnetic potential has the tetrad components

$$\begin{aligned}
&\vec{A}\cdot\vec{l} = -\frac{Q_{\mathrm{e}}r}{\Delta_r}, \quad \vec{A}\cdot\vec{n} = -\frac{Q_{\mathrm{e}}r}{2\Sigma}, \\
&\vec{A}\cdot\vec{m} = \vec{A}\cdot\vec{m}^* = 0, \\
&\vec{B}\cdot\vec{l} = -\frac{Q_{\mathrm{m}}r}{\Delta_r}, \quad \vec{B}\cdot\vec{n} = -\frac{Q_{\mathrm{m}}r}{2\Sigma}, \\
&\vec{B}\cdot\vec{m} = \vec{B}\cdot\vec{m}^* = 0. \qquad (7.2.4)
\end{aligned}$$

The spin coefficients are as follows:

$$
\begin{aligned}
\pi &\equiv -n_{\mu;\nu}m^{*\mu}l^{\nu} = \frac{i\sqrt{\Delta_\theta}a\sin\theta}{\sqrt{2}\Sigma}, \\
\epsilon &\equiv \frac{1}{2}(l_{\mu;\nu}n^{\mu}l^{\nu} - m_{\mu;\nu}m^{*\mu}l^{\nu}) = 0, \\
\mu &\equiv -n_{\mu;\nu}m^{*\mu}m^{\nu} = \frac{-\Delta_r}{2\Sigma\tilde{\Sigma}^*}, \\
\tau &\equiv l_{\mu;\nu}m^{\mu}n^{\nu} = \frac{-i\sqrt{\Delta_\theta}a\sin\theta}{\sqrt{2}\Sigma}, \\
\alpha &\equiv \frac{1}{2}(l_{\mu;\nu}n^{\mu}m^{*\nu} - m_{\mu;\nu}m^{*\mu}m^{*\nu}) = \pi - \beta^*, \\
\rho &\equiv l_{\mu;\nu}m^{\mu}m^{*\mu} = \frac{-1}{\tilde{\Sigma}^*}, \\
\gamma &\equiv \frac{1}{2}(l_{\mu;\nu}n^{\mu}n^{\nu} - m_{\mu;\nu}m^{*\mu}n^{\nu}) = \frac{1}{4\Sigma}\frac{d\Delta_r}{dr} + \mu, \\
\beta &\equiv \frac{1}{2}(l_{\mu;\nu}n^{\mu}m^{\nu} - m_{\mu;\nu}m^{*\mu}m^{\nu}) \\
&= \frac{1}{2\sqrt{2}\tilde{\Sigma}\sin\theta}\frac{d}{d\theta}(\sqrt{\Delta_\theta}\sin\theta).
\end{aligned}
\tag{7.2.5}
$$

Considering the azimuthal and time dependence of the fields in the form

$$
\exp[-i\{\omega t - (m - q_{\mathrm{e}}Q_{\mathrm{e}} - q_{\mathrm{m}}Q_{\mathrm{m}})\varphi\}],
$$

the directional derivatives are found as follows:

$$
\begin{aligned}
D &\equiv l^{\mu}\partial_\mu = \mathscr{D}_0, \\
\Delta &\equiv n^{\mu}\partial_\mu = \frac{-\Delta_r}{2\Sigma}\mathscr{D}_0^{\dagger}, \\
\delta &\equiv m^{\mu}\partial_\mu = \frac{\sqrt{\Delta_\theta}}{\sqrt{2}\tilde{\Sigma}}\mathscr{L}_0^{\dagger}, \\
\delta^* &\equiv m^{*\mu}\partial_\mu = \frac{\sqrt{\Delta_\theta}}{\sqrt{2}\tilde{\Sigma}^*}\mathscr{L}_0,
\end{aligned}
\tag{7.2.6}
$$

where

$$
\begin{aligned}
\mathscr{D}_n &= \partial_r + \frac{i\Xi K}{\Delta_r} + \frac{n}{\Delta_r}\frac{d\Delta_r}{dr}, \\
\mathscr{L}_n &= \partial_\theta + \frac{\Xi H}{\Delta_\theta} + \frac{n}{\sqrt{\Delta_\theta}\sin\theta}\frac{d}{d\theta}(\sqrt{\Delta_\theta}\sin\theta), \\
\mathscr{D}_n^\dagger &= \partial_r - \frac{i\Xi K}{\Delta_r} + \frac{n}{\Delta_r}\frac{d\Delta_r}{dr}, \\
\mathscr{L}_n^\dagger &= \partial_\theta - \frac{\Xi H}{\Delta_\theta} + \frac{n}{\sqrt{\Delta_\theta}\sin\theta}\frac{d}{d\theta}(\sqrt{\Delta_\theta}\sin\theta), \\
K &= am - \frac{\xi^2}{\Xi}\omega - q_\mathrm{e}Q_\mathrm{e}r - q_\mathrm{m}Q_\mathrm{m}r, \\
H &= \frac{m}{\sin\theta} - \frac{h}{\Xi\sin\theta}\omega.
\end{aligned}
\tag{7.2.7}
$$

The four Dirac equations in (7.2.1), with (7.2.4)-(7.2.6), reduce to

$$
\begin{aligned}
&\left(\mathscr{D}_0 + \frac{1}{\tilde{\Sigma}^*}\right)F_1 + \frac{\sqrt{\Delta_\theta}}{\sqrt{2}\tilde{\Sigma}^*}\mathscr{L}_{\frac{1}{2}}F_2 = +\frac{i\mu_q}{\sqrt{2}}G_1, \\
&\frac{\Delta_r}{2\Sigma}\mathscr{D}_{\frac{1}{2}}^\dagger F_2 - \frac{\sqrt{\Delta_\theta}}{2\tilde{\Sigma}}\left(\mathscr{L}_{\frac{1}{2}}^\dagger + \frac{ih}{\tilde{\Sigma}^*\sin\theta}\right)F_1 = -\frac{i\mu_q}{\sqrt{2}}G_2, \\
&\left(\mathscr{D}_0 + \frac{1}{\tilde{\Sigma}}\right)G_2 - \frac{\sqrt{\Delta_\theta}}{\sqrt{2}\tilde{\Sigma}^*}\mathscr{L}_{\frac{1}{2}}^\dagger G_1 = +\frac{i\mu_q}{\sqrt{2}}F_2, \\
&\frac{\Delta_r}{2\Sigma}\mathscr{D}_{\frac{1}{2}}^\dagger G_1 + \frac{\sqrt{\Delta_\theta}}{2\tilde{\Sigma}^*}\left(\mathscr{L}_{\frac{1}{2}} - \frac{ih}{\tilde{\Sigma}\sin\theta}\right)G_2 = -\frac{i\mu_q}{\sqrt{2}}F_1.
\end{aligned}
\tag{7.2.8}
$$

These equations simplify, if one chooses

$$\begin{aligned}
F_1 &= \exp[-i\{\omega t - (m - q_{\rm e}Q_{\rm e} - q_{\rm m}Q_{\rm m})\varphi\}] \\
&\qquad \times\{r - i(n + a\cos\theta)\}^{-1} f_1(r,\theta), \\
F_2 &= \exp[-i\{\omega t - (m - q_{\rm e}Q_{\rm e} - q_{\rm m}Q_{\rm m})\varphi\}] f_2(r,\theta), \\
G_1 &= \exp[-i\{\omega t - (m - q_{\rm e}Q_{\rm e} - q_{\rm m}Q_{\rm m})\varphi\}] g_1(r,\theta), \\
G_2 &= \exp[-i\{\omega t - (m - q_{\rm e}Q_{\rm e} - q_{\rm m}Q_{\rm m})\varphi\}] \\
&\qquad \times\{r + i(n + a\cos\theta)\}^{-1} g_2(r,\theta),
\end{aligned} \tag{7.2.9}$$

to the form

$$\begin{aligned}
&\left(\partial_r + \frac{i\Xi K}{\Delta_r}\right) f_1 + \frac{1}{\sqrt{2}}\left(\partial_\theta + \frac{\Xi H}{\Delta_\theta} + \frac{\cot\theta}{2} + \frac{1}{4\Delta_\theta}\frac{d\Delta_\theta}{d\theta}\right) f_2 \\
&\qquad = \frac{1}{\sqrt{2}}[i\mu_q r + \mu_q(n + a\cos\theta)] g_1, \\
&\Delta_r\left(\partial_r - \frac{i\Xi K}{\Delta_r} + \frac{1}{2\Delta_r}\frac{d\Delta_r}{dr}\right) f_2 \\
&-\sqrt{2}\left(\partial_\theta - \frac{\Xi H}{\Delta_\theta} + \frac{\cot\theta}{2} + \frac{1}{4\Delta_\theta}\frac{d\Delta_\theta}{d\theta}\right) f_1 \\
&\qquad = -\sqrt{2}[i\mu_q r + \mu_q(n + a\cos\theta)] g_2, \\
&\left(\partial_r + \frac{i\Xi K}{\Delta_r}\right) g_2 - \frac{1}{\sqrt{2}}\left(\partial_\theta - \frac{\Xi H}{\Delta_\theta} + \frac{\cot\theta}{2} + \frac{1}{4\Delta_\theta}\frac{d\Delta_\theta}{d\theta}\right) g_1 \\
&\qquad = \frac{1}{\sqrt{2}}[i\mu_q r - \mu_q(n + a\cos\theta)] f_2, \\
&\Delta_r\left(\partial_r - \frac{i\Xi K}{\Delta_r} + \frac{1}{2\Delta_r}\frac{d\Delta_r}{dr}\right) g_1 \\
&+\sqrt{2}\left(\partial_\theta + \frac{\Xi H}{\Delta_\theta} + \frac{\cot\theta}{2} + \frac{1}{4\Delta_\theta}\frac{d\Delta_\theta}{d\theta}\right) g_2 \\
&\qquad = -\sqrt{2}[i\mu_q r - \mu_q(n + a\cos\theta)] f_1,
\end{aligned} \tag{7.2.10}$$

The decoupled Dirac equations can be obtained from (7.2.10) by assuming

$$(f_1, f_2, g_1, g_2) = \big(R_{-1/2}(r)S_{-1/2}(\theta), R_{1/2}(r)S_{1/2}(\theta),$$

$$R_{1/2}(r)S_{-1/2}(\theta), R_{-1/2}(r)S_{1/2}(\theta)\big)\,, \qquad (7.2.11)$$

where $R_{-1/2}(r)$, i.e., $R(r)$ and $R_{1/2}(r)$ represent outgoing and ingoing waves, respectively. The radial outgoing wave equation can be found as

$$\sqrt{\Delta_r}\frac{d}{dr}\left(\sqrt{\Delta_r}\frac{dR}{dr}\right) - \frac{i\mu_q\Delta_r}{\lambda + i\mu_q r}\frac{dR}{dr}$$

$$+\left[\frac{1}{2\Delta_r}\left(2\Xi^2K^2 - i\Xi K\frac{d\Delta_r}{dr}\right) - 2i\omega r\right.$$

$$\left. -iq_{\rm e}Q_{\rm e} - iq_{\rm m}Q_{\rm m} + \frac{\mu_q K}{\lambda + i\mu_q r} - \mu_q^2 r^2 - \lambda^2\right]R = 0. \quad (7.2.12)$$

In the limit $n = 0$, $Q_{\rm m} = 0$, $\ell \to \infty$, the radial equation (7.2.12) reduces to the KN black hole case, as obtained in [72, 247].

7.3 Temperature and Tunneling Rate of the Inner Horizon

We introduce the tortoise coordinate transformation

$$\frac{d}{d\hat{r}} = \frac{\Delta_r}{\xi^2}\frac{d}{dr}. \qquad (7.3.1)$$

Near the horizons,

$$\hat{r} = \frac{1}{2\kappa_\pm}\ln(|r - r_\pm|) + \text{const.}, \qquad (7.3.2)$$

where the surface gravity of the horizons are given by

$$\kappa_{\pm} = \pm\frac{(r_+ - r_-)(r_\pm - r_1)(r_\pm - r_1^*)}{2\ell^2(r_\pm^2 + a^2 + n^2)}. \tag{7.3.3}$$

The surface gravity of the inner horizon κ_- is negative, since its direct is towards the singularity, not the horizon. This is opposite to κ_+ which is directed towards the outer horizon. The outer horizon of the KNKTN-AdS black hole is a future horizon for the observer outside the hole $r > r_+$, but the inner horizon is a "past horizon" for the observer inside the hole $r < r_-$. Hence, the inner horizon is a horizon of a white hole for the observer in the region $r < r_-$. Since the physical process near the white hole is a time reversal of the physical process near the black hole, we can expect "Hawking absorption" for the white hole as one expects Hawking radiation for the black hole.

The radial equation (7.2.12) with (7.3.1) reduces to the form

$$\begin{aligned}
\rho^2\frac{d^2R}{d\hat{r}^2} &+ \left[2r\Delta_r - \frac{\rho}{2}\frac{d\Delta_r}{dr} - \rho\mu_q\Delta_r\frac{\mu_q r + i\lambda}{\lambda^2 + \mu_q^2 r^2}\right]\frac{dR}{d\hat{r}} \\
&+\Delta_r\left[\frac{\Xi^2K^2}{\Delta_r} - \lambda^2 - \mu_q^2 r^2 + \frac{\mu_q\lambda K - i\mu_q^2 K r}{\lambda^2 + \mu_q^2 r^2}\right. \\
&\left.-i\left(2\omega r + q_{\mathrm{e}}Q_{\mathrm{e}} + q_{\mathrm{m}}Q_{\mathrm{m}} + \frac{\Xi K}{2\Delta_r}\frac{d\Delta_r}{dr}\right)\right]R = 0.
\end{aligned} \tag{7.3.4}$$

Near the horizons, (7.3.4) takes the form

$$\frac{d^2R}{d\hat{r}^2} + \frac{\Xi^2K^2}{\rho^2}R = 0, \tag{7.3.5}$$

which is the standard form of wave equation on the horizons. Solving this equation one can derive Hawking radiation near the outer horizon. In this chapter, we are interested in probing the case near the inner horizon

$(r < r_-)$. The solution of (7.3.5) is the radial wave function given by

$$\Psi_r = \exp[-i\omega t \pm i\omega\tilde{r}], \quad \tilde{r} = \frac{\omega - \omega_0}{\omega}\hat{r}, \tag{7.3.6}$$

where

$$\begin{aligned}\omega_0 &= j\Omega_- + q_{\mathrm{e}}V_{-0\mathrm{e}} + q_{\mathrm{m}}V_{-0\mathrm{m}} \\ &= \frac{\Xi a j}{r_-^2 + a^2 + n^2} + \frac{\Xi q_{\mathrm{e}} Q_{\mathrm{e}} r_-}{r_-^2 + a^2 + n^2} + \frac{\Xi q_{\mathrm{m}} Q_{\mathrm{m}} r_-}{r_-^2 + a^2 + n^2}.\end{aligned} \tag{7.3.7}$$

Thus on the inner horizon surface, the ingoing and outgoing waves are respectively represented by

$$\Psi_r^{\mathrm{out}} = \exp(-i\omega u), \tag{7.3.8}$$

$$\Psi_r^{\mathrm{in}} = \exp[-i\omega u - 2i(\omega - \omega_0)\hat{r}], \tag{7.3.9}$$

where $u = t - \tilde{r}$ is the retarded Eddington-Finkelstein coordinate. Because $r \to r_-$ corresponds to $\tilde{r} \to -\infty$ and $\tilde{r} \to 0$ as $r \to 0$, (7.3.8) and (7.3.9) are respectively the outgoing wave emitted by the inner horizon and the ingoing wave to the inner horizon. Since $\hat{r} \sim \frac{1}{2\kappa_-}\ln(r_- - r)$ when $r \to r_-$, the ingoing wave can be written as

$$\Psi_r^{\mathrm{in}}(r < r_-) = e^{-i\omega u}(r_- - r)^{-i(\omega-\omega_0)/\kappa_-}. \tag{7.3.10}$$

Due to the singularity on the inner horizon, the ingoing wave (7.3.10) cannot be extended straightforwardly to the region $r > r_-$. Considering this singularity as the center of a circle with radius $|r - r_-|$ and by analytical continuation rotating $-\pi$ along the upper-half in the complex r-plane, into the "one-way membrane" region between the inner and outer horizons, we have $(r_- - r) \to |r_- - r|e^{-i\pi} = (r - r_-)e^{-i\pi}$. Then, Ψ_r^{in} in the region

$r_- < r < r_+$ can be written as

$$\Psi_r^{\text{in}}(r > r_-) = e^{-i\omega u}(r - r_-)^{-i(\omega-\omega_0)/\kappa_-}e^{-\pi(\omega-\omega_0)/\kappa_-}. \tag{7.3.11}$$

Thinking of Sannan's work [202], it is possible to calculate the emission rate at the inner horizon. The total ingoing wave function can be put in the form

$$\Psi = N_\omega[Y(r_- - r)\Psi_r^{\text{in}}(r < r_-) + Y(r - r_-)\Psi_r^{\text{in}}(r > r_-)], \tag{7.3.12}$$

where Y is the Heaviside step function and N_ω represents the normalization factor. The normalization condition $(\Psi, \Psi) = \pm 1$ indicates that the inner horizon absorbs thermal radiation from the region $r < r_-$ whose thermal spectrum and temperature are respectively given by

$$N_\omega^2 = \frac{\Gamma_-}{1 - \Gamma_-} = \left[\exp\left(\frac{\omega - \omega_0}{T_-}\right) \pm 1\right]^{-1}, \tag{7.3.13}$$

$$T_- = \frac{-\kappa_-}{2\pi}. \tag{7.3.14}$$

The temperature of the inner horizon is found positive, which agrees with the findings of Refs. [189, 190, 191, 192]. The thermal radiation from the region $r < r_-$ to the inner horizon with temperature T_- is absorbed by the inner horizon and the corresponding quantum effect is named "Hawking absorption" [203, 204]. The inner horizon remains in thermal equilibrium with the thermal radiation inside the inner horizon similar as the outer horizon of the black hole is in thermal equilibrium with the thermal radiation outside the black hole. Thus the inner horizon not only absorbs thermal radiation at temperature T_- but emits as well thermal radiation at the same time at temperature T_-. This leads to interpret the inner horizon as a thermal system with temperature T_-. The outer horizon and the inner horizon radiations are separate and simultaneously ongoing processes in the KNKTN-AdS spacetime. From the exterior region of the black hole, an observer might detect the both types of radiation. In contrast to common beliefs, the most remarkable result is that the inner horizon is not a passive spectator but an active participant in the radiation processes [205, 206] of the KNKTN-AdS black hole. Hawking radiation then can be explained as follows: The inner horizon absorbs the positive energy particles created near the singularity, which transiting the "one-way membrane" region ($r_- < r < r_+$) arrive at the outer horizon and being scattered by the outer horizon, travel to infinity as Hawking radiation.

The tunneling rate at the inner horizon, represented by Γ_- in (7.3.13), is given by

$$\Gamma_- = \left|\frac{\Psi_r^{\text{in}}(r > r_-)}{\Psi_r^{\text{in}}(r < r_-)}\right|^2 = \exp[-2\pi(\omega - \omega_0)/\kappa_-]. \tag{7.3.15}$$

The ensuing temperature (7.3.14) is computed as usual by dividing the surface gravity by 2π, in accordance with the statistical Hawking temperature [74]. We assume the area theorem to be applicable on the inner horizon in the similar manner as on the outer horizon and obtain

$$\begin{aligned}\mathcal{A}_\pm &= \pm \int \left|\mathrm{g}_{\theta\theta}\mathrm{g}_{\varphi\varphi}\right|^{\frac{1}{2}}_{r=r_\pm} d\theta d\varphi \\ &= \pm\frac{4\pi(r_\pm^2 + a^2 + n^2)}{\Xi}.\end{aligned} \tag{7.3.16}$$

The inner horizon area $\mathcal{A}_-$ is negative because it is like the horizon of a white hole.

7.4 Inner Horizon Entropy and Bekenstein–Smarr Formula

In order to have a deep insight into the nature of spacetime and quantum theory, the mysterious relation between the gravity and entropy needs to be revealed. The statistical origin of the black hole entropy can be analyzed by exploiting the brick-wall model, which was initially proposed by 't Hooft [89]. But this model has some drawbacks. In this section, following Ref. [248], we compute the inner horizon entropy of the KNKTN-AdS black hole by using the improved brick-wall model, called the membrane model [112, 113]. We also calculate the BS formula in terms of the inner horizon parameters and find that the first law of black hole thermodynamics is tenable at the inner horizon. We derive a new BS formula for the KNKTN-AdS black hole.

7.4.1 Statistical Entropy

Substituting $R = e^{iS(r)}$ into (7.2.12) and using WKB approximation, we obtain

$$k_r^{\pm} = \frac{1}{2}\left[\frac{\lambda\mu_q}{\lambda^2 + \mu_q^2 r^2} \pm \sqrt{\left(\frac{\lambda\mu_q}{\lambda^2 + \mu_q^2 r^2}\right)^2 + \frac{4}{\Delta_r}F}\right], \tag{7.4.1}$$

where

$$F = \left(\frac{\Xi^2 K^2}{\Delta_r} + \frac{\lambda\mu_q}{\lambda^2 + \mu_q^2 r^2}K - \mu_q^2 r^2 - \lambda^2\right),$$

and the sign ambiguity of the square root is related to the "out-going" (k_r^+) or "in-going" (k_r^-) particles, respectively. Averaging the radial momentum,

$$k_r = \frac{k_r^+ - k_r^-}{2} = \sqrt{\frac{1}{4}\left(\frac{\lambda\mu_q}{\lambda^2 + \mu_q^2 r^2}\right)^2 + \frac{1}{\Delta_r}F}, \tag{7.4.2}$$

where the minus before the k_r^- is caused by a different direction.

According to (7.3.13), the number of particles in the l-th energy level is given by

$$\frac{\omega_l}{\exp[\beta_-(\omega-\omega_0)]\pm 1},$$

where ω_l is the degeneracy of the l-th energy level, $\beta_- = T_-^{-1}$, and

$$\omega_0 = m\Omega_- + q_{\rm e}V_{-0\rm e} + q_{\rm m}V_{-0\rm m},$$

as in (7.3.7). As Mann [98] illustrated, the Euclidean time variable should be periodic; hence, the Lorentzian time variable is periodic with period $8\pi n$. Thus, periodicity of the solutions to the wave equation (7.3.4) is desirable and we can set the system energy $E = \omega - \omega_0$. The wave numbers referring to the inner horizon can be written as

$$\begin{aligned} k_r = \frac{1}{\Delta_r}&\left[\xi^4\{E + m(\Omega_- - \Omega) + q_{\rm e}(V_{-0\rm e} - V_{0\rm e})\right. \\ &\left.+q_{\rm m}(V_{-0\rm m} - V_{0\rm m})\}^2 - (\mu_q^2 r^2 + \lambda^2)\Delta_r\right]^{\frac{1}{2}}, \qquad (7.4.3)\end{aligned}$$

where we have neglected the contributions of the terms with the factor $\lambda\mu_q(\lambda^2 + \mu_q^2 r^2)^{-1}$ for simplicity. The constraint of semi-classical quantum condition applied on k_r is

$$n\pi = \int_{r_- - \varepsilon}^{r_- - 2\varepsilon} k_r dr,$$

where n is a non-negative integer.

The free energy f in the theory of canonical ensemble is given by

$$\beta_- f = -\sum_E \ln(1 \pm e^{-\beta_- E}),$$

where "+" corresponds to a fermion field and "−" corresponds to a boson field.

According to semi-classical quantum theory, we get

$$\sum \to \int_0^\infty dE \mathrm{g}(E),$$

where $\mathrm{g}(E)$ is the density of states, i.e.

$$\mathrm{g}(E) = \omega' \frac{d\Gamma(E)}{dE},$$

ω' is the degeneracy of the fields (for scalar field and neutrino field, $\omega' = 1$; for Maxwell field, $\omega' = 2$). The state number is

$$\Gamma(E) = \sum_{m,\lambda} n_r(E, \lambda, m) = \int dm \int d\lambda \frac{1}{\pi} \int k_r(E, \lambda) dr.$$

Applying the quantum statistical mechanics, the expression for the free energy can be expressed by

$$\begin{aligned} f = & \frac{-1}{\pi} \int_0^\infty dE \int_{r_- - \varepsilon}^{r_- - 2\varepsilon} dr \int_0^{l_{\max}} dl \int_{-l}^{l} dm \frac{1}{\Delta_r} (e^{-\beta_- E} + 1)^{-1} \\ & \times \big[\xi^4 \{E + m(\Omega_- - \Omega) + q_{\mathrm{e}}(V_{-0\mathrm{e}} - V_{0\mathrm{e}}) \\ & + q_{\mathrm{m}}(V_{-0\mathrm{m}} - V_{0\mathrm{m}})\}^2 - \mu_q^2 r^2 \Delta_r - \Delta_r l(l+1)\big]^{\frac{1}{2}} \\ = & \frac{7}{180} \cdot \frac{\pi^3}{\beta_-^4} \cdot \frac{\ell^4 (r_-^2 + a^2 + n^2)^3}{(r_- - r_+)^2 (r_- - r_1)^2 (r_- - r_1^*)^2} \cdot \frac{\varepsilon}{\eta^2}, \end{aligned} \tag{7.4.4}$$

where the separation constant $\lambda = l(l+1)$, with l the angular quantum number. The extreme of integration with respect to l is performed so that $k_r > 0$ (the l reaches its maximum for $k_r = 0$). The ε is the ultraviolet regulator and satisfies $0 < \varepsilon \ll r_-$. This shows that the integral over the quantum number m does not diverge. Moreover, according to the membrane model, the black-hole entropy mainly comes from the vicinity of the horizon. So, we have considered $\lim_{r\to r_-}\Omega = \Omega_-$ in the m and r integrations. In evaluating the r-integration the median theorem have also been used ; so, $\varepsilon < \eta < 2\varepsilon$. With the standard formula, $S = \beta^2\frac{\partial F}{\partial \beta}$, we find the one componential entropy of the inner horizon as

$$S_{1-} = -\frac{7}{45}\cdot\frac{\pi^3}{\beta_-^3}\cdot\frac{\ell^4(r_-^2+a^2+n^2)^3}{(r_- - r_+)^2(r_- - r_1)^2(r_- - r_1^*)^2}\cdot\frac{\varepsilon}{\eta^2}. \tag{7.4.5}$$

We choose an appropriate cut-off distance ε to satisfy

$$\frac{1}{\varepsilon} = \frac{90\beta_-}{\Xi}.$$

Because ε and η in (7.4.4) are of the same order, we find that

$$\frac{\varepsilon}{\eta^2} \sim \frac{1}{\varepsilon} = \frac{90\beta_-}{\Xi}.$$

Since the wavefunction is of four components, the entropy of the inner horizon is found to be

$$S_- = 4S_{1-} = \frac{7}{8}\mathcal{A}_-, \tag{7.4.6}$$

where $\mathcal{A}_-$ is the area of the black-hole inner horizon, defined in (7.3.16). Thus, the inner horizon entropy is also proportional to the area of the inner horizon and cut-off factor $90\beta_-/\Xi$ is analogous to that in the calculation of the entropy of the outer horizon. Carrying out similar calculation, the entropy of the outer horizon is obtained as $S_+ = \frac{7}{8}\mathcal{A}_+$ for the KNKTN-AdS black hole, where $\mathcal{A}_+$ is the area of the outer horizon. The result agrees with that obtained in [248] for the NUT-KN black hole.

In the proceeding subsection we find that together with the inner horizon temperature T_-, the entropy S_- satisfies the familiar formula $T^{-1} = \frac{dS}{dm}$, that is,

$$T_- = \left(\frac{dS_-}{dM - \Omega_- dJ - V_- dQ - V'_- dn - \Theta_- d\Lambda} \right)^{-1} = -\frac{\kappa_-}{2\pi}, \qquad (7.4.7)$$

which suggests that the temperature or the entropy of the inner horizon is negative and another is positive. In several previous papers, the entropy was positive and the temperature was negative. However, there is no clear explanation in favor of negative temperature of the inner horizon. In fact, our understanding of the essence of the black hole entropy is still somewhat incomplete. Nevertheless, the negative entropy of the inner horizon can make possible the entropy of the black hole with two horizons to satisfy the Nernst theorem. In particular, if the entropy of the black hole is redefined as the sum of the contributions of the outer and inner horizons:

$$\begin{aligned} S_{\rm BH} &= S_+ + S_- \\ &= \frac{7}{8}(\mathcal{A}_+ + \mathcal{A}_-) = \frac{7}{2}\frac{\pi(r_+^2 - r_-^2)}{\Xi}, \end{aligned} \qquad (7.4.8)$$

it is obvious that the black hole entropy approaches zero as its temperature

$$T_+ = \frac{(r_+ - r_-)(r_+ - r_1)(r_+ - r_1^*)}{4\pi\ell^2(r_+^2 + a^2 + n^2)}$$

goes to absolute zero, $r_+ = r_-$. Thus the entropy defined now obeys the third law of thermodynamics and hence is a Planck absolute entropy.

7.4.2 New Form of Bekenstein-Smarr Formula

The inner horizon can have thermal character and the thermodynamic system of the black hole then is composed of two subsystems: the outer horizon and the inner horizon [78]. This is indeed suggested by (7.4.8), which demonstrates that the extensive quantity entropy is determined by the area of both the inner and outer horizons. We define the parameters M, J, z, n related to the mass, angular momentum, electric and magnetic charges, and NUT charge by the Komar integrals [92]

$$M' = \frac{M}{\Xi^2}, \quad J = \frac{aM}{\Xi^2}, \quad Q = \frac{z}{\Xi}, \quad n' = \frac{n}{\Xi}. \tag{7.4.9}$$

The cosmological constant Λ can be regarded as a thermodynamical variable parameter ([249] and references therein). Then, from $\Delta_r(r_-) = 0$ we obtain by simple algebraic manipulations a generalized Smarr formula in terms of classical entropy $S_- = \mathcal{A}_-/4$ for the KNKTN-AdS black hole's inner horizon (with omitting dashes in M' and n') as follows:

$$\begin{aligned} M^2 = & -\frac{S_-}{4\pi} - \frac{\pi}{4S_-}\{4J^2 + (Q^2 - 2n^2)^2\} + \frac{1}{2}(Q^2 - 2n^2) \\ & + \frac{J^2}{\ell^2} - \frac{S_-}{2\pi\ell^2}\Bigg\{Q^2 + 2n^2 - \frac{S_-}{\pi} - \frac{4\pi}{S_-}(Q^2 - 2n^2)n^2 \\ & + \frac{1}{\ell^2}\left(\frac{S_-^2}{2\pi^2} - \frac{4S_-}{\pi}n^2 + 8n^4\right)\Bigg\}. \end{aligned} \tag{7.4.10}$$

Considering the black hole thermodynamical fundamental relation for the inner horizon: $M = M(S_-, J, Q, n, \Lambda)$, we obtain the BS differential formula

$$dM = T_- dS_- + \Omega_- dJ + V_- dQ + V'_- dn + \Theta_- d\Lambda, \tag{7.4.11}$$

where

$$T_- = \left.\frac{\partial M}{\partial S_-}\right|_{JQn\Lambda}$$
$$= \frac{-1}{8\pi M}\left[1 + \frac{\pi^2}{S_-^2}\{4J^2 + (Q^2 - 2n^2)^2\}\right.$$
$$\left. + \frac{2}{\ell^2}\left\{Q^2 + 2n^2 - \frac{2S_-}{\pi} + \frac{1}{\ell^2}\left(\frac{3S_-^2}{2\pi^2} - \frac{8S_-}{\pi}n^2 + 8n^4\right)\right\}\right], \tag{7.4.12}$$

$$\Omega_- = \left.\frac{\partial M}{\partial J}\right|_{S_-Qn\Lambda}$$
$$= \frac{-\pi J}{MS_-}\left(1 - \frac{S_-}{\pi\ell^2}\right), \tag{7.4.13}$$
$$V_- = \left.\frac{\partial M}{\partial Q}\right|_{S_-Jn\Lambda}$$
$$= \frac{-\pi Q}{2MS_-}\left[Q^2 - 2n^2 - \frac{S_-}{\pi} + \frac{S_-}{\pi\ell^2}\left(\frac{S_-}{\pi} - 4n^2\right)\right], \tag{7.4.14}$$

$$V_-' = \left.\frac{\partial M}{\partial n}\right|_{S_-JQ\Lambda}$$
$$= \frac{-\pi n}{MS_-}\left[2n^2 - Q^2 + \frac{S_-}{\pi} + \frac{S_-}{\pi\ell^2}\left\{\frac{S_-}{\pi} - 2Q^2\right.\right.$$
$$\left.\left. + 8Q^2n^2 + \frac{2S_-}{\pi\ell^2}\left(4n^2 - \frac{S_-}{\pi}\right)\right\}\right], \tag{7.4.15}$$

$$\Theta_- = \left.\frac{\partial M}{\partial \Lambda}\right|_{S_-JQn}$$
$$= \frac{-1}{6M}\left[J^2 - \frac{S_-}{2\pi}\left\{Q^2 + 2n^2 - \frac{S_-}{\pi} - \frac{4\pi}{S_-}(Q^2 - 2n^2)n^2\right\}\right.$$
$$\left. + \frac{S_-\Lambda}{6\pi}\left(\frac{S_-^2}{\pi^2} - \frac{8S_-}{\pi}n^2 + 16n^4\right)\right], \tag{7.4.16}$$

This shows that the first law of black hole thermodynamics is also plausible at the inner horizon. Regarding M as a function of S_-, J, Q^2, n and Λ, one finds that it is a homogeneous function of degree 1/2. Euler's theorem then gives

$$\frac{1}{2}M = T_-S_- + \Omega_-J + \frac{1}{2}V_-Q + V'_-n - \Theta_-\Lambda, \tag{7.4.17}$$

called the BS integral formula for the inner horizon. One can verify that the relations (7.4.12), (7.4.13) and (7.4.14) for temperature, angular velocity and electric potential respectively, coincide with equations (7.3.14), (7.1.6) and (7.1.4) for the inner horizon case.

The above calculation with the outer horizon parameters yields for the outer horizon the following equations, analogous to (7.4.10), (7.4.11) and (7.4.17):

$$\begin{aligned} M^2 = & \frac{S_+}{4\pi} + \frac{\pi}{4S_+}\{4J^2 + (Q^2 - 2n^2)^2\} + \frac{1}{2}(Q^2 - 2n^2) \\ & + \frac{J^2}{\ell^2} + \frac{S_+}{2\pi\ell^2}\left\{Q^2 + 2n^2 + \frac{S_+}{\pi} + \frac{4\pi}{S_+}(Q^2 - 2n^2)n^2 \right. \\ & \left. + \frac{1}{\ell^2}\left(\frac{S_+^2}{2\pi^2} + \frac{4S_+}{\pi}n^2 + 8n^4\right)\right\}, \end{aligned} \tag{7.4.18}$$

$$dM = T_+dS_+ + \Omega_+dJ + V_+dQ + V'_+dn + \Theta_+d\Lambda, \tag{7.4.19}$$

$$\frac{1}{2}M = T_+S_+ + \Omega_+J + \frac{1}{2}V_+Q + V'_+n - \Theta_+\Lambda. \tag{7.4.20}$$

These expressions reduce to the corresponding KN-AdS expressions in the limit $n \to 0$ [92].

Thus the new BS formulae that receive contributions from both the inner and outer horizons are given as follows:

$$\begin{aligned} M &= T_+S_+ + \Omega_+J + \frac{1}{2}V_+Q + V'_+n - \Theta_+\Lambda \\ &+T_-S_- + \Omega_-J + \frac{1}{2}V_-Q + V'_-n - \Theta_-\Lambda, \end{aligned} \tag{7.4.21}$$

$$\begin{aligned} dM &= \frac{1}{2}T_+dS_+ + \frac{1}{2}\Omega_+dJ + \frac{1}{2}V_+dQ + \frac{1}{2}V'_+dn + \frac{1}{2}\Theta_+d\Lambda \\ &+\frac{1}{2}T_-dS_- + \frac{1}{2}\Omega_-dJ + \frac{1}{2}V_-dQ + \frac{1}{2}V'_-dn + \frac{1}{2}\Theta_-d\Lambda. \end{aligned} \tag{7.4.22}$$

In the limit $\ell \to 0$, (7.4.21) and (7.4.22) reduce to the NUT-KN black hole case [250].

7.5 Back-reaction of Radiation

We consider that the emitting particles have back-reaction on the space-time and assume the energy conservation, charge conservation, magnetic conservation, and angular momentum conservation, when a particle with energy ω_i, charge q_{ie}, magnet q_{im}, and angular momentum j_i tunnels out of the inner horizon and then out of the black hole. Then, after emission of the particle, the mass, charge, magnet parameters of the black hole will be replaced by $M - \omega_i$, $Q_{\mathrm{e}} - q_{\mathrm{ie}}$, $Q_{\mathrm{m}} - q_{\mathrm{im}}$, and a will be replaced by $a' = \frac{Ma - j_i}{M - \omega_i}$. So, we obtain the emission rate

$$\Gamma_{-i} = \exp[-2\pi(\omega_i - \omega_{0i})/\kappa_{-i}], \tag{7.5.1}$$

where, with replacing Ξa, Ξq_{e}, Ξq_{m} by a, q_{e}, q_{m},

$$\omega_{0i} = j_i\Omega_{-i} + q_{\mathrm{e}i}V_{-0\mathrm{e}i} + q_{\mathrm{m}i}V_{-0\mathrm{m}i}$$

$$= \frac{a' j_i}{r_{-i}^2 + a'^2 + n^2} + \frac{q_{\mathrm{e}i}(Q_{\mathrm{e}} - q_{\mathrm{e}i})r_{-i}}{r_{-i}^2 + a'^2 + n^2} + \frac{q_{\mathrm{m}i}(Q_{\mathrm{m}} - q_{\mathrm{m}i})r_{-i}}{r_{-i}^2 + a'^2 + n^2},$$

$$\begin{aligned} r_{\pm i} = {} & (M - \omega_i) \pm \sqrt{(M-\omega_i)^2 - a'^2 - z'^2 + n^2} \\ & \mp \frac{((M-\omega_i) \pm \sqrt{(M-\omega_i)^2 - a'^2 - z'^2 + n^2})^2}{2\ell^2\sqrt{(M-\omega_i)^2 - a'^2 - z'^2 + n^2}} \\ & \times[((M-\omega_i) \pm \sqrt{(M-\omega_i)^2 - a'^2 - z'^2 + n^2})^2 + a'^2], \end{aligned}$$

$$z' = (Q_{\mathrm{e}} - q_{\mathrm{e}i})^2 + (Q_{\mathrm{m}} - q_{\mathrm{m}i})^2,$$

$$\kappa_{-i} = -\frac{(r_{+i} - r_{-i})(r_{-i} - r_1)(r_{-i} - r_1^*)}{2\ell^2(r_{-i}^2 + a'^2 + n^2)}. \tag{7.5.2}$$

If we consider emission of many particles and think that they radiate one by one, the result gives

$$\Gamma_- = \prod_i \Gamma_i = \exp\left[\sum_i(-2\pi(\omega_i - \omega_{0i})/\kappa_{-i})\right]. \tag{7.5.3}$$

Considering the emission as a continuous procession, the sum in (7.5.3) could be replaced by integration

$$\Gamma_- = \exp[-2\pi\int(d\omega' - \Omega' dj' - V'_{\mathrm{e}0}dq'_{\mathrm{e}} - V'_{\mathrm{m}0}dq'_{\mathrm{m}})/\kappa'_-)]$$

$$= \exp[-2\pi\chi_-], \tag{7.5.4}$$

where

$$\chi_- = 2\ell^2 \int_{(0,0,0,0)}^{(\omega,j,q_\mathrm{e},q_\mathrm{m})} \Bigg[-\frac{r_-'^2 + (\frac{Ma-j'}{M-\omega'})^2 + n^2}{(r_+' - r_-')(r_-' - r_1')(r_-' - r_1^{*\prime})} d\omega' + \frac{\frac{Ma-j'}{M-\omega'}}{(r_+' - r_-')(r_-' - r_1')(r_-' - r_1^{*\prime})} dj' + \frac{(Q_\mathrm{e} - q_\mathrm{e}')r_-'}{(r_+' - r_-')(r_-' - r_1')(r_-' - r_1^{*\prime})} dq_\mathrm{e}' + \frac{(Q_\mathrm{m} - q_\mathrm{m}')r_-'}{(r_+' - r_-')(r_-' - r_1')(r_-' - r_1^{*\prime})} dq_\mathrm{m}' \Bigg]. \tag{7.5.5}$$

In the limit $\ell^2 \to \infty$, (7.5.5) reduces to the case of KNKTN black hole and is given by

$$\chi_- = 2 \int_{(0,0,0,0)}^{(\omega,j,q_\mathrm{e},q_\mathrm{m})} \Bigg[-\frac{r_-'^2 + (\frac{Ma-j'}{M-\omega'})^2 + n^2}{r_+' - r_-'} d\omega' + \frac{\frac{Ma-j'}{M-\omega'}}{r_+' - r_-'} dj' + \frac{(Q_\mathrm{e} - q_\mathrm{e}')r_-'}{r_+' - r_-'} dq_\mathrm{e}' + \frac{(Q_\mathrm{m} - q_\mathrm{m}')r_-'}{r_+' - r_-'} dq_\mathrm{m}' \Bigg]. \tag{7.5.6}$$

To calculate it, we make use of the inner horizon (classical) entropy $S_- = \mathcal{A}_-/4$ and obtain

$$\begin{aligned} \Delta S_- &= \pi[(r_-'^2 + a'^2) - (r_-^2 + a^2)] \\ &= \pi\Big[2(M-\omega)^2 - (Q_\mathrm{e} - q_\mathrm{e})^2 - (Q_\mathrm{m} - q_\mathrm{m})^2 - 2(M-\omega) \\ &\quad \times \sqrt{(M-\omega)^2 - (Q_\mathrm{e} - q_\mathrm{e})^2 - (Q_\mathrm{m} - q_\mathrm{m})^2 - a'^2 + n^2} \\ &\quad -2M^2 + Q_\mathrm{e}^2 + Q_\mathrm{m}^2 + 2M\sqrt{M^2 - Q_\mathrm{e}^2 - Q_\mathrm{m}^2 - a^2 + n^2} \Big], \end{aligned} \tag{7.5.7}$$

in which $\Delta S_- = S_-(M-\omega, Q_\mathrm{e} - q_\mathrm{e}, Q_\mathrm{m} - q_\mathrm{m}, a') - S_-(M, Q_\mathrm{e}, Q_\mathrm{m}, a)$ is the difference between the entropies of the inner horizon before and after the emission. Then we obtain

$$\begin{aligned}
\frac{\partial(\Delta S_-)}{\partial\omega} &= \frac{4\pi(r_-'^2 + a'^2)}{r_+' - r_-'},\\
\frac{\partial(\Delta S_-)}{\partial j} &= -\frac{4\pi a'}{r_+' - r_-'},\\
\frac{\partial(\Delta S_-)}{\partial q_{\rm e}} &= -\frac{4\pi(Q_{\rm e} - q_{\rm e})r_-'}{r_+' - r_-'},\\
\frac{\partial(\Delta S_-)}{\partial q_{\rm m}} &= -\frac{4\pi(Q_{\rm m} - q_{\rm m})r_-'}{r_+' - r_-'}.
\end{aligned} \tag{7.5.8}$$

Using (7.5.8) in (7.5.6), we get

$$\begin{aligned}
\chi_- \approx & -\frac{1}{2\pi}\int_{(0,0,0,0)}^{(\omega,j,q_{\rm e},q_{\rm m})}\left[\frac{\partial(\Delta S_-)}{\partial\omega'}d\omega'\right.\\
& \left. + \frac{\partial(\Delta S_-)}{\partial j'}dj' + \frac{\partial(\Delta S_-)}{\partial q_{\rm e}'}dq_{\rm e}' + \frac{\partial(\Delta S_-)}{\partial q_{\rm m}'}dq_{\rm m}'\right]\\
= & -\frac{1}{2\pi}\int d(\Delta S_-) = -\frac{1}{2\pi}\Delta S_-.
\end{aligned} \tag{7.5.9}$$

We now consider that ℓ^2 is a finite constant. In this case computing the integration (7.5.5) with above procedure is cumbersome. However, to make the calculation more simple, we do not need to perform the integration directly. Instead we work on it in the following way: making use of the inner horizon temperature of the black hole

$$\frac{1}{T_-'} = \left(-\frac{\kappa_-'}{2\pi}\right)^{-1} = \frac{4\pi\ell^2(r_-'^2 + a'^2 + n^2)}{(r_+' - r_-')(r_-' - r_1')(r_-' - r_1^{*\prime})}, \tag{7.5.10}$$

we get

$$\frac{1}{T_-'}(d\omega' - \Omega' dj' - V_{0\rm e}' dq_{\rm e}' - V_{0\rm m}' dq_{\rm m}') = dS_-', \tag{7.5.11}$$

which brings (7.5.5) in the form (7.5.9). This means, (7.5.9) is a natural result of the first law of black hole thermodynamics. Therefore, we obtain the inner horizon emission rate

$$\Gamma_- = e^{\Delta S_-}. \tag{7.5.12}$$

Applying analogous procedure, the emission rate of the outer horizon is found as

$$\Gamma_+ = e^{\Delta S_+}. \tag{7.5.13}$$

Thus, the total emitting rate is given by

$$\Gamma = \Gamma_+ \cdot \Gamma_- = e^{\Delta S_{\rm BH}}, \tag{7.5.14}$$

where

$$\Delta S_{\rm BH} = S_{\rm BH}(M-\omega, Q_{\rm e}-q_{\rm e}, Q_{\rm m}-q_{\rm m}, a') - S_{\rm BH}(M, Q_{\rm e}, Q_{\rm m}, a)$$

is the change of Bekenstein-Hawking entropy. The result supports the Parikh's work. Manifestly, the derived radiation spectrum deviates from purely thermal one, and is connected with the change of Bekenstein-Hawking entropy. Expanding the emission rate Γ in ω, $q_{\rm e}$, $q_{\rm m}$, and j, one can find

$$\Gamma = e^{-\beta(\omega-\omega_0)+\mathcal{O}(\omega,q_{\rm e},q_{\rm m},j)^2} = e^{-\beta'(\omega-\omega_0)}, \tag{7.5.15}$$

where the leading-order term is the Boltzman factor and β' is the inverse quantum-corrected temperature. The higher-order terms in ω, $q_{\rm e}$, $q_{\rm m}$, and j generate a deviation from a purely thermal spectrum. Moreover, in quantum mechanics the number of microstates of the initial and final states are the exponent of the initial and final entropies. So, the emitting rate is $\Gamma = (e^{S_f}/e^{S_i}) = e^{\Delta S}$ and it is consistent with our result. Therefore, satisfying the underlying unitary theory our result provides a might explanation to the black hole information puzzle.

7.6 Conclusions

In this chapter we have investigated the thermal character of the inner horizon of the KNKTN-AdS black hole and derived Hawking radiation via tunneling effect of both the inner and outer horizons. We have found that outside the black hole there occurs two simultaneous radiation processes which are caused by the pair creation effects at the both horizons of the hole. Thus the study gives the most remarkable result that the inner horizon (i.e., white hole horizon) is not a passive observer but an active participant in the radiation processes. Hawking radiation then can be explained as that the inner horizon absorbs the positive energy particles created near the singularity, which traveling the one-way intermediate horizon region reaches the outer horizon and being scattered by the outer horizon travels to infinity as Hawking radiation. In this study we have calculated the inner horizon temperature as positive ($T_{-} > 0$) and the statistical inner horizon entropy as $S_{-} = \sigma \mathcal{A}_{-}/8$ with $\sigma = 7$. Although the Euler characteristic is greater than two ($\sigma > 2$), the entropy satisfies the Bekenstein-Hawking area law as is found for the outer horizon of the NUT-KN black hole [248], if the cut-off factor is chosen properly. Because of the positive inner horizon temperature, there is no interpretative problem regarding the thermodynamical properties of the inner horizon radiation. The positive inner horizon temperature implies that the inner horizon entropy is negative, as explained in (7.4.7). The reason behind the negative inner horizon entropy is not clear and it is an open question.

Indeed, the essence of the black hole entropy is still not completely understood. Nevertheless, the negative entropy of the inner horizon makes the redefined entropy of the black hole to satisfy the Nernst theorem.

Our work suggests the inner horizon as an thermodynamic system and the KNKTN-AdS black hole's thermodynamics then is composed of two subsystems, the outer horizon and inner horizon. The corresponding BS formulae are obtained in (7.4.21) and (7.4.22). The black hole entropy decreases as it radiates but the total entropy of the system comprising the black hole and its surroundings remains constant. That means, the information is preserved.

The validity of our analysis lies in the fact that exactly the same method, which was used in the analysis of the radiation of the inner horizon, produces the well-known results for the radiation emitted by the outer horizon. The radiation effect of the inner horizon has much importance in its own right because it supports the idea that all horizons of spacetime emit radiation. The result of this chapter goes for the RNTN black hole case presented in chapter 4, when one sets $a = 0 = \Lambda$. If in addition $n = 0$, the result reduces to the RN black hole case [82]. The KN black hole result [81] is found for $n = 0 = \Lambda$. The study of this chapter agrees with the works of chapters 4, 5 and 6.

Hawking radiation connects classical general relativity, statistical physics, and quantum field theory in quantum black hole physics. So, a satisfactory quantum theory of gravity demands an intense investigation of black hole physics. In this regard, tunneling process of Hawking radiation deserves more attentions in a wider context.

Chapter 8

Discussion and Concluding Remark

Black holes are perhaps the most perfectly thermal objects in the universe, even though they are very cold for stellar mass black holes. Their thermal properties are not yet fully understood. They are described very accurately by a small number of macroscopic parameters (e.g., mass, angular momentum and charge), but the microscopic degrees of freedom that lead to their thermal behaviour have not yet been adequately identified. Although we have acquired an enormous amount of information about black holes and their thermal properties in the past 4 decades, it seems that there is even much more that we have yet to learn. In this thesis, we have investigated quantum effect, i.e. radiation of black holes which is commonly called Hawking radiation. We use semi-classical tunneling method such as null-geodesic method, Hamilton-Jacobi ansatz, and Damour-Ruffini method. Hawking radiation is one of very interesting phenomena where both of general relativity and quantum theory play a role at the same instant. It is derived by taking into account the quantum effects in the framework of general relativity. Hawking radiation is widely accepted by now because the same result is obtained by several different methods. There remain several aspects which have yet to be clarified. In this thesis, we have attempted to clarify some arguments in previous works and present more satisfactory derivations of Hawking radiation.

In chapter 2, we have reviewed basic facts and various properties of black holes. These are necessary preparation to discuss Hawking radiation. We have demonstrated the black hole solutions and their types as a result of general relativity, and Penrose diagrams which are useful to understand the global structure of black hole spacetime. A part of energy can be extracted from a rotating black hole by the Penrose process. We also have discussed analogies between black hole physics and thermodynamics, and explained a method to derive the black hole entropy which was suggested by Bekenstein. In order to understand the properties of black holes, it is very useful to consider black hole physics in terms of well-known thermodynamics. However, the corresponding relationships are no more than analogies in classical theory. If we would like to establish the complete correspondence between black hole physics and thermodynamics, namely, to demonstrate that black holes actually have entropy and temperature, we need to explain black hole radiation. Although Bekenstein suggested that black holes can have entropy, the complete corresponding relationships was not established because the mechanism of black hole radiation was not explained exclusively.

In chapter 3, we have discussed several previous works on Hawking radiation derived by using quantum effects in black hole physics. Following Hawking's original derivation, we have computed the expectation value of the particle number by using the Bogoliubov transformations. As is well known, the result corresponds with the black body spectrum with a certain temperature. By defining the temperature as the black hole temperature, it is demonstrated that a black hole behaves as a black body and we can thus explain the black hole radiation. The existence of black hole radiation suggests that the Hawking area theorem is violated. However, the second law of black hole physics holds in a suitably generalized form. These considerations lead to understand the complete corresponding relationships between black hole physics and thermodynamics, and suggest that the radiation-dominated tiny black hole will eventually evaporate at some point. We also have briefly studied some representative derivations of Hawking radiation and reviewed the tunneling method as an alternative description of the quantum radiation from black holes. The method is first formulated and considered for the case of stationary black holes, and then a foundation is furnished in terms of analytic continuation throughout complex spacetime. The two main implementations of the tunneling approach, which are the null geodesic method and the Hamilton-Jacobi method, are reviewed. They provide equivalent result in the investigation of black hole evaporation.

Since the discovery of quantum black hole thermal radiance by Hawking [2, 3], it became pretty clear that something remarkable concerning the interface of gravity, quantum theory and thermodynamics was at work. In the usual picture, a radiating black hole loses energy and therefore shrinks, evaporating away to a fate which is still debated. From the recognition that quantum field theory implied a thermal spectrum, many new concepts came out. Among these concepts the most impressive probably is 't Hooft's idea of a dimensional reduction in quantum gravity and the associated holographic description [251, 252, 253], and the principle of black hole complementarity aimed to reconcile the apparent loss of unitarity implied by the Hawking process with the rest of physics as seen by external observers [254]. There were also other issues more practical regarding these matters, and some of them bewildered scientists since the very beginning and that have been only partly resolved. A key issue is that the original derivation of Hawking's radiation has application only to stationary black holes, but the picture above uses quasi-stationary arguments. In actual fact, an evaporating black hole is non-stationary. However, a surprising aspect of the semi-classical result is that the radiation induced by the changing metric of the collapsing star approaches a steady outgoing flux at large times, which implies a drastic violation of energy conservation. This indeed signifies that one cannot neglect the back-reaction problem, which has not been solved yet in a satisfactory way. The other key issues are to deal with the final state of the evaporation process, the thermal nature and the related information loss paradox, the Bekenstein-Hawking entropy and the associated micro-states counting, the trans-Planckian problem, and so on. In order to address some of these questions, there had begun to appear some alternative derivations and descriptions of Hawking's emission process over the years [235, 255, 256, 257, 258, 259, 260, 261, 262], and one of these being the so-called tunneling method.

The original tunneling method, introduced by Parikh and Wilczek to study the black hole radiance, is the *null geodesic method*, which is subsequently applied to verify the thermal properties of a static, spherically symmetric Schwarzschild black hole in the scalar sector. Despite the merits of the seminal work by Parikh and Wilczek, there is a couple of unpleasant features of their null geodesic method: (i) it strongly relies on a very specific choice of (regular-across-the-horizon) coordinates, and (ii) it turns upside down the relationship between Hawking's radiation and back-reaction. In the spirit of general relativity, it should be clear how irrelevant is the choice of coordinates (according to point (i)): physical observables being invariant with respect to the group of diffeomorphisms (the hole temperature is such an observable), there is no reason why Painlevé-Gullstrand coordinates [188] should be favourable with respect to other (equally well-behaved) coordinates. Regarding the latter point, we note that apparently there cannot be Hawking's radiation without back-reaction in the null geodesic description. However, careful watching shows that the discovery of Hawking's radiation justifies back-reaction and makes the treatment of Hawking radiation's self-gravity commendable. The so-called *Hamilton-Jacobi method* can cope with both issues. The null geodesic method is only suitable for the reversible process, but the factual emission process is irreversible; so there is possible to lose information. However, the Hamilton-Jacobi method can be suitable for the irreversible process and there are very few information lost in the emitting process. There is proposed another method by Liu [71], based on the Damour-Ruffini method [7], to model Hawking radiation as a tunneling process. When energy conservation and the particles' back-reaction are taken into account, the same conclusion as the previous works can be derived from it.

In some recent derivations thermal characters of the inner horizon have been employed; however, the understanding of possible role that may play the inner horizons of black holes in black hole thermodynamics is still somewhat incomplete. Motivated by this problem we investigate Hawking radiation of the Reissner-Nordström-Taub-NUT black hole in chapter 4 by considering thermal characters of both the outer and inner horizons [83]. We apply Damour-Ruffini method and the thin film brick wall model to calculate the temperature and the entropy of the inner horizon of the RNTN black hole. The inner horizon admits thermal character with positive temperature and entropy proportional to its area, and it thus may contribute to the total entropy of the black hole in the context of Nernst theorem. Considering conservations of energy and charge and the back-reaction of emitting particles to the spacetime, the emission spectra are obtained for both the inner and outer horizons. The total emission rate is the product of the emission rates of the inner and outer horizons. It deviates from the purely thermal spectrum and can bring some information out. Thus, the result may be treated as an explanation to the information loss paradox.

The radiation emitted by the inner horizon of the RNTN black hole is directed towards the singularity $r = 0$ and the observer at rest with respect to the inner horizon must be situated inside the two-sphere $r = r_-$. Hence, the roles of the ingoing and the outcoming modes interchange. The inner horizon emits particles inside the inner horizon with a positive temperature. When real particles with energy ω are emitted towards the singularity from the inner horizon, it is necessary to maintain a local energy balance that antiparticles with energy $-\omega$ are emitted away from the singularity through the inner horizon. The process is analogous to the one which takes place at the outer horizon according to the Hawking effect—at the outer horizon antiparticles go in and particles come out. This is true at the inner horizon as well. The real particle remains inside the inner horizon and finally meets with the singularity, while the antiparticle enters the intermediate region between the horizons. One may speculate on the possibility that it travels across the intermediate region and finally comes out from the white hole horizon, if the backscattering effects are neglected. However, the situation is quite complicated because the vacuum states corresponding to a freely falling observer near the inner horizon of the black hole and the white hole horizon are completely different. The analysis in [87] predicts that not only does the black hole horizon emit thermal radiation with a black body spectrum but thermal radiation is emitted by the white hole horizon as well. Thus outside the black hole there exists two simultaneous radiation processes: the normal black hole radiation, and the "white hole radiation" which is caused by the pair creation effects at the inner horizon. The white hole radiation contains only antiparticles with negative energy and this may be understood as an absorption of energy by the white hole horizon. However, this feature contradicts with the classical results in a similar way as does the evaporation process at black hole horizons.

The surface gravity of the inner horizon κ_- is negative, since it is directed to the singularity, not to the horizon, opposite to κ_+ which is directed to the outer horizon. The outer horizon of the RNTN black hole is a future horizon for the observer outside the hole $r > r_+$, while the inner horizon is a "past horizon" for the observer inside the hole $r < r_-$. This means that the inner horizon is a horizon of a white hole for the observer in the region $r < r_-$. Because the physical process near the white hole is a time reversal of the physical process near the black hole, we can expect "Hawking absorption" for the white hole as one expects Hawking radiation for the black hole.

We find that the temperature of the inner horizon is $T_- = \frac{-\kappa_-}{2\pi} > 0$ (positive). Thus there exists some thermal radiation from the region $r < r_-$ to the inner horizon with temperature T_-. This thermal radiation is absorbed by the inner horizon and the corresponding quantum effect is termed "Hawking absorption." Similar as the outer horizon of the black hole is in thermal equilibrium with the thermal radiation outside the black hole, the inner horizon is in thermal equilibrium with the thermal radiation inside the inner horizon. The inner horizon absorbs thermal radiation at temperature T_{-}, and at the same time it emits thermal radiation at temperature T_{-}. Thus, the inner horizon is a thermal system with temperature T_{-}. The radiations of the outer horizon and the inner horizon are separate and simultaneously ongoing processes in the spacetime, and an observer situated at the exterior region of the black hole observes the both types of radiation. Then the most remarkable result is that, in contrast to common beliefs, the inner horizon is not a passive spectator but an active participant in the radiation processes [205, 206] of the black hole. We can explain Hawking radiation as follows. The inner horizon absorbs the positive energy particles created near the singularity. Transiting the "one-way membrane" region $r_{-} < r < r_{+}$, these particles arrive at the outer horizon. Being scattered by the outer horizon, they travel to infinity as Hawking radiation.

The temperature or the entropy of the inner horizon is negative and another is positive. The positive temperature implies that the inner horizon entropy is negative. It is not clear why the inner horizon entropy is negative. In fact, our understanding of the essence of the black hole entropy is still incomplete. However, the negative entropy of the inner horizon can make possible the entropy of the black hole with two horizons to satisfy the Nernst theorem. The emission process in this analyze is an reversible one. In this picture, by the process of entropy flux, the two horizons and the outside spacetime approach an thermal equilibrium. As the black hole radiates, its entropy decreases but the total entropy of the system remains constant, and the information is preserved. However, the existence of the negative heat capacity makes an evaporating black hole a highly unstable system, and the thermal equilibrium between the black hole and the outside becomes unstable (there will exist difference in temperature). The process is then irreversible and the underlying unitary theory is not satisfied, i.e., information does not conserve during the evaporation. Further, our study is still a semi-classical analysis in which the radiation is treated as point particles. This type of approximation can only be valid in the low energy regime. To properly address the information loss problem, a better understanding of physics at the Planck scale is a necessary prerequisite, particularly that of the last stages of the endpoint of Hawking evaporation. The procedure of this chapter could be applied to any black hole with two horizons to obtain Hawking radiation via tunneling phenomenon of both horizons. The radiation effect of the inner horizon has much importance because it supports the idea that all horizons of spacetime emit radiation.

In chapter 5 we apply the null-geodesic method to the Taub-NUT-Reissner-Nordström-AdS spacetime [90], which contains a wider range of spacetimes. We also discuss in particular the cases for the Schwarzschild-AdS, Taub-NUT-AdS, Reissner-Nordström-AdS, and Taub-NUT-Reissner-Nordström-AdS black hole spacetimes. In this study we investigate the Hawking radiation of charged and magnetized massive particles via tunneling effect with treating the background spacetime as dynamical. To describe across-horizon phenomena in the null-geodesic method, it is necessary to choose coordinates which, unlike Schwarzschild coordinates, are not singular at the horizon. General Painlevé coordinate transformations [188] are used to eliminate the coordinate singularity from the metric. The crucial features of these coordinates are that they are stationary and nonsingular through the horizon. Hence, it is possible to define an effective "vacuum" state of a quantum field by requiring that it annihilate modes which carry negative frequency with respect to time. A state of this type will look essentially empty (in any case, nonsingular) to a freely falling observer as the observer passes through the horizon. This vacuum differs strictly from the standard Unruh vacuum, defined by requiring positive frequency with respect to the Kruskal coordinate [9]. However, the difference is only in transients, and does not affect the late-time radiation.

Taking into account the particle's self-gravitation and the conservation of energy, electric charge and magnetic charge, we obtain that the emission rate is connected with the change in Bekenstein-Hawking entropy and depended on the emitted particle's energy, electric charge and magnetic charge. The result shows that the Hawking thermal radiation actually deviates from perfect thermality and is consistent with an underlying unitary theory. The result is fully in accordance with the previous literature. We derive the expected Hawking temperature and find, in contrast to a common black hole, that the entropy is not just a quarter area at the horizon of NUT charged black holes, which is consistent with the finding of Refs. [93, 95, 97, 98, 99, 100]. The result can also be treated as a quantum-corrected radiation temperature and it depends not only on the black hole background but also on the radiation particle's energy and charges. The result of this chapter agrees with that of chapter 4 obtained by Liu's method [71] which is based on the Damour-Ruffini method [7].

In chapter 6 we apply Hamilton-Jacobi ansatz in the background of Demiański-Newman black holes to investigate Hawking radiation of charged and magnetized scalar as well as fermion particles [102]. We divide the emission time into a series of infinite small pieces. In each of small segments the process can be treated as a quasi-static one with the background spacetime as fixed. There exists equilibrium temperature in each piece and the Hamilton-Jacobi method can be applied there. In different piece the instantaneous event horizon is different. If I_i be the action in the i-th tiny time piece after the particle tunneled across the instantaneous horizon and $\Delta I_i = I_i - I_{i-1}$, the last action is found as $I = \sum \Delta I_i \sim \int dI$. The Demiański-Newman black hole [103] is a five-parameter stationary axisymmetric solution of the Einstein-Maxwell equations, which is interesting in that it generalizes the well-known Kerr-Newman spacetime with two intriguing parameters the gravitomagnetic and magnetic monopoles. In the stationary pure vacuum limit, the Demiański-Newman metric reduces to the combined Kerr-NUT and Taub-NUT solutions. It is interesting that the spacetimes with the NUT charge are not asymptotically flat but asymptotically locally flat [93, 100, 101] and they possess several special properties. As discussed in [44], tunneling and temperature of Taub-NUT black holes can be formally carried out and the physical interpretation is less problematic in the context of the Hamilton-Jacobi ansatz than the null-geodesic method. The NUT charged black holes have been of particular interest in AdS/CFT conjecture [93, 94, 95]. In AdS backgrounds, Lorentzian sector of these spacetimes' boundary metric is similar with the Gödel metric [104]. In recent years the thermodynamics of various Taub-NUT spacetimes has become a subject of intense study. Entropy of these spacetimes is not just a quarter area at the horizon and their free energy can sometimes be negative [93, 97, 98, 99, 100, 101, 105, 106]. Taking into account self-gravitation interaction and unfixed background spacetime, we find the spectrum of radiation not accurately thermal. The calculation gives an emission probability of $\Gamma \sim e^{\Delta S_{BH}}$ with $\Delta S_{BH} = -\frac{\omega-\omega_o}{T} + \mathcal{O}(\omega, q, p, j)^2$. Then from comparison with the purely thermal spectrum, a quantum-corrected temperature found as $T' = T(M, Q, P, J) + \frac{1}{2\pi} f(\omega, q, p, j)$, where $f(\omega, q, p, j)$, given by (6.4.4), is dependent not only on the black hole background but also on the radiation particle's energy, charges and angular momentum. In particular, $T' \approx \frac{1}{8\pi M}(1 + \frac{\omega}{M})$ for the Schwarzschild case. Thus the black hole temperature increases after emission of a particle. This causes the black hole to emit further. Our study shows that the black hole emits tunneling radiation spectrum of massive and massless (scalar or fermion) particles at the same temperature in the semi-classical limit in which the WKB approximation is applicable. However, when dealing with the Hawking radiation of fermions tunneling, there is a subtle technical issue in selecting an appropriate ansatz for the Dirac field consistent with the choice of matrices γ^μ, and failure to make such a choice results to a breakdown in the method. We also calculate the change of total entropy of the system including black hole and radiating particles. The result shows that the change in total entropy is $\Delta S > 0$ (indicating the process as irreversible) but very small and can be neglected. This has some difference from null-geodesic method [19] in which $\Delta S = 0$. It also suggests that the probing of radiating particles of the black hole is connected with the change of the black hole entropy.

We find that the result obtained by using Hamilton-Jacobi method is in agreement with that obtained in chapter 4 by Damour-Ruffini method and in chapter 5 by the null geodesic method. However, the physical picture in Hamilton-Jacobi method is more clear. There are some differences between the two methods, as mentioned earlier. Although the null geodesic method strongly relies on a very specific choice of coordinates, the Hamilton-Jacobi method can directly be applied to rotating black holes without converting the metric to the corotating frame. Moreover, the factual emission process is irreversible and the null geodesic method is only suitable for the reversible process. The Hamilton-Jacobi method, on the contrary, can be suitable for the irreversible process as well and there is very few information lost in the emitting process. Further, to conserve the symmetry of the spacetime in null geodesic method, the particle should be an ellipsoid shell during the tunneling process. It implies that a should be chosen as a constant. However, this assumption needs not be considered in the Hamilton-Jacobi method and a can be substituted with $\tilde{a} = \frac{J-j}{M-\omega}$. The work of this chapter might not only be theoretically interesting in Hawking radiation, but also meaningful to study the dynamical process of black hole physics.

In chapter 7 we investigate by Hawking radiation of electrically and magnetically charged Dirac particles from a more general black hole called dyonic Kerr-Newman-Kasuya-Taub-NUT-Anti-de Sitter black hole. We consider thermal characters of both the outer and inner horizons. The work of this chapter is a generalization of the work of chapter 4 in which our study is concerned with analysis of charged scalar particles' Hawking radiation by the Klein-Gordon equation in the background of the Reissner-Nordström-Taub-NUT black hole. In the present chapter charged fermions' dynamics in the KNKTN-AdS spacetime are described by Dirac equations with using the Newman-Penrose formalism [246]. We apply Damour-Ruffini method [7] and membrane method [112, 113], which is the modified form of the brick-wall model, proposed by 't Hooft [89]. As in the case of the RNTN black hole in chapter 4, the inner horizon of the KNKTN-AdS black hole admits thermal character with positive temperature and entropy proportional to its area. The inner horizon entropy contributes to the total entropy of the black hole in the context of Nernst theorem. We have found that outside the black hole there occurs two simultaneous radiation processes which are caused by pair creation effects at both horizons of the hole. Thus the study gives the most remarkable result that the inner horizon (i.e., white hole horizon) is not a passive observer but an active participant in the radiation processes. Hawking radiation then can be explained as that the inner horizon absorbs the positive energy particles created near the singularity, which traveling the one-way intermediate horizon region reaches the outer horizon and being scattered by the outer horizon travels to infinity as Hawking radiation.

In this study we have calculated the inner horizon temperature as positive ($T_- > 0$) and the statistical inner horizon entropy as $S_- = \sigma\mathcal{A}_-/8$ with $\sigma = 7$. Although the Euler characteristic is greater than two ($\sigma > 2$), the entropy satisfies the Bekenstein-Hawking area law as is found for the outer horizon of the NUT-KN black hole [248], if the cut-off factor is chosen properly. Because of the positive inner horizon temperature, there is no interpretative problem regarding the thermodynamical properties of the inner horizon radiation. The positive inner horizon temperature implies that the inner horizon entropy is negative, as explained in (7.4.7). The reason behind the negative inner horizon entropy is not clear and it is an open question. Indeed, the essence of the black hole entropy is still not completely understood. Nevertheless, the negative entropy of the inner horizon makes the redefined entropy of the black hole to satisfy the Nernst theorem. The KNKTN-AdS black hole's thermodynamics then is composed of two subsystems, the outer horizon and inner horizon. The corresponding Bekenstein-Smarr formulae are obtained in (7.4.21) and (7.4.22). The black hole entropy decreases as it radiates but the total entropy of the system comprising the black hole and its surroundings remains constant. That means, the information is preserved. The total emission rate is obtained as the product of the emission rates of the inner and outer horizons. It deviates from the purely thermal spectrum with the leading term exactly the Boltzman factor and can bring some information out. The result thus can be treated as an explanation to the information loss paradox. The validity of our analysis lies in the fact that exactly the same method, which was used in the analysis of the radiation of the inner horizon, produces the well-known results for the radiation emitted by the outer horizon. The radiation effect of the inner horizon has much importance in its own right because it supports the idea that all horizons of spacetime emit radiation. The result of this chapter goes for the Reissner-Nordström-Taub-NUT black hole case presented in chapter 4, when one sets $a = 0 = \Lambda$. The study of this chapter agrees with the works of chapters 4, 5 and 6.

In this thesis we have investigated the black hole tunneling radiation and black hole thermodynamics. We have exploited three different techniques to calculate tunneling radiation from different black hole backgrounds: the null-geodesic method, the Hamilton-Jacobi ansatz, and the Damour-Ruffini method. The results indicate that the tunneling method can be seen to be robust in the sense that it works effectively with an extensive range of horizons. Among the three methods, however, the physical picture in Hamilton-Jacobi ansatz is more clear. The Hamilton-Jacobi ansatz is a result of complex path analysis and it ignores self gravitation effects but it can model massive particle emission. Kerner and Mann extended the Hamilton-Jacobi method to model spin-1/2 fermions tunneling from the black hole [60], which was further improved by viewing the Hawking radiation as a series of infinite small quasi-static emission process [70].

The future areas of research would be to consider higher order calculations in WKB in both the scalar as well as fermionic fields (in order to calculate grey body effects). Particularly, it would be worth to investigate fermion emission from rotating spacetimes beyond lowest order to observe whether a coupling term between angular velocity of the black hole and the spin of the fermions can be found. Unfortunately such a coupling term was not seen to the lowest order of WKB. If such a coupling term could be found then it would be a discovery of new physics and would show how fermion emission varies from scalar particle emission. The method can also be applied to other types of particles by using various wave equations. It would also be interesting to investigate the possibility of calculating a density matrix for the emitted particles from a tunneling approach in order to compute correlations between particles. Another interesting case is to extend the tunneling method to model other types of fermions (e.g. fermions' with spin-3/2).

In the null geodesic method the s-wave is massless and follows null geodesics of the Painlevé form of the black hole metric. This method does not model fermion emissions from the black hole. With this method it is possible to calculate the self interaction effect resulting from energy conservation of the system. However, it is possible to ignore the self interaction by doing a perturbative expansion in terms of the particle's energy as long as this energy is much smaller than the energy of the system (i.e. ADM mass). The third technique proposed by Liu [71] is based on Damour-Ruffini method and it can be applied to model scalar particles' tunneling as well as fermions' tunneling. When energy conservation and the particles' back-reaction are taken into account, the same conclusion as the previous works can be obtained from it.

Chapter 9

Appendix

A. Conformal transformations

In this appendix we introduce the notion of conformal transformations (following Ref. [263]). We use the notation:

- $\mathbf{ds}$: The metric, $\mathbf{ds} = \mathrm{g}_{\mu\nu}\mathbf{dx}^{\mu} \otimes \mathbf{dx}^{\nu}$;
- $\mathrm{g}_{\mu\nu}$: The (components of) metric tensor;
- g : The determinant of the metric tensor, g$=\det(\mathrm{g}_{\mu\nu})$;
- $\mathbf{ds}(\mathbf{u},\mathbf{v}) = \mathrm{g}_{\mu\nu}u^{\mu}v^{\nu} = \mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$.

Consider two manifolds M and N with metrics $\widetilde{\mathbf{ds}}$ and $\mathbf{ds}$ respectively. Then, a smooth function $f : M \mapsto N$ is defined as a conformal transformation if, for some nonzero function Ω,

$$\Omega^{-2}\widetilde{\mathbf{ds}} = f^{*}\mathbf{ds}, \qquad (A.1)$$

where f^{*} is a pull-back. If such a map exists, then manifolds M and N are said to be conformally equivalent.

For a local set of coordinates x' so that $x'^{\mu} = f^{\mu}(x)$, or for short, $x' = x'(x)$, the pull-back f^* is defined on a function h by

$$(f^*h)(x) = (h \circ f)(x) = h(x'(x)), \tag{A.2}$$

i.e. the composition of h with f, and on a one-form $\boldsymbol{\alpha} = \alpha_\mu \mathbf{dx}'^{\mu}$ by

$$(f^*\boldsymbol{\alpha})(\mathbf{v}) = \boldsymbol{\alpha}(f_*\mathbf{v}) \tag{A.3}$$

for all vectors $\mathbf{v}$. Here f_* is the push-forward. In terms of the local coordinates and $\mathbf{v} = v^{\mu}\frac{\partial}{\partial x'^{\mu}}$, we have

$$f_*\mathbf{v} = v^{\beta}\frac{\partial x'^{\mu}}{\partial x^{\beta}}\frac{\partial}{\partial x'^{\mu}}, \tag{A.4}$$

i.e., f_* can be considered as the linear map, with the Jacobian

$$(f_*)^{\mu}_{\beta} = \frac{\partial x'^{\mu}}{\partial x^{\beta}}, \tag{A.5}$$

hence,

$$(f^*\boldsymbol{\alpha})(\mathbf{v}) = (\alpha_\nu \mathbf{dx}'^{\nu})\left(v^{\beta}\frac{\partial x'^{\mu}}{\partial x^{\beta}}\frac{\partial}{\partial x'^{\mu}}\right) = v^{\beta}\frac{\partial x'^{\mu}}{\partial x^{\beta}}\alpha_{\mu}. \tag{A.6}$$

Since this is valid for any $\mathbf{v}$, we see that $f^*\boldsymbol{\alpha}$ is the one-form on M given by

$$f^*\boldsymbol{\alpha} = \frac{\partial x'^{\mu}}{\partial x^{\beta}}\alpha_{\mu}\mathbf{dx}^{\beta}. \tag{A.7}$$

Thus, the one-form, $\boldsymbol{\alpha}$, on N is pulled back to a one-form on M. This is why f^* is called a pull-back. For $\boldsymbol{\alpha} = \mathbf{dx}'^{\nu}$ the pull-back is only the chain rule

$$f^*\mathbf{dx}'^{\nu} = \frac{\partial x'^{\nu}}{\partial x^{\beta}}\mathbf{dx}^{\beta}. \tag{A.8}$$

Conformal transformations rescale the metric and relate manifolds where the metric is the same up to a rescaling. If $\Omega = 1$ and $f(x) = x'$, then

$$\widetilde{\mathbf{ds}} = f^*\mathbf{ds} = \frac{\partial x'^\alpha}{\partial x^\nu}\frac{\partial x'^\beta}{\partial x^\mu}\mathrm{g}_{\alpha\beta}\mathbf{dx}^\mu \otimes \mathbf{dx}^\nu. \tag{A.9}$$

This is just the metric at $f(x)$ instead of at x, pulled back to x. If this metric happens to be equal to the original one, i.e.

$$\widetilde{\mathbf{ds}} = \mathbf{ds}, \tag{A.10}$$

then $f(x)$ is called an isometry. That is, isometries are conformal transformations with $\Omega = 1$.

Consider two vectors $\mathbf{v}$ and $\mathbf{u}$. The lengths and the angles of these vectors will be preserved under isometries. Using conformal transformations, we get

$$\widetilde{\mathbf{ds}}(\mathbf{v}, \mathbf{v}) = \Omega^2\mathbf{ds}(\mathbf{v}, \mathbf{v}), \tag{A.11}$$

and also find that

$$\begin{aligned}\widetilde{\angle}(\mathbf{v}, \mathbf{u}) &= \frac{\widetilde{\mathbf{ds}}(\mathbf{v}, \mathbf{u})}{\sqrt{\widetilde{\mathbf{ds}}(\mathbf{v}, \mathbf{v})\widetilde{\mathbf{ds}}(\mathbf{u}, \mathbf{u})}} \\ &= \frac{\mathbf{ds}(\mathbf{v}, \mathbf{u})}{\sqrt{\mathbf{ds}(\mathbf{v}, \mathbf{v})\mathbf{ds}(\mathbf{u}, \mathbf{u})}} = \angle(\mathbf{v}, \mathbf{u}),\end{aligned} \tag{A.12}$$

that is, angles are preserved under conformal transformations.

B. Killing Vectors and Null Hypersurfaces

The total energy-momentum vector P^μ on a 3-dimensional spacelike hypersurface Σ is defined by

$$P^\mu = \int_\Sigma T^{\mu\nu} d\Sigma_\nu, \qquad (B.1)$$

where $T_{\mu\nu}$ is the energy-momentum tensor. This definition, however, loses a physical meaning in the curved space. The global energy or momentum conservation laws cannot be maintained in general. Nevertheless, when there exist particular vectors, the corresponding conservation laws can be maintained.

In order to find these laws, we consider a quantity given by

$$P_\xi(\Sigma) = \int_\Sigma \xi_\mu T^{\mu\nu} d\Sigma_\nu, \qquad (B.2)$$

where ξ^μ is an arbitrary vector. This is a scalar quantity. We consider the volume V enclosed by two surfaces Σ and Σ' (see Fig. 3.2 in section 3.2). Applying the Gauss theorem, we find

$$\begin{aligned} P_\xi(\Sigma') - P_\xi(\Sigma) &= \int_V \nabla_\nu (\xi_\mu T^{\mu\nu}) dV \\ &= \int_V [(\nabla_\nu \xi_\mu) T^{\mu\nu} + \xi_\mu (\nabla_\nu T^{\mu\nu})] dV \\ &= \frac{1}{2} \int_V (\nabla_\nu \xi_\mu + \nabla_\nu \xi_\mu) T^{\mu\nu} dV, \end{aligned} \qquad (B.3)$$

where we have used the local conservation law of the energy-momentum tensor, $\nabla_\nu T^{\mu\nu} = 0$, and the fact that $T^{\mu\nu}$ is a symmetric tensor. It is obvious that the quantity $P_\xi(\Sigma)$ is conserved, if the vector ξ_μ satisfies

$$\nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu = 0, \tag{B.4}$$

and there is thus a corresponding symmetry in the system. The particular vector ξ^μ and the corresponding equation (B.4) are respectively called the "Killing vector" and the "Killing equation." Equivalently, when there are symmetries in the system, there exist the corresponding conserved quantities (Noether's theorem) and the corresponding Killing vectors. The conserved quantities are identified at asymptotic infinity.

We now discuss hypersurfaces and for this purpose define $\mathscr{S}(x)$ as a smooth function of the spacetime coordinates x^μ. Consider a family of hypersurfaces

$$\mathscr{S} = \text{constant}. \tag{B.5}$$

The vector normal to the hypersurface is given by

$$n = F(x)(\mathrm{g}^{\mu\nu}\partial_\nu \mathscr{S})\frac{\partial}{\partial x^\mu}, \tag{B.6}$$

where $F(x)$ is an arbitrary nonzero function. Then the hypersurface $\mathscr{S}$ is called a "null hypersurface," if the relation

$$n^2 = 0 \tag{B.7}$$

is satisfied. For example, let us consider the case of a Schwarzschild background, described by the metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{B.8}$$

where $r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$. In the ingoing Eddington-Finkelstein coordinates (v, r, θ, φ), it is rewritten as

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2drdv + r^2 d\Omega^2, \qquad (B.9)$$

where $v = t + r_*$ is the advanced time. Then for the surface defined by

$$\mathscr{S} = r - 2M, \qquad (B.10)$$

we obtain, from (B.6),

$$n = F(r) \left[\left(1 - \frac{2M}{r}\right) \frac{\partial}{\partial r} + \frac{\partial}{\partial v} \right], \qquad (B.11)$$

and

$$n^2 = \mathrm{g}^{\mu\nu} \partial_\mu \mathscr{S} \partial_\nu \mathscr{S} F^2(r) = \left(1 - \frac{2M}{r}\right) F^2(r), \qquad (B.12)$$

which shows that $r = 2M$ is a null hypersurface. Then relation (B.11) becomes

$$n \mid_{r=2M} = F(r) \frac{\partial}{\partial v}. \qquad (B.13)$$

We therefore define $\mathscr{N}$ as a null hypersurface with a normal vector n, if (B.7) is satisfied for (B.6) with $\mathscr{S} = \mathscr{N}$. A vector t_N is tangent to $\mathscr{N}$ when $n \cdot t_N = 0$. However, the relation $n \cdot t_N = 0$ is satisfied because $\mathscr{N}$ is null. Thus, the vector n is itself a tangent vector, i.e., we have

$$n^\mu = \frac{dx^\mu}{d\lambda}, \qquad (B.14)$$

where $x^\mu(\lambda)$ is geodesic. Further, it is known from the definition that a Killing horizon is a null hypersurface $\mathscr{N}$ with a Killing vector ξ normal to $\mathscr{N}$.

C. The First Integral by Carter

In this appendix, we present the first integral, derived by Carter [161], for the Kerr-Newman metric given by

$$ds^2 = -\frac{\Delta - a^2\sin^2\theta}{\Sigma}dt^2 + 2\frac{\Delta - (r^2+a^2)}{\Sigma}a\sin^2\theta dt d\varphi$$
$$+\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\varphi^2, \qquad (C.1)$$

where notations are the same as in section 2.2. We write the metric as follows:

$$ds^2 = \Sigma d\theta^2 - 2a\sin^2\theta dr d\tilde{\varphi}$$
$$+2drdu + \frac{1}{\Sigma}[(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\sin^2\theta d\tilde{\varphi}^2$$
$$-\frac{2a}{\Sigma}(2Mr - Q^2)\sin^2\theta d\tilde{\varphi}du - \left(1 - \frac{2Mr - Q^2}{\Sigma}\right)du^2, \qquad (C.2)$$

where u is the retarded time. This metric agrees with the Kerr-Newman metric (C.1) under the transformations given by

$$\begin{cases} du = dt + \frac{r^2+a^2}{\Delta}dr, \\ d\tilde{\varphi} = d\varphi + \frac{a}{\Delta}dr. \end{cases} \qquad (C.3)$$

In the background with the metric (C.2), Carter considered the behavior of a particle of mass μ and electrical charge e. For this case, the general form of the Hamilton-Jacobi equation is described by

$$\frac{\partial S}{\partial \lambda} = \frac{1}{2}\mathrm{g}^{ij}\left(\frac{\partial S}{\partial x^i} - eA_i\right)\left(\frac{\partial S}{\partial x^j} - eA_j\right), \tag{C.4}$$

where λ is an affine parameter must be related to the proper time τ by

$$\tau = \mu\lambda, \tag{C.5}$$

A_μ is the gauge field (the electrical potential) and S is the Jacobi action.

If there exists a separable solution, we can write S in terms of the already known constants of the motion as follows:

$$S = -\frac{1}{2}\mu^2\lambda - Eu + J\tilde{\varphi} + S_\theta + S_r, \tag{C.6}$$

where E and J are given by

$$p_u = -E, \quad p_{\tilde{\varphi}} = J, \tag{C.7}$$

where p_μ is the momentum component in the direction of each coordinate variable. Here, S_θ and S_r are respectively functions of θ and r only. In this case, the first integral can be given by

$$p_\theta^2 + \left(aE\sin\theta - \frac{J}{\sin\theta}\right)^2 + a^2\mu^2\cos^2\theta = \mathscr{K}, \tag{C.8}$$

$$\Delta p_r^2 - 2[(r^2+a^2)E - aJ + eQr]p_r + \mu^2 r^2 = -\mathscr{K}, \tag{C.9}$$

where $\mathscr{K}$ is a constant.

We now write the relations (C.8) and (C.9) by using the Kerr-Newman metric (C.1) [264]. For an electrically neutral particle, we have $e = 0$. Differentiating (C.6), we find

$$dS = -\frac{1}{2}\mu^2 d\lambda - Edu + Jd\tilde{\varphi} + \frac{\partial S_\theta}{\partial\theta}d\theta + \frac{\partial S_r}{\partial r}dr. \qquad (C.10)$$

Using the transformations (C.3) into (C.10), we obtain

$$\begin{aligned} dS = &-\frac{1}{2}\mu^2 d\lambda - E\left(dt + \frac{r^2+a^2}{\Delta}dr\right) \\ &+J\left(d\varphi + \frac{a}{\Delta}dr\right) + \frac{\partial S_\theta}{\partial\theta}d\theta + \frac{\partial S_r}{\partial r}dr \\ = &-\frac{1}{2}\mu^2 d\lambda - Edt + Jd\varphi + \frac{\partial S_\theta}{\partial\theta}d\theta \\ &+\left(-\frac{r^2+a^2}{\Delta}E + \frac{a}{\Delta}J + \frac{\partial S_r}{\partial r}\right)dr \end{aligned} \qquad (C.11)$$

The momenta conjugate to θ and r are respectively

$$p_\theta = \frac{\partial S_\theta}{\partial\theta}, \quad p_r = \frac{\partial S_r}{\partial r}. \qquad (C.12)$$

Using (C.12) into the relation (C.11), we obtain

$$\begin{aligned} dS = &-\frac{1}{2}\mu^2 d\lambda - Edt + Jd\varphi + p_\theta d\theta \\ &+\left(-\frac{r^2+a^2}{\Delta}E + \frac{a}{\Delta}J + p_r\right)dr. \end{aligned} \qquad (C.13)$$

Comparison between (C.10) and (C.13) results in a new radial momentum p'_r given by

$$p'_r = p_r - \frac{r^2+a^2}{\Delta}E + \frac{a}{\Delta}J, \qquad (C.14)$$

from which we can also write

$$p_r = p'_r + \frac{1}{\Delta}\left[(r^2+a^2)E - aJ\right]. \qquad (C.15)$$

Defining a new constant q related to the others by

$$q = \mathscr{K} - (J - aE)^2, \qquad (C.16)$$

we obtain from (C.8)

$$p_\theta^2 + \left(aE\sin\theta - \frac{p_\varphi}{\sin\theta}\right)^2 + a^2\mu^2\cos^2\theta = q + (p_\varphi - aE)^2, \qquad (C.17)$$

which gives

$$q = \cos^2\theta\left[a^2(\mu^2 - E^2) + \frac{p_\varphi^2}{\sin^2\theta}\right] + p_\theta^2. \qquad (C.18)$$

This Carter's kinetic constant is used in (2.6.20) in section 2.6. Substituting the value of $\mathscr{K}$ from (C.16) into (C.9), we obtain

$$p_r^2 - 2\left[(r^2+a^2)E - aJ\right]p_r + \mu^2 r^2$$

$$= -q - p_\varphi^2 + 2aEp_\varphi - a^2E^2. \qquad (C.19)$$

Inserting (C.15) into (C.19), one can find

$$E^2\left[r^4 + a^2(r^2 + 2Mr - Q^2)\right] - 2E(2Mr - Q^2)ap_\varphi$$

$$-(r^2 - 2Mr + Q^2)p_\varphi^2 - (\mu^2 r^2 + q)\Delta = (p'_r\Delta)^2, \qquad (C.20)$$

D. Bogoliubov Transformations

In this appendix, we show that the inverse transformations of (3.2.18) and (3.2.19):

$$a_i = \sum_j \left(b_i \alpha_{ij} + b_j^\dagger \beta_{ij}^* \right), \tag{D.1}$$

$$a_i^\dagger = \sum_j \left(b_i \beta_{ij} + b_j^\dagger \alpha_{ij}^* \right), \tag{D.2}$$

give the Bogoliubov transformations (3.2.20) and (3.2.21):

$$b_i = \sum_j \left(\alpha_{ij}^* a_j - \beta_{ij}^* a_j^\dagger \right), \tag{D.3}$$

$$b_i^\dagger = \sum_j \left(\alpha_{ij} a_j^\dagger - \beta_{ij} a_j \right). \tag{D.4}$$

Inserting both (D.1) and (D.2) into the right-hand side of (D.3), we obtain

$$\begin{aligned}
&\sum_j \left(\alpha_{ij}^* a_j - \beta_{ij}^* a_j^\dagger \right) \\
&= \sum_{j,k} \left[\alpha_{ij}^* \left(\alpha_{jk} b_k + \beta_{ik}^* b_k^\dagger \right) - \beta_{ij}^* \left(\beta_{jk} b_k + \alpha_{ik}^* b_k^\dagger \right) \right] \\
&= \sum_k \left[\sum_j \left(\alpha_{ij}^* \alpha_{jk} - \beta_{ij}^* \beta_{jk} \right) b_k + \sum_k \left(\alpha_{ij}^* \beta_{jk}^* - \beta_{ij}^* \alpha_{jk}^* \right) b_k^\dagger \right].
\end{aligned} \tag{D.5}$$

Then the orthonormal condition (3.2.6) for $\{f_i\}$ and $\{f_i^*\}$:

$$\rho \left(f_i, f_j^* \right) = \frac{1}{2} i \int_\Sigma \left(f_i \nabla_\mu f_j^* - f_j^* \nabla_\mu f_i \right) d\Sigma^\mu = \delta_{ij}, \tag{D.6}$$

yields the following relations:

$$\rho\left(f_i^*, f_j\right) = -\delta_{ij}, \tag{D.7}$$

$$\rho\left(f_i, f_j\right) = \rho\left(f_i^*, f_j^*\right) = 0. \tag{D.8}$$

The orthonormal condition (3.2.12) is also satisfied for $\{p_i\}$ and $\{p_i^*\}$, giving

$$\rho\left(p_i, p_j^*\right) = \delta_{ij}. \tag{D.9}$$

Substituting in (D.9) the relations between $\{p_i\}$ and $\{f_i\}$, which are

$$p_i = \sum_k \left(\alpha_{ik} f_k + \beta_{ik} f_k^*\right), \tag{D.10}$$

$$p_j^* = \sum_l \left(\alpha_{jl}^* f_l^* + \beta_{jl}^* f_l\right), \tag{C.11}$$

we obtain

$$\begin{aligned}
\rho\left(p_i, p_j^*\right) &= \rho\left(\sum_k \left(\alpha_{ik} f_k + \beta_{ik} f_k^*\right),\ \sum_l \left(\alpha_{jl}^* f_l^* + \beta_{jl}^* f_l\right)\right) \\
&= \sum_{k,l} \left[\alpha_{ik}\alpha_{jl}^* \rho(f_k, f_l^*) + \alpha_{ik}\beta_{jl}^* \rho(f_k, f_l)\right. \\
&\qquad \left. + \beta_{ik}\alpha_{jl}^* \rho(f_k^*, f_l^*) + \beta_{ik}\beta_{jl}^* \rho(f_k^*, f_l)\right].
\end{aligned} \tag{D.12}$$

With (D.6)–(D.8), we obtain from (D.12) that

$$\begin{aligned}
\rho(p_i, p_j^*) &= \sum_{k,l} \left[\alpha_{ik}\alpha_{jl}^* \delta_{kl} + \beta_{ik}\beta_{jl}^*(-\delta_{kl})\right] \\
&= \sum_k \left(\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*\right).
\end{aligned} \tag{D.13}$$

Hence, from (D.9), we find that

$$\sum_k (\alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk}) = \delta_{ij}. \qquad (D.14)$$

In the similar we consider the case of $\rho(p^*_i, p^*_j) = 0$ and obtain

$$\sum_k (\beta^*_{ik}\alpha^*_{jk} - \alpha^*_{ik}\beta^*_{jk}) = 0. \qquad (D.15)$$

Inserting (D.14) and (D.15) into (D.5), we obtain

$$\sum_k \left[\sum_j \left(\alpha^*_{ij}\alpha_{jk} - \beta^*_{ij}\beta_{jk}\right) b_k + \sum_j \left(\alpha^*_{ij}\beta^*_{jk} - \beta^*_{ij}\alpha^*_{jk}\right) b^\dagger_k \right]$$
$$= \sum_k \delta_{ik} b_k = b_i. \qquad (D.12)$$

We can thus reproduce b_i from the right-hand side of (D.3). Similarly $b^\dagger_i$ can be reproduced from the right-hand side of (D.4). This confirms that the relations (D.3) and (D.4) are the inverse transforms of (D.1) and (D.2).

E. Klein-Gordon Equation in Schwarzschild Background

In this appendix, we present the solutions of the Klein-Gordon equation (3.2.5),

$$\mathrm{g}^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = 0. \qquad (E.1)$$

Since $\nabla_\nu\Phi = \partial_\nu\Phi$ for a scalar field Φ, the equation (E.1) becomes

$$\nabla_\mu(\mathrm{g}^{\mu\nu}\partial_\nu\Phi) = 0. \qquad (E.2)$$

where ∂_ν and ∇_μ respectively stand for an ordinary derivative and a covariant derivative. Also, for a vector field A^μ,

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu_{\nu\mu} A^\nu, \tag{E.3}$$

where $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol and

$$\begin{aligned}\Gamma^\mu_{\nu\mu} &= \frac{1}{2}\mathrm{g}^{\mu\rho}(\partial_\mu \mathrm{g}_{\rho\nu} + \partial_\nu \mathrm{g}_{\rho\mu} - \partial_\rho \mathrm{g}_{\nu\mu}) \\ &= \frac{1}{2}(\mathrm{g}^{\mu\rho}\partial_\mu \mathrm{g}_{\rho\nu} + \mathrm{g}^{\mu\rho}\partial_\nu \mathrm{g}_{\rho\mu} - \mathrm{g}^{\mu\rho}\partial_\rho \mathrm{g}_{\nu\mu}) \\ &= \frac{1}{2}\mathrm{g}^{\mu\rho}\partial_\nu \mathrm{g}_{\rho\mu},\end{aligned} \tag{E.4}$$

since $\mathrm{g}^{\mu\rho}\partial_\rho \mathrm{g}_{\nu\mu} = \mathrm{g}^{\mu\rho}\partial_\mu \mathrm{g}_{\rho\nu}$ (exchanging μ and ρ).

We are now to calculate $\partial_\nu \mathrm{g}_{\rho\mu}$. Defining g by

$$\begin{aligned}\mathrm{g} \equiv \det(\mathrm{g}_{\mu\rho}) &= \exp(\ln\det \mathrm{g}_{\mu\rho}) \\ &= \exp(\mathrm{Tr}\ln \mathrm{g}_{\mu\rho}),\end{aligned} \tag{E.5}$$

the small variation of g is given by

$$\begin{aligned}\delta\mathrm{g} &= \exp\left[\mathrm{Tr}\{\ln(\mathrm{g}_{\mu\rho} + \delta\mathrm{g}_{\mu\rho})\}\right] - \exp\left[\mathrm{Tr}(\ln \mathrm{g}_{\mu\rho})\right] \\ &\approx \exp\left[\mathrm{Tr}(\ln \mathrm{g}_{\mu\rho} + \mathrm{g}^{\mu\nu}\delta\mathrm{g}_{\nu\rho})\right] - \exp\left[\mathrm{Tr}(\ln \mathrm{g}_{\mu\rho})\right] \\ &= \exp\left[\mathrm{Tr}(\ln \mathrm{g}_{\mu\rho})\right]\exp\left[\mathrm{Tr}(\mathrm{g}^{\mu\nu}\delta\mathrm{g}_{\nu\rho})\right] - \exp\left[\mathrm{Tr}(\ln \mathrm{g}_{\mu\rho})\right],\end{aligned} \tag{E.6}$$

where we have used the Taylor expansion for a matrix X

$$\ln(X + \delta X) \approx \ln X + X^{-1}\delta X. \qquad (E.7)$$

Performing the Taylor expansion for $\exp[\mathrm{Tr}(\mathrm{g}^{\mu\nu}\delta \mathrm{g}_{\nu\rho})]$ and retaining the first order of terms, we find

$$\begin{aligned}
\delta \mathrm{g} &\approx \exp[\mathrm{Tr}(\ln \mathrm{g}_{\mu\rho})](1 + \mathrm{Tr}\mathrm{g}^{\mu\nu}\delta \mathrm{g}_{\nu\rho}) - \exp[\mathrm{Tr}(\ln \mathrm{g}_{\mu\rho})] \\
&= \exp[\mathrm{Tr}(\ln \mathrm{g}_{\mu\rho})]\mathrm{Tr}(\mathrm{g}^{\mu\nu}\delta \mathrm{g}_{\nu\rho}) \\
&= \mathrm{g} \cdot \mathrm{Tr}(\mathrm{g}^{\mu\nu}\delta \mathrm{g}_{\nu\rho}) \\
&= \mathrm{g} \cdot \mathrm{g}^{\mu\nu}\delta \mathrm{g}_{\nu\mu}, \qquad (E.8)
\end{aligned}$$

Replacing ν with ρ in the last equality of (E.8), we obtain

$$\delta \mathrm{g} = \mathrm{g} \cdot \mathrm{g}^{\mu\rho}\delta \mathrm{g}_{\rho\mu} \qquad (E.9)$$

Hence, we find

$$\partial_\nu \mathrm{g} = \mathrm{g} \cdot \mathrm{g}^{\mu\rho}\partial_\nu \mathrm{g}_{\rho\mu} \quad \Leftrightarrow \quad \partial_\nu \mathrm{g}_{\rho\mu} = (\mathrm{g} \cdot \mathrm{g}^{\mu\rho})^{-1}\partial_\nu \mathrm{g} \qquad (E.10)$$

We substitute (E.10) into (E.4)and obtain

$$\begin{aligned}
\Gamma^\mu_{\nu\mu} &= \frac{1}{2}\mathrm{g}^{\mu\rho}\frac{1}{\mathrm{g} \cdot \mathrm{g}^{\mu\rho}}\partial_\nu \mathrm{g} \\
&= \frac{1}{2\mathrm{g}}\partial_\nu \mathrm{g} \\
&= \frac{1}{\sqrt{-\mathrm{g}}}\partial_\nu(\sqrt{-\mathrm{g}}), \qquad (E.11)
\end{aligned}$$

From (E.3) and (E.11), we obtain

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + \frac{1}{\sqrt{-\mathrm{g}}}\partial_\nu(\sqrt{-\mathrm{g}})A^\nu$$

$$= \frac{1}{\sqrt{-\mathrm{g}}}\partial_\mu(\sqrt{-\mathrm{g}}A^\mu), \tag{E.12}$$

replacing ν by μ. So, the Klein-Gordon equation (E.2) is written as

$$\frac{1}{\sqrt{-\mathrm{g}}}\partial_\mu(\sqrt{-\mathrm{g}}\cdot \mathrm{g}^{\mu\nu}\partial_\nu\Phi) = 0. \tag{E.13}$$

From the metric (2.2.1) of the Schwarzschild background, we have

$$(\mathrm{g}_{\mu\nu}) = \begin{pmatrix} -\left(1-\frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & \left(1-\frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{E.14}$$

and

$$\mathrm{g} = -r^4\sin^2\theta. \tag{E.15}$$

The Klein-Gordon equation (E.13) with (E.14) and (E.15) takes the form

$$\Bigg[-\frac{r}{r-2M}\frac{\partial^2}{\partial^2 t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\} - \frac{1}{r^2}\left(-\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\Bigg]\Phi = 0 \tag{E.16}$$

Defining a quadratic angular momentum by

$$\hat{L}^2 = -\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}, \tag{E.17}$$

the equation (E.16) can be written as

$$\left[-\frac{r}{r-2M}\frac{\partial^2}{\partial^2 t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\}-\frac{1}{r^2}\hat{L}^2\}\right]\Phi=0. \quad (E.18)$$

We write Φ as

$$\Phi=(Ae^{-i\omega t}+A^*e^{i\omega t})R(r)S(\theta,\varphi) \quad (E.19)$$

and substitute it into (E.18). The result is

$$\left[-\frac{r}{r-2M}(i\omega)^2+\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\}-\frac{1}{r^2}\hat{L}^2\}\right] \times(Ae^{-i\omega t}+A^*e^{i\omega t})R(r)S(\theta,\varphi)=0 \quad (E.16)$$

Dividing both sides by $(Ae^{-i\omega t}+A^*e^{i\omega t})$ and then separating variables, equation (E.16) can be put in the following form

$$\frac{r^2}{R(r)}\left[\frac{r}{r-2M}\omega^2+\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\}\right]R(r) =\frac{1}{S(\theta,\varphi)}\hat{L}^2S(\theta,\varphi)=\lambda, \quad (E.17)$$

where λ is a separation constant. Setting $\lambda=l(l+1)$, we obtain

$$\hat{L}^2S(\theta,\varphi)=l(l+1)S(\theta,\varphi), \quad (E.18)$$

$$\left[\frac{r}{r-2M}\omega^2+\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\}-\frac{l(l+1)}{r^2}\right]R(r)=0. \quad (E.19)$$

We expand $S(\theta,\varphi)$ as follows:

$$S(\theta,\varphi)=\sum_m B_{lm}Y_{lm}(\theta,\varphi), \quad (E.20)$$

where B_{lm} is an integration constant and $Y_{lm}(\theta,\varphi)$ is the spherical harmonics. In equation (E.19), we put

$$\tilde{R}(r_*) = rR(r), \tag{E.21}$$

where r_* is the tortoise coordinate defined by

$$r_* \equiv r + 2M \ln \left| \frac{r}{2M} - 1 \right| . \tag{E.22}$$

From (E.22), we get

$$\frac{\partial}{\partial r} = \left(\frac{r}{r - 2M} \right) \frac{\partial}{\partial r_*}. \tag{E.23}$$

With (E.21)–(E.23), equation (E.19) becomes

$$\frac{1}{r} \left[\frac{r}{r - 2M} \omega^2 + \left(\frac{r}{r - 2M} \right) \frac{\partial^2}{\partial r_*^2} - \frac{2M}{r^3} - \frac{1}{r^2} l(l+1) \right] \tilde{R}(r_*) = 0. \tag{E.24}$$

Dividing both sides of (E.24) by $\frac{1}{r}\left(\frac{r}{r-2M}\right)$, we find

$$\frac{\partial}{\partial r_*^2} \tilde{R}(r_*) + \left[\omega^2 - \frac{1}{r^2} \left\{ \frac{2M}{r} + l(l+1) \right\} \left(1 - \frac{2M}{r} \right) \right] \tilde{R}(r_*) = 0. \tag{E.25}$$

In the asymptotic region, $r \to \infty$, equation (E.25) becomes

$$\frac{\partial^2}{\partial r_*^2} \tilde{R}(r_*) + \omega^2 \tilde{R}(r_*) = 0. \tag{E.26}$$

Its solution is given by

$$\tilde{R}(r_*) = C_{\omega l} e^{-i\omega r_*} + C_{\omega l}^* e^{i\omega r_*}, \tag{E.27}$$

where $C_{\omega l}$ is an integration constant. Substituting (E.21) into (E.27), we obtain

$$R(r) = \frac{1}{r} (C_{\omega l} e^{-i\omega r_*} + C_{\omega l}^* e^{i\omega r_*}). \tag{E.28}$$

Using (E.20) and (E.28) in (E.19), we find

$$\Phi = \sum_{\omega,l,m} \frac{1}{r} \Big[AC_{\omega l} e^{-i\omega(t+r_*)} + AC^*_{\omega l} e^{-i\omega(t-r_*)}$$

$$+A^* C_{\omega l} e^{i\omega(t-r_*)} + A^* C^*_{\omega l} e^{i\omega(t+r_*)} \Big] B_{lm} Y_{lm}(\theta,\varphi). \qquad (E.29)$$

We define

$$v = t + r_*, \qquad (E.30)$$

$$u = t - r_*, \qquad (E.31)$$

where v and u are respectively called the advanced time and the retarded time. If one puts together the integration constants in (E.29) except for the normalization constant $\frac{1}{\sqrt{2\pi\omega}}$, one can write the partial waves as

$$f_{\omega' lm} = \frac{F_{\omega'}(r)}{r\sqrt{2\pi\omega'}} e^{i\omega' v} Y_{lm}(\theta,\varphi), \qquad (E.32)$$

$$p_{\omega lm} = \frac{P_{\omega}(r)}{r\sqrt{2\pi\omega}} e^{i\omega u} Y_{lm}(\theta,\varphi). \qquad (E.33)$$

Here $F_{\omega'}(r)$ and $P_{\omega}(r)$ are not integration constants but rather "integration variables" which depend on r, because we want to take into account the small effect of r arising from the approximation by setting $r \to \infty$.

F. Calculation of Bogoliubov Coefficients

The Bogoliubov coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$, as given in (3.2.34) and (3.2.35), are respectively

$$\alpha_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}} \int_{-\infty}^{\infty} dv e^{-i\omega' v} p_{\omega}, \qquad (F.1)$$

$$\beta_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}}\int_{-\infty}^{\infty} dv e^{i\omega' v} p_{\omega}. \qquad (F.2)$$

If the partial wave function p_ω be such that $p_\omega^{(2)} \sim 0$ for $v > v_o$ and

$$p_\omega^{(2)} \sim \frac{P_\omega^-}{r\sqrt{2\pi\omega}} \exp\left[-i\frac{\omega}{\kappa}\ln\left(\frac{v_0 - v}{CD}\right)\right], \quad \text{for} \quad (v \leq v_0), \qquad (F.3)$$

as represented in (3.2.41), we find that the corresponding $\alpha_{\omega\omega'}^{(2)}$ and $\beta_{\omega\omega'}^{(2)}$ are given by (3.2.42) and (3.2.43), i.e.,

$$\alpha_{\omega\omega'}^{(2)} \approx \frac{1}{2\pi} P_\omega^- (CD)^{\frac{i\omega}{\kappa}} e^{-i\omega' v_o} \left(\sqrt{\frac{\omega'}{\omega}}\right) \Gamma\left(1 - \frac{i\omega}{\kappa}\right) (-i\omega')^{-1+\frac{i\omega}{\kappa}}, \qquad (F.4)$$

$$\beta_{\omega\omega'}^{(2)} \approx -i\alpha_{\omega(-\omega')}^{(2)}. \qquad (F.5)$$

Using (F.3) in (F.1),

$$\alpha_{\omega\omega'}^{(2)} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}(r)} \int_{-\infty}^{v_0} dv \frac{P_\omega(r)}{r\sqrt{2\pi\omega}} \exp\left[-i\frac{\omega}{\kappa}\ln\left(\frac{v_0 - v}{CD}\right)\right] e^{-i\omega' v}, \qquad (F.6)$$

where integration variables $F_{\omega'}(r)$ and $P_\omega(r)$, which take the small effect of r into account, are collectively rewritten as $P_\omega(r)$. Near the horizon $r = 2M$ we use $P_\omega^- = P_\omega(2M)$. So, the relation (F.6) becomes

$$\alpha_{\omega\omega'}^{(2)} = \frac{1}{2\pi} P_\omega^- (CD)^{i\frac{\omega}{\kappa}} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv (v_0 - v)^{-i\frac{\omega}{\kappa}} e^{-i\omega' v}. \qquad (F.7)$$

Integrating over the variable defined as $v_o - v = x$, we find

$$\begin{aligned}\alpha_{\omega\omega'}^{(2)} &= \frac{1}{2\pi} P_\omega^- (CD)^{i\frac{\omega}{\kappa}} \sqrt{\frac{\omega'}{\omega}}\, e^{-i\omega' v_0} \int_0^{\infty} dx x^{-i\frac{\omega}{\kappa}} e^{-(-i\omega')x} \\ &= \frac{1}{2\pi} P_\omega^- (CD)^{\frac{i\omega}{\kappa}} \sqrt{\frac{\omega'}{\omega}} e^{-i\omega' v_0} \Gamma\left(1 - \frac{i\omega}{\kappa}\right) (-i\omega')^{1-\frac{i\omega}{\kappa}}, \end{aligned} \qquad (F.8)$$

where we have used the formula of the gamma function,

$$\Gamma(\varepsilon)t^{-\varepsilon} = \int_0^\infty ds s^{\varepsilon-1} e^{-ts}. \qquad (F.9)$$

We can show in the similar way the relation (F.5) for $\beta^{(2)}_{\omega\omega'}$.

G. The Unruh Temperature

In this appendix we utilize the ideas of Hawking's original work of deriving black hole radiation (described in section 3.2) in Rindler spacetime and obtain the temperature of Unruh radiation by simply comparing the solutions to the Klein-Gordon equation for massless particles from the points of views of inertial and uniformly accelerated observers [87].

Let us consider the thermal radiation of the Rindler horizon found by Unruh in 1976 [9]. Rindler horizons are such horizons of spacetime which appear in the rest frame of a uniformly accelerated observer.

Suppose an observer in flat two-dimensional Minkowski spacetime experiences a constant positive proper acceleration a in the X direction. Then

$$\left(\frac{d^2X}{d\tau^2}\right)^2 = a^2, \quad \left(\frac{d^2T}{d\tau^2}\right)^2 = 0, \qquad (G.1)$$

where X and T, respectively, are the minkowskian space and time coordinates. The scalar product of proper velocities and accelerations are given by

$$\left(\frac{dX}{d\tau}\right)^2 - \left(\frac{dT}{d\tau}\right)^2 = -1, \qquad (G.2)$$

$$\left(\frac{d^2X}{d\tau^2}\right)^2 - \left(\frac{d^2T}{d\tau^2}\right)^2 = a^2. \qquad (G.3)$$

These equations are solved by

$$\frac{dT}{d\tau} = \cosh(a\tau), \quad \frac{dX}{d\tau} = \sinh(a\tau). \tag{G.4}$$

Integrating (G.4) the world line of the observer may be written in the parametrized form:

$$T(\tau) = \frac{1}{a}\sinh(a\tau), \quad X(\tau) = \frac{1}{a}\cosh(a\tau). \tag{G.5}$$

We define the Rindler coordinates t and x such that

$$x = \frac{1}{a}, \quad t = a\tau. \tag{G.6}$$

Then the metric of two-dimensional Minkowski spacetime can be written as

$$ds^2 = -x^2 dt^2 + dx^2. \tag{G.7}$$

In Fig. 9.1 below the world line of a uniformly accelerated observer has been shown. The figure also show the Rindler horizon of the accelerated observer. The Rindler spacetime has four regions, labelled as I, II, III and IV. Because this diagram is very similar to the Kruskal diagram of Schwarzschild spacetime, one would expect Rindler spacetime to have physical properties similar to those of Schwarzschild spacetime. In fact, it is obvious that the causal features of the regions II and IV are, respectively, analogous to those of a black and a white hole.

Consider massless particles in the rest frame of an accelerated observer. In general, the Klein-Gordon equation may be written as in (3.2.5):

$$\mathrm{g}^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = 0. \tag{G.8}$$

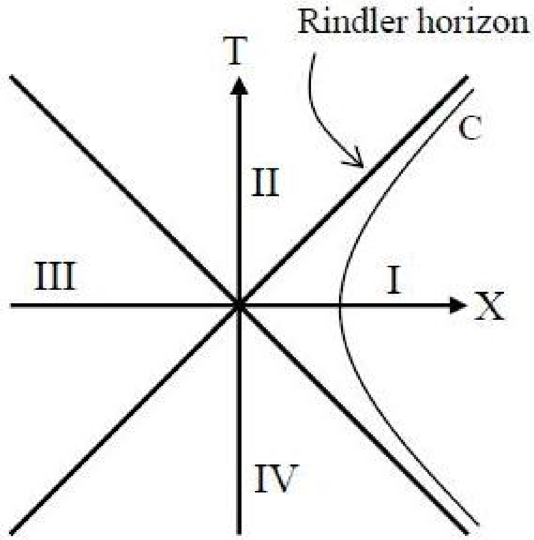

Figure 9.1: Rindler spacetime. The curve C describes the wordline of a uniformly accelerated observer.

When spacetime metric is that of Eq.(G.7), Eq. (G.8) takes the form

$$\left(-\frac{1}{x^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}+\frac{1}{x}\frac{\partial}{\partial x}\right)\Phi=0. \tag{G.9}$$

Defining

$$x^*=\ln x, \tag{G.10}$$

we obtain from (G.9)

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^{*2}}\right)\Phi=0, \tag{G.11}$$

which has orthonormal solutions

$$u_\omega=N_\omega e^{-i\omega U}, \tag{G.12}$$

where N_ω is an appropriate normalization constant and

$$U = t - x^*. \qquad (G.13)$$

From the point of view of an accelerated observer in the region I, these solutions represent particles with energy ω propagating to the positive X-direction. On the contrary, the corresponding solutions to the massless Klein-Gordon equation

$$\left(-\frac{\partial^2}{\partial T^2} + \frac{\partial^2}{\partial X^2}\right)\Phi = 0, \qquad (G.14)$$

written from the point of view of an inertial observer at rest with respect to the Minkoski coordinates T and X, can be given by

$$u'_\omega = N_\omega e^{-i\omega\tilde{u}}, \quad \tilde{u} = T - X. \qquad (G.15)$$

These solutions also represent particles with energy ω propagating to the positive X-direction.

From (G.5) and (G.13),

$$U = -\ln(-\tilde{u}), \qquad (G.16)$$

and hence the Bogoliubov transformation

$$u_\omega = \sum_{\omega'} \left(A'_{\omega\omega'} u'_{\omega'} + B'_{\omega\omega'} u'^*_{\omega'}\right) \qquad (G.17)$$

between the orthonormal solutions u_ω and u'_ω may be written as

$$e^{i\omega ln(-\tilde{u})} = \sum_{\omega'} \left(A'_{\omega\omega'} e^{-i\omega'\tilde{u}} + B'_{\omega\omega'} e^{i\omega'\tilde{u}}\right), \qquad (G.18)$$

where the Bogoliubov coefficients $A'_{\omega\omega'}$ and $B'_{\omega\omega'}$ are expressible as Fourier integrals:

$$A'_{\omega\omega'} = \frac{1}{2\pi}\int_{-\infty}^{0} d\tilde{u} e^{i\omega \ln(-\tilde{u})} e^{i\omega' \tilde{u}}, \qquad (G.19)$$

$$B'_{\omega\omega'} = \frac{1}{2\pi}\int_{-\infty}^{0} d\tilde{u} e^{i\omega \ln(-\tilde{u})} e^{-i\omega' \tilde{u}}. \qquad (G.20)$$

The limit of integration is from $-\infty$ to 0, since we are considering particles in the region-I where $\tilde{u} < 0$.

In Fig. 9.2 γ_+ and γ_- are respectively closed contours circulating the shaded regions in the upper and the lower half of the complex plane. The value of the integral in (G.19) is zero along the arcs of the contour γ_+ in the limit where $R \to \infty$ and $r \to 0$. Therefore the integral from $-\infty$ to 0 along the real axis may be transformed into an integral from $+\infty$ to 0 along the imaginary axis. Analogous result holds also for the integral in (G.20) along the path γ_-, except that now the integral from $-\infty$ to 0 along the real axis may be transformed to an integral from $-\infty$ to 0 along the imaginary axis. The integrals along the imaginary axis lead directly to

$$|A'_{\omega\omega'}| = e^{\pi\omega} |B'_{\omega\omega'}|. \qquad (G.21)$$

From the well-known relation between the Bogoliubov coefficients,

$$\sum_{\omega'} \left(|A'_{\omega\omega'}|^2 - |B'_{\omega\omega'}|^2 \right) = 1, \qquad (G.22)$$

one can find that when the field is in vacuum from the point of view of an inertial observer, the number of particles with energy ω, from the point of

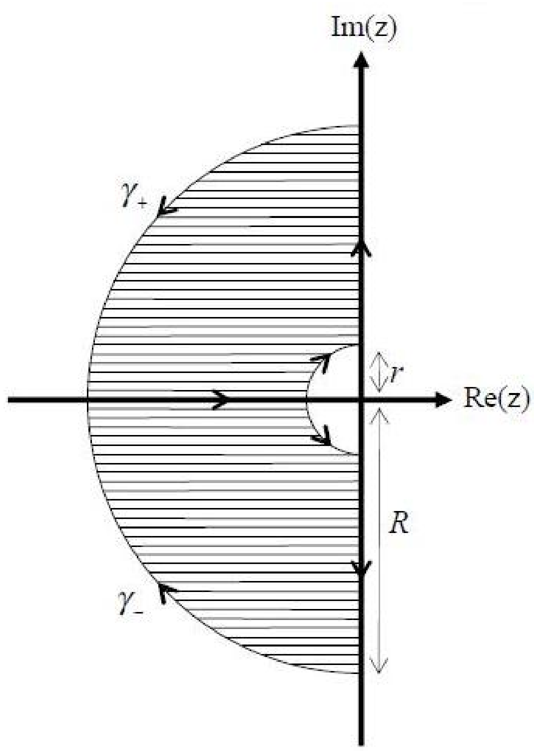

Figure 9.2: Integration contours in the complex plane.

view of an accelerated observer, is

$$n_\omega = \sum_{\omega'} |B'_{\omega\omega'}|^2 = \frac{1}{e^{2\pi\omega} - 1}. \tag{G.23}$$

This is the Planck spectrum at the temperature $T_0 = \frac{1}{2\pi}$ and it is related to the temperature experienced by an observer located at a given point in space by the Tolman relation [265]:

$$T = \frac{T_0}{\sqrt{g_{00}}}. \tag{G.24}$$

Thus a uniformly accelerated observer detects particles coming out from the Rindler horizon with the black-body spectrum corresponding to the characteristic temperature

$$T_U = \frac{1}{2\pi x} = \frac{a}{2\pi}, \tag{G.25}$$

even when the field is in vacuum from the point of view of an inertial observer. This result is known as the Unruh effect. It is one of the most remarkable outcomes of quantum field theory.

Bibliography

[1] J.D. Bekenstein, "Black Holes and Entropy", Phys. Rev. D **7**, (1973) 2333–2346.

[2] S.W. Hawking, "Particle Creation By Black Holes", Commun. Math. Phys. **43**, (1975) 199–220.

[3] S.W. Hawking, "Black-hole explosions" Nature (London) **248**, (1974) 30–31.

[4] G.W. Gibbons and S.W. Hawking, "Cosmological Event Horizons, Thermodynamics, And Particle Creation", Phys. Rev. D **15**, (1977) 2738–2751.

[5] G.W. Gibbons and S.W. Hawking, "Action Integrals and Partition Functions in Quantum Gravity", Phys. Rev. D **15**, (1977) 2752.

[6] J.B. Hartle and S.W. Hawking, "Wave Function of the Universe.", Phys. Rev. D **28**, (1983) 2960–2975.

[7] T. Damour and R. Ruffini, "Black-hole evaporation in the Klein-Sauter-Heisenberg-Euler formulation", Phys. Rev. D **14**, (1976) 332–334.

[8] R.M. Wald Ann. Phys. **82**, (1974) 548–556.

[9] W.G. Unruh, "Notes on black hole evaporation", Phys. Rev. D **14**, (1976) 870–892.

[10] D. Page, "Particle Emission Rates from a Black Hole: Massless Particles from an Uncharged, Nonrotating Hole", Phys. Rev. D **13**, (1976) 198–206.

[11] D. Page, "Particle Emission Rates from a Black Hole. 2. Massless Particles from a Rotating Hole", Phys. Rev. D **14**, (1976) 3260–3273.

[12] N.D. Birrel and P.C.W. Davies,*Quantum Fields in Curved Space*, Cambridge University Press, (1982).

[13] M.H. Ali, "Quantum Nonthermal Radiation of Nonstationary Kerr-Newman-de Sitter Black Holes", Gen. Relativ. Gravit. **36**, (2004) 1171–1181.

[14] M.H. Ali, "Superradiance of dyonic black holes", Gen. Relativ. Gravit. **37**, (2005) 977–983.

[15] P. Kraus and F. Wilczek,"A Simple Stationary Line Element for the Schwarzschild Geometry, and Some Applications" [gr-qc/9406042] (1994).

[16] P. Kraus and F. Wilczek, "Self-Interaction Correction to Black Hole Radiance", Nucl. Phys. B **433**, (1995) 403. [gr-qc/9408003]

[17] P. Kraus and F. Wilczek, "Effect of Self-Interaction on Charged Black Hole Radiance", Nucl. Phys. B **437**, (1995) 231–242. [hep-th/9411219]

[18] P. Kraus and E. Keski-Vakkuri, "Microcanonical D-branes and Back Reaction", Nucl.Phys. B **491**, (1997) 249–262. [hep-th/9610045]

[19] M.K. Parikh and F. Wilczek, "Hawking Radiation as tunnelling", Phys. Rev. Lett. **85**, (2000) 5042. [arXiv:hep-th/9907001]

[20] V.A. Berezin, A. Boyarsky and A.Yu. Neronov, "On the Mechanism of Hawking Radiation", Gravitation & Cosmology **5**, (1999) 16–22.

[21] G.E. Volovik, "Simulation of Panleve-Gullstrand black hole in thin 3He-A film", Pisma Zh. Eksp. Teor. Fiz. **69**, (1999) 662–668; JETP Lett. **69**, (1999) 705–713.

[22] A. Calogeracos and G.E. Volovik, "Rotational quantum friction in superfluids: Radiation from object rotating in superfluid vacuum", JETP Lett. **69**, (1999) 281–287; Pisma Zh. Eksp. Teor. Fiz. **69**, (1999) 257–262.

[23] M.K. Parikh, "New Coordinates for de Sitter Space and de Sitter Radiation", Phys. Lett. B **546**, (2002) 189. [arXiv:hep-th/0204107]

[24] M.K. Parikh, "A Secret Tunnel Through The Horizon", Int. J. Mod. Phys. D **13**, (2004) 2351–2354. [arXiv:hep-th/0405160]

[25] M.K. Parikh, "Energy Conservation and Hawking Radiation", [arXiv:hep-th/0402166]

[26] K. Srinivasan and T. Padmanabhan , "Particle Production and Complex Path Analysis", Phys. Rev. D **60**, (1999) 24007. [gr-qc-9812028]

[27] S. Shankaranarayanan, K. Srinivasan and T. Padmanabhan, "Method of complex paths and general covariance of Hawking radiation" Mod. Phys. Lett. A **16**, (2001) 571–578. [arXiv:gr-qc/0007022v2]

[28] S. Shankaranarayanan, T. Padmanabhan and K. Srinivasan, "Hawking radiation in different coordinate settings: Complex paths approach" Class. Quant. Grav. **19**, (2002) 2671–2688. [arXiv:gr-qc/0010042v4]

[29] T. Padmanabhan, "Entropy of Horizons, Complex Paths and Quantum tunnelling", Mod. Phys. Lett. A **19**, (2004) 2637–2643. [gr-qc/0405072]

[30] E.C. Vagenas, "Are Extremal 2D Black Holes Really Frozen?", Phys. Lett. B **503**, (2001) 399. [arXiv:hep-th/0012134]

[31] A.J.M. Medved, "Radiation via tunnelling from a de Sitter cosmological horizon", Phys. Rev. D **66**, (2002) 124009. [arXiv:hep-th/0207247]

[32] S. Shankaranarayanan, "Temperature and entropy of Schwarzschild-de Sitter spacetime" Phys. Rev. D **67**, (2003) 084026. [arXiv:gr-qc/0301090v2]

[33] A.J.M. Medved and E. Vagenas, "On Hawking radiation as tunnelling with back-reaction", Mod. Phys. Lett. A **20**, (2005) 2449–2454. [arXiv:gr-qc/0504113]

[34] M. Arzano, A. Medved and E. Vagenas, "Hawking Radiation as tunnelling through the Quantum Horizon", JHEP **0509** (2005) 037. [hep-th/0505266]

[35] M. Angheben, M. Nadalini, L. Vanzo and S. Zerbini, "Hawking Radiation as tunnelling for Extremal and Rotating Black Holes", JHEP **0505** (2005) 014. [arXiv:hep-th/0503081]

[36] Qing-Quan Jiang and Shuang-Qing Wu, "Hawking radiation of charged particles as tunnelling from Reissner-Nordström-de Sitter black holes with a global monopole", Phys. Lett. B **635**, (2006) 151–155, Erratum-ibid. **639**, (2006) 684. [arXiv:hep-th/0511123]

[37] Qing-Quan Jiang, Shuang-Qing Wu and Xu Cai, "Hawking radiation as tunnelling from the Kerr and Kerr-Newman black holes", Phys. Rev. D **73**, (2006) 064003. [hep-th/0512351]

[38] Jing-Yi Zhang and Zheng Zhao, "Hawking radiation of charged particles via tunnelling from the Reissner-Nordström black hole", JHEP **0510**, (2005) 055.

[39] Jing-Yi Zhang and Zheng Zhao, "Charged particles' tunnelling from the Kerr-Newman black hole", Phys. Lett. B **638**, (2006) 110–113. [gr-qc/0512153]

[40] Yapeng Hu, Jingyi Zhang and Zheng Zhao, "The relation between Hawking radiation via tunnelling and the laws of black hole thermodynamics" [gr-qc/0601018]

[41] Zhao Ren, Li Huai-Fan and Zhang Sheng-Li, "Canonical Entropy of charged black hole" [gr-qc/0608123]

[42] Liu Zhao, "Tunnelling through black rings", Commun. Theor. Phys. **47**, (2007) 835–842. [hep-th/0602065]

[43] S.-Q. Wu and Q.-Q. Jiang, "Remarks on Hawking radiation as tunnelling from the BTZ black holes", JHEP **0603** (2006) 079. [hep-th/0602033]

[44] R. Kerner and R.B. Mann, “Tunnelling, Temperature and Taub-NUT Black Holes”, Phys. Rev. D **73** (2006) 104010. [arXiv:gr-qc/0603019]

[45] S.-Q. Wu and Q.-Q. Jiang, “Hawking Radiation of Charged Particles as tunnelling from Higher Dimensional Reissner-Nordström-de Sitter Black Holes”, [hep-th/0603082]

[46] B.D. Chowdhury, “Tunnelling of Thin Shells from Black Holes: An Ill Defined Problem”, Pramana **70**, (2008) 593–612. [hep-th/0605197]

[47] E.T. Akhmedov, V. Akhmedova, T. Pilling and D. Singleton, “Thermal radiation of various gravitational backgrounds”, Int. J. Mod. Phys. A **22**, (2007) 1705–1715. [hep-th/0605137]

[48] E.T. Akhmedov, V. Akhmedova and D. Singleton, “Hawking temperature in the tunnelling picture”, Phys. Lett. B **642** (2006) 124–128. [hep-th/0608098]

[49] V. Akhmedova, T. Pilling, A. de Gill, and D. Singleton, “Temporal contribution to gravitational WKB-like calculations”, [arXiv:0804.2289]

[50] Jun Ren, Jingyi Zhang and Zheng Zhao, “Tunnelling Effect and Hawking Radiation from a Vaidya Black Hole”, Chin. Phys. Lett. **23** (2006) 2019–2022. [gr-qc/0606066]

[51] P. Mitra, “Hawking temperature from tunnelling formalism”, Phys. Lett. B **648**, (2007) 240–242. [hep-th/0611265]

[52] ShuZheng Yang and DeYou Chen, “A new method to study Hawking tunnelling radiation of the charged particles from Reissner-Nordström black hole”, Int. J. Theor. Phys. **46**, (2007) 1747–1752.

[53] B. Chatterjee, A. Ghosh and P. Mitra, "Tunnelling from black holes and tunnelling into white holes", Phys. Lett. B **661**, (2008) 307–311. [arXiv:0704.1746]

[54] R.D. Criscienzo, M. Nadalini, L. Vanzo, S. Zerbini and G. Zoccatelli, "On the Hawking radiation as tunnelling for a class of dynamical black holes", Phys. Lett. B **657**, (2007) 107–111. [arXiv: 0707.4425]

[55] R. Kerner and R.B. Mann, "Tunnelling From Gödel Black Holes", Phys. Rev. D **75**, (2007) 084022.

[56] M.H. Ali, "Hawking radiation via tunneling from hot NUT-Kerr-Newman-Kasuya spacetime", Class. Quantum Grav. **24** (2007) 5849–5860.

[57] M.H. Ali, "Charged Particles Tunneling from Hot-NUT-Kerr-Newman-Kasuya Spacetime", Int. J. Theor. Phys. **47**, (2008) 2203–2217.

[58] R. Banerjee and B.R. Majhi "Quantum tunnelling and Back Reaction", Phys. Lett. B **662**, (2008) 62–65. [arXiv:0801.0200]

[59] R. Banerjee, B.R. Majhi and S. Samanta, "Noncommutative Black Hole Thermodynamics", [arXiv:0801.3583]

[60] R. Kerner and R.B. Mann, "Fermions Tunnelling from Black Holes", Class. Quant. Grav. **25**, (2008) 095014. [arXiv:0710.0612]

[61] Yuichi Sekiwa, "Decay of the cosmological constant by Hawking radiation as quantum tunnelling", [arXiv:0802.3266]

[62] A.J.M. Medved, "A Brief Editorial on de Sitter Radiation via tunnelling", [arXiv:0802.3796]

[63] G.E. Volovik, “On de Sitter radiation via quantum tunnelling”, [arXiv:0803.3367]

[64] Ran Li and Ji-Rong Ren, “Dirac particles tunnelling from BTZ black hole”, Phys. Lett. B **661**, (2008) 370–372. [arXiv:0802.3954]

[65] R.D. Criscienzo and L. Vanzo, “Fermion tunnelling from Dynamical Horizons”, [arXiv:0803.0435]

[66] Ran Li and Ji-Rong Ren, “Hawking radiation of Dirac particles via tunnelling from Kerr black hole”, [arXiv:0803.1410]

[67] R. Kerner and R.B. Mann, “Charged Fermions Tunnelling from Kerr-Newman Black Holes”, Phys. Lett. B **665**, (2008) 277–283. [arXiv:0803.2246]

[68] De-You Chen, Qing-Quan Jiang and Xiao-Tao Zu, “Hawking radiation of Dirac particles via tunnelling from rotating black holes in de Sitter spaces”, [arXiv:0804.0131]

[69] S. Iso, H. Umetsu and F. Wilczek, “Anomalies, Hawking Radiations and Regularity in Rotating Black Holes”, Phys. Rev. D **74**, (2006) 044017. [hep-th/0606018]

[70] C. Ding, “Back-Reaction of Black Hole Radiation from Hamilton-Jacobi Method”, Int. J. Theor. Phys. **53**, (2014) 694–705. arXiv:1302.0353v2 [gr-qc]

[71] W.B. Liu, Acta Phys. Sin. **56**, (2007) 6164.

[72] S. Zhou and W. Liu, Phys. Rev. D **77**, (2008) 104021.

[73] W.G. Unruh and R.M. Wald, Phys. Rev. D **25**, (1982) 942.

[74] V.P. Frolov and D.N. Page, Phys. Rev. Lett. **71**, (1993) 3902.

[75] R.M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, p. 151, University of Chicago Press, Chicago (1994).

[76] H. Lee, S.W. Kim and W.T. Kim, Phys. Rev. D **54**, (1996) 6559.

[77] R.M. Wald, Phys. Rev. D **56**, (1997) 6467.

[78] Z. Zhao, Int. J. Theor. Phys. **38**, (1999) 1539.

[79] M.H. Ali, "Planck absolute entropy of Demiański-Newman black holes", Astroparticle Physics **22**, (2004) 227–233.

[80] M.H. Ali and A. Banu, "Hawking Absorption and Planck Absolute Entropy of Kerr-Newman Black Hole", Int. J. Theor. Phys. **45**, (2006) 651–661.

[81] J. Ren, Int. J. Theor. Phys. **48**, (2009) 2088.

[82] J. Ren, Int. J. Theor. Phys. **48**, (2009) 431.

[83] M.H. Ali and K. Sultana, "Charged Particles' Hawking Radiation via Tunneling of Both Horizons from Reissner-Nordström-Taub-NUT Black Holes", Int. J. Theor. Phys. **52**, (2013) 2802–2817.

[84] S.Q. Wu and X. Cai, Nuovo Cimento B **115**, (2000) 143.

[85] S.Q. Wu, Phys. Lett. B **608**, (2005) 251.

[86] S.Q. Wu, arXiv:hep-th/0411002

[87] A. Peltola and J. Mäkelä, Int. J. Mod. Phys. D **15**, (2006) 817–844.

[88] X. Li and Z. Zhao, Phys. Rev. D **62**, (2000) 104001.

[89] G. 't Hooft, Nucl. Phys. B **256**, (1985) 727.

[90] M.H. Ali and K. Sultana, "Tunneling of Charged Massive Particles from Taub-NUT-Reissner-Nordström-AdS Black Holes", Int. J. Theor. Phys. **53**, (2014) 1441–1453.

[91] S.W. Hawking, C.J. Hunter and M. Taylor, Phys. Rev. D, Part. Fields **59**, (1999) 064005.

[92] M.M. Caldarelli, G. Cognola and D. Klemm, Class. Quantum Gravity **17**, (2000) 399.

[93] S.W. Hawking, C.J. Hunter and D.N. Page, Phys. Rev. D **59**, (1999) 044033.

[94] A. Chamblin, R. Emparan, C.V. Johnson, and R.C. Myers, Phys. Rev. D **59**, (1999) 064010.

[95] R.B. Mann, Phys. Rev. D **60**, (1999) 104047.

[96] C.W. Misner, "Taub-NUT space as a counter example to almost anything", In: Ehlers, J. (ed.) Relativity Theory and Astrophysics 1. Lectures in AppliedMathematics, vol. **8**, p. 160. Am.Math. Soc., Providence (1967).

[97] R. Emparan, C.V. Johnson and R. Myers, Phys. Rev. D **60**, (1999) 104001.

[98] R.B. Mann, Phys. Rev. D **61**, (2000) 084013.

[99] R. Clarkson, L. Fatibene and R.B. Mann, Nucl. Phys. B **652**, (2003) 348. [arXiv:hep-th/0210280]

[100] S.W. Hawking and C.J. Hunter, Phys. Rev. D **59**, (1999) 044025.

[101] D. Astefanesei, R.B. Mann and Radu, J. High Energy Phys. **01**, (2005) 049.

[102] M.H. Ali and K. Sultana, "Tunneling and temperature of Demiański-Newman black holes", Int. J. Theor. Phys. DOI:10.1007/s10773-014-2154-1 (in press).

[103] J.A. Aguilar-Sánchez, A.A. García and V.S. Manko: Gravitation & Cosmology **7**, (2001) 149.

[104] E. Radu and M. Astefanesei, Int. J. Mod. Phys. D **11**, (2002) 715.

[105] M.H. Dehghani and A. Khodam-Mohammadi, Phys. Rev. D **73**, (2006) 124039.

[106] A. Khodam-Mohammadi and M. Monshizadeh, Phys. Rev. D **79**, (2009) 044002.

[107] S. Mignemi, Phys. Rev. D **51**, (1995) 934.

[108] D. Kramer and G. Neugebauer, Phys. Lett. A **75**, (1980) 259.

[109] A. Tomimatsu and M. Kihara, Prog. Theor. Phys. **67**, (1982) 1406.

[110] M. Ahmed and M.H. Ali, "U(1)-Gauge Theory of Fermions in Space-times with Horizons", Int. J. Theor. Phys. **38**, (1999) 933–938.

[111] M.H. Ali and K. Sultana, "Charged Dirac Particles' Hawking Radiation via Tunneling of Both Horizons and Thermodynamics Properties of Kerr-Newman-Kasuya-Taub-NUT-AdS Black Holes", Int. J. Theor. Phys. **52**, (2013) 4537–4556.

[112] C.J. Gao, and Y.G. Shen, Chin. Phys. Lett. **18**, (2001) 1167.

[113] X. Li and Z. Zhao, Chin. Phys. Lett. **18**, (2001) 463.

[114] A. Einstein, Preuss. Akad. Wiss. Berlin, Sitzber. (1915) 778.

[115] A. Einstein, Preuss. Akad. Wiss. Berlin, Sitzber. (1915) 844.

[116] W. Israel, Phys. Rev. **164**, (1967) 1776.

[117] B. Carter, Phys. Rev. Lett. **26**, (1971) 331.

[118] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-Time, (Cambridge University Press, 1975), 9.3.

[119] R. Ruffini and J.A. Wheeler, Phys. Today, **24**, (1971) 30.

[120] R. Penrose, Riv. Nuovo Cim.**1**, (1969) 252.

[121] K. Schwarzschild, Sitzber. Deut. Akad. Wiss. Berlin, Kl. Math.-Phys. Tech., (1916) 189.

[122] H. Reissner, Ann. Phys. (Leipzig) **50**, (1916) 106.

[123] G. Nordström, Proc. Kon. Ned. Akad. Wet. **20**, (1918) 1238.

[124] R.P. Kerr, Physical Review Letters **11** (5) (1963) 237.

[125] E. Newman, et al., Journal of Mathematical Physics **6** (6) (1965) 918.

[126] R. Penrose, "Conformal treatment of infinity," in: B.S. DeWitt and C.M. DeWitt (eds), Relativity, Groups and Topology,(Gordon and Breach, New York, 1964), 565.

[127] P.K. Townsend, arXiv: gr-qc/9707012.

[128] J.A. Wheeler, Phys. Rev. **97**, (1955) 511.

[129] T. Regge and J.A. Wheeler, Phys. Rev. **108**, (1957) 1063.

[130] A.S. Eddington, Nature **113**, (1924) 192.

[131] D. Finkelstein, Phys. Rev. **110**, (1958) 965.

[132] M.D. Kruskal, Phys. Rev. **119**, (1960) 1743.

[133] G. Szekeres, Publ. Mat. Debrecen **7**, (1960) 285.

[134] Chandrima Ganguly and Soumitra SenGupta, arXiv:1401.6826v2 [hep-th] (2014).

[135] J.-P. Lasota, E. Gourgoulhon, M. Abramowicz, A. Tchekhovskoy and R. Narayan, Phys. Rev. D **89**, (2014) 024041.

[136] M. Bejger, T. Piran, M. Abramowicz and F. Håkanson, Phys. Rev. Lett. **109**, (2012) 121101.

[137] A.A. Abdujabbarov, B.J. Ahmedov, S.R. Shaymatov and A.S. Rakhmatov, Astrophys. Space Sci. **334**, (2011) 237. arXiv:1105.1910v1 [astro-ph.SR]

[138] S.S. Komissarov, Journal of the Korean Physical Society, **54**, No. 6, (2009) 2503.

[139] M. Bhat, S. Dhurandhar and N. Dadhich, J. Astrophys. Astr. **6**, (1985) 85.

[140] S.W. Hawking, Phys. Rev. Lett. **6**, (1971) 1344.

[141] D. Christodoulou, Phys. Rev. Lett. **25**, (1970) 1596.

[142] J.M. Bardeen, W.H. Press and S.A. Teukolsky, Astrophys. J. **178**, (1972) 347.

[143] R.M. Wald, Astrophys. J. **191** (1974) 231.

[144] W.C. Misner, Phys. Rev. Lett. **28**, (1972) 994.

[145] W.H. Press and S.A. Teukolsky, Nature **238**, (1972) 211.

[146] A.A. Starobinskii, "Amplification of Waves during Reflection from a Rotating Black Hole," Zh. Eksp. Teor. Fiz. **64**, (1973) 48–57 (English transl.: Soviet Phys.–JETP **37**, (1973) 28–32).

[147] Ya.B. Zel'dovich, "Amplification of Cylindrical Electromagnetic Waves Reflected from a Rotating Body," Zh. Eksp. Teor. Fiz. **62**, (1972) 2076–2081 (English transl.: Soviet Phys.–JETP **35**, (1972) 1085–1087).

[148] W. Unruh, Phys. Rev. Lett. **31**, (1973) 1265.

[149] R. Guven, Phys. Rev. D **6**, (1977) 1706.

[150] J.M. Greif, Junior thesis, Princeton University, *unpublished* (1969).

[151] J.M. Bardeen, B. Carter and S.W. Hawking, Commun. Math. Phys. **31**, (1973) 161.

[152] R.M. Wald: General Relativity, (The University of Chicago Press, 1984), 12.5.

[153] J.D. Bekenstein, Phys. Rev. D **9** (1974) 3292.

[154] C.E. Shannon, Bell Syst. Tech. J. **27**, (1948) 379.

[155] C.E. Shannon and W. Weaver, *The Mathematical Theory of Communications*, (University of Illinois Press, Urbana, 1949).

[156] M. Tribus and E.C. McIrvine, Sci. Amer. **225** (No. 3), (1971) 179.

[157] E.T. Jaynes, Phys. Rev. **106**, (1957) 620.

[158] E.T. Jaynes, Phys. Rev. **108**, (1957) 171.

[159] L. Brillouin, *Science and Information Theory* (Academic, New York, 1956), Chaps. 1, 9, 12–14.

[160] D. Christodoulou and R. Ruffini, Phys. Rev. D **4**, (1971) 3552.

[161] B. Carter, Phys. Rev. **174**, (1968) 1559.

[162] J.D. Bekenstein, Lett. Nuovo Cimento **4**, (1972) 737.

[163] L.D. Landau and E. M. Lifshitz, Statistical Physics (Addison-Wesley, Reading, Mass., 1969).

[164] W. Greiner, L. Neise and H. Stöcker, THERMODYNAMICS AND STATISTICAL MECHANICS, (Springer-Verlag New York, Inc. 1995), Chapter 5.

[165] B.S. DeWitt, Phys. Rept. C **19**, (1975) 295.

[166] S.W. Hawking, Mon. Not. Roy. Astron. Soc. **152**, (1971) 75.

[167] R. Bertlmann and E. Kohlprath, Ann. of Phys. **288**, (2001) 137.

[168] S.W. Hawking, Commun. Math. Phys. **25**, (1972) 152.

[169] W.G. Unruh, “Particle Detectors and Black Hole Evaporation,” Ann. N.Y. Acad. Sci. **302**, (1977) 186.

[170] P.C.W. Davies, J. Phys. A **8**, (1975) 609.

[171] P. Candelas and D.W. Sciama, Phys. Rev. Lett. **38**, (1977) 1372.

[172] T.H. Boyer, Phys. Rev. D **21**, (1980) 2137.

[173] T.H. Boyer, Phys. Rev. D **29**, (1984) 1089.

[174] D.W. Sciama, P. Candelas and D. Deutsch, Adv. Phys. **30**, (1981) 327.

[175] An extensive review is given by S. Takagi, Prog. Theor. Phys. **88**, (1986) 1.

[176] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space*, Cambridge University Press, N.Y., (1982).

[177] J.F. Donoghue and B.R. Holstein, "Temperature measured by a uniformly accelerated observer," Am. J. Phys. **52**, (1984) 730.

[178] P.M. Alsing and P.W. Milonni, "Simplified derivation of the Hawking-Unruh temperature for an accelerated observer in vacuum," arXiv:quant-ph/0401170v2 (2004).

[179] L.C.B. Crispino, A. Higuchi and G.E.A. Matsas, "The Unruh effect and its applications," Rev. Mod. Phys. **80**, (2008) 787–838.

[180] R. Penrose and R. Floyd, Nature **229**, (1971) 193.

[181] G. Denardo and R. Ruffini, Phys. Lett. B **45**, (1973) 259.

[182] G. Denardo, L. Hively and R. Ruffini, Phys. Lett. B **50**, (1974) 270.

[183] N. Deruelle and R. Ruffini, Phys. Lett. B **52**, (1974) 437.

[184] T. Damour and R. Ruffini, Phys. Rev. Lett. **35**, (1975) 463.

[185] N. Deruelle and R. Ruffini, Phys. Lett. B **57**, (1975) 248.

[186] T. Nakamura and H. Sato, Phys. Lett. B **61**, (1976) 371.

[187] P. Kraus and F. Wilczek, "Effect of Self-Interaction on Charged Black Hole Radiance", Nucl. Phys. B **437**, (1995) 231. [hep-th/9411219]

[188] P. Painlevé, C.R. Acad Sci. (Paris), **173**, (1921) 677; A. Gullstrand, Ark. Mat. Astron. Fys. **16**, (1922) 1.

[189] T. Padmanabhan, Gen. Relativ. Gravit. **34**, (2002) 2029.

[190] T. Padmanabhan, Class. Quantum Gravity **19**, (2002) 5387.

[191] T. Padmanabhan, arXiv:gr-qc/0311036

[192] T. Padmanabhan, arXiv:gr-qc/0202080

[193] M. Karlovini, and R. von Unge, Phys. Rev. D **72**, (2005) 104013.

[194] A.N. Aliev, Phys. Rev. D **77**, (2008) 044038.

[195] K. Lee, V.P. Nair, and E.J. Weinberg, Phys. Rev. Lett. **68**, (1992) 1100.

[196] Z. Zhao, Acta Phys. Sin. **30**, (1981) 1508.

[197] P.A.M. Dirac, Phys. Rev. **74**, (1948) 817.

[198] S. Cherkis, and A. Hashimoto, J. High Energy Phys. **11**, (2002) 036.

[199] R. Clarkson, A.M. Ghezelbash, and R.B. Mann, J. High Energy Phys. **04**, (2004) 063.

[200] R. Clarkson, A.M. Ghezelbash, and R.B. Mann, J. High Energy Phys. **08**, (2004) 025.

[201] D. Chen, and X. Zu, Acta Phys. Pol. B **39**, (2008) 1329.

[202] S. Sannan, Gen. Relativ. Gravit. **20**, (1988) 239.

[203] Z. Zhao, Y.X. Gui, and L. Liu, J. Beijing Norm. Univ. (Nat. Sie. Ed.) **19**, (1983) 73. (in Chinese)

[204] Z. Zhao, J.Y. Zhu, and W.B. Liu, Chin. Phys. Lett. **16**, (1999) 698.

[205] J. Mäkelä, and P.Repo, Phys. Rev. D **57**, (1998) 4899.

[206] J. Mäkelä, P.Repo, M. Luomajoki, and J. Piilonen, Phys. Rev. D **64**, (2001) 024018.

[207] G.H. Tian, and Z. Zhao, Nucl. Phys. B **636**, (2002) 418.

[208] J.W. York, Phys. Rev. D **31**, (1985) 775.

[209] C.O. Lousto, and N. Sanchez, Phys. Lett. B **212**, (1988) 411.

[210] D.J. Qi, Int. J. Theor. Phys. **52**, (2013) 1455.

[211] G. 't Hooft, Nucl. Phys. B **79**, (1974) 276.

[212] J.S. Schwinger, Phys. Rev. **144**, (1966) 1087.

[213] A.N. Aliev, Phys. Rev. D **75**, (2007) 084041.

[214] M.A. Rahman, and M.I. Hossain, Astrophys. Space Sci. **345**, (2013) 325.

[215] G.M. Zhao, and P.Z. Li, Sci. China, Phys. Mech. Astron. **55**, (2012) 568.

[216] R. Zhao, L.C. Zhang, and H.F. Li, Gen. Relativ. Gravit. **42**, (2010) 975.

[217] S. Zhou, and W.B. Liu, Int. J. Theor. Phys. **47**, (2008) 617.

[218] S.W. Hawking, Phys. Rev. D **72**, (2005) 084013.

[219] J. Preskill, arXiv:hep-th/9209058 (1992).

[220] S.W. Hawking, 17th International Conference on General Relativity and Gravitation, Dublin, 2004, World Scientific, Singapore Florides, P., Nolan, B., Ottewil, A. (eds.) (2005).

[221] C.G. Callan, and J.M. Maldacena, Nucl. Phys. B **472**, (1996) 591. [arXiv:hep-th/9602043]

[222] L.D. Landau, and E.M. Lifshitz, *Quantum Mechanics* (Nonrelativistic Theory), Course of Theoretical Physics, vol. **2**. Pergamon, New York (1975).

[223] H.L. Liu, Z.X. Liu, J.S. Hou, and S.Z. Yang, Int. J. Theor. Phys. **47**, (2008) 2960.

[224] U.A. Gillani, M. Rehman, and K. Saifullah, JCAP **06**, (2011) 016.

[225] H. Ding, and W.B. Liu, Front. Phys. **6**, (2011) 106.

[226] U.A. Gillani, and K. Saifullah, Phys. Lett. B **699**, (2011) 15.

[227] T.I. Singh, I.A. Meitei, and K.Y. Singh, Astrophys. Space Sci. **345**, (2013) 177.

[228] K. Jan, and H. Gohar, Astrophys. Space Sci. **350**, (2014) 279.

[229] L. Vanzo, G. Acquaviva, and R.D. Criscienzo, Class. Quant. Grav. **28**, (2011) 183001.

[230] J. Zhang, Y. Hu, and Z. Zhao, Mod. Phys. Lett. A **21**, (2006) 1865.

[231] S.W. Gao, Z. Peng, and Z. Zhao, J. Beijing Norm. Univ. (Nat. Sci.) **37**, (2001) 493.

[232] W.H. Zurek, Phys. Rev. Lett. **49**, (1982) 1683.

[233] D.N. Page, Phys. Rev. Lett. **50**, (1983) 1013.

[234] I.D. Lawrie, 2nd ed. A Unified Grand Tour of Theoretical Physics. IOP Publishing, Bristol (2002).

[235] M. Visser, Int. J. Mod. Phys. D **12**, (2003) 649.

[236] J.-Y.Zhang, and Z. Zhao, Phys. Lett. B **618**, (2005) 14.

[237] J.-Y.Zhang, and Z. Zhao, Mod. Phys. Lett. A **20**, (2005) 1673.

[238] J.-Y.Zhang, and Z. Zhao, Nucl. Phys. B **725**, (2005) 173.

[239] J.M. Maldacena, Adv. Theor. Math. Phys. **2**, (1998) 231.

[240] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, Phys. Lett. B **428**, (1998) 105.

[241] E. Witten, Adv. Theor. Math. Phys. **2**, (1998) 253.

[242] S.W. Hawking, and D.N. Page, Commun. Math. Phys. **87**, (1983) 577.

[243] C.W. Misner, J. Math. Phys. **4**, (1963) 924.

[244] K. Ghosh, arXiv:gr-qc/0208055

[245] G.W. Gibbons, and M.J. Perry, Phys. Rev. D **22**, (1980) 313.

[246] S. Chandrasekhar, Proc. Phys. Soc. Lond. **349**, (1976) 571.

[247] D.N. Page, "Dirac equation around a charged, rotating black hole," Phys. Rev. D **14**, (1976) 1509–1510.

[248] X.H. Ge, and Y.G. Shen, Class. Quantum Gravity **20**, (2003) 3593.

[249] Y. Sekiwa, "Thermodynamics of de Sitter black holes: Thermal cosmological constant," Phys. Rev. D **73**, (2006) 084009

[250] R. Zhao, Y. Wu, and L. Zhang, Bulg. J. Phys. **32**, (2005) 1–9.

[251] G.'t Hooft, "Dimensional reduction in quantum gravity," arXiv:gr-qc/9310026 (1993)

[252] G.'t Hooft, Nucl. Phys. Proc. Suppl. **43**, (1995) 1–11.

[253] L. Susskind, J. Math. Phys. **36**, (1995) 6377–6396.

[254] L. Susskind, L. Thorlacius, and J. Uglum, Phys. Rev. D **48**, (1993) 3743–3761.

[255] T. Jacobson, Phys. Rev. D **48**, (1993) 728741.

[256] M. Visser, Phys. Rev. Lett. **80**, (1998) 3436–3439.

[257] C. Barcelo, S. Liberati, S. Sonego, and M. Visser, Phys. Rev. Lett. **97**, (2006) 171301.

[258] R. Schutzhold, and W.G. Unruh, Phys. Rev. D **78**, (2008) 041504.

[259] C. Kiefer, Class. Quantum Grav. **18**, (2001) L151.

[260] C. Vaz, C. Kiefer, T.P. Singh, and L. Witten, Phys. Rev. D **67**, (2003) 024014.

[261] W.G. Unruh, and R. Schutzhold, Phys. Rev. D **71**, (2005) 024028.

[262] C. Kiefer, J. Muller-Hill, T.P. Singh, and C. Vaz, Phys. Rev. D **75**, (2007) 124010.

[263] Ø. Grøn and S. Hervik, Einstein's General Theory of Relativity. (Springer, New York, 2007).
[*http://gujegou.free.fr/general_relativity.pdf*]

[264] K. Umetsu, arXiv:1003.5534v1 [hep-th] (2010).

[265] The information about the changes caused by general relativity in the usual thermodynamical relations can be found in: L.D. Landau and E.M. Lifshitz, Statistical Physics (Pergamon Press Ltd., Oxford, 1980).