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An Analytical and Numerical Study on Fixed Point Theorems

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An Analytical and Numerical Study on Fixed Point Theorems

M. PHIL. THESIS

This Thesis is submitted to the Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh in partial fulfillment of the requirements for the degree of Master of Philosophy in Mathematics

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DEDICATED

To

My Late Grandfather
Md. Rawshon Ali

STATEMENT OF ORIGINALITY

I declare that the works in my Master of Philosophy thesis entitled **"An Analytical and Numerical Study on Fixed Point Theorems"** is original and accurate to the best of my knowledge. I also declare that the materials contained in my research work have not been previously published or written by any person for any kind of degree.

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CERTIFICATE

I have the pleasure to certify that the Master of Philosophy thesis entitled **"An Analytical and Numerical Study on Fixed Point Theorems"** submitted by Md. Asaduzzaman in fulfillment of the requirement for the degree of Master of Philosophy in Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh has been completed under my supervision. I believe that the research work is an original one and it has not been submitted elsewhere for any kind of degree.

I wish him a bright future and every success in life.

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CONTENTS

ABSTRACT

Fixed point theory has fascinated hundreds of researchers since 1922 with the celebrated Banach's fixed point theorem. There exists a vast literature on the topic and this is a very active field of research at present. Let X be a set and let T be a mapping from X into X. A fixed point of T is an element $x \in X$ such that $T(x) = x$. In mathematics, a fixed point theorem is a result saying that a function T will have at least one fixed point, under some conditions on T that can be stated in general terms. In other words, a fixed point theorem is a theorem that asserts that every function that satisfies some given property must have a fixed point. Fixed point theorems give the conditions under which maps (single or multivalued) have solutions. Fixed point theory is a beautiful mixture of analysis, topology, and geometry. If we have an equation whose explicit solution is not so easy to find, in that case we rewrite the equation in the form $T(x) = x$ to find its solution by applying any suitable fixed point theorem. This method can be applied not only to numerical equations but also to equations involving vectors or functions. In particular, fixed point theorems are often used to prove the existence of solutions to differential equations. Fixed point theorems also play a fundamental role in demonstrating the existence of solutions to a wide variety of problems arising in social sciences, biology, chemistry, economics, engineering, physics and mathematics. For instance, the Banach [18, 40], Brouwer [18, 49] and Kakutani [18] fixed point theorems have been among the most-used tools in economics and game theory. Over the last 50 years the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena.

There are many ways of developing the concept of fixed point theory as well as Metric fixed point theory. This thesis investigates some effective and quantitative aspects of Metric fixed point theory in the light of different fixed point iterative schemes and their convergence.

The main purpose of this thesis is to present that part of Metric fixed point theory in different fixed point iterative schemes analytically and numerically, which in recent years has shown itself to be most useful for its applications. Though this thesis is developed as branch of pure Mathematics it is presented in such a way that it has

immediate application to any branch of applied Mathematics, which requires the basic theory of fixed point as a foundation for its mathematical apparatus and it should be found useable as a source of reference of the more advanced mathematician.

For convenience, this work is divided into five individual chapters. In our chapter-1, we have presented several known definitions and some fundamental results of fixed point theories, which are used as the tools of our work. Perhaps the most well known result in the theory of fixed points is Banach's contraction mapping principle. So, it will befit to commence this thesis with a discussion of contraction mappings.

In our chapter-2, first we have identified some fixed point problems on contraction mapping and non-expansive mapping and tried to solve these problems in our own fashion. The problem 2.2.1 is an existence and uniqueness theorem for ordinary non-linear differential equations. Although, in [18] the problem 2.2.1 and the problem 2.2.3 are known as Cauchy-Lipschitz theorem and Implicit function theorem respectively, moreover here we treated these theorems as fixed point problems on contraction mapping. In this chapter, secondly we have chosen some open problems on non-expansive mapping and shrinking mapping (special type of non-expansive mapping) and tried to solve these open problems. In the end of this chapter, we gave a vast discussion on asymptotically non-expansive mappings.

In our chapter-3, we have defined different fixed point iterative schemes and gave some convergence theorems on these iterative schemes. The main feature of this chapter is to establish the convergence theorem of Kranoselskii's iterative scheme and Noor iterative scheme on arbitrary Banach space B whenever the operator $T: B \to B$ satisfies at least one of the following three Zamfirescu conditions

(z_1) $\|Tx - Ty\| \le a\|x - y\|$;

(z_2) $\|Tx - Ty\| \le b[\|x - Tx\| + \|y - Ty\|]$;

(z_3) $\|Tx - Ty\| \le c[\|x - Ty\| + \|y - Tx\|]$;

where, a, b & c are real numbers satisfying $0 \le a < 1, 0 \le b < 1, c < 1/2$. Here, we have also stated and proved some convergence theorems of Mann iterative scheme on Banach space B whereas the operator $T: B \to B$ satisfies the contractive condition

$$\| T(x) - T(y) \| \le k \max\{c \| x - y \|, [\| x - T(x) \| + \| y - T(y) \|], [\| x - T(y) \| + \| y - T(x) \|]\}$$

for all $x, y \in S$, where $k, c \geq 0, 0 \leq k < 1$.

Recently, in a series of papers [9-17], B. E. Rhoades and S.M. Soltuz, proved that Mann and Ishikawa iteration schemes are equivalent for several classes of mappings such as Lipschitzian, strongly pscudocontractive, strongly hemicontractive, strongly accretive, strongly successively pscudocontractive, strongly successively hemicontractive mapping and Krishna Kumar in [28] proved that Mann and Ishikawa schemes are equivalent for the class of uniformly pscudocontractive operators. In [51], the following open question was given: "are Krasnoselskii's iteration and Mann iteration equivalent for enough large classes of mappings?" We gave a positive answer to this question: if Krasnoselskii's iteration converges, then Mann (and the corresponding Ishikawa iteration) also converges and conversely, dealing with maps satisfying Zamfirescu condition (Z). Note that B. E. Rhoades and S.M. Soltuz have already given a positive answer in [16] for the class of pseudocontractive maps. In the fact of this above discussion, in our chapter-4, we have shown that the equivalence of Mann iterative scheme to the Ishikawa iterative scheme, Krasnoselskii's iterative scheme to Mann iterative scheme, Mann iterative scheme to Ishikawa iterative scheme to Noor iterative scheme and Mann iterative scheme to Multi step iterative schemes for the class of Zamfirescu operator, which is described over the Banach space. In the end of this chapter, we have also shown that the equivalence of the $T-$stability of Mann iterative scheme to the $T-$stability of Ishikawa iterative scheme and the $T-$stability of Mann iterative scheme to $T-$stability of Ishikawa iterative scheme to $T-$stability of Noor iterative scheme for the same situation.

Finally in our chapter-5, we have studied the rate of convergence of different fixed point iterative schemes. Here, we have studied the rate of convergence of different fixed point iterative schemes in two different contexts. First one is theoretical approchement of the rate of convergence. In this case, we have compared the rate of convergence of different fixed point iterative schemes theoretically. And the second one is Empirical approachment of the rate of convergence. In this case, we have compared the rate of convergence of different fixed point iterative schemes numerically or practically. After completing this type of study, we have suggested that the Krasnoselskii's iterative scheme converges to a fixed point faster than Mann, Ishikawa, Noor and Newton-Raphson iterative schemes for a certain type of operator.

LIST OF FIGURES AND THEIR SIGNIFICANCES

CHAPTER-1

MATHEMATICAL PRELIMINARIES AND SOME FUNDAMENTAL RESULTS

CHAPTER-1

MATHEMATICAL PRELIMINARIES AND SOME FUNDAMENTAL RESULTS

1.1 Introduction

The main aim of this Chapter is to state some basic definitions with examples and some fundamental results of fixed point theory to keep this thesis in sequence and for the convenience of references. For the brevity, most of the theorems are stated without proof. However, some results are given with short proof with adequate references. Although, all of these fixed point theorems are well established by different mathematicians, moreover, we again describe these fixed point theorems for convenience of readers and acquiring complete knowledge on the fixed point theory. The notations and terminologies used in the dissertation are also fixed in this chapter. In the study of Functional analysis and Topology, Metric spaces play a very important role, and Metric spaces have gained considerable importance after the famous Banach's fixed point theorem (Contraction mapping principle).

1.2 Some basic definitions of Mathematical analysis

Definition 1.2.1, (see [56]). A *vector space or linear space* is a set X together with two operations, addition and scalar multiplication such that for all $x, y \in X$ and all $\alpha \in \mathbf{R}$ (set of real numbers) both $x+y$ and αx are in X, and for all $x, y, z \in X$ and all $\alpha, \beta \in \mathbf{R}$ the following properties are satisfied:

(i) $x+y=y+x$;

(ii) $(x+y)+z=x+(y+z)$;

(iii) $(\alpha+\beta)x=\alpha x+\beta x$;

(iv) $\alpha(x+y)=\alpha x+\alpha y$;

(v) $\alpha(\beta x)=(\alpha\beta)x$;

(vi) there exists a $0 \in X$ such that for all $x' \in X$, $0+x'=x'$;

(vii) there exists a $-x \in X$ such that for all $x \in X$, $x+(-x)=0$;

(viii) there exists a $1 \in \mathrm{R}$ such that for all $x \in X$, $1.x=x$.

Definition 1.2.2, (see [55]). Let X be a non empty set. The mapping $d: X \times X \rightarrow \mathbf{R}$ is said to be a ***metric*** on X if it satisfies the following conditions:

(*a*) $d(x,y) \geq 0, \ \forall x, y \in X$;

(*b*) $d(x,y) = 0$ *iff* $x = y, \ \forall \ x, y \in X$;

(*c*) $d(x,y) = d(y,x), \ \forall \ x, y \in X$;

(*d*) $d(x,y) \leq d(x,z) + d(z,y), \ \forall \ x, y, z \in X$.

Here the set X together with the metric d i.e., the order pair (X,d) is called a ***metric space.***

Example. If $X = \mathbf{R}$ (Set of real numbers) and the metric define by

$$d(x,y) = | x - y |, \ \forall x, y \in X.$$

Then (X,d) form a metric space.

Definition 1.2.3, (see [56]). Let X be a vector space. A function $\|\cdot\| : X \rightarrow [0,\infty)$ is called a ***norm*** if and only if for all $x, y \in X$ and all $\alpha \in \mathbf{R}$, the following rules hold:

(a) $\|x + y\| \leq \|x\| + \|y\| \ \ \forall \ x, y \in X$;

(b) $\|\alpha x\| = |\alpha| \|x\|$ if $x \in X$ and $\alpha \in \mathbf{R}$;

(c) $\|x\| = 0 \Rightarrow x = 0$.

The pair $(X, \|\cdot\|)$ is then called a ***normed linear space***. A normed linear space $(X, \|\cdot\|)$ defines a metric space (X,d) with d defined by $d(x, y) = \|x - y\|$.

Definition 1.2.4, (see [55]). For each positive integer k, let $\mathbf{R}^k$ be the set of all ordered $k-$tuples $x = (x_1, x_2, \ldots\ldots\ldots\ldots, x_k)$, where $x_1, x_2, \ldots\ldots\ldots\ldots, x_k$ are real numbers, called the coordinates of x. The elements of $\mathbf{R}^k$ are called points, or vector, especially when $k > 1$. If $y = (y_1, y_2, \ldots\ldots\ldots\ldots, y_k)$ and if α is a real number, put

$$x + y = (x_1 + y_1,\ x_2 + y_2, \ldots\ldots\ldots\ldots, x_k + y_k)$$

and $$\alpha x = (\alpha x_1, \alpha x_2, \ldots\ldots\ldots\ldots, \alpha x_k)$$

so that $x + y \in \mathbf{R}^k$ and $\alpha x \in \mathbf{R}^k$. This defines addition of vectors, as well as multiplication of a vector by a real number. These two operations satisfy the commutative, associative, and distributive laws and make R^k into a vector space over the real field. The zero element of R^k is the point **0**, all of whose coordinates are 0. We also define the so-called "inner product" of x and y by $< x, y >= \sum_{i=1}^{k} x_i y_i$ and the norm of x by $\|x\| = (< x, x >)^{1/2} = \left(\sum_{i=1}^{k} x_i^{\,2} \right)^{1/2}$. The vector space $\mathbb{R}^k$ with the above inner product and norm is called ***Euclidean*** k ***– space.***

Definition 1.2.5, (see [56]). A vector space X is called an ***inner product space or unitary space*** if to each ordered pair of vectors x *and* $y \in X$ there is associated a number $< x, y >$, the so-called ***inner product or scalar product*** of x *and* y, such that the following rules hold:

(a) $< y, x >= \overline{< x, y >}$ (The bar denotes complex conjugation.);

(b) $< x + y, z >=< x, z > + < y, z > \ \forall \ x, y, z \in X$;

(c) $< \alpha x, y >= \alpha < x, y > \forall x, y \in X$ *and* α is a scalar;

(d) $< x, x > \geq 0 \ \ \forall x \in X$;

(e) $< x, x >= 0$ *iff* $x = 0$.

Definition 1.2.6, (see, [55]). A sequence $\{x_n\}$ in a metric space X is said to ***converge*** if there is a point $x \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(x_n, x) < \varepsilon$.

In this case we also say that the sequence $\{x_n\}$ is converges to $x \in X$, or that $x \in X$ is the limit of $\{x_n\}$, and we write $x_n \to x$, or $\lim_{n \to \infty} x_n = x$.

If the sequence $\{x_n\}$ does not converge, then it is said to ***diverge.***

A sequence $\{x_n\}$ in a metric space X is said to be a ***Cauchy sequence*** if for every $\varepsilon > 0$ there is an integer N such that $d(x_n, x_m) < \varepsilon$ if n *and* $m \geq N$.

A metric space X is said to be a ***complete space*** if every Cauchy sequence in X is convergent.

Definition 1.2.7, (see [56]). Let (H, d) be a metric space. If this metric space is complete, i.e., if every Cauchy sequence converges in H, where the metric d is defined by the inner product of the space, then the space (H, d) is called a ***Hilbert Space.***

Example. For any fixed n, the set C^n of all n-tuples $x = (x_1, x_2, \ldots\ldots\ldots, x_n)$, where $x_1, x_2, \ldots\ldots\ldots, x_n$ are complex numbers, is a Hilbert space if addition and scalar multiplication are defined componentwise, as usual, and if $< x, y >= \sum_{i=1}^{n} x_i y_i$, where $y = (y_1, y_2, ..., y_n)$.

Definition 1.2.8, (see [56]). A normed linear space B, which is complete as a metric space and the metric is defined by its norm, is called a ***Banach Space.***

Examples. Every Hilbert space, R^k Euclidean space and the set of all complex numbers $\mathbb{C}$ with the norm $\|x\| = |x|$ are Banach Spaces.

A Banach space B is said to be ***uniformly convex*** if, $\|x_n\| \le 1$, $\|y_n\| \le 1$ and $\|x_n + y_n\| \to 2$ as $n \to \infty$ imply $\|x_n - y_n\| \to 0$, $\forall\ x_n, y_n \in B$.

Definition 1.2.9, (see [44]). Let X be a non-empty set. A class τ of subsets of X is a ***topology*** on X iff τ satisfies the following axioms:

(a) X & ϕ (null set) belong to τ.

(b) The union of any number of sets in τ belongs to τ.

(c) The intersection of any two sets in τ belongs to τ.

Here the set X together with τ i.e., the pair (X, τ) is called a ***topological space.***

Example. If $X = \{a,b,c,d,e\}$ and $\tau = \{X, \phi, \{a\}, \{c, d\}, \{a,c,d\}, \{b,c,d,\}\}$, then τ is a topology on X and the pair (X, τ) is a topological space.

A set in a topological space is called a ***relatively compact*** if its closure is complete.

Definition 1.2.10, (see [44]). A function $H : X \to Y$ between two topological spaces X and Y is called a ***homeomorphism*** if it has the following properties: (i) H is a bijection, (ii) H is continuous and (iii) the inverse function H^{-1} is continuous. If such a function exists we say X and Y are ***homeomorphic***. The homeomorphisms form an equivalence relation on the class of all topological spaces.

Example. Let $X = (-1, 1)$ and $Y = \mathbb{R}$ (set of real numbers). Then the function $H : X \to Y$ defined by $H(x) = \tan(\frac{\pi x}{2})$ is one-one, onto and continuous i.e. H is bijection and continuous. Furthermore, the inverse function H^{-1} is also continuous. Hence the set of real numbers $\mathbb{R}$ and the open interval $(-1, 1)$ are homeomorphic.

Definition 1.2.11, (see [18]). The ***Standard n-simplex*** S^n is the closed convex hull of $v^0, v^1, ..., v^n$ in $\mathbb{R}^{n+1}$ i.e., $S^n = \{x \in \mathbb{R}^{n+1} : v^T x = 1\}$. For $i \in \mathbb{N}$ let S_i^n denote the face of S^n

opposite v^i, i.e.,$S_i^n = \{x \in S^n : x_i = 0\}$. The boundary of S^n is $\partial S^n = \bigcup_{i \in N_0} S_i^n$.

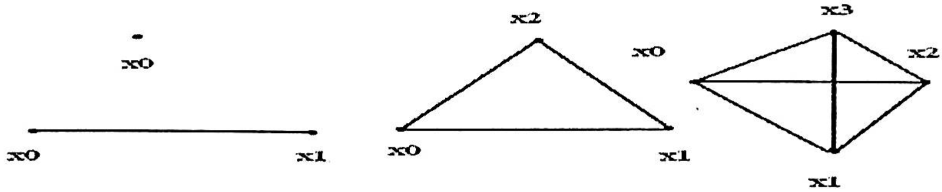

Figure-1.1: Example of simplexes for$n = 1, 2, \& 3$.

n	n-Simplex
0	Point
1	Line segment
2	Triangle
3	Tetrahedron

Definition 1.2.12, (see [18]). A ***triangulation*** G of n-Simplex S^n is a finite collection of closed n-Simplexes, together with all their closed faces, that form a partition of S^n, i.e. S^n is their disjoint union. This is equivalent to the conditions:

(i) The closed n-Simplexes cover S^n;

and (ii) If two closed n-Simplexes meet; their intersection is a common face.

Let G be a triangulation of S^n with each vertex of G labelled with an integer in $\mathbf{N}_0$ such that no vertex in S_i^n is labeled i. Then G is said to have an ***admissible labeling***. Here the simplex in G whose vertices carry all the labels in $\mathbf{N}_0$ is said to be a ***completely labelled simplex.***

Theorem 1.2.13 (Knaster-Kuratowski-Mazurkiewicz(K-K-M)lemma),(see [18])

Statement. *Let* C_i, $i \in N_0$, *be a family of closed subsets of simplex* S^n *satisfying the following conditions:*

(i) $S^n = \bigcap_{i \in N_0} C_i$; *and*

(ii) *If* $I \subseteq N_0$ *is non-empty and* $J = N_0 - I$, *then* $\bigcap_{i \in I} S^n \subseteq \bigcup_{i \in N_0} C_i$.

Then $\bigcap_{i \in N_0} C_i$ *is non-empty.* ■

Theorem 1.2.14 (Sperner's lemma), (see [18])

Statement. *Let G be a triangulation of S^n with an admissible labeling. Then there is a completely labeled simplex in G.* ■

Note. Theorem 1.2.13 and Theorem 1.2.14 are used as the helping tools to realize the Theorem 1.4.2 (Brouwer fixed point theorem).

Definition 1.2.15, (see [18]). An additive abelian group A together with a homomorphism $d: A \to A$ is called a ***differential group*** if $d^2 = 0$ *i.e.,* $A \xrightarrow{d} A \xrightarrow{d} A$.

If (A, d) is a differential group *i.e.,* $d^2 = 0$. Then it is clear that $\text{Im}(d) \subseteq Ker(d)$. Here d is called the differential or boundary operator of A. The elements of A are called chains. The elements of $\text{Im}(d)$ are called boundaries and the elements of $Ker(d)$ are called cycles.

The Homology group of the differential group (A, d) is defined to be the factor group of cycles modulo boundaries i.e., $\dfrac{Ker(d)}{\text{Im}(d)}$ is called the ***homology group*** of the differential group (A, d) and is denoted by $H(A)$ *i.e.* $H(A) = \dfrac{Ker(d)}{\text{Im}(d)}$. The elements of $H(A)$ are the cosets $z + \text{Im}(d)$ where $z \in Ker(d)$. Then $z + \text{Im}(d) \in H(d)$ is called the ***homology classes*** and we write $cls(z) = z + \text{Im}(d)$.

The nth homology group with integral coefficients of a complex $\mathcal{X}$ in a Euclidean space is denoted by $H_n(\mathcal{X})$. If S^n is a n-sphere, then $H_n(S^n) = \mathbb{Z}$ (group of integers).

1.3 Fundamental concept on fixed point

Definition 1.3.1, (see [18]). Consider a mapping T of a set M into itself (or into some set containing M). Then the solution of the equation $T(x) = x$ is called a ***fixed point*** (sometimes shortened to ***fixpoint***, also known as an ***invariant point***) of the mapping T in M for all $x \in M$. Briefly, the point $x \in M$ is called a ***fixed point*** of the mapping $T: M \to M$ iff $T(x) = x$. Geometrically, the intersecting point of the curve $y = T(x)$ and the straight line $y = x$ is a fixed point. We write $F(T)$ for the set of fixed point of T, where $F(T) = \{x : T(x) = x\}$.

Example. Let the mapping T be defined on the real numbers by $T(x) = x^2 - 3x + 4$, then $x = 2$ is a fixed point of T, because $T(2) = 2$. Consider $y = T(x)$ and we obtain the following figure.

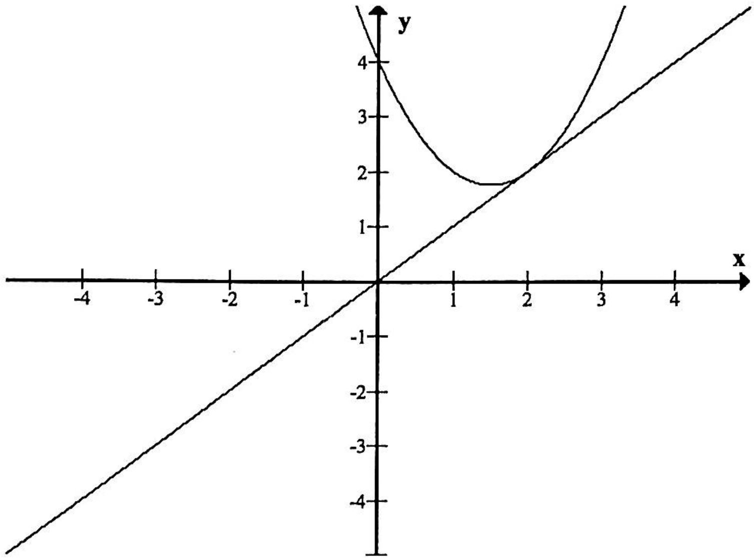

Figure-1.2: Fixed point of $T(x) = x^2 - 3x + 4$.

Not all functions have fixed points: for example, if T is a function defined on the real numbers as $T(x) = x + 1$, then it has no fixed points, since x is never equal to $x+1$ for any real number. In graphical terms, a fixed point means the point $(x, T(x))$ is on the line $y = x$, or in other words the graph of T has a point in common with that line. The example $T(x) = x + 1$, is a case where the graph and the line are a pair of parallel lines. Points which come back to the same value after a finite number of iterations of the function are known as periodic points; a fixed point is a periodic point with period equal to one.

Application of fixed point. In many fields, equilibrium or stability are fundamental concepts that can be described in terms of fixed points. For example, in economics, Nash equilibrium of a game is a fixed point of the game's best response correspondence. In compilers, fixed point computations are used for whole program analysis, which are often required to do code optimization. The vector of page rank values of all web pages is the fixed point of a linear transformation derived from the World Wide Web's link structure. Logician Saul Kripke makes use of fixed points in his influential theory of truth. He shows how one can generate a partially defined truth predicate (one which remains undefined for problematic sentences like "This sentence is not true"), by recursively defining "truth" starting from the segment of a language which contains no occurrences of the word, and continuing until the process ceases to yield any newly well-defined sentences. (This will take a denumerable infinity of steps.) That is, for a language L, let L-prime is the language generated by adding to L, for each sentence S in L, the sentence "*S* is true." A fixed point is reached when L-prime is L; at this point sentences like "This sentence is not true" remain undefined, so, according to Kripke, the theory is suitable for a natural language which contains its own truth predicate.

Definition 1.3.2, (see [18]). An ***attractive fixed point*** of a function T is a fixed point x_0 of T such that for any value of x in the domain that is close enough to x_0, the iterated function sequence $x, T(x), T(T(x)), T(T(T(x))),.....,$ converges to x_0. How close is "close enough" is sometimes a subtle question. The natural cosine function ("natural" means in radians, not degrees or other units) has exactly one fixed point, which is attractive. In this case, "close enough" is not a stringent criterion at all to demonstrate this, start with any real number and repeatedly press the cos key on a calculator. It quickly converges to about 0.739085133, which is a fixed point. That is where the graph of the cosine function intersects the line $y = x$.

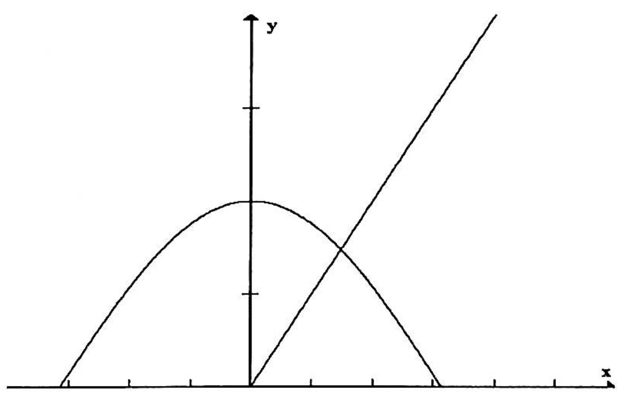

Figure-1.3: The fixed point iteration $x_{n+1} = \cos x_n$ with initial value $x_0 = -1$.

Not all fixed points are attractive: for example, $x = 0$ is a fixed point of the function $T(x) = 2x$, but iteration of this function for any value other than zero rapidly diverges. However, if the function T is continuously differentiable in an open neighbourhood of a fixed point x_0, and $|T'(x_0)| < 1$, attraction is guaranteed.

Attractive fixed points are a special case of a wider mathematical concept of attractors. An attractive fixed point is said to be a ***stable fixed point*** if it is also Lyapunov stable.

A fixed point is said to be a ***neutrally stable fixed point*** if it is Lyapunov stable but not attracting. The center of a linear homogeneous differential equation of the second order is an example of a neutrally stable fixed point.

Theorem 1.3.3, (see [18], Theorem 1.1.3). Let M be a metric space. Suppose that T is a continuous mapping of M into a compact subset of M and that, for each $\varepsilon > 0$, there exists $x(\varepsilon)$ such that $d(Tx(\varepsilon), \ x(\varepsilon)) < \varepsilon$. (1)

Then T has a fixed point.

Definition 1.3.4, (see [18]). The points $x(\varepsilon)$ satisfying (1) of Theorem 1.3.3 is called ε***-fixed points*** for T.

1.4 Theorems guaranteeing fixed points

There are numerous theorems in different parts of mathematics that guarantee that functions, if they satisfy certain conditions, have at least one fixed point. These are amongst the most basic qualitative results available: such fixed-point theorems that apply in generality provide valuable insights. Here first we discuss about the convergence of fixed point and then we give some fixed point theorems.

1.4.1 Convergence of fixed point, (see [18]). A formal definition of convergence can be stated as follows. Suppose p_n as n goes from 0 to ∞ is a sequence that converges to p, with $p_n \neq 0 \quad \forall\, n$. If positive constants λ and α exist with $\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$, then p_n as n goes from 0 to ∞ converges to p of order α, with asymptotic error constant λ.

There is a nice checklist for checking the convergence of a fixed point p for a function $T(x) = x$.

1) First check that, $T(p) = p$

2) Check for linear convergence. Start by finding $|T'(p)|$. If

$\lvert T'(p)\rvert \in (0, 1]$	then we have linear convergence
$\lvert T'(p)\rvert > 1$	series diverges
$\lvert T'(p)\rvert = 0$	then we have at least linear convergence and maybe something better, we should check for quadratic

3) If we find that we have something better than linear we should check for quadratic convergence. Start by finding $|T''(p)|$. If

$\|T''(p)\| \neq 0$	then we have quadratic convergence provided that $T''(p)$ is continuous
$\|T'(p)\| = 0$	then we have something even better than quadratic convergence
$\|T'(p)\|$ does not exist	then we have convergence that is better than linear but still not quadratic

Theorem 1.4.2 (Brouwer fixed point theorem), (see [18, 49])

The Brouwer fixed point theorem is one of the early achievements of algebraic topology and is the basic of the more general fixed point theorems that are important in Functional analysis as well as numerical analysis. This theorem is named after Dutch Mathematician L. E. J. Brouwer (1910).

Statement. *A continuous mapping of a convex, closed set into itself necessarily has a fixed point.*

Examples.

1. A continuous mapping that maps the set [0, 1] into itself has a fixed point.
2. A continuous mapping that maps a disk into itself has a fixed point.
3. A continuous mapping that maps a spherical ball into itself has a fixed point.

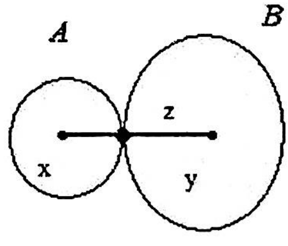

Figure-1.4: Geometrical interpretation of Brouwer fixed point theorem.

Application. The Brouwer fixed point theorem forms the starting point of a number of more general fixed point theorems, such as Kakutani's fixed point theorem, Lefschetz fixed point theorem. These two theorems we will describe later. We also see that Sperner's lemma $\Rightarrow$ K-K-M lemma $\Rightarrow$ Brouwer fixed point theorem.

Now, we describe the application of Brouwer fixed point theorem.

This theorem is used to

(i) Establish the different types of fixed point theorems and lemmas;

(ii) Find the solution of the differential equations;

(iii) Establish the Jordan Curve Theorem;

(iv) Establish the Existence of equilibrium points, which is very important in theoretical Economics;

(v) Establish the proof of the existence of a winner in the game of Hex;

(vi) Establish the Cake Cutting algorithm;

(vii) Establish the different theory of Numerical analysis. ■

1.5 Contraction Mappings

Definition 1.5.1, (see [18, 55, 59]). Let M be a metric space. A mapping $T: M \to M$ is called a ***contraction mapping*** if $\exists$ a positive real number $0 \le \lambda < 1$ such that

$$d(Tx, Ty) \le \lambda d(x, y), \quad \forall\ x,\ y \in M$$

where $d(x, y)$ denotes the metric between x and y and $Tx = T(x)$.

If M is a normed space, then T is contraction if

$$\|Tx - Ty\| \le \lambda \|x - y\|.$$

If T is linear, this reduces to $\|Tx\| \le \lambda \|x\|$ $\forall x \in M$. Thus, a linear operator $T: M \to M$ is contraction if its norm satisfies

$$\|T\| = \sup_{x \ne 0} \frac{\|Tx\|}{\|x\|} = \sup_{x=1} \|Tx\|.$$

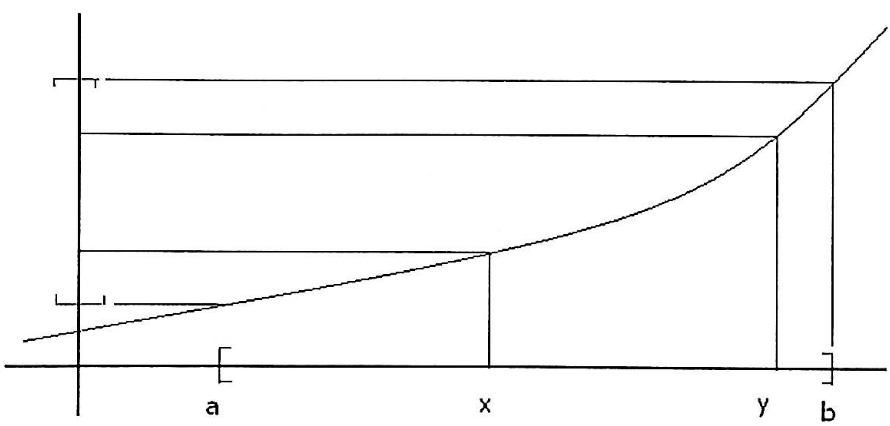

Figure-1.5: Contraction Mapping.

Example. Consider the cosine function on [0, 1]. Graphs of $y = \cos x$ & $y = x$ show there is one intersecting point in [0, 1], which means cosine function has a fixed point in [0, 1]. We will show this point can be obtained through iteration.

Since cosine is a decreasing function. Therefore, for $0 \le x \le 1$ we have $\cos 1 \le \cos x \le 1$, with $\cos x \approx .54$ i.e., $\cos : [0, 1] \to [0, 1]$. For $x, y \in [0, 1]$ the Mean-value theorem tells us

$$\cos x - \cos y = \cos'(t)(x-y) = (-\sin t)(x-y).$$

for some t between x and y. Thus $|\cos x - \cos y| = |\sin t||x-y|$.

Since $0 \le t \le 1$ and sine is increasing on this interval (it increases from 0 up to $\frac{\pi}{2} \approx 1.57 > 1$) we have $|\sin t| = \sin t \le \sin 1 \approx .84147$.

Therefore, $|\cos x - \cos y| \le .84147|x-y|$

So, cosine is a contraction mapping on [0, 1], which is complete. Hence, there is a unique $a \in [0, 1]$ such that $\cos a = a$.

A beautiful application of contraction mappings to the construction of fractals (interpreted as fixed points in a metric space whose points are compact subsets of the plane).

Theorem 1.5.2 (Banach fixed point theorem), (see [18, 40])

Banach fixed point theorem is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also its significance lies in its vast applicability in a number of branches of mathematics. This theorem was first stated by Polish Mathematician Stefan Banach in 1922. He established this Theorem as a part of his doctoral thesis. It is also known as Contraction mapping theorem. Here we state and prove this celebrated theorem.

Statement. *Let (X,d) be non-empty complete metric space and $T: X \to X$ be a contraction mapping on X, i.e. there is a non negative real number $0 \le q < 1$ such that*

$$d(Tx,Ty) \le q d(x,y), \qquad \forall x, y \in X$$

Then the mapping T admits one and only one (unique) fixed point in X. For any $x_0 \in X$ the sequence of iterates $x_0, T(x_0), T(T(x_0)), \ldots\ldots\ldots\ldots$ converges to the fixed point of T.

Proof. Recalling the notation $T: X \to X$ is a contraction mapping with contraction constant q. We want to show that T has a unique fixed point, which can be obtained as a limit through iteration of T.

To show that T has at most one fixed point in X, let a and a' be two different fixed points of T, i.e. $Ta = a$ and $Ta' = a'$. Then

$$d(a,a') = d(Ta,Ta') \leq qd(a,a'). \tag{1}$$

If $a \neq a'$ then $d(a,a') > 0$, so we can divide (1) by $d(a,a')$ and we get $1 \leq q$, which is false by our assumption. Thus $a = a'$. Hence the fixed point of a contraction mapping is unique.

Next we want to show, for any $x_0 \in X$, that the recursively defined iterates $x_n = Tx_{n-1}$ for $n \geq 1$ converge to a fixed point of T. The key idea is that iterating the function several times contracts distances by an increasing power of the contraction constant. This bring points together through iteration at a geometric rate, and that will be enough to force convergence of this iterates because X is complete.

For any $n \geq 1$, $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq qd(x_{n-1}, x_n)$.

Therefore,

$$d(x_n, x_{n+1}) \leq qd(x_{n-1}, x_n) \leq q^2 d(x_{n-2}, x_{n-1}) \leq q^3 d(x_{n-3}, x_{n-2}) \leq .... \leq q^n d(x_0, x_1).$$

$$i.e.\ d(x_n, x_{n+1}) \leq q^n d(x_0, x_1).$$

Using this expression on the right as an upper bound on $d(x_n, x_{n+1})$ shows the x_n's are getting consecutively close at a geometric rate. This implies the x_n's are Cauchy. Now for any $m > n$, by triangle inequality we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq q^n d(x_0, x_1) + q^{n+1} d(x_0, x_1) + ... + q^{m-1} d(x_0, x_1) \\ &= (q^n + q^{n+1} + ... + q^{m-1}) d(x_0, x_1) \\ &\leq (q^n + q^{n+1} + ... + q^{m-1} + ...) d(x_0, x_1) \\ &\leq \frac{q^n}{1-q} d(x_0, x_1). \end{aligned}$$

This is true when $m = n$ too. So for given $\varepsilon > 0$, pick $N \geq 1$ such that $(q^N/(1-q)) d(x_0, x_1) < \varepsilon$. Then for any $m \geq n \geq N$,

$$d(x_n, x_m) \leq \frac{q^n}{1-q} d(x_0, x_1) \leq \frac{q^N}{1-q} d(x_0, x_1) < \varepsilon.$$

This proves $\{x_n\}$ is a Cauchy sequence. Since X is complete, then the sequence $\{x_n\}$ converge in X. Set $x = \lim_{n \to \infty} x_n$ in X.

Now the function $T: X \to X$ is continuous, because it is contraction by our assumption, so from $x_n \to x$ we get $Tx_n \to Tx$. But $Tx_n = x_{n+1}$, so $Tx_n \to x$ as $n \to \infty$. Then Tx and x both are limits of $\{x_n\}_{n\geq 0}$. From the uniqueness of limits, $Tx = x$. This concludes the proof of our contraction mapping theorem or Banach fixed point theorem.

Application. A standard application of Banach fixed point theorem is the proof of the Picard-Lindelof theorem about the existence and uniqueness of solution to certain ordinary differential equations. The sought for solution of the differential equation is expressed as a fixed point of suitable integral operator which transforms continuous functions into continuous functions. The Banach fixed point theorem is then used to show that this integral operator has a unique fixed point. This theorem is also used to prove the implicit function theorem inverse function theorem. ■

Theorem 1.5.3 (Krasnoselskii's fixed point theorem), (see [18, 30])

Two main results of fixed point theory are Schauder's fixed point theorem and the contraction mapping principle (Banach's fixed point theorem). In 1932 Russian Mathematician Mark Alexandrovich Krasnoselskii combined them and forms a new theorem. This theorem is known as Krasnoselskii's fixed point theorem.

Statement. *Let K be a closed convex non-empty subset of a Banach space B. Suppose that T & S be maps of K into B and that*

(i) $Tx + Sy \in K, \ \forall\, x, y \in K$;

(ii) *S is compact and continuous;*

(iii) *T is contraction mapping.*

Then there exists a y in K such that $Ty + Sy = y$.

Application. Krasnoselskii's fixed point theorem is used to obtain existence results for multiple positive solutions of various types of boundary value problems. It is also applied to study the existence of periodic solutions of periodic systems of ordinary differential equations. ■

Theorem 1.5.4(Kannan's fixed point theorem), (see [18, 37])

Statement. Let $T: X \to X$ be a mapping where, (X,d) is a complete metric space and T satisfies the condition

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$$

where $0 < \beta < 1/2$ and $x, y \in X$. Then T has a unique fixed point in X. ■

1.6 Fixed point property

A mathematical object A has the fixed point property if every suitably well-behaved mapping from A to itself has a fixed point. It is a special case of the fixed morphism property. The term is most commonly used to describe topological spaces on which every continuous mapping has a fixed point. But another use is in order theory, where a partially ordered set P is said to have the fixed point property if every increasing functions on P has a fixed point.

Definition 1.6.1, (see [18, 59]). Let A be an object in the concrete category $\mathcal{C}$. Then A has the ***fixed point property*** if any morphism (i.e., every function) $f : A \to A$ has a fixed point. The most common usage is when $\mathcal{C}$ = Topology is the category of topological spaces. Then a topological space X has the fixed point property if every continuous mapping of the space X into itself has a fixed point. Of course, if a topological space has the fixed point property and any other topological space homeomorphic to the first will also possess the fixed point property. In other words, the fixed point property is a topological property.

Examples.

1. A closed interval [a, b] has the fixed point property.
2. A space with only one point has the fixed point property.
3. A retract of a space with fixed point property has the fixed point property.
4. The set of extended real numbers have the fixed point property, as they are homeomorphic to the closed interval [0, 1].
5. The closed unit ball with the subspace topology has the fixed point property.
6. Every simply-connected plane continuum has the fixed point property.
7. The set of real numbers $\mathbb{R}$ does not have the fixed point property.
8. An open interval (a, b) does not have the fixed point property.

Definition 1.6.2, (see [18]). Let S is a set which is neither compact nor contractible. Then the set S is called ***a set with the lack of fixed point property***.

Example. If a subset of R^k (Euclidean space) is not compact then we can usually produce a fixed point free mapping by moving all points towards a missing limit point, or towards infinity. Thus we see that sets such as an open interval or open ball, or half-line lack of fixed point property.

Note. The definition 1.6.2 is not always true, which we will realize by Theorem 1.8.3 and Theorem 1.8.4.

Theorem 1.6.3 (Schauder fixed point theorem), (see [18, 26, 39])

The Schauder fixed point theorem is an extension of the Brouwer fixed point theorem to topological vector spaces, which may be of infinite dimension. It asserts that if K is a compact, convex subset of a topological vector space V and T is a continuous mapping of K into itself, then T has a fixed point. It was conjectured and proved for special cases, such as Banach spaces, by Juliusz Schauder in 1930. The full result was proved by Robert Cauty in 2001[39].

Statement. *Any compact convex non-empty subset Y of normed space X has a fixed point property.*

Application. Perhaps the most important principle of Analysis is the Schauder fixed point theorem, a well known result expounded in the texts of Functional analysis with an immense amount of applications. The Schauder fixed point theorem is especially useful to establish the existence of solutions to differential and integral equations. ■

Theorem 1.6.4(Schauder-Tychonoff fixed point theorem), (see [18, 23])

In 1935, the Soviet mathematician H. Tychonoff gave a generalization of the Schauder theorem for locally convex vector spaces [32]. This result is usually termed the Schauder-Tychonoff theorem. Now, we state this theorem:

Statement. *Any compact convex non-empty subset of a locally convex space has a fixed point property.*

Application. The Schauder-Tychonoff theorem is an important tool in the investigation of the solutions of ordinary differential equations (ODE) on the interval $[0, \infty)$. ■

Theorem 1.6.5 (Rothe's fixed point theorem), (see [18])

This theorem is given by Mathematician Rothe's in 1937. Now, we state this theorem below:

Statement. *Let B be normed space, D the closed unit ball in B and ∂D (boundary of D) the unit sphere in B. Let T be a continuous compact mapping of D into B such that $T(\partial D) \subset D$. Then T has a fixed point.*

Application. This theorem is used to develop the concept of Schauder fixed theorem. ■

1.7 Retraction Mappings

Definition 1.7.1, (see [18]). Let X and Y be two sets. We say that X is ***retract*** of Y if $X \subset Y$ and there exists a continuous mapping r of Y in X such that $r = I$ on X. Here r is called a ***retraction mapping***. A retract X of a set Y with the fixed point property also has the fixed point property. This is because if $r : Y \to X$ is a retraction and $f : X \to X$ is any continuous function, then the composition $i \circ f \circ r : Y \to Y$ (where $i : X \to Y$ is inclusion) has a fixed point. That is, there is $x \in X$ such that $f \circ r(x) = x$.. Since $x \in X$ we have that $r(x) = x$ and therefore $f(x) = x$.

Examples.

1. A closed convex non-empty subset of a Hilbert space is a retract of any larger subset.
2. A closed convex non-empty subset of a Banach space is also a retract of any larger subset.
3. For $n \geq 1$, S^{n-1} ($(n-1)$-sphere) is not a retract of B^n (closed n-ball).

1.8 Contractible Spaces

Definition 1.8.1, (see [18]). A topological space X is called ***contractible*** to a point x_0 in X if $\exists$ a continuous function $f(x, t)$ on $X \times [0, 1]$ to X such that $f(x, 0) = x$ and $f(x, 1) = x_0$. If X is contractible, then the nth homology group $H_n(X) = \{e\}$ (trivial group).

Examples.

1. The closed unit n-ball B^n is contractible.
2. The Whitehead manifold is contractible.
3. Any star domain of an Euclidean space is contractible.
4. The unit sphere in Hilbert is contractible.
5. The house with two rooms is standard example of a space which is contractible.
6. Spheres of any finite dimension are not contractible.

Theorem 1.8.2 (Lefschetz fixed point theorem), (see [18])

In mathematics, the Lefschetz fixed-point theorem is a formula that counts the number of fixed points of a continuous mapping from a compact topological space X to itself by means of traces of the induced mappings on the homology groups of X. It is named after Solomon Lefschetz, who first stated it in 1926. Now we state this theorem:

Statement. *If X is a compact locally contractible metric space, all of whose homology groups are trivial. Then X has the fixed point property.*

Application. The Lefschetz fixed point theorem is used to generalizes the Brouwer fixed point theorem. ■

1.8.3 (Kinoshita fixed point theorem), (see [18, 45])

According to Brouwer fixed point theorem every compact, closed and convex subset of a Euclidean space has the FPP (fixed point property). Compactness alone does not imply the FPP and convexity is not even a topological property so it makes sense to ask how to topologically characterize the FPP. In 1932 Borsuk asked whether compactness together with contractibility could be a necessary and sufficient condition for the FPP to hold. The problem was open for 20 years until the conjecture was disproved by S. Kinoshita who found an example of a compact contractible space without the FPP in [45]. Now we state this theorem:

Statement. *There exists a compact, contractible subset of* $\mathbb{R}^3$ *which lacks the fixed point property.*

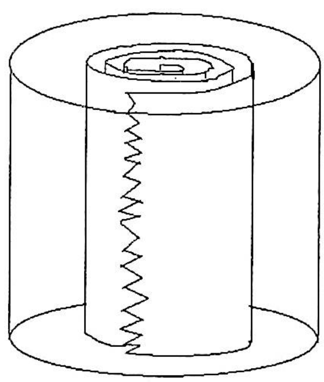

Figure-1.6

Example. Consider a vertical cylinder of unit height based on the edge of a horizontal closed disc and a vertical sheet of unit height with infinite length which spirals out from the axis of the cylinder, approaching closer and closer to the cylinder. Then the set formed by this situation has the lakes of fixed point property although this set is compact and contractible.

Application. This theorem is used to develop the concept of fixed point property. ■

Theorem 1.8.4, (see [18])

This theorem is stated depending on the following figures:

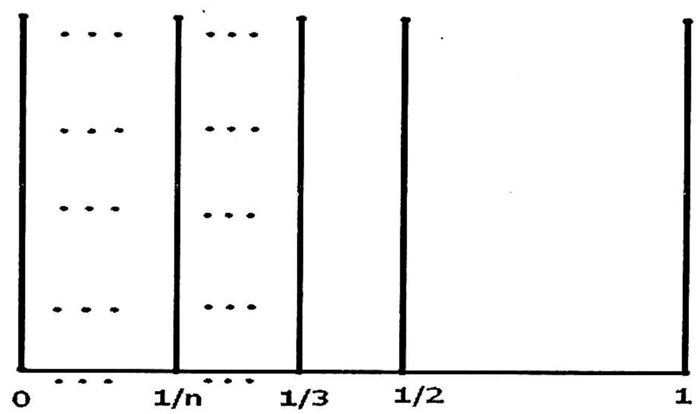

Figure-1.7

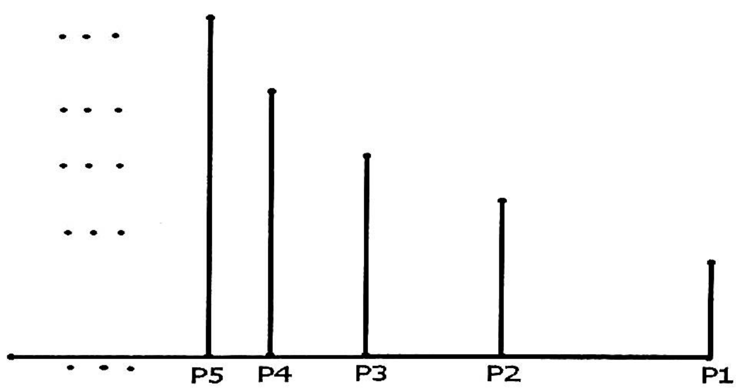

Figure-1.8

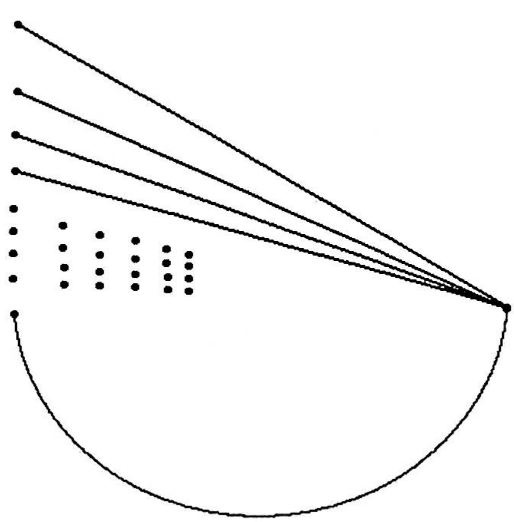

Figure-1.9

Statement. *The sets shown in Figures-1.7, 1.8, 1.9 have the fixed point property, although the set in Figure-1.8 is not compact and the set in Figure-1.9 is neither compact nor contractible.*

This theorem is proved by D. R. Smart [18] in 1974.

Application. This theorem is used to develop the concept of fixed point property. ■

1.9 Non-expansive Mappings

Definition 1.9.1, (see [18, 59]). Let M be a metric space. A mapping $T: M \to M$ is called a ***non-expansive*** if

$$d(Tx, Ty) \le d(x, y), \quad \forall\ x,\ y \in M$$

where $d(x, y)$ denotes the metric between x and y and $Tx = T(x)$.

If M is a normed space, then T is ***non-expansive*** if

$$\|Tx - Ty\| \le \|x - y\|.$$

If T is linear, this reduces to $\|Tx\| \le \|x\|\ \forall x \in M$. Thus, a linear

operator $T: M \to M$ is ***non-expansive*** if its norm satisfies $\|T\| \le 1$.

The non-expansive mapping $T: M \to M$ is called ***strictly non-expansive*** if

$$d(Tx, Ty) = d(x, y) \Rightarrow x = y\ \forall\ x, y \in M\ .$$

If M is a normed space, then the condition for *strictly non-expansive* reduces to

$$\|Tx - Ty\| \le \|x - y\| \text{ iff } x \ne y.$$

If T is linear, this reduces to $\|Tx\| \le \|x\|\ \forall \textit{non zero}\ x \in M$.

Examples. Contraction mapping, isometrics and orthogonal projections all are non-expansive mappings. A fixed point of a non-expansive mapping need not be unique.

Definition 1.9.2, (see [24, 46]). A mapping T on a complete metric space (X, d) is said to be ***diametrically contractive*** if $\delta(TA) < \delta(A)$ for all closed subsets A with $0 < \delta(A) < \infty$. (Here $\delta(A) = \sup\{d(x, y) : x, y \in A\}$ is the diameter of $A \subset X$.)

Example. If $M = [0, 5]$ and $T: M \to \mathbf{R}$ defined by $Tx = \begin{cases} x+1 & \textit{if } x \le 3 \\ 4 & \textit{if } x > 3 \end{cases}$, then T is a diametrically contractive mapping.

In [46] S. Dhompongsa and H. Yingtaweesittikul established the following result for diametrically contractive mapping.

Theorem 1.9.3, (see [ ], Theorem 2.2). *Let $\mathcal{F}(X)$ be the collection of nonempty closed subsets of a Banach space X and let $F(T)$ denote the set of fixed points of T. Recall that $TA = \bigcup_{a\in A} Ta$. Let M be a weakly compact subset of X and let $T: M \to \mathcal{F}(X)$, $Tx \cap M = \phi$ for all $x \in M$ and $\delta(TA \cup A) < \delta(A)$ for all closed sets A with $\delta(A) > 0$. Then T has a unique fixed point.* ■

In [24] H. K. Xu established the following result for diametrically contractive mapping.

Theorem 1.9.4, (see [24], Theorem 2.3). *Let M be a weakly compact subset of a Banach space X and let $T: M \to M$ be diametrically contractive, then T has a fixed point.* ■

Definition 1.9.5, (see [10]). Let $T: B \to B$ be a mapping from a Banach space B into itself and L be a pre assigned constant. Then the mapping T is called a ***Lipschitzian mapping*** if it satisfies the following contractive condition:

$$\|Tx - Ty\| \le L\|x - y\|, \ \forall \ x, y \in B.$$

If $L < 1$ then this Lipschitzian mapping is called a contraction mapping.

Example. Let $B = [0, 2]$ be a Banach space and $T: B \to B$ be defined by

$$T(x) = \frac{2}{x+1} \ \forall \ x \in B.$$

Then, T is a Lipschitzian mapping with Lipschitz constant $L \le 1$.

1.10 Set-valued or Multi-valued functions

Definition 1.10.1, (see [18, 44]). A ***set-valued (multi-valued) function*** ϕ from the set X to the set Y is some rule that associates one or more points in Y with each point in X. Formally it can be seen just as an ordinary function from X to the power set of Y, written as $\phi: X \to 2^Y$. A Set-valued function $\phi: X \to 2^Y$ is said to have a ***closed graph*** if the set $\{< x, y > | y \in \phi(x)\}$ is closed subset of $X \times Y$ in the product topology. The point $a \in X$ is a fixed point of ϕ if $a \in \phi(a)$.

Examples.

1. The complex logarithm function is a set valued function.
2. Inverse trigonometric functions are set valued function.

Theorem 1.10.2 (Kakutani's fixed point theorem), (see [18])

In Mathematical analysis, the Kakutani's fixed point theorem is a fixed point theorem for set-valued functions. This theorem is generalization of Brouwer fixed point theorem. This theorem was developed by Shizuo Kakutani in 1941. Now, we state this theorem:

Statement. *Let S be a non-empty, compact and convex subset of some Euclidian space R^k. Let $\phi : S \to 2^S$ be a set-valued function on S with a closed graph and the property that $\phi(x)$ is non-empty and convex for all $x \in S$. Then ϕ has a fixed point.*

Example. Let $\phi(x)$ be a set-valued function defined on the closed interval $[0, 1]$ that maps a point x to the closed interval $\left[1-\frac{x}{2}, 1-\frac{x}{4}\right]$. Then $\phi(x)$ satisfies all the assumptions of Kakutani's fixed point theorem and must have a fixed point.

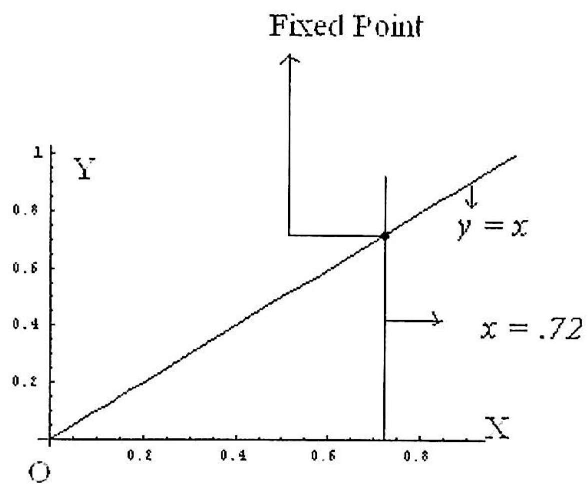

Figure-1.10: Fixed point of $\phi(x) \in \left[1-\frac{x}{2}, 1-\frac{x}{4}\right]$.

Application. This theorem is used to

(i) Develop the game theory (Mathematician John Nash used the Kakutani's fixed point theorem to prove the major results in game theory. This work would later earn him a Nobel prize in Economics),

(ii) Establish the Equilibrium theory in Economics. ■

1.11 T-Stability

Definition 1.11.1, (see [1, 32]). Let B be a Banach space, T be a self map of B, and assume that $x_{n+1} = f(T, x_n)$ defines some iteration schemes involving T. For example, $f(T, x_n) = T(x_n)$. Suppose that $F(T)$, the fixed point set of T, is nonempty and that the sequence $\{x_n\}$ converges to a fixed point p of T Let $\{y_n\}$ be an arbitrary sequence in B and define $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$ for $n = 0,1,2,\ldots$. If $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = p$, then the iteration process $x_{n+1} = f(T, x_n)$ is said to be T- ***stable***.

Examples. Picard's iteration scheme, Mann iterative scheme and Ishikawa iterative scheme all are T - stable.

Theorem 1.11.2, (see [60], Theorem 1). *Let (X, d) be a nonempty complete metric space and T be a self-map of X with $F(T) \neq \phi$. If there exist numbers $L \geq 0,\ 0 \leq h < 1$, such that*

$$d(Tx, q) \leq Ld(x, Tx) + hd(x, q) \qquad (1)$$

for each $x \in X,\ q \in F(T)$, and, in addition,

$$\lim_{n\to\infty} d(y_n, Ty_n) = 0 \qquad (2)$$

Then, Picard's iteration scheme is T-stable. ■

Corollary 1.11.3, (see [60], Corollary 1). *Let (X, d) be a nonempty complete metric space and T be a self-map of X satisfying the following: there exists $0 \leq h < 1$, such that, for each $x, y \in X$,*

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Then, Picard's iteration scheme is T-stable. ■

Corollary 1.11.4, (see [60], Corollary 2). *Let (X, d) be a nonempty complete metric space and T a self-map of X satisfying*

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y)$$

for all $x, y \in X$, where $L \geq 0,\ 0 \leq a < 1$. Suppose that T has a fixed point p. Then, T is Picard T-stable. ■

1.12 Common fixed points

Definition 1.12.1, (see [18]). Let A be a family of mappings T of some set into itself. If $Tx = x$ for all T in A, then we say that x is a ***common fixed point*** for A or for the mapping T in A. The set of common fixed points of a family A is given by $\bigcap_{T \in A} F(T)$, where $F(T)$ is the set of fixed point of T.

Theorem 1.12.2 (Downward induction fixed point theorem), (see [18])

Statement. *Suppose that*

(i) there is a non-empty compact(convex) set M_0, invariant under a family of operators F;

(ii) if M_1 is any compact (convex) set invariant under F and if M_1 has more than one point then M_1 contains a strictly smaller compact (convex) invariant set. Then there is a common fixed point for F. ■

1.13 Continuation fixed point theorem

Let $\mathcal{M}$ be a region in a normed space $\mathcal{N}$ and U_t $(0 \le t \le 1)$ be a family of mappings from $\mathcal{M}$ into $\mathcal{N}$ such that U_t has no fixed points on the boundary $\partial \mathcal{M}$. This means that as t changes, fixed point cannot escape from $\mathcal{M}$ through $\partial \mathcal{M}$. Thus if U_0 satisfies suitable conditions (which ensure a fixed point for U_0) we expect that U_1 must have a fixed point.

Definition 1.13.1, (see [18]). Let U_0 and U_1 be mappings of a set $\mathcal{L}$ into $\mathcal{N}$. We say that U_0 is ***fp-homotopic*** to U_1 on $\mathcal{L}$ if there exists a family of mappings U_t $(0 \le t \le 1)$ of $\mathcal{L}$ into $\mathcal{N}$ such that

(i) $U_t(x) = U(x, t)$ is continuous on $\mathcal{L} \times [0, 1]$,

(ii) $U(\mathcal{L} \times [0, 1])$ is contained in a compact subset of $\mathcal{L}$,

(iii) $U_t(x) \neq x$ for all $x \in \partial \mathcal{L}$.

Theorem 1.13.3 (Continuation fixed point theorem), (see [18])

Statement. *If (a) a condition on $\mathcal{M}$, (b) a condition on U_0, and (c) U_1 is fp-homotopic to U_0 on $\partial \mathcal{M}$, then U_1 has a fixed point.*

Application. Continuation fixed point theorem is applicable for solving non-linear problems. ■

Definition 1.13.4, (see [18]). Let F be a simplified mapping (or, a differentiable mapping) of $\mathcal{M}$ into $\mathbb{R}^n$ (n-Euclidian space). The ***degree*** of F with respect to $\mathcal{M}$ at a point x in $\mathbb{R}^n - F(\partial \mathcal{M})$ is the algebraic number of times that (almost all) points are covered, in the region U_i containing x. This integer is written deg $(F, \mathcal{M}, x)$, or deg $(F, \mathcal{M}, U_i)$. The main properties of degree are given below:

1. deg $(F, \mathcal{M}, x)$ is an integer, defined if $x \notin F(\partial \mathcal{M})$.
2. If $F = I$, then deg $(F, \mathcal{M}, x) = 1$ *if* $x \in \mathcal{M}$ and

   deg $(F, \mathcal{M}, x) = 0$ *if* $x \notin$ closure of $\mathcal{M}$.
3. deg $(F, \bigcup \mathcal{M}_i, x) = \sum$ deg$(F, \mathcal{M}_i, x)$ if the $\mathcal{M}_i$ are disjoint regions and both sides of the equation are defined.
4. If deg $(F, \mathcal{M}, x) \neq 0$ then $x \in F(\mathcal{M})$.

Theorem 1.13.5 (Leray-Schauder fixed point theorem), (see [18])

Statement. *If deg* $(I - T_0, \mathcal{M}, x) \neq 0$ *and* T_1 *is fp-homotopic to* T_0 *(that is, homotopic under a compact homotopy with no fixed point on* $\partial \mathcal{M}$*), then* T_1 *has a fixed point in* $\mathcal{M}$. ■

CHAPTER-2

FIXED POINT PROBLEMS FOR CONTRACTION AND NON-EXPANSIVE MAPPINGS

CHAPTER-2

FIXED POINT PROBLEMS FOR CONTRACTION AND NON-EXPANSIVE MAPPINGS

2.1 Introduction

The study of non-expansive mappings has been one of the main features in recent developments of fixed point theory: see for instance [18, 22, 24, and 38]. Non-expansive mappings are a special case of contraction mappings. So, the study of contraction mappings also plays an important role in the developments of fixed point theory. In this chapter, first we identify some fixed point problems which are related with contraction and non-expansive mappings and then we try to solve these problems. Although, most of these problems we solved our own fashion, moreover some problems are available in literature. But it is true that, we are the first who identifies these problems as fixed point problems related with contraction mappings or non-expansive mappings. Some problems were to be open since many years. We try to solve these problems.

Here, first we give some fixed point problems which are related with the contraction mappings and then we described some fixed point problems which are related with the non-expansive mappings.

2.2 Problems related with Contraction mappings

Contraction mapping is a very essential topic of fixed point theory after invention Banach's fixed point theorem 1.5.2. We have already defined contraction mapping with example in our first chapter. From this point of view, here we will study the fixed point theory of contraction mappings which we have identified as fixed point problems related with contraction mapping and try to solve these problems.

Problem 2.2.1. *Let T be a continuous function and satisfies a Lipschitz with condition respect to y: $|T(t,y)-T(t,z)| \le K|y-z|$ in some neighbourhood N_1 of a point (a,b). Then the differential equation with initial condition*

$$\frac{dy}{dt} = T(t,y), \qquad T(a) = b, \tag{1}$$

has a unique solution in some neighbourhood of a.

Solution. We observe that the differential equation (1) is equivalent to the integral equation

$$y(t) = b + \int_a^t T(x,\ y(x))\,dx \qquad (2)$$

We consider a set F of functions and a mapping U in F. The image Uy of a function y with values $y(x)$ will be given by

$$(Uy)(t) = b + \int_a^t T(x,\ y(x))\,dx \qquad (3)$$

Now, we find a set of functions which is mapped into itself by U. For this first we choose a compact neighbourhood N_2 of (a,b), inside N_1; then T is bounded on N_2, say $|T(x,y)| \le L \quad \forall\ (x,y) \in N_2$. If y is a function with graph in N_2, then we have

$$|Uy(t) - b| = \left|\int_a^t T(t, y(t))\,dt\right| \le L|t-a|.$$

This means that if y is a continuous function defined for $|t-a| \le d$, for which $|y(t) - b| \le Ld$, then Uy satisfies the same conditions. We must choose d small enough for the rectangular figure-2.1 $R = \overline{N}(a,d) \times \overline{N}(b, Ld)$ to be in N_2. We then define F to be set of continuous functions with graphs in R, and our arguments shows that F is mapped into itself by U.

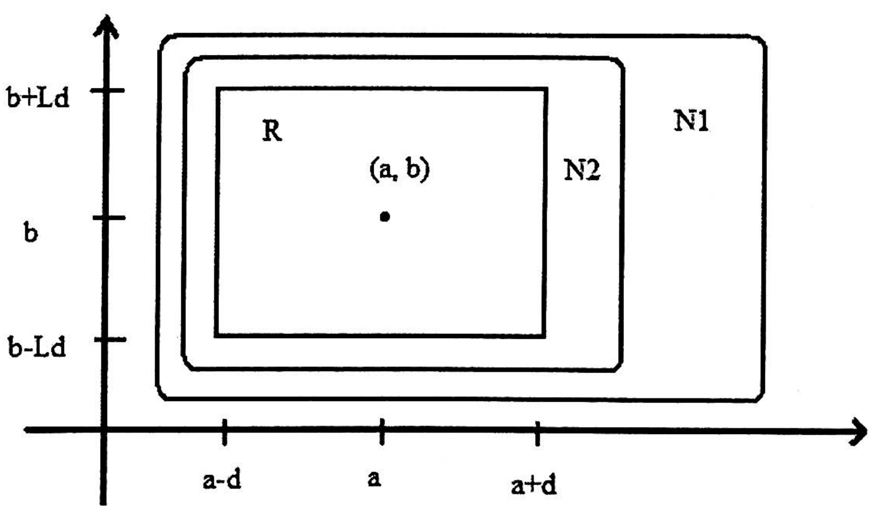

Figure-2.1

To ensure that U is a contraction mapping we should also arrange, in choosing d, that $dK < 1$. Then we have, for all y and z in F

$$|Uy(t)-Uz(t)| = \left|\int_a^t (T(x,y(x)) - T(x,z(x)))\,dx\right|$$
$$\leq d \sup |T(x,y(x)) - T(x,z(x))|$$
$$\leq d \sup |y(x) - z(x)|.$$

Thus

$$\|Uy - Uz\| = \sup_t |Uy(t) - Uz(t)|$$
$$\leq dK \sup |y(x) - z(x)| = dK\|y - z\|,$$

and since $dK < 1$, therefore U is a contraction mapping. Thus by Banach's fixed point Theorem 1.5.2, we can say that U has a unique fixed point in F. This means that there is a unique function in F which is a solution of the differential equation (1). Since any solution of the differential equation (1) is in F (for d is sufficiently small), there is a unique solution of the differential equation (1). This completes our problem. ■

Lemma 2.2.2, (see [18]). *If* $|\partial f/\partial y| \leq \frac{1}{2}$ *at all points between* (x, y) *and* (x, z) *then*

$$|f(x, y) - f(x, z)| \leq \frac{1}{2}|y - z|$$ ■

Problem 2.2.3. *Let* N *be a neighbourhood of a point* (a, b) *in* R^2. *Suppose that* T *is a continuous function of* x *and* y *in* N *and that* $\frac{\partial T}{\partial y}$ *exists in* N *and is continuous in* (a,b). *Now, if*

(*i*) $\frac{\partial T}{\partial y}(a, b) \neq 0$,

(*ii*) $T(a, b) = 0$.

Then, there exists a unique continuous function y_0 *on some neighbourhood of* a, *such that* $T(x, y_0(x)) = 0$.

Solution. We use the notation D_T for $\partial T(a, b)/\partial y$. Now, we will look for a fixed point of a mapping defined

$$Sz(x) = z(x) - D_T^{-1} T(x, z(x)).$$

(This mapping is suggested by the idea of finding $y_0(x)$ by Newton's method.) It is clear that if y is a fixed point we must have $T(x, y(x)) \equiv 0$. We will find a set of functions M such that S maps M into M and that S is a contraction mapping in M. Within N we choose a closed rectangle $R = \overline{N}(a, \varepsilon) \times \overline{N}(b, \delta)$ small enough to give

$$\left| D_T^{-1} \frac{\partial T}{\partial y}(x,\ y) - 1 \right| < \frac{1}{2}, \quad \forall\ (x,\ y) \in R,$$

$$\left| D_T^{-1} T(x,\ b) \right| < \frac{1}{2}\delta, \quad \forall\ |x| \le \varepsilon.$$

Now, we write $C = C(\overline{N}(a,\ \varepsilon))$ and put

$$M = \{ y \in C \mid y(a) = b,\ \|y - \beta\| \le \delta \}$$

where β the function is identically equal to b. Clearly S maps M into C. We have

$$\|S\beta - \beta\| = \left\| D_T^{-1} T(x,\ b) \right\| < \frac{1}{2}\delta.$$

For $(x,\ y)$ in R we have

$$\left| \frac{\partial}{\partial y}(y - D_T^{-1}T(x,\ y)) \right| = \left| (1 - D_T^{-1} \frac{\partial}{\partial y} T(x,\ y)) \right| < \frac{1}{2}.$$

Thus by lemma 2.2.2, if y and z are in M then

$$|Sy(x) - Sz(x)| \le \frac{1}{2}|y(x) - z(x)|, \quad \forall\ x \in \overline{N}(a,\ \varepsilon),$$

so that $\|Sy - Sz\| \le \frac{1}{2}\|y - z\|$. Thus S is a contraction mapping.

Also

$$\begin{aligned} \|Sy - \beta\| &\le \|Sy - S\beta\| + \|S\beta - \beta\| \\ &\le \frac{1}{2}\|y - \beta\| + \|T\beta - \beta\| \\ &\le \frac{1}{2}\delta + \frac{1}{2}\delta \\ &= \delta \end{aligned}$$

so that S maps M into M. Since M is complete. Therefore, by Banach fixed point theorem we can say, S has a unique fixed point in M. Thus our problem has a unique solution which can be calculated by successive approximations, using the operator S and starting from any member of M. ■

Problem 2.2.4. *Extend Banach fixed point Theorem 1.5.2 to the case where T^k is a contraction mapping for some integer $k > 1$. i.e., If T be a mapping of a complete metric space M into M such that T^k is a contraction mapping for some integer $k > 1$, then T has a unique fixed point in M. Moreover, the fixed point of T can be obtained by iteration of T starting from any $x_0 \in M$.*

Solution. Since T^k is a contraction mapping. Therefore, by Banach fixed point Theorem 1.5.2, we can say that T^k has a unique fixed point in M.

Let x be the unique fixed point of T^k, *i.e.,*

$$T^k(x) = x. \tag{1}$$

Now, we have

$$T^k(T(x)) = T(T^k(x)) = T(x) \quad \text{[By the equation, (1)]}$$

This implies that, $T(x)$ is a fixed point of T^k. But, the fixed point of T^k is unique and which is x. So $T(x)$ will be a fixed point of T^k if and only if $T(x) = x$. Therefore, x is a fixed point of T.

We now show that x is a unique fixed point of T.

Suppose that z is another fixed point of $T. \therefore\ T(z) = z.$ Since T^k is contraction mapping, therefore T is so. i.e., for some $0 < \lambda < 1$ we have $d(T(x), T(z)) \le \lambda d(x, z)$.

Now $d(x, z) = d(T(x), T(z)) \le \lambda d(x, z).$

This implies that $d(x, z) = 0,$ *i.e.,* $x = z.$ Which proves the uniqueness of the fixed point x of T.

Now, its remain to show that for any $x_0 \in M$ the points $T^n(x_0)$ converge to x as $n \to \infty$. Consider the iterates $T^n(x_0)$ as n runs through a fixed congruence class modulo k. That is, fix $0 \le r \le k-1$ and look at the points $T^{km+r}(x_0)$ as $m \to \infty$. Since

$$T^{km+r}(x_0) = T^{km}(T^r(x_0)) = (T^k)^m(T^r(x_0)),$$

these points can be viewed (for fixed r) as iterates of T^k starting at the point $y_0 = T^r(x_0)$. Since T^k is a contraction, these iterates of T^k (from any initial point, such as y_0) must tend to x by the Banach fixed point Theorem 2.5.2. This is independent of the value of r in the range $\{1, 2, \ldots\ldots\ldots, k-1\}$, so all k sequences $\{T^{km+r}(x_0)\}_{m\ge1}$ tend to x as $m \to \infty$. This shows $T^n(x_0) \to x$ *as* $n \to \infty$. This completes our problem. ■

2.2.5 Example for the problem 2.2.4.

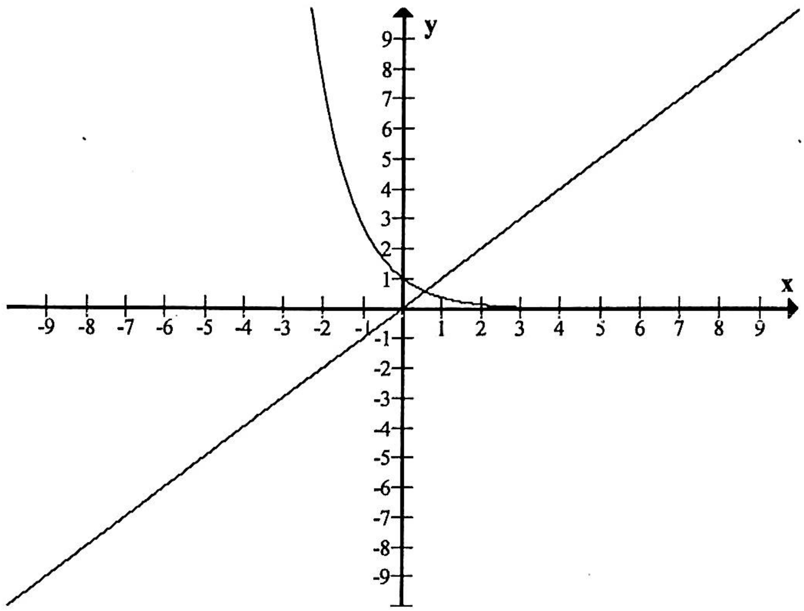

Figure-2.2

From the figure 2.2 it is clear that the graphs of $y = e^{-x}$ and $y = x$ intersect at a unique real value of x, i.e., the function defined by $T(x) = e^{-x}$ has a unique fixed point.

However, the function defined by $T(x) = e^{-x}$ is not a contraction mapping. For instance, $|T(-2) - T(0)| \approx 6.38 > |-2-0| = 2$, so the contraction mapping Theorem 1.5.2 does not justify for finding the fixed point of T by iteration. But the second iteration $S(x) = T^2(x) = e^{-e^{-x}}$ is a contraction mapping. This will be proved by the following justification.

By Mean-value theorem we have,

$$S(x) - S(z) = S'(t)(x - z)$$

for some t between x and z, where $|S'(t)| = \left|e^{-e^{-t}}.e^{-t}\right| = e^{-(t+e^{-t})} \le e^{-1}$

(since $t + e^{-t} \ge 0$ for all real t).

Therefore, $|S(x) - S(z)| = |S'(t)(x - z)| \le e^{-1}|x - z|$

Hence $T^2(x) = S(x) = e^{-e^{-x}}$ is a contraction mapping with contraction constant $e^{-1} = 1/e < 1$.

So, by our Problem 2.2.4 the solution of the equation $e^x = x$ can be found by iteration of $T(x) = e^{-x}$ starting with any real number. ■

Problem 2.2.6. *If T is a contraction mapping of a Banach space B into itself, then show that the equation $Tf - f = g$ has a unique solution f for each g in B. Also show that $T - I$ and $(T-I)^{-1}$ are uniformly continuous.*

Solution. The given equation can be written as

$$Tf = f + g \tag{1}$$

Now, we have to show that the equation (1) has a unique solution f for each g in B. It is clear from the equation (1) that f must be a solution of this equation, because $f \to f + g$ defines a map from B into B for each g in B.

If possible let this f has two different values f_1 & f_2. Then

$$Tf_1 = f_1 + g \tag{2}$$

$$\text{and} \quad Tf_2 = f_2 + g \tag{3}$$

Since, T is a contraction mapping from B into itself.

Therefore,

$$d(Tf_1, Tf_2) \le kd(f_1, f_2) \text{ for some } 0 < k < 1.$$

$$\Rightarrow d(f_1 + g, f_2 + g) \le kd(f_1, f_2) \quad \text{[By (1) and (2)]}$$

$$\Rightarrow d(f_1, f_2) \le kd(f_1, f_2)$$

$$\Rightarrow d(f_1, f_2) \le \frac{1}{1-k}$$

This implies that $f_1 = f_2$, i.e. f_1 & f_2 are not different. This contradicts our assumption.

So, we can say that the given equation has a unique solution f for each g in B.

Since, T is contraction mapping then, $T - I$ is so.

$$\text{i.e., } d((T-I)x, (T-I)y) \le kd(x, y) \text{ for some } 0 < k < 1.$$

Hence, by the definition of uniformly continuous mappings it is clearing that $T - I$ uniformly continuous.

Now, for all $x, y \in B$ and $0 < k < 1$ we have,

$$\begin{aligned} d((T-I)x, (T-I)y) &= |(T-I)x - (T-I)y| = |(Tx - Ty) - (x - y)| \\ &= |(x-y) - (Tx - Ty)| \\ &\ge |x - y| - |Tx - Ty| \\ &= d(x, y) - d(Tx, Ty) \\ &\ge d(x, y) - kd(x, y) = (1-k)d(x, y) \end{aligned}$$

$$\text{i.e., } d((T-I)x, (T-I)y) \ge (1-k)d(x, y).$$

This implies that $(T-I)^{-1}$ is uniformly continuous.

This completes our problem. ■

2.3 Problems related with Non-expansive mappings

In the previous section we saw that the fixed point theory for contraction mapping is extremely nice, even from a computational point of view. There exist a large number of results which in some sense extend the contraction mapping principle, and in this section as well as the next ones we will consider some relevant topics. One of the most natural ways to try to extend the contraction mapping principle is to consider the limiting case when the Lipschitz constant is allowed to be 1, in which case we end up with the nonexpansive mappings from definition 1.8.1.

The fixed point theory of non-expansive mappings is very different from that of contraction mappings, and the study of these mappings has been one of the main research areas of nonlinear functional analysis since the 1950s.The most famous result in the theory of nonexpansive mappings is probably the following theorem, which was proved independently by F.E. Browder [21] and W.A. Kirk [57].

Nonexpansive self mappings of nonempty complete metric spaces do not in general have fixed points, as for example consider $T:R\to R$, with $T(x)=x+1$ and one consequently considers various geometric conditions on the space in order to ensure the existence of a fixed point. And when fixed points exist, they are in general not unique, since, the identity mapping is nonexpansive. Here we will study the fixed point theory of nonexpansive mappings which we have identified as fixed point problems related with Non-expansive mapping.

2.3.1 Browder fixed point Theorem, (see, [18, 19]). *If M be a bounded closed convex subset of a Hilbert space H, then any non-expansive mapping T of M into M has a fixed point.* ■

By the Browder fixed point theorem 2.3.1, it is clear that, if M is bounded closed convex subset of a Hilbert space H and T is a non-expansive mapping, then T has a fixed point. This result remains true for the case of uniformly convex Banach space; see [20], [22] and [57]. In connection with these results D. R. Smart raised the some questions (see [18], p. 36). We described these questions by problems 2.3.3, 2.3.5 and 2.3.7 and tried to solve them.

Definition 2.3.2, (see [18, 56]). A Banach space is called ***reflexive*** if it coincides with the dual of its dual space in topological and algebraic senses. Reflexive Banach spaces are often characterized by their geometric properties. Let B be a Banach space and $B^{**} = (B^*)^*$ denotes the second dual space of B. The canonical map $x \mapsto \hat{x}$ defined by $\hat{x}(f) = f(x),\ f \in B^*$ gives an isometric linear isomorphism (embedding) from B into B^{**}. The space B is called ***reflexive*** if this map is surjective. For example, finite-dimensional Banach spaces and Hilbert spaces are reflexive.

If a Banach space D is isomorphic to a reflexive Banach space B, then D is reflexive. The promised geometric property of reflexive Banach spaces is the following: if C is a closed non-empty convex subset of the reflexive space B, then for every $x \in B$ there exists a $c \in C$ such that $\|x - c\|$ minimizes the distance between x and points of C. Let B be a Banach space. The following are equivalent.

1. The space B is reflexive.
2. The dual of B is reflexive.
3. The closed unit ball of B is compact in the weak topology.
4. Every bounded sequence in B has a weakly convergent subsequence.
5. Every continuous linear functional on B attains its maximum on the closed unit ball in B. (James' theorem)

Problem 2.3.3. Let T be a non-expansive mapping of a nonempty bounded closed and convex subset K of a reflexive Banach space B into itself. Now, if $\sup_{y \in F}\|y - Ty\| \le \delta(F)/2$ for every nonempty bounded closed convex subset F of K, containing more than one element and mapped into itself by T, then T has a unique fixed point in K.

Solution. We have B is a reflexive Banach space if and only if every decreasing sequence of nonempty bounded closed convex subsets of B has a nonempty intersection.

Let $\mathcal{K}$ be the family of all closed convex bounded subsets of K, mapped into itself by T. Obviously $\mathcal{K}$ is nonempty. Applying Zorn's lemma, we get a minimal element S in $\mathcal{K}$, S being minimal with respect to being nonempty, bounded closed and convex and invariant under T. If S contains only one element, then that element is a fixed point of T. If not, let S contain more than one element.

Now for $x, y \in S$, we have

$$\|Tx - Ty\| \leq \frac{\|x - Tx\|}{2} + \frac{\|y - Ty\|}{2} \leq \sup_{y \in S}\|y - Ty\|.$$

Hence, $T(S)$ is contained in the closed sphere C with Tx as centre and $\sup_{y\in S}\|y - Ty\|$ as radius. Also $S \cap C$ is invariant under T. Therefore, by the minimality of S it follows that $S \subset C$ i.e., $\|Tx - y\| \leq \sup_{y \in S}\|y - Ty\|$, for every $y \in S$. Hence, for any arbitrary but fixed $x \in S$, we have

$$\sup_{y \in S}\|Tx - y\| \leq \sup_{y \in S}\|y - Ty\|. \qquad (1)$$

Let $S' = \{z \in S : \sup_{y \in S}\|z - y\| \leq \sup_{y \in S}\|y - Ty\|\}$. Obviously S' is closed, convex and nonempty ($Tx \in S'$). Again if $z \in S'$, then $z \in S$ and hence $Tz \in S'$ by (1). Hence S' is invariant under T. Also

$$\delta(S') \leq \sup_{y \in S}\|y - Ty\| < \delta(S). \text{ [By hypothesis]}$$

Hence S' is a proper subset of S, which contradicts the minimality of S.

Hence S has only one element which is a fixed point of T. The unicity of the fixed point follows from the fact that if $x = Tx,\ y = Ty$ then

$$\|x - y\| = \|Tx - Ty\| \leq \frac{\|x - Tx\|}{2} + \frac{\|y - Ty\|}{2} = 0 \textit{ i.e., } x = y.$$

This completes our problem. ■

Definition 2.3.4 (see, [18, 56]). A ***strictly convex space*** is a normed topological vector space $(V, \|\ \|)$ for which the unit ball is a strictly convex set. Put another way, a strictly convex space is one for which, given any two points x and y in the boundary $\partial\mathcal{B}$ of the unit ball $\mathcal{B}$ of V, the affine line $L(x, y)$ passing through x and y meets $\partial\mathcal{B}$ *only* at x and y. A ***strictly convex Banach space*** is a Banach space which has the following properties:

1. A Banach space $(V, \|\ \|)$ is strictly convex if and only if the modulus of convexity δ for $(V, \|\ \|)$ satisfies $\delta(2) = 1$.

2. A Banach space $(V, \| \|)$ is strictly convex if and only if $x \neq y$ and $\|x\| = \|y\| = 1$ together implies that $\|x + y\| < 2$.

3. A Banach space $(V, \| \|)$ is strictly convex if and only if $x \neq y$ and $\|x\| = \|y\| = 1$ together implies that $\|\alpha x + (1-\alpha)y\| < 1 \ \forall \ 0 < \alpha < 1$.

4. A Banach space $(V, \| \|)$ is strictly convex if and only if $x \neq 0$ and $y \neq 0$ and $\|x + y\| = \|x\| + \|y\|$ together implies that $x = cy$ for some constant $c > 0$.

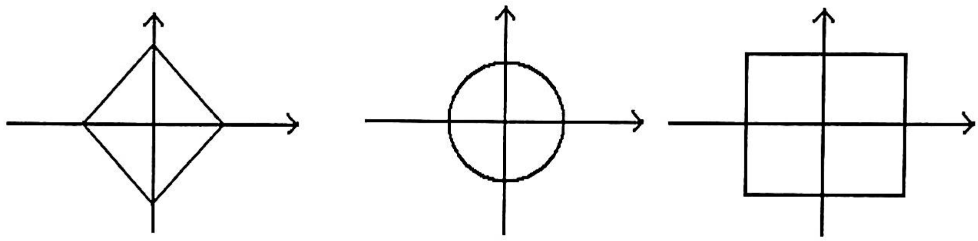
Figure-2.3(a) **Figure-2.3(b)** **Figure-2.3(c)**

The unit ball in the figure-2.3(b) is strictly convex, while the other two balls in the figures 2.3(a) and 2.3(c) are not because, they contain a line segment as part of their boundary.

Problem 2.3.5. *If M be a bounded closed convex subset of a strictly convex Banach space B, then any non-expansive mapping T of M into M has a fixed point.*

Solution. From Zorn's lemma, minimal element M_0 exists in the collection of all nonempty convex and closed subsets of M, each of them is mapped into itself by T. We show that M_0 consists of a single point. We assume that $diam\, M_0 > 0$.

Since, every convex and bounded set in strictly convex Banach space B has normal structure, and then M_0 has normal structure.

i.e., $\exists\ x \in M_0,$ *such that* $\sup\{\|x - y\| : y \in M_0\} = q < diam\ M_0$.

We denote convex closed hull of set $T(M_0)$ with $\overline{co}T(M_0) = M_1$. Since $T(M_0) \subset M_0$ then

$$M_1 = \overline{co}T(M_0) \subset \overline{co}M_0 = M_0$$

and $T(M_1) \subset T(M_0) \subset \overline{co}T(M_0) = M_1$.

The minimality of M_0 implies $M_1 = M_0$.

We define a set

$$C = (\bigcap_{y \in M_0} D(y;\, q)) \bigcap M_0.$$

That is nonempty since $x \in C$, that is convex (every balls are convex sets) and closed set as intersection of convex and closed sets.

We define a set

$$C_1 = (\bigcap_{y \in T(M_0)} D(y;\, q)) \bigcap M_0.$$

Since $T(M_0) \subset M_0$ then $C_1 \supset C$. If $z \in C_1$ then

$$T(M_0) \subset D(z;\, q) \ \&\ M_0 = M_1 = \overline{co}T(M_0) \subset D(z;\, q)$$

(Because $D(z;\, q)$ is closed and convex set) therefore $C \supset C_1$. It follows that $C = C_1$.

We choose $z \in C$ and $y \in T(M_0)$. Then there exists $x \in M_0$ such that $y = T(x)$.

Therefore,

$$\|T(z) - y\| = \|T(z) - T(x)\| \leq \|z - x\| < q,$$

i.e., $T(z) \in C_1$. Since $C = C_1$ then $T(z) \in C$ *or* $T(C) \subset C$. The minimality of M_0 implies $C = M_0$. But $diam\, C < q < diam\, M_0$.

From obtained a contradiction and we conclude that $diam\, M_0 = 0$ & $M_0 = \{x^*\}$.

Therefore, $T(x^*) = x^*$. This completes our problem. ■

Definition 2.3.6, (see [56]). Let K be a bounded subset of a Banach space B. A point $x_0 \in M$ is said to be ***non-diametral*** point of K if

$$\sup\{\|x - x_0\| : x \in K\} < \delta(K), \text{ where } \delta(K) = \sup\{\|x - y\| : x, y \in K\},$$

is the diameter of K.

A bounded closed convex subset M of a Banach space B is said to have ***normal structure*** if for each closed convex subset H of M which contains more than one point there exists an $x \in H$, which is non-diametral point of H.

Evidently, a bounded closed convex subset M of a Banach space B has ***normal structure*** if and only if for each closed convex subset H of M which contains more than one point there exists an $x \in H$ and $\alpha(H),\ 0 < \alpha(H) < 1$, such that $\sup\{\|x - y\| : y \in H\} = r_x(H) \leq \alpha(H)\delta(H)$.

Problem 2.3.7. *Let M be a non empty weakly compact convex subset of the Banach* *space B and M has normal structure. Let T_1 and T_2 be mappings of M into itself satisfying:*

(a) *for each closed convex subset F of M invariant under T_1 and T_2 there exists some $\alpha_1(F)$, $0 \le \alpha_1(F) < 1$, such that*

$$\|T_1x - T_2y\| \le \max\{\frac{1}{2}(\|x - T_1x\| + \|y - T_2y\|), \frac{1}{3}(\|x - T_2y\| + \|y - T_1x\|), \frac{1}{3}(\|x - y\| + \|x - T_1x\| + \|y - T_2y\|), r_x(F), \alpha_1\delta(F)\}$$

for all $x, y \in F$;

(b) *$T_1C \subset C$ if and only if $T_2C \subset C$ for each closed convex subset C of M;*

(c) *for each closed convex subset D of M invariant under T_1 and T_2 there exists some $\alpha_2(D)$, $\frac{1}{2} \le \alpha_2(D) < 1$, such that either*

$$\sup_{z\in D}\|z - T_1z\| \le \max\{r(D), \alpha_2\delta(D)\}, \text{ where } r(D) = \inf\{r_x(D) : x \in D\};$$

$$\text{or, } \sup_{z\in D}\|z - T_2z\| \le \max\{r(D), \alpha_2\delta(D)\}, \text{ where } r(D) = \inf\{r_x(D) : x \in D\}.$$

Then there exists a common fixed point of T_1 and T_2.

Solution. Let $\mathcal{G}$ denote the family of all nonempty closed convex subsets of M, each of which is mapped into itself by T_1 *and* T_2. Ordering $\mathcal{G}$ by set inclusion, by weak compactness of M and Zorn's lemma, we obtain a minimal element F of M. By the definition of normal structure, there exists $x_0 \in F$ such that

$$\sup\{\|x_0 - y\| : y \in F\} = r_{x_0}(F) \le \alpha_3\delta(F)$$

for some α_3, $0 < \alpha_3 < 1$.

Without loss of generality assume that

$$\sup_{z\in F}\|z - T_2z\| \le \max\{r(F), \alpha_2\delta(F)\}$$

for some α_2, $\frac{1}{2} \le \alpha_2 < 1$. If

$$\|T_1x - T_2y\| \le \max\{\frac{1}{2}(\|x - T_1x\| + \|y - T_2y\|), \frac{1}{3}(\|x - T_2y\| + \|y - T_1x\|), \frac{1}{3}(\|x - y\| + \|x - T_1x\| + \|y - T_2y\|), r_x(F)\}$$

for all $x, y \in F$.

Let $\beta = \max\{\alpha_2, \alpha_3\}$ and $F_\delta = \{x \in F : r_x(F) \leq \beta\,\delta(F)\}$.

Otherwise, by hypothesis (a) there exists $\alpha_1(F), 0 \leq \alpha_1(F) < 1$, such that

$$\|T_1 x - T_2 y\| \leq \alpha_1\ \delta(F) \text{ for some } x,\ y \in F.$$

Let $\beta = \max\{\alpha_1, \alpha_2, \alpha_3\}$ and $F_\delta = \{x \in F : r_x(F) \leq \beta\,\delta(F)\}$.

As $x_0 \in F_\delta$, then F_δ is nonempty. Evidently, F_δ is convex. Since $x \to r_x(F)$ is continuous, then F_δ is closed. Let $x \in F_\delta$. Then

$$\begin{aligned}\|T_1 x - T_2 y\| \leq \max\{&\frac{1}{2}(\|x - T_1 x\| + \|y - T_2 y\|), \frac{1}{3}(\|x - T_2 y\| + \|y - T_1 x\|),\\ &\frac{1}{3}(\|x - y\| + \|x - T_1 x\| + \|y - T_2 y\|),\ r_x(F),\ \alpha_1\delta(F)\}\\ \leq\ &\beta\delta(F)\ \ for\ y \in F.\end{aligned}$$

This gives that $T_1(F)$ is contained in a spherical ball $\overline{U}$ centered at $T_1 x$ and of radius $\beta\delta(F)$, *i.e.*, $T_2(F) \subset \overline{U}$, whence $T_2(F \cap \overline{U}) \subset F \cap \overline{U}$ and by hypothesis (b) $T_1(F \cap \overline{U}) \subset F \cap \overline{U}$. By the minimality of F, we obtain $F \subset \overline{U}$.

Hence $r_x(F) \leq \beta\delta(F)$, and this implies $T_1 x \in F_\delta$.

Therefore, $T_1(F_\delta) \subset F_\delta$ and by hypothesis (ii) $T_2(F_\delta) \subset F_\delta$.

Hence, $F_\delta \in \mathcal{G}$. But $\delta(F_\delta) \leq \beta\delta(F) < \delta(F)$, which contradicts the minimality of F.

Hence F contains a unique point x_0 such that $T_1 x_0 = x_0 = T_2 x_0$. This completes our problem.

■

Problem 2.3.8. *If H be a Hilbert space and T be a non-expansive mapping of H into itself, then the set of fixed point of T is either empty or closed and convex.*

Solution. We know that a non-expansive mapping of a complete space into itself need not have a fixed point and it is also clear that every Hilbert space is a complete space. So, we can say that the non-expansive mapping $T : H \to H$ need not have a fixed point. Hence in this case the set of fixed point of T may be empty set.

Now, we show that if the set of fixed points of T is non-empty, then it must be closed and convex. If there is only one fixed point, then there is nothing to show.

Consequently, let x and y be two fixed points of T and put

$$z = \alpha x + (1 - \alpha)y \tag{1}$$

where $0 < \alpha < 1$.

Consider the inequalities

$$\begin{aligned}\|Tz - x\| &= \|Tz - Tx\|, \quad [Since\ x\ is\ a\ fixed\ point\ of\ T] \\ &\le \|z - x\|, \quad [Since\ T\ is\ a\ nonexpansive\ mapping] \\ &= (1-\alpha)\|x - y\|, \quad [By\ (1)] \end{aligned} \tag{2}$$

$$\begin{aligned}\|Tz - y\| &= \|Tz - Ty\|, \quad [Since\ y\ is\ a\ fixed\ point\ of\ T] \\ &\le \|z - y\|, \quad [Since\ T\ is\ a\ nonexpansive\ mapping] \\ &= \alpha\|x - y\|, \quad [By\ (1)] \end{aligned} \tag{3}$$

Adding (2) and (3), we get

$$\|Tz - x\| + \|Tz - y\| \le \|x - y\|. \tag{4}$$

But, by triangle inequality we have

$$\|Tz - x\| + \|Tz - y\| \ge \|x - y\|. \tag{5}$$

Now, combining (4) and (5), we have

$$\|Tz - x\| + \|Tz - y\| = \|x - y\|. \tag{6}$$

This means that

$$\|Tz - x\| = \|z - x\| = (1-\alpha)\|x - y\|, \tag{7}$$

$$and\ \|Tz - y\| = \|z - y\| = \alpha\|x - y\|. \tag{8}$$

Applying the parallelogram law,

$$\|a - b\|^2 + \|a + b\|^2 = 2\|a\|^2 + 2\|b\|^2$$

With $a = Tz - x\ and\ b = z - x$ we get,

$$\|Tz - z\|^2 + \|Tz - x + z - x\|^2 = 2\|Tz - x\|^2 + 2\|z - x\|^2.$$

Therefore,

$$\begin{aligned}\|Tz - z\|^2 &= 2\|Tz - x\|^2 + 2\|z - x\|^2 - \|Tz - x + z - x\|^2 \\ &= 2\|z - x\|^2 + 2\|z - x\|^2 - \|Tz - Tx + z - x\|^2 \\ &= 4\|z - x\|^2 - \|(Tz - Tx) + (z - x)\|^2 \\ &\le 4\|z - x\|^2 - (\|Tz - Tx\| + \|z - x\|)^2 \\ &\le 4\|z - x\|^2 - (\|z - x\| + \|z - x\|)^2, \quad [Since\ T\ is\ a\ nonexpansive\ mapping] \\ &= 4\|z - x\|^2 - 4\|z - x\|^2 = 0.\end{aligned}$$

This implies that $\|Tz - z\| = 0$ and consequently $Tz = z$, i.e., z is a fixed point of T. Hence, by the definition of a convex set we can say that, the set of fixed point of T is a convex set.

It remains to show that, the set of fixed points of T is closed. Let $\{z_n\}$ be a sequence of fixed points, and let z be its limit. Now, we will show that z is also a fixed point of T.

We have

$$\|z_n - Tz\| = \|Tz_n - Tz\| \leq \|z_n - z\|, \text{ [Since } T \text{ is a nonexpansive mapping].}$$

Taking limit on both sides as $n \to \infty$ and we get,

$$\lim_{n\to\infty}\|z_n - Tz\| = 0,$$

and consequently z_n converges to Tz. Hence $Tz = z$, which shows that z is also a fixed point of T.

So, by the definition of a closed set we can say that, the set of fixed points of T is a closed set. This completes our proof. ■

2.3.9 Geometrical interpretation of the problem 2.3.8. A geometrical interpretation of the problem is given in following figure-2.4:

The sets A and B are defined by

$$A = \{x : \|Tz - x\| \leq (1-\alpha)\|x - y\|\},$$
$$B = \{x : \|Tz - y\| \leq \alpha\|x - y\|\}.$$

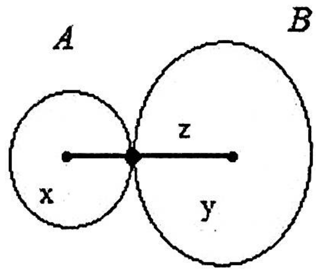

Figure-2.4

The image of z by T must lie in the intersection of A and B, which reduces to the point z. Thus Tz must be equal to z.

Note that it is strict convexity of the balls A and B that implies $Tz = z$.

Definition 2.3.10, (see [18]). Let M be a closed ball of radius n in a normed space $\mathcal{N}$. Then the ***radical retraction*** r ***onto*** M is defined by

$$rx = \begin{cases} x & if\ x \in M \\ nx/\|x\| & if\ x \notin M \end{cases}$$

such that (a) r is a continuous retraction of $\mathcal{N}$ into M,

(b) if $rx \in M^0$ (interior of M) then $rx = x$,

(c) if $x \notin M$ then $rx \in \partial M$ (boundary of M).

Definition 2.3.11, (see [18]). Let M be a closed convex subset of a normed space $\mathcal{N}$ such that $0 \in M^0$. Then the ***Minkowski functional***

$$g(x) = \inf\{c : x \in cM\}$$

is a continuous real function on $\mathcal{N}$ such that

$$(i)\quad g(cx) = cg(x) \quad for \quad c \geq 0;$$
$$(ii)\quad g(x+y) \leq g(x) + g(y);$$
$$(iii)\quad 0 \leq g(x) < 1\ if\ x \in M^0;$$
$$(iv)\quad g(x) > 1\ if\ x \notin M;$$
$$(v)\quad g(x) = 1\ if\ x \in \partial M.$$

The ***radical retraction of $\mathcal{N}$ into*** M is defined by

$$rx = x/\max(1,\ g(x))$$

such that r has the properties (a), (b) and (c) of the definition 2.3.10.

Lemma 2.3.12, (see [18], Lemma 4.2.2). Let $T: M \to N$ be compact and let $r: N \to P$ be continuous, then rT is compact. ■

Problem 2.3.13. *If M is a closed convex subset of a Hilbert space H, then prove the following statements.*

(i) The radial retraction of H onto M is not non-expansive.

(ii) The metric retraction of H onto M is non-expansive.

(iii) If T is a contraction mapping of M into H such that $T(\partial M) \subset M$, then T has a fixed point.

(iv) In (iii) we can replace 'contraction' by non-expansive if M is a closed ball.

Solution. (i) Let r be the radial retraction of H onto M. Then by definition 2.3.11, we have $r\,x = x/\max(1,\ g(x))$.

$$\therefore\ r\,y = y/\max(1,\ g(y)).$$

Since, $g(x)$ & $g(y)$ are Minkowski functional then, we getting

$$d(rx, ry) = |rx - ry|$$
$$= |x/\max(1, g(x)) - y/\max(1, g(y))| = |x - y| = d(x, y)$$
$$i.e., d(rx, ry) = d(x, y).$$

But, we have a mapping $r : H \to M$ is non-expansive if

$$d(rx, ry) \le d(x, y) \ \forall x, y \in H.$$

Hence we can say that, the radial retraction of H onto M is not non-expansive.

(ii) By the definition of metric retraction, this is defined in our definition 1.7.1 we have

$$d(rx, ry) = |rx - ry| \le |x - y| = d(x, y)$$
$$i.e., d(rx, ry) \le d(x, y) \ \forall x, y \in H.$$

This shows that $r : H \to M$ is non-expansive.

Hence we can say that, the metric retraction of H onto M is non-expansive.

(iii) By our assumption T is a contraction mapping of M into H. Hence T is also continuous.

Now, let $r : H \to M$ be a compact retraction mapping. Then by Lemma 2.3.12 rT is compact and by Schauder fixed point theorem 1.6.3 rT has a fixed point p (say). i.e., $rTp = p$.

If $p \in \partial M$, then $Tp \in M$. [By our assumption]

Therefore, $p = rTp = Tp$. This implies that $Tp = p$.

Hence p is a fixed point of T.

(iv) In our question (iii) if we consider that M is a closed ball then, it is clear that the non-expansive mapping $T : M \to H$ will convert to a contraction mapping. This proves (iv). Hence our problem is complete.

■

2.4 Shrinking mapping or Contractive mapping

Definition 2.4.1, (see [18]). Let (M, d) be a metric space and T be a mapping of M into itself. Then the mapping T is called a shrinking mapping or contractive mapping if $d(Tx, Ty) < d(x, y)$ for all $x, y \in M$ and $x \neq y$.

Thus a shrinking mapping is a non-expansive mapping, but need not be a contraction mapping. It is also clear that a shrinking mapping can have at most one fixed point.

Example.

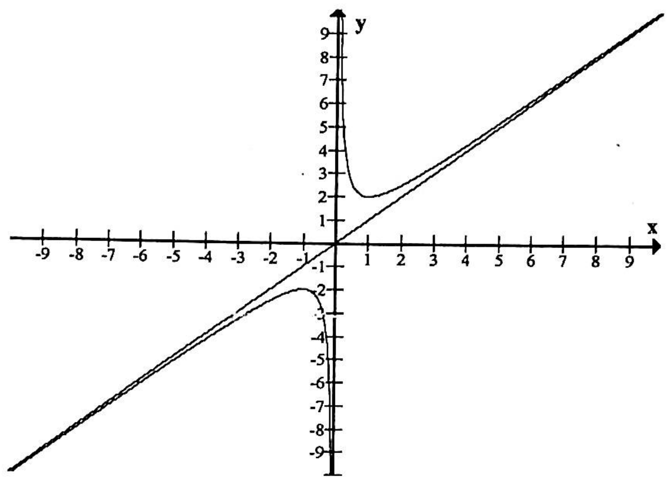

Figure: 2.5

Let $M = [1, \infty)$, which is a complete metric space. Set $T : M \to M$ *by* $T(x) = x + \dfrac{1}{x}$.

So when $x \neq y$ *in* $[1, \infty)$, then we have

$$|T(x) - T(y)| = \left|x - y + \frac{1}{x} - \frac{1}{y}\right| = \left|(x-y)(1 - \frac{1}{xy})\right| = |x - y|\left|1 - \frac{1}{xy}\right| < |x - y|$$

i.e., $|T(x) - T(y)| < |x - y|$.

Hence $T : M \to M$ is a shrinking mapping.

Since $T(x) > x \; \forall x \in M$. Therefore, it is clear from the figure-2 there are no fixed points of T in $[1, \infty)$.

Now, we state and prove a theorem which shows that under what condition a shrinking mapping have a unique fixed point.

Theorem 2.4.2. *Let M be compact metric space. If $T : M \to M$ such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in M$ and $x \neq y$ i.e., $T : M \to M$ is a shrinking mapping in M, then T has a unique fixed point in M and the fixed point can be found as the limit of $T^n(x_0)$ as $n \to \infty$ for any $x_0 \in M$.*

Proof. To show T has a unique fixed point in M, suppose T has two fixed points $x_1 \neq x_2$. Then $d(x_1, x_2) = d(Tx_1, Tx_2) < d(x_1, x_2)$. This is impossible, so $x_1 = x_2$.

Next we prove T has a fixed point in M. Let $F : M \to [0, \infty)$ by $F(x) = d(x, Tx)$. This measures the distance between a point and its T-value. A fixed point of T is where F takes the value 0.

Since F is continuous and M is compact, F takes on its minimum value: there is an $m \in M$ such that $F(m) \le F(x)\ \forall\ x \in M$. This point m is a fixed point for T. Indeed, if $Tm \neq m$ then

$$F(Tm) = d(Tm, T(Tm)) < d(m, Tm) = F(m),$$

which contradicts the minimality of $F(m)$. Hence $Tm = m$ and $F(m) = 0$.

Finally, we show for any $x_0 \in M$ that the sequence $x_n = T^n(x_0)$ converges to m as $n \to \infty$. We will show $d(x_n, m) \to 0$ *as* $n \to \infty$. We don't have the uniform control coming from a contraction constant. Instead we will exploit compactness.

If for any $k \ge 0$ we have $x_k = m$ then $x_{k+1} = T(x_k) = Tm = m$, and more generally $x_n = m\ \forall n \ge k$, so $x_n \to m$ in the sense that the sequence eventually equals m for all large n. Now, we may assume instead that $x_n \neq m\ \ \forall\ n$. Then

$$0 < d(x_{n+1}, m) = d(T(x_n), Tm) < d(x_n, m),$$

so the sequence of real numbers $d(x_n, m)$ is decreasing and positive. Thus it has a limit $l = \lim_{n\to\infty} d(x_n, m) \ge 0$. We want to show that $l = 0$. By compactness of M, the sequence $\{x_n\}$ has a convergent subsequence x_{n_i}, say $x_{n_i} \to y \in X$.

Then, by continuity of $T, T(x_n) \to T(y)$, which says $x_{n+1_i} \to y$ *as* $i \to \infty$. Since $d(x_n, m) \to l$ *as* $n \to \infty$, $d(x_{n_i}, m) \to l$ and $d(x_{n_i+1}, m) \to l$. By continuity of the metric, $d(x_{n_i}, m) \to d(y, m)$ and $d(x_{n_i+1}, m) = d(T(x_{n_i}), m) \to d(T(y), m)$. Having already shown these limits are l. Therefore, we have $d(y, m) = l = d(T(y), m)$. (1)

Since $d(T(y), m) = d(T(y), T(m))$, *if* $y \neq a$ then $d(T(y), m) < d(y, m)$, but then by (1) we get $l < l$, which is impossible. So $y = m$, this means $l = d(y, m) = 0$. That shows $d(x_n, m) \to 0$ *as* $n \to \infty$. This completes our theorem. ■

It is clear from the Theorem 2.4.2 that if M is compact and $T : M \to M$ is a shrinking mapping, then T has a unique fixed point. By the Browder fixed point theorem 2.3.1, the same conclusion holds provided M is closed unit ball of a Hilbert space and T is shrinking mapping. In connection with these results D. R. Smart raised the following question [see, [18] p. 39]: "Dose every shrinking mapping of the closed unit ball in a Banach space has a fixed point?" The aim of the Problem 2.4.3 is to give negative answer to this problem. Furthermore, our mapping has an additional property that it is affine.

Problem 2.4.3. *If B be a Banach space and T be an affine shrinking mapping of the closed unit ball β of B into itself, then T does not have any fixed point.*

Solution. Let $B = C_0$ be a Banach space of all real sequences such that $x = \{x_1, x_2, ..., x_n, ...\}$ $\lim_{n\to\infty} x_n = 0$ i.e., x converging to 0, and whose norm is defined by $\|x\| = \max\{|x_n|\}$.

Now, we define our map T as follows: Let $\{a_1, a_2, ...\}$ be any sequence of positive real numbers such that (i) each a_j is less than 1, and (ii) the sequence of partial products, $P_n = \prod_{j=1}^{n} a_j$, is bounded away from zero. (One such sequence is defined by $a_n = (2^n + 1)/(2^n + 2)$.)

Now, if $x = \{x_1, x_2, ..., x_n, ...\} \in C_0$, we let $T(x) = \{1, a_1x_1, a_2x_2, a_3x_3, ...\}$. Then it is clear that $\|x\| \le 1$, and $\|T(x)\| \le 1$. (In fact, $\|T(x)\| = 1$, if $\|x\| \le 1$.) Thus T takes the unit ball in C_0 to itself.

Since, $T(tx + (1-t)y) = tT(x) + (1-t)T(y)$, $\forall\, x, y \in C_0$ *and* $0 < t < 1$.

Therefore, T is affine.

We have,

$$\|T(x) - T(y)\| = \max\{|a_n(x_n - y_n)|\} = a_j|x_j - y_j|, \textit{ for some } j, \textit{ and if } x \ne y.$$

$$< |x_j - y_j| \le \max\{|x_n - y_n|\} = \|x - y\|.$$

i.e., $\|T(x) - T(y)\| < \|x - y\|$ *if* $x \ne y$.

Hence, T is a shrinking mapping.

Finally, suppose $x = \{x_1, x_2, ..., x_n, ...\}$ is a fixed point of T. Then

$$x_1 = 1, \qquad x_2 = a_1x_1 = a_1,$$

$$x_3 = a_2x_2 = a_1a_2, \qquad x_4 = a_3x_3 = a_1a_2a_3, \textit{ etc.}$$

and these numbers are bounded away from zero by the way that the sequence $\{a_1, a_2, ...\}$ was chosen. Thus, x is not in C_0. This is a contradiction. This completes our problem. ■

2.4.4 In [4] B. Fischer made the following conjecture:

Suppose S and T are mapping of the complete metric space to itself, with either S or T continuous, satisfying the inequality

$$d(Sx, TSy) \leq c\ diam\{x, Sx, Sy, TSy\} \quad (1)$$

for all $x, y \in X$, where $0 \leq c < 1$. Then S and T have a unique common fixed point.

This conjecture has been open even for compact X. Now, in our Theorem 2.4.5 we shall show that the above conjecture is true for $c < 1/2$ but false for $c \geq 1/2$.

Problem 2.4.5. *If X is complete, and* $S : X \to X, T : X \to X$ *are two mappings with the property (1), where* $c < 1/2$, *then S and T have a unique common fixed point. On the other hand, neither the mapping* $S : X \to X$ *nor the mapping* $T : X \to X$ *have any fixed point such that*

$$d(Sx, TSy) \geq 1/2\ diam\{x, Sx, Sy, TSy\}.$$

Thus, if $c < 1/2$ *we do not need any continuity assumption, and for* $c \geq 1/2$ *even the simultaneous continuity of S and T and the compactness of X do not help.*

Solution. To prove the first part of our problem, we let $x_0 \in X$ be arbitrary and let

$$x_n = \begin{cases} (TS)^{n/2} x_0, & if\ n\ is\ even \\ S(TS)^{(n-1)/2} x_0, & if\ n\ is\ odd. \end{cases}$$

Now, by inequality (1) we have

$$d(x_{2n+1}, x_{2n}) = d(STSx_{2n-2}, TSx_{2n-2}) \leq c\, diam\{Sx_{2n-2}, TSx_{2n-2}, STSx_{2n-2}\}$$
$$= c\, diam\{x_{2n-1}, x_{2n}, x_{2n+1}\} \leq c(d(x_{2n}, x_{2n-1}) + d(x_{2n+1}, x_{2n}))$$
$$i.e.,\ d(x_{2n+1}, x_{2n}) \leq (c/(1-c))d(x_{2n}, x_{2n-1}) \qquad \forall\ n \geq 1. \quad (2)$$

Similarly,

$$d(x_{2n+2}, x_{2n+1}) = d(Sx_{2n}, TSx_{2n}) \leq c\, diam\{x_{2n}, x_{2n+1}, x_{2n+2}\}$$
$$\leq c(d(x_{2n+1}, x_{2n}) + d(x_{2n+2}, x_{2n+1}))$$
$$i.e.,\ d(x_{2n+1}, x_{2n}) \leq (c/(1-c))d(x_{2n+1}, x_{2n}) \qquad \forall\ n \geq 1. \quad (3)$$

Since $c < 1/2$, we have $c/(1-c) < 1$, and so (2) and (3) imply that the sequence $\{x_n\}$ is a Cauchy sequence and thus, by completeness, $x_n \to z,\ as\ n \to \infty \forall z \in X$.

Using again (1) we get,

$$d(Sz, x_{2n+2}) \leq c\ diam\{z, Sz, x_{2n+1}, x_{2n+2}\} \leq c(d(Sz, z) + d(z, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}))$$

Letting here $n \to \infty$ we obtain

$$d(Sz, z) \leq cd(Sz, z)\ i.e., d(Sz, z) = 0.$$

This implies that $Sz = z$. But

$$d(z, Tz)d(Sz, TSz) \leq c\ diam\{z, Sz, TSz\} = c\,d(z, Tz)\ i.e., d(z, Tz) = 0.$$

This implies that $Tz = z$. Hence z is a common fixed point of S and T. The uniqueness of the common fixed point follows easily from the inequality (1).

After this let us prove that the conjecture is false for $c = 1/2$ and hence also $c \geq 1/2$.

Now, let $X = \{A, B, C, D\}$ with $d(A, D) = d(B, C) = d(B, D) = d(A, C) = 1$ and $d(A, B) = d(C, D) = 2$, (see the figure-2.6) and let S and T be the two mapping indicated below:

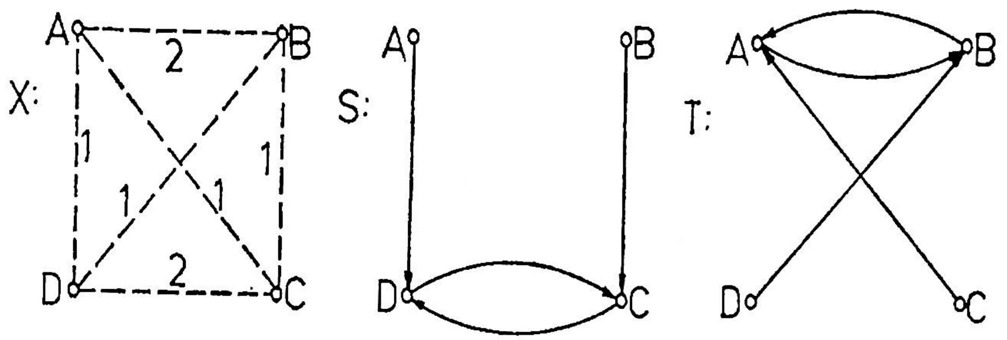

Figure-2.6

Neither S nor T have any fixed point. However, $Sx \in \{D, C\}$, $TSy \in \{A, B\}$ and so $d(Sx, TSy) = 1$ for every $x, y \in X$; furthermore

(a) $d(x, Sx) = 2,\quad if\ \ x = C\ \ or\ \ x = D;$

(b) $d(Sx, Sy) = 2,\quad if\ \ x + A\ \ and\ \ y \in \{B, D\}\ or\ x = B\ and\ \ y \in \{A, C\};$

(c) $d(x, TSy) = 2,\quad if\ \ x = A\ \ and\ \ y \in \{A, C\}\ \ or\ \ x = B\ and\ \ y \in \{B, D\};$

i.e., in any case $diam\{x, Sx, Sy, TSy\} = 2$ and so (1) holds foe every $x, y \in X$ with $c = 1/2$. This completes our theorem. ■

2.5 Asymptotically non-expansive mappings

Let K be a nonempty closed convex bounded subset of a Banach space B. In 1955, Krasnoselskii [31] proved first that a sum $T + S$ of two mappings T and S has a fixed point in K, when $T : K \to B$ is a contraction and $S : K \to B$ is compact (that is, a continuous mapping which maps bounded sets into relatively compact sets) and satisfies the condition that $Tx + Sy \in K,\ \forall\ x, y \in K$. In [31] M.A. Nashed and J.S.W. Wong generalized Krasnoselskii's theorem to sum $T + S$ of a nonlinear contraction mapping

$T : K \to B$ and a compact mapping $S : K \to B$. Subsequently, J. Reinermann [27] extended Krasnoselskii's theorem to a sum $T + S$ of a non-expansive mapping T and a strongly continuous mapping S when the underlying spaces B is uniformly convex Banach space. The study of asymptotically non-expansive mappings concerning the existence of fixed points have become attractive to the authors working in nonlinear analysis, since the asymptotically non-expansive mappings include non-expansive as well as contraction mappings. W.A. Kirk [59] introduced the concept of asymptotically non-expansive mappings in Banach spaces and proved a theorem on the existence of fixed points for such mappings in uniformly convex Banach spaces.

Let B denote a Hausdorff locally convex linear topological space with a family $(d_\alpha)_{\alpha \in J}$ of seminorms which defines the topology on B, where J is any index set.

Definition 2.5.1, (see [18, 36]). Let K be a nonempty subset of B. If T maps K into B, then T is called an ***asymptotically non-expansive*** if for all $x, y \in K$

$$d_\alpha(T^n x - T^n y) < k_n d_\alpha(x - y), \ \forall \ x, y \in K,$$

for each $\alpha \in J$ and for $n = 1, 2, \ldots$, where $\{k_n\}$ is a sequence of real numbers such that $\lim_{n \to \infty} k_n = 1$.

It is assumed that $k_n \geq 1$ and $k_n \geq k_{n+1}$ for $n = 1, 2, \ldots$. We gives the following definition:

Definition 2.5.2, (see [18, 36]). If T and S map K into B, then T is called a ***uniformly asymptotically regular*** with respect to S if, for each $\alpha \in J$ and $\eta > 0$, there exists $N = N(\alpha, \eta)$ such that

$$d_\alpha(T^n x - T^{n-1} + Sx) < \eta \quad \text{for all } n \geq N \text{ and for all } x \in K.$$

Example. Let $B = \mathrm{R}$ and $K = [0, 1]$. We define a map $T : K \to B$ by $Tx = 1 + x$ for all $x \in K$. Then $T^2 x = T(1 + x) = 2 + x$. By induction, we prove that $T^n x = n + x$.

Again, we define a map $S : K \to B$ by $Sx = -1$ for all $x \in K$.

Therefore, $|T^n x - T^{n-1} x + Sx| = 0$. Hence T is uniformly asymptotically regular with respect to S.

In [36] P. Vijayarju eastablished the following fixed point theorems for a sum of nonexpansive and continuous mappings in locally convex spaces.

Theorem 2.5.3, (see [36], THEOREM 2.1). Let K be a nonempty compact convex subset of B. Let T be an asymptotically non-expansive self-mapping of K. Let S be a continuous mapping of K into B. Suppose that T is uniformly asymptotically regular self-mapping of K with respect to the mapping S and that $T^n x + Sy \in K$ for all $x, y \in K$ and $n = 1,2,...$. Then $T + S$ has a fixed point in K. ■

Theorem 2.5.4, (see [36], THEOREM 2.2). Let K be a nonempty compact convex subset of B. Let T be a non-expansive mapping of K into B and S be a continuous mapping of K into B such that $Tx + Sy \in K$ for all $x, y \in K$. Then $T + S$ has a fixed point in K. ■

Theorem 2.5.5, (see [36], THEOREM 2.5). Let K be a nonempty complete bounded convex subset of B. Let T be an asymptotically non-expansive self-mapping of K. Suppose that S is a continuous mapping of K into B such that $S(K)$ is contained in some compact subset M of K. Assume further that T is a uniformly asymptotically regular with respect to S and that $T^n B + Sy$ in K for all $x, y \in K$ and $n = 1, 2,....$ If $(I - T - S)(K)$ is closed, then $T + S$ has a fixed point in K. ■

Theorem 2.5.6(see [36], THEOREM 2.6). Let K be a nonempty complete bounded convex subset of B. Let T be a non-expansive mapping of K into B. Suppose that S is a continuous mapping of K into B such that $S(K)$ is contained in a compact subset M of K and $TX + Sy \in K$ for all $x, y \in K$. If $(I - T - S)(K)$ is closed, then $T + S$ has a fixed point in K. ■

continuous mapping of K into B. Suppose that T is uniformly asymptotically regular self-mapping of K with respect to the mapping S and that $T^n x + Sy \in K$ for all $x, y \in K$ and $n = 1, 2, \ldots$ Then $T + S$ has a fixed point in K. ■

Theorem 2.5.4, (see [36], THEOREM 2.2). Let K be a nonempty compact convex subset of B. Let T be a non-expansive mapping of K into B and S be a continuous mapping of K into B such that $Tx + Sy \in K$ for all $x, y \in K$. Then $T + S$ has a fixed point in K. ■

Theorem 2.5.5, (see [36], THEOREM 2.5). Let K be a nonempty complete bounded convex subset of B. Let T be an asymptotically non-expansive self-mapping of K. Suppose that S is a continuous mapping of K into B such that $S(K)$ is contained in some compact subset M of K. Assume further that T is a uniformly asymptotically regular with respect to S and that $T^n B + Sy$ in K for all $x, y \in K$ and $n = 1, 2, \ldots.$ If $(I - T - S)(K)$ is closed, then $T + S$ has a fixed point in K. ■

Theorem 2.5.6(see [36], THEOREM 2.6). Let K be a nonempty complete bounded convex subset of B. Let T be a non-expansive mapping of K into B. Suppose that S is a continuous mapping of K into B such that $S(K)$ is contained in a compact subset M of K and $TX + Sy \in K$ for all $x, y \in K$. If $(I - T - S)(K)$ is closed, then $T + S$ has a fixed point in K. ■

CHAPTER-3

SOME FIXED POINT ITERATIVE SCHEMES

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SOME FIXED POINT ITERATIVE SCHEMES

3.1 Introduction

The importance of metrical fixed point theory consists mainly in the fact that for most nonlinear functional equations $y = f(x)$ we can equivalently transform them in a fixed point problem $Tx = x$ and then apply an appropriate fixed point theorem to get information on the existence or existence and uniqueness of the fixed point, that is, of the solution of the original equation. Most of the fixed point theorems also provide a method for constructing such a solution. These methods are usually iterative methods. The Brouwer fixed point theorem was one of the early major achievements of algebraic topology. This celebrated theorem has been generalized in several ways. Nowadays, the Brouwer and Kakutani fixed point theorems have become the most often used tools in economics, game theory and Numerical analysis.

A fundamental principle in mathematical sciences is iteration. As the name suggests, a process is repeated until an answer is achieved. Iteration scheme is used to find roots of equation, solution of linear and nonlinear system of equations and solution of differential equations. All the numerical iteration process are formulated to compare with the fixed point iteration process and to establish the fixed point iteration process, different types of fixed point theorems are used as very important tools. So, we can say that numerical iterative scheme is a great achievement of fixed point theorems.

In the previous Chapter we have discussed the different fixed point problems relating with contraction and non-expansive mappings. In the present Chapter first we give the definition of different fixed point iterative schemes and finally we state and prove their convergence theorems for some fixed operators. These convergence theorems shows that all fixed point iterative schemes strongly converges to the fixed point.

3.2 Fixed point iterative scheme

In numerical analysis, fixed point iterative scheme is a method of computing fixed points of iterated functions.

Definition 3.2.1, (see [18, 43, 59]). Let T be a mapping from a non-empty set X into itself i.e., $T: X \to X$. Then the iterative scheme $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_n = T(x_{n-1}), \qquad \text{(i)}$$

for $x_0 \in X$ and $n = 1, 2, 3, \ldots\ldots$, is called ***fixed point iterative scheme.***

Theorem 3.2.2, (see [59]). *Assume that $T(x)$ is a continuous function and that $\{x_n\}_{n=0}^{\infty}$ is a sequence generated by fixed point iteration (i). If $\lim_{n\to\infty} x_n = p$ then p is a fixed point of $T(x)$.* ■

Theorem 3.2.3, (see, [59]). *Assume that $T \in c[a, b]$, i.e., $T(x)$ is continuous on $[a, b]$. Then we have the following conclusions:*

(i) If the range of the mapping $Y = T(x)$ satisfies $Y \in [a, b]\ \forall\ x \in [a, b]$ then T has a fixed point in $[a, b]$.

(ii) Furthermore, suppose that $T'(x)$ is defined over (a, b) and that a positive constant $k < 1$ exists with $|T'(x)| \le k\ \forall\ x \in (a, b)$, than T has a unique fixed point p in $[a, b]$. ■

Theorem 3.2.4, (see [59]). *Assume that the following hypotheses hold:*

(a) p is a fixed point of a function T;

(b) $T, T' \in c[a, b]$;

(c) k is a positive constant;

(d) $p_0 \in (a, b)$; and

(e) $T(x) \in [a, b]\ \forall\ x \in [a, b]$.

Then we have the following conclusions:

(i) If $|T'(x)| \le k < 1\ \forall x \in [a, b]$, then the iteration $x_n = T(x_{n-1})$ will converge to the unique fixed point $p \in (a, b)$. In this case, p is said to be an attractive fixed point.

(ii) If $|T'(x)| > 1\ \forall x \in [a, b]$, then the iteration $x_n = T(x_{n-1})$ will not converge to p. In this case, p is said to be a repelling fixed point and the iteration exhibits local divergence. ■

Remarks on the Theorem 3.2.3. 1. It is assumed that $x_0 \ne p$ in statement (ii). 2. Because T is continuous on an interval containing p, it is permissible to use the simpler criterion $|T'(x)| \le k < 1 \; \forall x \in [a, b]$ and $|T'(x)| > 1 \; \forall x \in [a, b]$ in statement (i) and (ii), respectively.

3.2.5 Algorithm of fixed point iteration. To find a solution of the equation $x = T(x)$ by starting with $x = x_0$ and iterating by the formulae $x_n = T(x_{n-1})$ until we get our desirable solution.

Example. Find the solution to $x = \sqrt{2x}$ by using the starting approximation $x_0 = 0.1$.

Solution. Here the fixed points of the given equation are $x = 0 \;\&\; x = 2$, i.e. $x = 0 \; or \; x = 2$, must be a solution of the given equation.

Now, we solve the given equation by using the iteration formulae $x_n \sqrt{2x_{n-1}}$ with the starting value $x_0 = 0.1$ and after fifteenth iteration we get our desired results $x_{15} = 2.0000$. ■

3.3 Picard iterative Scheme

Actually, another form of fixed point iterative scheme is Picard iterative scheme. Moreover, for the convenience of our discussion again we describe below this iterative scheme.

Definition 3.3.1, (see [51]). Let $T : X \to X$ be a given operator and X be a Metric space or Normed linear space or Banach space. Then the sequence $\{p_n\}_{n=0}^{\infty}$ defined by

$$p_{n+1} = Tp_n \qquad \text{(ii)}$$

for $n = 0, 1, 2, \ldots \; and \; p_0 \in X$ is called ***Picard iterative Scheme.***

Theorem 3.3.2, (see [43], Theorem 2.1). *Let $p = \xi$ be a root of the equation $f(p) = 0$ and let I be an interval containing the point $p = \xi$. Let $T(p)$ and $T'(p)$ be continuous in I where $T(p)$ is defined by the equation $p = T(p)$ which is equivalent to $f(p) = 0$. Let $p = \xi$ be a fixed point of T. Then if $|T'(p)| < 1$ for all p in I, the sequence of approximation $\{p_n\}_{n=0}^{\infty}$ defined by the Picard iterative scheme (ii) converges to the root $p = \xi$ (fixed point of T), provided that the initial approximation p_0 is chosen in I.* ■

Theorem 3.3.3, (see [47]). *Let X be a Banach space and $T : X \to X$ be a map for which there exist the real numbers a, b and c satisfying $0 < a < 1$, $0 < b, c < 1/2$ such that for each pair x, y in X at least one of the following is true:*

(z_1) $\|Tx - Ty\| \leq a\|x - y\|$;

(z_2) $\|Tx - Ty\| \leq b[\|x - Tx\| + \|y - Ty\|]$;

(z_3) $\|Tx - Ty\| \leq c[\|x - Ty\| + \|y - Tx\|]$.

Then T have a unique fixed point p and the Picard iteration (ii), $\{p_n\}_{n=0}^{\infty}$ defined by $p_{n+1} = Tp_n$, $n = 0,1,2,...$ converge to p for any $p_0 \in X$. ■

3.4 Kranoselskii's iterative Scheme

Definition 3.4.1, (see [16, 30]). Let $T: X \to X$ be a given operator and X be a Metric space or Normed linear space or Banach space. Then the sequence $\{u_n\}_{n=0}^{\infty}$ defined by

$$u_{n+1} = (1-\lambda)x_n + \lambda Tx_n \tag{iii}$$

for $n = 0,1,2, \ldots$, $u_0 \in X$ and $\lambda \in (0, 1)$ is called ***Kranoselskii's iterative Scheme.***

Theorem 3.4.2. *Let E be an arbitrary Banach space, X a closed convex subset of E, and $T: X \to X$ an operator satisfying the Zamfirescu condition **Z**, i.e., there exist the real numbers a, b & c satisfying $0 \leq a < 1, 0 \leq b < 1, c < 1/2$ such that for each pair x, y in X, at least one of the following is true:*

(z_1) $\|Tx - Ty\| \leq a\|x - y\|$;

(z_2) $\|Tx - Ty\| \leq b[\|x - Tx\| + \|y - Ty\|]$;

(z_3) $\|Tx - Ty\| \leq c[\|x - Ty\| + \|y - Tx\|]$.

Let $\{u_n\}_{n=0}^{\infty}$ be defined by (iii) and $u_0 \in X$ with $\lambda \in (0, 1)$. Then $\{u_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Proof. By theorem 3.3.3, we know that T has a unique fixed point in X, say u. Consider $x, y \in X$. Since T is a Zamfirescu operator, therefore, at least one of the conditions (z_1), (z_2) and (z_3) is satisfied by T.

If (z_2) holds, then

$$\begin{aligned}
\|Tx - Ty\| &\leq b[\|x - Tx\| + \|y - Ty\|] \\
&\leq b[\|x - Tx\| + [\|y - x\| + \|x - Tx\| + \|Tx - Ty\|]]
\end{aligned}$$

$$\Rightarrow (1-b)\|Tx - Ty\| \leq b\|x - y\| + 2b\|x - Tx\|$$

$$\Rightarrow \|Tx - Ty\| \leq \frac{b}{(1-b)}\|x - y\| + 2\frac{b}{(1-b)}\|x - Tx\| \tag{1}$$

If (z_3) holds, then similarly we obtain

$$\|Tx - Ty\| \le \frac{c}{(1-c)}\|x-y\| + 2\frac{c}{(1-c)}\|x - Tx\| \tag{2}$$

Let us denote

$$\delta = \max\left\{a, \frac{b}{(1-b)}, \frac{c}{(1-c)}\right\} \tag{3}$$

Then we have, $0 \le \delta < 1$ and in view of (z_1), (1) and (2) we get the following inequality

$$\|Tx - Ty\| \le \delta\|x-y\| + 2\delta\|x - Tx\| \quad \text{holds } \forall\ x, y \in X \tag{4}$$

Now let $\{u_n\}_{n=0}^{\infty}$ be the Kranoselskii's iteration Scheme defined by (iii) and $u_0 \in X$ arbitrary. Then

$$\begin{aligned}\|u_{n+1} - u\| &= \|(1-\lambda)u_n + \lambda Tu_n - (1-\lambda+\lambda)u\| \\ &= \|(1-\lambda)(u_n - u) + \lambda(Tu_n - u)\| \\ &\le (1-\lambda)\|u_n - u\| + \lambda\|Tu_n - u\| \end{aligned} \tag{5}$$

Take, $x = u$ and $y = u_n$ in (4) we obtain

$$\|Tu_n - u\| \le \delta\|u_n - u\| \tag{6}$$

where δ is given by (3).

Now, combining (5) and (6), we get

$$\begin{aligned}\|u_{n+1} - u\| &\le (1-\lambda)\|u_n - u\| + \lambda\delta\|u_n - u\| \\ &= (1-\lambda+\lambda\delta)\|u_n - u\| \\ &= (1-(1-\delta)\lambda)\|u_n - u\| \end{aligned}$$

$$i.e., \|u_{n+1} - u\| \le (1-(1-\delta)\lambda)\|u_n - u\|, \quad n = 0,1,2,.... \tag{7}$$

Inductively we obtain,

$$\|u_{n+1} - u\| \le \prod_{k=0}^{n}(1-(1-\delta)\lambda)^k\|u_0 - u\|, \quad n = 0,1,2,.... \tag{8}$$

As $\delta < 1$ and $\lambda \in (0, 1)$ it results that

$$\lim_{n\to\infty}\prod_{k=0}^{n}(1-(1-\delta)\lambda)^k\|u_0 - u\| = 0.$$

This by (8) implies that,

$\lim_{n\to\infty}\|u_{n+1} - u\| = 0$. i.e., $\{u_n\}_{n=0}^{\infty}$ is converges strongly to u.

This completes our theorem. ■

3.5 Newton-Raphson iterative Scheme

Definition 3.5.1, (see [35, 43]). Let $T: X \to X$ be a given operator and X be a Metric space or Normed linear space or Banach space. Then the sequence $\{w_n\}_{n=0}^{\infty}$ defined by

$$w_{n+1} = w_n - \frac{Tw_n}{T' w_n} \qquad \text{(iv)}$$

for $n = 0,1,2,\ldots$, $w_0 \in X$ is called ***Newton-Raphson iterative Scheme***. Here T' denote the first derivative of T.

Theorem 3.5.2, (see [35], Theorem 2.5.1). *Let $w = \alpha$ be a root of the equation $T(w) = 0$ in the interval $[\alpha - A, \alpha + A]$ containing the point $w = \alpha$, where A be some positive integer. Let $T''(w)$ be continuous on $[\alpha - A, \alpha + A]$. Let $w = \alpha$ be a fixed point of T. Then the sequence of approximation $\{w_n\}_{n=0}^{\infty}$ defined by the Newton-Raphson iterative scheme (iv) converges to the root $w = \alpha$ (fixed point of T), provided that $T'(w) \neq 0$ and the initial approximation w_0 is sufficiently close to α.* ■

3.6 Mann iterative Scheme

Definition 3.6.1, (see [52, 54]). Let $T: X \to X$ be a given operator and X be a Metric space or Normed linear space or Banach space. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1 - a_n)x_n + a_n T x_n \qquad \text{(v)}$$

for $n = 0,1,2,\ldots$ *and* $x_0 \in X$ is called ***Mann iterative Scheme***. Here $\{a_n\}$ is a sequence of non-negative numbers satisfying (a) $a_0 = 1$, (b) $0 \le a_n < 1$ and (c) $\sum_{n=1}^{\infty} a_n = \infty$, *i.e.*, $\{a_n\}$ is divergent.

Definition 3.6.2, (see [6]). Let S be a non-empty bounded closed convex subset of a Banach space B. Let T be a mapping from S into S i.e., $T: S \to S$. The contractive definition is defined as follows:

$$\| T(x) - T(y) \| \le k \max\{c \| x - y \|, [\| x - T(x) \| + \| y - T(y) \|], [\| x - T(y) \| + \| y - T(x) \|]\} \qquad (1)$$

for all $x, y \in S$, where $k, c \ge 0, 0 \le k < 1$.

Theorem 3.6.3. *Let S be a non-empty bounded closed convex subset of a Banach space B and T be a map from S into S satisfying the contractive definition (3.6.2). Let $\{x_n\}$*

be a sequence in S defined by (v) *(Mann iterative scheme). Now, if* $\{x_n\}$ *converges, then it converges to a fixed point of* T.

Proof. Suppose that, $\lim_{n\to\infty} x_n = r$, where r is any finite number. Now, we are to show that r is the fixed point of T i.e., $F(r) = r$.

By (v) we have, $x_{n+1} = (1-a_n)x_n + a_n T(x_n)$

$$i.e., x_{n+1} - x_n = a_n[T(x_n) - x_n]$$

Since, $\lim_{n\to\infty} x_n = r$ and therefore,

$$\lim_{n\to\infty} a_n[T(x_n) - x_n] = 0, \quad i.e., \lim_{n\to\infty}[T(x_n) - x_n] = 0.$$

This gives $\| x_n - T(x_n) \| \to 0 \quad as \quad n \to \infty$.

By definition 3.6.2 we have,

$$\| T(x_n) - T(r) \| \le k \max\{c \| x_n - r \|, [\| x_n - T(x_n) \| + | \\ | r - T(r) \|], [\| x_n - T(r) \| + \| r - T(x_n) \|]\}$$

$$\text{But, } \| r - T(r) \| \le \| r - x_n \| + \| x_n - T(x_n) \| + \| T(x_n) - T(r) \|$$

$$\| x_n - T(r) \| \le \| x_n - T(x_n) \| + \| T(x_n) - T(r) \|$$

$$\| r - T(x_n) \| \le \| r - x_n \| + \| x_n - T(x_n) \|.$$

Therefore,

$$\| T(x_n) - T(r) \| \le k \max\{c \| x_n - r \|, [2 \| x_n - T(x_n) \| + \\ \| r - x_n \| + \| T(x_n) - T(r) \|], [2 \| x_n - T(x_n) \| + \\ \| r - x_n \| + \| T(x_n) - T(r) \|]\}$$

$$= k \max\{c \| x_n - r \|, [2 \| x_n - T(x_n) \| + \| r - x_n \| + \| T(x_n) - T(r) \|]\}$$

Now, since $x_n \to r$ and $\| x_n - T(x_n) \| \to 0 \quad as \quad n \to \infty$.

$$\therefore \lim_{n\to\infty} \| T(x_n) - T(r) \| = 0$$

Also we have, $\| r - F(r) \| \le \| r - x_n \| + \| x_n - F(x_n) \| + \| F(x_n) - F(r) \|$.

So, $\lim_{n\to\infty} \| r - T(r) \| = 0$. This implies that, $r - T(r) = 0$. $\therefore T(r) = r$.

This completes our proof. ■

Theorem 3.6.4. *Let* S *be a non-empty closed convex subset of a uniformly convex Banach space* B. *Let* $T : S \to S$ *satisfying the contractive definition (3.6.2) and such that* $T(S)$ *is relatively compact. If* $F(T)$ *the fixed point set of* T *is non-empty, then Mann* *iterative process* (v) *with* $\{a_n\}$ *satisfying (a), (b) & (c) converges to a fixed point of* T.

Proof. Suppose, $r \in F(T)$ then,

$$\| T(x_n) - r \| = \| T(x_n) - T(r) \| \le k \max\{c \| x_n - r \|, [\| x_n - T(x_n) \| + \| r - T(r) \|], [\| x_n - T(r) \| + \| r - T(x_n) \|]\}$$

$$\le k \max\{c \| x_n - r \|, [\| x_n - r \| + \| r - T(x_n) \| + \| r - T(r) \|], [\| x_n - r \| + \| r - T(r) \| + \| r - T(x_n) \|]\}$$

$$= k \max\{c \| x_n - r \|, [\| x_n - r \| + \| r - T(x_n) \|], [\| x_n - r \| + \| r - T(x_n) \|]\}$$

$[\because T(r) = r]$

Therefore, it follows that

$$\| T(x_n) - r \| = \| T(x_n) - T(r) \| \le k \| x_n - r \| \quad (2)$$

Now, $\| x_{n+1} - r \| = \| (1 - a_n)x_n + a_n T(x_n) - r \|$

$$= \| (1 - a_n)(x_n - r) + a_n (T(x_n) - r) \|$$

$$\le (1 - a_n) \| x_n - r \| + a_n \| T(x_n) - r \| \quad (3)$$

Now, combining (2) & (3) we get,

$$\| x_{n+1} - r \| \le \| (1 - a_n) \| x_n - r \| + k a_n \| T(x_n) - r \|$$

$$= \| (1 - a_n + k a_n) \| x_n - r \|$$

i.e., $\| x_{n+1} - r \| \le \| x_n - r \|$ [Since $0 \le k < 1$ and $0 \le a_n \le 1$] (4)

Therefore, $\{\| x_n - r \|\}$ is a monotone decreasing positive sequence, and hence converges to a real number a. Suppose, $a > 0$.

Since, $\| x_n - T(x_n) \| \le \| x_n - r \| + \| T(x_n) - r \| \le \| x_n - r \| + k \| x_n - r \| = (1 + k) \| x_n - r \|$

i.e., $\| x_n - T(x_n) \| \le (1 + k) \| x_n - r \|$ [Using (2)] (5)

So $\{\| x_n - T(x_n) \|\}$ is a bounded sequence.

Now, $\| r - x_{n+1} \| = \| r - (1 - a_n)x_n - a_n T(x_n) \|$

$$= \| (1 - a_n)(r - x_n) + a_n r - a_n T(x_n) \|$$

$$= \| (1 - a_n)(r - x_n) + (1 - a_n)(r - T(x_n)) + (2a_n - 1)(r - T(x_n)) \|$$

$$\le (1 - a_n) \| (r - x_n) + r - T(x_n) \| + (2a_n - 1) \| r - T(x_n) \|$$

$$\le (1 - a_n) \| (r - x_n) + (r - T(x_n)) \| + (2a_n - 1) \| r - x_n \|$$ [Using (2)]

$$\leq (1-a_n)\| r-x_n \| . \frac{\| (r-x_n)+(r-T(x_n)) \|}{\| r-x_n \|} + (2a_n - 1)\| r-x_n \| \qquad (6)$$

Now, we define $L = \frac{r-x_n}{\| r-x_n \|}$ & $M = \frac{r-T(x_n)}{\| r-x_n \|}$.

Then, $\| L+M \| = \frac{\| (r-x_n)+(r-T(x_n)) \|}{\| r-x_n \|}$

$$\| L \| = \frac{\| r-x_n \|}{\| r-x_n \|} \leq 1 \quad i.e., \quad \| L \| \leq 1$$

$$\| M \| = \frac{\| r-T(x_n) \|}{\| r-x_n \|} = \frac{|-1| . \| T(x_n)-r \|}{|-1| . \| x_n - r \|} \leq \frac{k \| x_n - r \|}{\| x_n - r \|} \leq 1 \quad i.e., \quad \| M \| \leq 1 \text{ [Using (2)]}$$

and $\| L-M \| = \frac{\| r-x_n-r+T(x_n) \|}{\| r-x_n \|} = \frac{\| T(x_n)-x_n \|}{\| r-x_n \|}$

So, from (6) we get

$$\| r-x_{n+1} \| \leq (1-a_n)\| r-x_n \| . \| L+M \| + (2a_n - 1)\| r-x_n \| \qquad (7)$$

So, we get

$$\lim_{n\to\infty} \| L+M \| = \lim_{n\to\infty} \frac{\| (r-x_n)+(r-T(x_n)) \|}{\| r-x_n \|}$$

$$\leq \lim_{n\to\infty} \frac{\| r-x_n \| + \| r-T(x_n) \|}{\| r-x_n \|} \quad \text{[By triangle inequality]}$$

$$= \lim_{n\to\infty} \frac{\| r-x_n \|}{\| r-x_n \|} + \lim_{n\to\infty} \frac{\| r-T(x_n) \|}{\| r-x_n \|}$$

$$= \lim_{n\to\infty} \frac{\| r-x_n \|}{\| r-x_n \|} + \lim_{n\to\infty} \frac{|-1| . \| T(x_n)-r \|}{\| x_n - r \|}$$

$$\leq \lim_{n\to\infty} \frac{\| r-x_n \|}{\| r-x_n \|} + \lim_{n\to\infty} \frac{k \| x_n - r \|}{\| x_n - r \|} \quad \text{[Using (2)]}$$

$$\leq 2$$

i.e., $\lim_{n\to\infty} \| L+M \| \leq 2$ *i.e.,* $\| L+M \| \to 2$ *as* $n \to \infty$

Therefore, from (7) we get

$$\| r-x_{n+1} \| \leq \| r-x_n \| \qquad (8)$$

Hence, we can also say that $\{\| r-x_n \|\}$ is a monotone decreasing positive sequence. Since, B is uniformly convex and $\| L \| \leq 1, \| M \| \leq 1$ & $\| L+M \| \to 2$. Then, we get

$$\| L-M \| \to 0. \text{ i.e., } \lim_{n\to\infty} \| L-M \| = 0 \Rightarrow \lim_{n\to\infty} \frac{\| x_n - T(x_n) \|}{\| r - x_n \|} = 0.$$

But, $\lim_{n\to\infty} \| r - x_n \| \neq 0$. Because, if $\lim_{n\to\infty} \| r - x_n \| = 0$ then, $x_n \to r$ and which contradicts the definition of a. Hence, $\lim_{n\to\infty} \| x_n - T(x_n) \| = 0$

Since, $T(S)$ is relatively compact, there exists a subsequence $\{T(x_{n_i})\}$ of $\{F(x_n)\}$such that $\lim_i [T(x_{n_i})] = d \in B$, (say)

But, $\| x_{n_i} - d \| \leq \| x_{n_i} - T(x_{n_i}) \| + \| T(x_{n_i}) - d \| \to 0$ as $i \to \infty$.

Now, $\| T(d) - d \| \leq \| T(d) - T(x_{n_i}) \| + \| T(x_{n_i}) - d \|$

$$\leq k \max\{c \| d - x_{n_i} \|, [\| x_{n_i} - T(x_{n_i}) \| + \| d - T(d) \|], [\| x_{n_i} - T(d) \| + \| d - T(x_{n_i}) \|]\} + \| T(x_{n_i}) - d \|$$

$$\leq k \max\{c \| d - x_{n_i} \|, [\| x_{n_i} - T(x_{n_i}) \| + \| d - T(d) \|], [\| x_{n_i} - d \| + \| d - T(d) \| + \| d - T(x_{n_i}) \|]\} + \| T(x_{n_i}) - d \|$$

So, $\| T(d) - d \| \leq k \max\{\| d - T(d) \|, \| d - T(d) \|\}$ *as* $i \to \infty$.

Hence, $\| T(d) - d \| \leq k \| d - T(d) \|$.

This is a contradiction, because $0 \leq k < 1$ and $T(d) = d$.

Hence d is a fixed point of F *i.e.*, $d \in F(T)$.

Now, replacing r by d in (4), it follows that $\{\| x_n - d \|\}$ is monotone decreasing sequence in n.Since, $\lim_i [x_{n_i}] = d$. Then this implies that $\lim_n [x_n] = d$.

Hence, we can say that $\{x_n\}$ converges to a fixed point of T.

This completes our proof. ■

The following results was obtained by B.E. Rhoades ([3], Theorem 4) and Vasile Berinde ([53], Theorem 2.1) about the convergence of Mann iterative scheme.

Theorem 3.6.4, (see [3], Theorem 4). *Let* E *be a uniformly convex Banach space,* K *a closed convex subset of* E *and* $T: K \to K$ *a Zamfirescu operator, i.e., there exist the real numbers* a, b & c *satisfying* $0 \leq a < 1, 0 \leq b < 1, c < 1/2$ *such that for each pair* x, y *in* K *, at least one of the following is true:*

(z_1) $\|Tx - Ty\| \leq a\|x - y\|$;

(z_2) $\|Tx - Ty\| \leq b[\|x - Tx\| + \|y - Ty\|]$;

(z_3) $\quad \|Tx - Ty\| \le c[\|x - Ty\| + \|y - Tx\|]$.

Let $\{x_n\}_{n=0}^{\infty}$ *be defined by (v) and* $x_0 \in K$, *with* $\{a_n\}$ *satisfying*

$$(a)\ a_0 = 1;\ (b)\ 0 < a_n < 1\ for\ n \ge 1;\ (c)\ \sum_{n=1}^{\infty} a_n(1 - a_n) = \infty.$$

Then $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the fixed point of* T. ■

Theorem 3.6.5, (see [53], Theorem 2.1). *Let* E *be an arbitrary Banach space,* K *a closed convex subset of* E, *and* $T: K \to K$ *an operator satisfying condition **Z**, i.e., there exist the real numbers* $a, b \,\&\, c$ *satisfying* $0 \le a < 1,\ 0 \le b < 1,\ c < 1/2$ *such that for each pair* $x,\ y$ *in* X, *at least one of the following is true:*

(z_1) $\quad \|Tx - Ty\| \le a\|x - y\|$;

(z_2) $\quad \|Tx - Ty\| \le b[\|x - Tx\| + \|y - Ty\|]$;

(z_3) $\quad \|Tx - Ty\| \le c[\|x - Ty\| + \|y - Tx\|]$.

Let $\{x_n\}_{n=0}^{\infty}$ *be defined by (v) and* $x_0 \in K$, *with* $\{a_n\} \subset [0, 1]$ *satisfying* $\sum_{n=1}^{\infty} a_n = \infty$

Then $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the fixed point of* T. ■

3.7 Ishikawa iterative scheme

Definition 3.7.1, (see [41]). Let $T: X \to X$ be a given operator and X be a Metric space or Normed linear space or Banach space. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left.\begin{aligned} x_{n+1} &= (1 - a_n)x_n + a_n T y_n \\ y_n &= (1 - b_n)x_n + a_n T x_n \end{aligned}\right\} \qquad \text{(vi)}$$

for $n = 0, 1, 2, \ldots$ *and* $x_0 \in X$ is called ***Ishikawa iterative Scheme***. Here $\{a_n\}$ and $\{b_n\}$ are sequences of non-negative numbers such that (a) $0 \le a_n < 1$ and $0 \le b_n < 1$, (b) $\limsup_{n\to\infty}(b_n) < 1$.

Now, we state some convergence theorems of Ishikawa iterative scheme, which are given by B.E. Rhoades [3], Vasile Berinde [50], Kalishankar, Tiwary and S.C. Debnath [29].

Theorem 3.7.2, (see [3], Theorem 8). *Let* E *be a uniformly convex Banach space,* K *a closed convex subset of* E *and* $T: K \to K$ *be an operator satisfying Zamfirescu* *condition* **Z,** *i.e., there exist the real numbers* $a, b \& c$ *satisfying* $0 \le a < 1, 0 \le b < 1, c < 1/2$ *such that for each pair* x, y *in* K, *at least one of the following is true:*

(z_1) $\|Tx - Ty\| \le a\|x - y\|$;

(z_2) $\|Tx - Ty\| \le b[\|x - Tx\| + \|y - Ty\|]$;

(z_3) $\|Tx - Ty\| \le c[\|x - Ty\| + \|y - Tx\|]$.

Let $\{x_n\}_{n=0}^{\infty}$ *be defined by (vi) and* $x_0 \in K$, *with* $\{a_n\}$ *and* $\{b_n\}$ *are sequences of numbers in* $[0, 1]$ *satisfying* $\sum_{n=1}^{\infty} a_n(1-a_n) = \infty$.

Then $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the fixed point of* T. ■

Theorem 3.7.3, (see [50], Theorem 2). *Let* E *be an arbitrary Banach space,* K *a closed convex subset of* E, *and* $T : K \to K$ *an operator satisfying Zamfirescu condition* **Z,** *i.e., there exist the real numbers* $a, b \& c$ *satisfying* $0 \le a < 1, 0 \le b < 1, c < 1/2$ *such that for each pair* x, y *in* K, *at least one of the following is true:*

(z_1) $\|Tx - Ty\| \le a\|x - y\|$;

(z_2) $\|Tx - Ty\| \le b[\|x - Tx\| + \|y - Ty\|]$;

(z_3) $\|Tx - Ty\| \le c[\|x - Ty\| + \|y - Tx\|]$.

Let $\{x_n\}_{n=0}^{\infty}$ *be the Ishikawa iterative scheme defined by (vi) and* $x_0 \in K$, *with* $\{a_n\}$ *and* $\{b_n\}$ *are sequences of positive numbers satisfying* $\sum_{n=1}^{\infty} a_n = \infty$.

Then $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the fixed point of* T. ■

Theorem 3.7.4, (see [29], Theorem 1). *Let* S *be a non-empty bounded closed convex subset of a Banach space* B *and* T *be a map from* S *into* S *satisfying the contractive definition*

$$\| T(x) - T(y) \| \le \max\{\| x - y \|, \frac{1}{2}[\| x - T(x) \| + \| y - T(y) \|], \frac{1}{2}[\| x - T(y) \| + \| y - T(x) \|]\}.$$

Let $\{x_n\}$ *be a sequence in* S *defined by (vi) (Ishikawa iterative scheme). Now, if* $\{x_n\}$ *converges, then it converges to a fixed point of* T. ■

Theorem 3.7.5, (see [29], Theorem 2). *Let S be a non-empty closed convex subset of a uniformly convex Banach space B. Let $T : S \to S$ satisfying the contractive definition*

$$\| T(x)-T(y) \| \le \max\{\| x-y \|, \frac{1}{2}[\| x-T(x)\| + \| y-T(y)\|], \frac{1}{2}[\| x-T(y)\| + \| y-T(x)\|]\}.$$

and such that $T(S)$ is relatively compact. If $F(T)$ the fixed point set of T is non-empty, then Ishikawa iterative process (vi), $\{x_n\}$ converges to a fixed point of T. ■

3.8 Noor iterative scheme

Definition 3.8.1, (see [33, 34]). Let $T : X \to X$ be a given operator and X be a Metric space or Normed linear space or Banach space. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left.\begin{aligned} x_{n+1} &= (1-a_n)x_n + a_n T y_n \\ y_n &= (1-b_n)x_n + b_n T z_n \\ z_n &= (1-c_n)x_n + c_n T x_n, \end{aligned}\right\} \qquad \text{(vii)}$$

for $n = 0,1,2,....$ *and* $x_0 \in X$ is called ***Noor iterative scheme.*** Where the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ *and* $\{c_n\}_{n=0}^{\infty} \subset [0, 1]$ are convergent, such that

$$\lim_{n\to\infty} a_n = 0,\ \lim_{n\to\infty} b_n = 0,\ \lim_{n\to\infty} c_n = 0 \text{ and } \sum_{n=1}^{\infty} a_n = \infty \qquad \text{(vii.a)}$$

Theorem 3.8.2. *Let X be an arbitrary Banach space, B be a nonempty closed convex subset of X and $T : B \to B$ be an operator satisfying the Zamfirescu condition **Z**, i.e., there exist the real numbers a, b & c satisfying $0 \le a < 1$, $0 \le b < 1$, $c < 1/2$ such that for each pair x, y in B, at least one of the following is true:*

(z_1) $\|Tx-Ty\| \le a\|x-y\|$;

(z_2) $\|Tx-Ty\| \le b[\|x-Tx\| + \|y-Ty\|]$;

(z_3) $\|Tx-Ty\| \le c[\|x-Ty\| + \|y-Tx\|]$.

Let $p \in F(T)$ be a fixed point of T, where $F(T)$ denotes the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration defined by (vii) *and* (vii.a) *and $x_0 \in B$, where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences of positive numbers in $[0, 1]$ with $\{a_n\}$ satisfying* (vii.a). *Then The Noor iterative scheme strongly converges to the fixed point $p \in F(T)$.*

Proof. By theorem 3.3.3, we know that T has a unique fixed point in B, say p. Consider $x, y \in B$. Since T is a Zamfirescu operator, therefore, at least one of the conditions (z_1), (z_2) and (z_3) is satisfied by T.

If (z_2) holds, then

$$\begin{aligned}\|Tx-Ty\| &\le b[\|x-Tx\|+\|y-Ty\|]\\ &\le b[\|x-Tx\|+[\|y-x\|+\|x-Tx\|+\|Tx-Ty\|]]\end{aligned}$$

$$\Rightarrow (1-b)\|Tx-Ty\| \le b\|x-y\|+2b\|x-Tx\|$$

$$\Rightarrow \|Tx-Ty\| \le \frac{b}{(1-b)}\|x-y\|+2\frac{b}{(1-b)}\|x-Tx\| \tag{1}$$

If (z_3) holds, then similarly we obtain

$$\|Tx-Ty\| \le \frac{c}{(1-c)}\|x-y\|+2\frac{c}{(1-c)}\|x-Tx\| \tag{2}$$

Let us denote

$$\lambda = \max\left\{a, \frac{b}{(1-b)}, \frac{c}{(1-c)}\right\} \tag{3}$$

Then we have, $0 \le \lambda < 1$ and in view of (z_1), (1) and (2) we get the following inequality

$$\|Tx-Ty\| \le \lambda\|x-y\|+2\lambda\|x-Tx\| \quad \text{holds } \forall\ x, y \in B \tag{4}$$

Now, let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration defined by (vii) *and* (vii.a) *and* $x_0 \in B$ arbitrary. Then

$$\begin{aligned}\|x_{n+1}-p\| &= \|(1-a_n)x_n + a_nTy_n - (1-a_n+a_n)p\|\\ &= \|(1-a_n)(x_n-p)+a_n(Ty_n-p)\|\\ &\le (1-a_n)\|x_n-p\|+a_n\|Ty_n-p\|\end{aligned} \tag{5}$$

With $x = p$ and $y = y_n$ from (4) we obtain

$$\|Ty_n - p\| \le \lambda\|y_n - p\| \tag{6}$$

where λ is given by (3).

Further we have

$$\begin{aligned}\|y_n-p\| &= \|(1-b_n)x_n + b_nTz_n - (1-b_n+b_n)p\|\\ &\le (1-b_n)\|x_n-p\|+b_n\|Tz_n-p\|\end{aligned} \tag{7}$$

Again by (4), this time with $x = p$ and $y = z_n$ we find that

$$\|Tz_n - p\| \le \lambda\|z_n - p\| \tag{8}$$

Combining (6), (7) and (8) we obtain,

$$\|Ty_n - p\| \le \lambda[(1-b_n)\|x_n - p\| + \lambda b_n\|z_n - p\|]$$

Now, (9)

$$\|z_n - p\| = \|(1-c_n)x_n + c_n Tz_n - (1-c_n+c_n)p\|$$
$$\le (1-c_n)\|x_n - p\| + c_n\|Tx_n - p\|$$

From, (9) and (10) we get, (10)

$$\|Ty_n - p\| \le \lambda[(1-b_n)\|x_n - p\| + \lambda b_n[(1-c_n)\|x_n - p\| + c_n\|Tx_n - p\|]]$$

Again by (4), with $x = p$ and $y = x_n$ we find that (11)

$$\|Tx_n - p\| \le \lambda\|x_n - p\|$$

Now, combining (5), (11) and (12) we obtain, (12)

$$\|x_{n+1} - p\| \le (1-a_n)\|x_n - p\| + a_n[\lambda[(1-b_n)\|x_n - p\| + \lambda b_n[(1-c_n)\|x_n - p\| + \lambda c_n\|x_n - p\|]]]$$
$$= [1-(1-\lambda)(1+\lambda b_n + \lambda^2 b_n c_n)a_n]\|x_n - p\|$$

Since, $[1-(1-\lambda)(1+\lambda b_n + \lambda^2 b_n c_n)a_n] \le [1-(1-\lambda)a_n]$ (13)

So, from (13) we get,

$$\|x_{n+1} - p\| \le [1-(1-\lambda)a_n]\|x_n - p\|, \qquad n = 0,1,2,.... \tag{14}$$

By (14) we inductively obtain

$$\|x_{n+1} - p\| \le \prod_{n=0}^{\infty}[1-(1-\lambda)a_k]\|x_0 - p\| \quad n = 0,1,2,... \tag{15}$$

Using the fact that $0 \le \lambda < 1$, $a_n, b_n, c_n \in [0, 1]$ and $\sum_{n=0}^{\infty} a_n = \infty$, we obtain that,

$$\lim_{n\to\infty}\prod_{n=0}^{\infty}[1-(1-\lambda)a_k] = 0 \tag{16}$$

Now, from (15) and (16), we obtain

$$\lim_{n\to\infty}\|x_{n+1} - p\| = 0$$

i.e., $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point p.

This completes our proof. ■

Definition 3.8.3, (see [1, 5, 32]). Let $F(T) = \{p \in X : T(p) = p\}$, $p \in F(T)$. Consider

$$\eta_n = \|x_{n+1} - (1-a_n)x_n - a_n Tx_n\| \tag{17.a}$$
$$\mu_n = \|x_{n+1} - (1-a_n)x_n - a_n Ty_n\| \tag{17.b}$$
$$\xi_n = \|x_{n+1} - (1-a_n)x_n - a_n Ty_n\| \tag{17.c}$$

If $\lim_{n\to\infty}\eta_n = 0$, $\lim_{n\to\infty}\mu_n = 0$ *and* $\lim_{n\to\infty}\xi_n = 0$, then the iterative schemes (v), (vi) and (vii) respectively are said to be *T-stable*.

Lemma 3.8.4. Let X be a Banach space, B be a nonempty, convex subset of X and $T: B \to B$ be a Zamfirescu operator. If the Noor (respectively Ishikawa and Mann) iterative scheme converges, then $\lim_{n\to\infty} \xi_n = 0$ (respectively $\lim_{n\to\infty} \eta_n = 0$ and $\lim_{n\to\infty} \mu_n = 0$).

Proof of Lemma 3.8.4. Let $\lim_{n\to\infty} x_n = x^*$. Then from (17.c) we have

$$0 \le \xi_n = \|x_{n+1} - (1-a_n)x_n - a_n Tx_n\|$$
$$= \|x_{n+1} - x_n + a_n(x_n - Tx_n)\|$$
$$\le \|x_{n+1} - x_n\| + a\|x_n - Tx_n\|$$
$$\le \|x_{n+1} - x^*\| + \|x_n - x^*\| + a_n\|x_n - x^*\| + a\|x^* - Tx_n\|$$
$$\to 0 \quad \text{as } n \to \infty.$$

i.e., $\lim_{n\to\infty} \xi_n = 0$.

■

Now, we state and prove a Stablity theorem for Noor iterative scheme.

Theorem 3.8.5. *Let* X *be an arbitrary Banach space,* B *be a nonempty closed convex subset of* X *and* $T: B \to B$ *be an operator satisfying the condition* **Z**, *which is defined in 4.2.2 i.e.,* $T: B \to B$ *be a Zamfirescu operator. Let* $p \in F(T)$ *be a fixed point of* T, *where* $F(T)$ *denotes the set of fixed points of* T. *Let* $\{x_n\}_{n=0}^{\infty}$ *be the Noor iteration defined by* (vii) *and* (vii.a) *and* $x_0 \in B$, *where* $\{a_n\}, \{b_n\}$ *and* $\{c_n\}$ *are sequences of positive numbers in* $[0, 1]$ *with* $\{a_n\}$ *satisfying* (vii.a). *Then the Noor iterative scheme is T-stable.*

Proof. From the definition 3.8.3, we can say that *the Noor iteration defined by* (vii) *and* (vii.a) will be *T-stable* if $\lim_{n\to\infty} \xi_n = 0$, where $\xi_n = \|x_{n+1} - (1-a_n)x_n - a_n Tx_n\|$.

Now, from the lemma 3.8.4, we observed that if the *Noor iteration defined by* (vii) *and* (vii.a) converges to a fixed point of T then $\lim_{n\to\infty} \xi_n = 0$. But in our theorem 3.1 we have already proved that the *Noor iteration defined by* (vii) *and* (vii.a) is strongly convergent to a fixed point of T.

So, by combining our theorem 3.8.2 and lemma 3.8.4 we obtain $\lim_{n\to\infty} \xi_n = 0$ and this prove that *the Noor iterative scheme is T-stable.*

This completes our theorem.

■

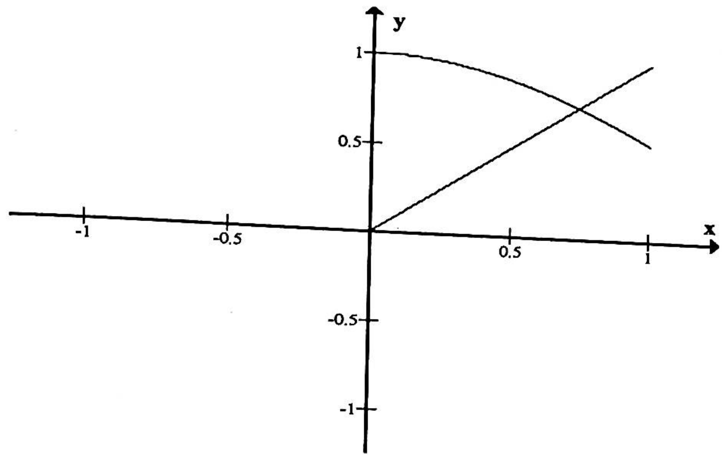

Figure-3.1

Example 3.8.6. Let $X = R$ (set of all real numbers), $B = [0, 1]$ and $T : B \to B$ be a Zamfirescu operator defined by $Tx = \cos(x)$. Then it is clear from figure-3.1 that $p = .739 \in B$ is a fixed point of T. Now, let us choose the sequences $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ & $\{c_n\}_{n=0}^{\infty}$ such that $a_n = \frac{1}{n+1}, b_n = \frac{1}{n+2}$ & $c_n = \frac{1}{n+3}$ respectively and $x_0 = 0.2 \in B$ (arbitrary). Then, all conditions of our theorem 3.8.2 are satisfied. A few steps of Noor iteration scheme calculating by MATLAB programming are given below:

Iteration Number(n)	Approximated value obtained by Noor iteration Scheme (x_n)	Iteration Number(n)	Approximated value obtained by Noor iteration Scheme (x_n)
$n=1$	0.552163335379582	$n=9$	0.72455871969895
$n=2$	0.63964429783674	$n=10$	0.72664614741441
$n=3$	0.67586208709879	....	....
$n=4$	0.69472497853525	$n=100$	0.73876771961646
$n=5$	0.70593959759843	....	....
$n=6$	0.71321174442310	$n=150$	0.73892290777246
$n=7$	0.71822682756174	....	....
$n=8$	0.72184831852270	$n=282$	0.739

Table-3.1

Under the same condition, if we choose, $x_0 = 0.9 \in B$, then we obtain the following results:

Iteration Number(n)	Approximated value obtained by Noor iteration Scheme (x_n)	Iteration Number(n)	Approximated value obtained by Noor iteration Scheme (x_n)
$n=1$	0.78933005771077	$n=9$	0.74283687263002
$n=2$	0.76527527920347	$n=10$	0.74229610349214
$n=3$	0.75560017767153	....	....
$n=4$	0.75062298329500	$n=100$	0.73916548378602
$n=5$	0.74768373276426	....	....
$n=6$	0.74578572342660	$n=150$	0.73912687797785
$n=7$	0.74448048233644	....	....
$n=8$	0.74353983009123	$n=282$	0.739

<u>**Table-3.2**</u>

3.9 The multi-step iterative scheme

Definition 3.9.1, (see [15]). Let $T: X \to X$ be a given operator and X be a Metric space or Normed linear space or Banach space. Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\left.\begin{aligned} y_n^{q-1} &= (1-b_n^{q-1})x_n + b_n^{q-1}Tx_n, \\ y_n^i &= (1-b_n^i)x_n + b_n^i Ty_n^{i+1}, \quad i = 1,2,........,q-2; \\ x_{n+1} &= (1-a_n)x_n + a_n Ty_n^1 \end{aligned}\right\} \quad \text{(viii)}$$

where, $\{a_n\} \subset (0,\ 1)$, $\{b_n^i\} \subset [0,\ 1)$, $1 \le i \le q-1$ for $n = 0,1,2,...$ *and* $x_0 \in X$ is called ***multi step iterative scheme.***

3.10 Halpern iterative scheme

Definition 3.10.1, (see [52]). Let $T: X \to X$ be a given operator and X be a Metric space or Normed linear space or Banach space. Then the sequence $\{r_n\}_{n=0}^{\infty}$ defined by

$$r_{n+1} = (1-a_n)u + a_n Tr_n \quad \text{(ix)}$$

for $n = 0,1,2,.....$, $r_0 \in X$, $u \in X$ *and* $\{a_n\}_{n=0}^{\infty} \subset [0,1]$ is called ***Halpern iterative scheme.***

CHAPTER-4

EQUIVALENCE OF ITERATIVE SCHEMES IN BANACH SPACES

CHAPTER-4

EQUIVALENCE OF ITERATIVE SCHEMES IN BANACH SPACES

4.1 Introduction

The first theorem on fixed point was established by Polish Mathematician Stefan Banach [40] in 1922. This theorem is known as Banach fixed point theorem or Contraction mapping theorem. Banach fixed point theorem has been applied to many different areas. About this theorem we have already discussed in our Chapter-1 in the art 1.5.2.

The Mann iterative scheme, known as one-step iterative scheme is invented in 1953, by W.R. Mann [54] was used to prove the convergence of the sequence to a fixed point of many valued mapping for which the Banach fixed point theorem 1.5.2 failed. Later, in 1974 Ishikawa [41] devised a new iteration scheme known as two-step iterative scheme to establish the convergence of Lipschitzian pscudocontractive map when Mann iteration scheme failed to converge. M.A. Noor [33, 34] introduced and analyzed three-step iterative scheme to study the approximate solutions of variational inclusions (inequalities) in Hilbert spaces by using the techniques of updating the solution and the auxiliary principle. B. Xu and M.A. Noor [8] studied the convergence of Noor iterative scheme to fixed point of an asymptotically non-expensive self map defined in a closed, bounded and convex subset of a uniformly convex Banach space. A bulk of literature now exist around the theme of establishing the convergence of the Mann iteration for certain classes of mapping and then showing that the Ishikawa and Noor iterations also converges. In fact, proving the convergence of Ishikawa and Noor iterations the convergence of the corresponding Mann iteration can be obtained. Indeed, in many cases, if Mann iterative sequence for mapping T converges then, Ishikawa and Noor iterative sequences also converge for that mapping. But this cannot be proved in general. In the light of this fact, recently, in a series of papers [9-17], B. E. Rhoades and S.M. Soltuz, proved that Mann and Ishikawa iteration schemes are equivalent for several classes of mapping such as Lipschitzian, strongly pscudocontractive, strongly hemicontractive, strongly accretive, strongly successively pscudocontractive, strongly successively hemicontractive mapping and Krishna Kumar in [28] proved that Mann and Ishikawa schemes are equivalent for the class of uniformly pscudocontractive operators. In [51], the following open question was given: "are Krasnoselskii's iteration and Mann iteration equivalent for enough large classes of mappings?"

We shall give a positive answer to this question: if Krasnoselskii's iteration converges, then Mann (and the corresponding Ishikawa iteration) also converges and conversely, dealing with maps satisfying Zamfirescu's condition (Z). Note that B. E. Rhoades and S.M. Soltuz have already given a positive answer in [16] for the class of pseudocontractive maps.

In the present chapter, we will show that the equivalencies of Mann iterative scheme to the Ishikawa iterative scheme, Krasnoselskii's iterative scheme to Mann iterative scheme, Mann iterative scheme to Ishikawa iterative scheme to Noor iterative scheme and Mann iterative scheme to Multi step iterative schemes for the class of Zamfirescu operator, which is described over the Banach space. In end of this chapter, we will show that the equivalencies of the T – stability of Mann iterative scheme to the T – stability of Ishikawa iterative scheme and the T – stability of Mann iterative scheme to T – stability of Ishikawa iterative scheme to T – stability of Noor iterative scheme for the same situation.

4.2 Zamfirescu operator

In 1972, T. Zamfirescu [47] obtained a very interesting fixed point theorem which is stated as follows:

Theorem 4.2.1(Z), (see [47]). *Let X be a Banach space and $T: X \to X$ be a map for which there exist the real numbers a, b and c satisfying $0<a<1,\ 0<b,c<1/2$ such that for each pair x, y in X at least one of the following is true:*

$$(z_1) \quad \|Tx - Ty\| \le a\|x - y\|;$$

$$(z_2) \quad \|Tx - Ty\| \le b[\|x - Tx\| + \|y - Ty\|];$$

$$(z_3) \quad \|Tx - Ty\| \le c[\|x - Ty\| + \|y - Tx\|].$$

Then T have a unique fixed point p and the Picard iteration (ii), $\{p_n\}_{n=0}^{\infty}$ defined by $p_{n+1} = Tp_n, \quad n = 0,1,2,...$ converge to p for any $p_0 \in X$. ∎

Definition 4.2.2, (see [47]). Let X be a Banach space. Then the operator $T: X \to X$ is called Zamfirescu operator if it satisfies one of the conditions (z_1), (z_2) and (z_3) The class of Zamfirescu operator T is one of the most studied class of quasi-contractive type operators, for which all important fixed point iteration schemes, i.e., Picard [51], Mann [54] and Ishikawa [41] iterations, are known to converge to the unique fixed point of T. Zamfirescu showed in [47] that an operator satisfying condition **Z** has a unique fixed point that can be approximated using the Picard iteration. Later, B.E. Rhoades [3] proved that the Mann and Ishikawa iterations can also be used to approximate fixed points of Zamfirescu operator. The class of operators satisfying condition **Z** is independent, see B.E. Rhoades [3], of the class of strictly (strongly) pseudocontractive operators, extensively studied by several authors in the last years. The set of fixed points of the operator T is denoted by $F(T) = \{p \in X : Tp = p\}$.

4.3 Equivalence of Mann and Ishikawa iterative schemes

Theorem 4.3.1. *Let X be an arbitrary Banach space, B be a nonempty closed convex subset of X and $T: B \to B$ be an operator satisfying condition **Z** i.e., $T: B \to B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T where $F(T)$ denotes the set of fixed points of T. Let $\{u_n\}_{n=0}^{\infty}$ be the Mann iteration defined by (v) and $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by (vi). Then the following assertions are equivalent:*

(i) The Mann iterative scheme converges to p;

(ii) The Ishikawa iterative scheme converges to p.

Proof. First we show that (ii) $\Rightarrow$ (i).

Suppose that the Ishikawa iteration scheme converges to p. Then it is clear that this p is a fixed point of T i.e., $Tp = p$.

By setting, $b_n = 0 \ \forall \ n \in \mathrm{N}$ (set of all natural numbers), in (vi) we obtain the convergence of Mann iterative scheme (v).

Conversely, we shall show that (i) $\Rightarrow$ (ii) i.e., the convergence of Mann iterative scheme to the fixed point p implies the convergence of Ishikawa iterative scheme to the fixed point p.

Now, by Theorem 4.2.1(Z) we know that T has a unique fixed point in B, say p. Consider $x,\ y \in B$. Since T is a Zamfirescu operator, therefore, at least one of the conditions (z_1), (z_2) and (z_3) is satisfied by T.

If (z_2) holds, then

$$\begin{aligned} \|Tx - Ty\| &\le b[\|x - Tx\| + \|y - Ty\|] \\ &\le b[\|x - Tx\| + [\|y - x\| + \|x - Tx\| + \|Tx - Ty\| \end{aligned} \tag{1}$$

$$\Rightarrow \|Tx - Ty\| \le \frac{b}{(1-b)}\|x - y\| + \frac{2b}{(1-b)}\|x - Tx\|$$

If (z_3) holds, then similarly we obtain

$$\|Tx - Ty\| \le \frac{c}{(1-c)}\|x - y\| + \frac{2c}{(1-c)}\|x - Tx\| \tag{2}$$

Let us denote $\lambda = \max\left\{a,\ \frac{b}{(1-b)},\ \frac{c}{(1-c)}\right\}$ (3)

Then we have, $0 \le \lambda < 1$ and in view of (z_1), (1) and (2) we get the following inequality

$\|Tx - Ty\| \le \lambda\|x - y\| + 2\lambda\|x - Tx\|$ holds $\forall\ x,\ y \in B$. (4)

Since $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iterative scheme defined by (vi) and $\{u_n\}_{n=0}^{\infty}$ be the Mann iterative scheme defined by (v). Therefore, we get

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &= \|(1-a_n)x_n + a_nTy_n - (1-a_n)u_n + a_nTu_n\| \\ &= \|(1-a_n)(x_n - u_n) + a_n(Ty_n - Tu_n)\| \\ &\le (1-a_n)\|x_n - u_n\| + a_n\|Ty_n - Tu_n\| \end{aligned} \tag{5}$$

Now, according to the supposition Mann iterative scheme $\{u_n\}_{n=0}^{\infty}$ converge to the fixed point p, *i.e.*, $\lim_{n\to\infty} u_n = p$. This implies that

$\lim_{n\to\infty}[Tu_n - u_n] = Tp - p = 0$. [Since, p is a fixed point of T]

$$\Rightarrow \lim_{n\to\infty} Tu_n = \lim_{n\to\infty} u_n = p \tag{6}$$

$$\Rightarrow \lim_{n\to\infty} Tu_n = p \tag{7}$$

Again, by Mann iterative scheme we have

$$\begin{aligned} u_{n+1} &= (1-a_n)u_n + a_nTu_n \\ \Rightarrow u_{n+1} - u_n &= a_n[Tu_n - u_n] \end{aligned} \tag{8}$$

From, (6) and (8) we can write,

$$\lim_{n\to\infty}[u_{n+1}-u_n]=0 \Rightarrow \lim_{n\to\infty}u_{n+1}=p \tag{9}$$

Now, combining (5), (6) and (7) we obtain,

$$\|x_{n+1}-u_{n+1}\| \le (1-a_n)\|x_n-p\|+a_n\|y_n-p\| \tag{10}$$

By setting, $x=p$ & $y=y_n$ in (4) we get,

$$\|Ty_n-p\| \le \lambda\|y_n-p\| \tag{11}$$

where, λ is given by (3).

We have,

$$\begin{aligned}\|y_n-p\| &= \|(1-b_n)x_n+b_nTx_n-(1-b_n-b_n)p\| \\ &= \|(1-b_n)(x_n-p)+b_n(Tx_n-p)\| \\ &\le (1-b_n)\|x_n-p\|+b_n\|Tx_n-p\|\end{aligned} \tag{12}$$

Again, by setting $x=p$ & $y=x_n$ in (4) we get,

$$\|Tx_n-p\| \le \lambda\|x_n-p\| \tag{13}$$

From (12) and (13) we obtain,

$$\|y_n-p\| \le (1-b_n)\|x_n-p\|+b_n\lambda\|x_n-p\| \tag{14}$$

Now, combining (10), (11) and (14) we obtain,

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le (1-a_n)\|x_n-p\|+\lambda a_n[(1-b_n)\|x_n-p\|+\lambda b_n\|x_n-p\|] \\ &= [1-(1-\lambda)a_n(1-\lambda b_n)]\|x_n-p\| \\ &\le [1-(1-\lambda)^2a_n]\|x_n-p\|\end{aligned} \tag{15}$$

$$[\text{Since}, [1-(1-\lambda)a_n(1-\lambda b_n)] \le [1-(1-\lambda)^2a_n]]$$

By (15), we inductively obtain,

$$\|x_{n+1}-u_{n+1}\| \le \prod_{k=0}^{n}[1-(1-\lambda)^2a_k]\|x_0-p\|, \text{ where, } n=0,1,2,\ldots. \tag{16}$$

Using the fact that $0\le\lambda<1$, $a_k\in[0,1]$ & $\sum_{n=0}^{\infty}a_n=\infty$ we obtain,

$$\lim_{n\to\infty}\prod_{k=0}^{n}[1-(1-\lambda)^2a_k]=0 \tag{17}$$

Comparing, (16) and (17) we get,

$$\lim_{n\to\infty}\|x_{n+1}-u_{n+1}\|=0 \Rightarrow \lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}u_{n+1} \Rightarrow \lim_{n\to\infty}x_{n+1}=p. \text{ [By equation (9)]}$$

From, which we can say that, $\{x_n\}_{n=0}^{\infty}$ converges to the fixed point p. i.e., the Ishikawa iterative scheme converges to p.

This completes our proof. ■

Now, we give an example, which proves our theorem 4.3.1 numerically.

Example 4.3.2. Let $X = \mathrm{R}$ (set of all real numbers), $B = [0, 2]$ and $T : B \to B$ be a Zamfirescu operator defined by $Tx = \dfrac{x+1}{2}$. T has a fixed point $p = 1 \in B$. Now, let us choose the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ such that $a_n = \dfrac{1}{n+1}$ and $b_n = \dfrac{1}{n+2}$ respectively, and $u_0 = x_0 = 0.1 \in B$. Then, all conditions of our theorem 4.3.1 are satisfied.

4.4 Equivalence of Krasnoselskii's and Mann iterative schemes

Lemma 4.4.1, (see [66]). *Let $\{\alpha_n\}$ be a nonnegative sequence which satisfies the following inequality* $\alpha_{n+1} \le (1-\lambda_n)\alpha_n + \sigma_n$, (1)

where $\lambda_n \in (0, 1)$, $\forall n \ge n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, *and* $\sigma_n = 0(\lambda_n)$. *Then* $\lim_{n\to\infty} \alpha_n = 0$. ■

Theorem 4.4.2. *Let X be an arbitrary Banach space, B be a nonempty closed convex subset of X and $T : B \to B$ be an operator satisfying condition **Z** i.e., $T : B \to B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T where $F(T)$ denotes the set of fixed points of T. Let $\{u_n\}_{n=0}^{\infty}$ be the Krasnoselskii's iteration defined by* (iii) *and $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (v). *If $x_0 = u_0 \in B$, then the following assertions are equivalent:*

(a) The Krasnoselskii's iterative scheme converges to p;

(b) The Mann iterative scheme converges to p.

Proof. First we will prove *(a)* ⇒ *(b)*. That is, if Krasnoselskii's iteration converges to p, then Mann iteration does converge to p.

Suppose that the Krasnoselskii's iterative scheme converges to p. Use the equation (4) of the theorem 4.3.1 and $x = x_n$ and $y = u_n$ to obtain

$$\begin{aligned}
&\|u_{n+1} - x_{n+1}\| \\
&= \|u_n - x_n - a_n u_n + a_n x_n + a_n u_n - \lambda u_n + \lambda T x_n - a_n T u_n + a_n T u_n - a_n T x_n\| \\
&= \|(1-a_n)(u_n - x_n) + (a_n - \lambda)u_n - (a_n - \lambda)u_n T u_n + a_n (T u_n - T x_n)\| \\
&\le (1-a_n)\|u_n - x_n\| + |a_n - \lambda|\|u_n - T u_n\| + a_n \|T u_n - T x_n\| \\
&\le (1-a_n)\|u_n - x_n\| + |a_n - \lambda|\|u_n - T u_n\| + a_n \delta \|u_n - x_n\| + 2a_n \delta \|u_n - T u_n\| \\
&= (1 - a_n(1-\delta))\|u_n - x_n\| + (|a_n - \lambda| + 2a_n\delta)\|u_n - T u_n\| \ .
\end{aligned}$$

Denote $\alpha_n = \|u_n - x_n\|$, $\lambda_n = a_n(1-\delta) \subset (0,1)$ and $\sigma_n = (|a_n - \lambda| + 2a_n\delta)\|u_n - Tu_n\|$.

Since $\lim_{n\to\infty}\|u_n - p\| = 0$, T satisfies condition ***(Z)*** and $p \in F(T)$, therefore from the equation (4) of the theorem 4.3.1 we have,

$$\begin{aligned}
0 &\le \|u_n - T u_n\| \\
&\le \|u_n - p\| + \|p - T u_n\| \\
&\le (\delta + 1)\|u_n - p\| \to 0 \ \ as \ \ n \to \infty,
\end{aligned}$$

Hence $\lim_{n\to\infty}\|u_n - Tu_n\| = 0$, that is $\sigma_n = 0(\lambda_n)$. Lemma 4.4.1 leads to $\lim_{n\to\infty}\|u_n - x_n\| = 0$.

Thus $\|p - x_n\| \le \|u_n - x_n\| + \|u_n - p\| \to 0 \ as \ n \to \infty$.

This shows that the Mann iterative scheme converges to p .

Now, we will prove *(b)* $\Rightarrow$ *(a)*. That is, if Mann iteration converges to p then Krasnoselskii's iteration does converge to p .

Suppose that the Mann iterative scheme converges to p .

Use the equation (4) of the theorem 4.3.1 and $x = x_n$ and $y = y_n$ to obtain

$$\begin{aligned}
&\|x_{n+1} - u_{n+1}\| = \|u_{n+1} - x_{n+1}\| \\
&= \|u_n - x_n - \lambda u_n + \lambda x_n - \lambda x_n + a_n x_n + \lambda T u_n - \lambda T x_n + \lambda T x_n - a_n T x_n\| \\
&\le (1-\lambda)\|x_n - u_n\| + |a_n - \lambda|\|x_n - T x_n\| + \lambda \|T x_n - T u_n\| \\
&\le (1-\lambda)\|x_n - u_n\| + |a_n - \lambda|\|x_n - T x_n\| + \lambda\delta\|x_n - u_n\| + 2\lambda\delta\|x_n - T x_n\| \\
&= (1 - \lambda(1-\delta))\|x_n - u_n\| + (|a_n - \lambda| + 2\lambda\delta)\|x_n - T x_n\| \ .
\end{aligned}$$

Denote $\alpha_n = \|x_n - u_n\|$, $\lambda_n = \lambda(1-\delta) \subset (0,1)$ and $\sigma_n = (|a_n - \lambda| + 2\lambda\delta)\|x_n - Tx_n\|$.

Since $\lim_{n\to\infty}\|x_n - p\| = 0$, T satisfies condition ***(Z)*** and $p \in F(T)$, therefore from the equation (4) of the theorem 4.3.1 we have,

$$
\begin{aligned}
0 &\le \|x_n - Tx_n\| \\
&\le \|x_n - p\| + \|p - Tx_n\| \\
&\le (\delta + 1)\|x_n - p\| \to 0 \ \ as \ \ n \to \infty,
\end{aligned}
$$

Hence $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$; that is $\sigma_n = 0\ (\lambda_n)$. Lemma 4.4.1 leads to $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Use $0 \le \|p - u_n\| \le \|x_n - p\| + \|u_n - x_n\|$ to deduce $\lim_{n\to\infty} u_n = p$.

This shows that the Krasnoselskii's iterative scheme converges to p .

This completes our theorem. ■

Theorems 4.3.1 and 4.4.2 lead to the following corollary.

Corollary 4.4.3. *Let X be an arbitrary Banach space, B be a nonempty closed convex subset of X and $T : B \to B$ be an operator satisfying condition $\boldsymbol{Z}$ i.e., $T : B \to B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T where $F(T)$ denotes the set of fixed points of T. Let $\{u_n\}_{n=0}^{\infty}$ be the Krasnoselskii's iteration defined by* (iii), *$\{x_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (v) *and $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by* (vi). *If $x_0 = u_0 \in B$, then the following assertions are equivalent:*

(a) the Mann iteration (v) *converges to $p \in F(T)$;*

(b) the Ishikawa iteration (vi) *converges to $p \in F(T)$;*

(c) the Krasnoselskii's iteration (iii) *converges to $p \in F(T)$.* ■

4.5 Equivalence of Mann, Ishikawa and Noor iterative schemes

Theorem 4.5.1. *Let X be an arbitrary Banach space, B be a nonempty closed convex subset of X and $T : B \to B$ be an operator satisfying condition $\boldsymbol{Z}$ i.e., $T : B \to B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T where $F(T)$ denotes the set of fixed points of T. Let $\{u_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (v), *$\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by* (vi) *and $\{p_n\}_{n=0}^{\infty}$ be the Noor iteration defined by* (vii). *Then the following assertions are equivalent:*

(a) The Mann iterative scheme converges to $p \in F(T)$;

(b) The Ishikawa iterative scheme converges to $p \in F(T)$;

(c) The Noor iterative scheme converges to $p \in F(T)$.

Proof. We prove our theorem in the following three steps: step-1: $(a) \Leftrightarrow (b)$, step-2: $(b) \Leftrightarrow (c)$ and step-3: $(c) \Leftrightarrow (a)$.

Step-1: In this step we first prove that $(b) \Rightarrow (a)$.

Suppose that the Ishikawa iteration scheme converges to p. Then it is clear that this p is a fixed point of T. i.e., $Tp = p$.

By setting, $b_n = 0 \ \forall \ n \in \mathbb{N}$ (set of all natural numbers), in (vi) we obtain the convergence of Mann iterative scheme (v).

Conversely, we prove that $(a) \Rightarrow (b)$ i.e. the convergence of Mann iterative scheme to the fixed point p implies the convergence of Ishikawa iterative scheme to the fixed point p.

Now, by Theorem 4.2.1(Z) we know that T has a unique fixed point in B, say p. Consider $x, y \in B$.

Since T is a Zamfirescu operator, therefore, at least one of the conditions (z_1), (z_2) and (z_3) is satisfied by T.

If (z_2) holds, then

$$\begin{aligned}\|Tx - Ty\| &\leq b[\|x - Tx\| + \|y - Ty\|] \\ &\leq b[\|x - Tx\| + [\|y - x\| + \|x - Tx\| + \|Tx - Ty\| \end{aligned} \tag{1}$$

$$\Rightarrow \|Tx - Ty\| \leq \frac{b}{(1-b)}\|x - y\| + \frac{2b}{(1-b)}\|x - Tx\|$$

If (z_3) holds, then similarly we obtain

$$\|Tx - Ty\| \leq \frac{c}{(1-c)}\|x - y\| + \frac{2c}{(1-c)}\|x - Tx\| \tag{2}$$

Let us denote $\lambda = \max\left\{a, \frac{b}{(1-b)}, \frac{c}{(1-c)}\right\}$ (3)

Then we have, $0 \leq \lambda < 1$ and in view of (z_1), (1) and (2) we get the following inequality

$\|Tx - Ty\| \leq \lambda\|x - y\| + 2\lambda\|x - Tx\|$ holds $\forall \ x, y \in B$. (4)

Since $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iterative scheme defined by (vi) and $\{u_n\}_{n=0}^{\infty}$ be the Mann iterative scheme defined by (v), therefore, we get

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &= \|(1-a_n)x_n + a_nTy_n - (1-a_n)u_n + a_nTu_n\| \\ &= \|(1-a_n)(x_n-u_n) + a_n(Ty_n - Tu_n)\| \\ &\le (1-a_n)\|x_n-u_n\| + a_n\|Ty_n - Tu_n\| \end{aligned} \quad (5)$$

Now, according to the supposition Mann iterative scheme $\{u_n\}_{n=0}^{\infty}$ converge to the fixed point p. *i.e.*, $\lim_{n\to\infty} u_n = p$. This implies that

$\lim_{n\to\infty}[Tu_n - u_n] = Tp - p = 0$. [Since, p is a fixed point of T]

$$\Rightarrow \lim_{n\to\infty} Tu_n = \lim_{n\to\infty} u_n = p \quad (6)$$

$$\Rightarrow \lim_{n\to\infty} Tu_n = p \quad (7)$$

Again, by Mann iterative scheme we have

$$\begin{aligned} u_{n+1} &= (1-a_n)u_n + a_nTu_n \\ \Rightarrow u_{n+1} - u_n &= a_n[Tu_n - u_n] \end{aligned} \quad (8)$$

From, (6) and (8) we can write,

$$\lim_{n\to\infty}[u_{n+1}-u_n] = 0 \Rightarrow \lim_{n\to\infty} u_{n+1} = p \quad (9)$$

Now, combining (5), (6) and (7) we obtain,

$$\|x_{n+1}-u_{n+1}\| \le (1-a_n)\|x_n - p\| + a_n\|y_n - p\| \quad (10)$$

By setting, $x = p$ & $y = y_n$ in (4) we get,

$\|Ty_n - p\| \le \lambda\|y_n - p\|$, where, λ is given by (3). (11)

We have,

$$\begin{aligned}\|y_n - p\| &= \|(1-b_n)x_n + b_nTx_n - (1-b_n-b_n)p\| \\ &= \|(1-b_n)(x_n-p) + b_n(Tx_n - p)\| \\ &\le (1-b_n)\|x_n - p\| + b_n\|Tx_n - p\| \end{aligned} \quad (12)$$

Again, by setting $x = p$ & $y = x_n$ in (4) we get,

$$\|Tx_n - p\| \le \lambda\|x_n - p\| \quad (13)$$

From (12) and (13) we obtain,

$$\|y_n - p\| \le (1-b_n)\|x_n - p\| + b_n\lambda\|x_n - p\| \quad (14)$$

Now, combining (10), (11) and (14) we obtain,

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le (1-a_n)\|x_n-p\|+\lambda a_n[(1-b_n)\|x_n-p\|+\lambda b_n\|x_n-p\|] \\ &= [1-(1-\lambda)a_n(1-\lambda b_n)]\|x_n-p\| \\ &\le [1-(1-\lambda)^2 a_n]\|x_n-p\| \end{aligned} \tag{15}$$

$$[\text{Since}, [1-(1-\lambda)a_n(1-\lambda b_n)] \le [1-(1-\lambda)^2 a_n]]$$

By (15), we inductively obtain,

$$\|x_{n+1}-u_{n+1}\| \le \prod_{k=0}^{n}[1-(1-\lambda)^2 a_k]\|x_0-p\|, \text{ where, } n=0,1,2,\ldots \tag{16}$$

Using the fact that $0 \le \lambda < 1$, $a_k \in [0, 1]$ & $\sum_{n=0}^{\infty} a_n = \infty$ we obtain,

$$\lim_{n\to\infty}\prod_{k=0}^{n}[1-(1-\lambda)^2 a_k]=0 \tag{17}$$

Comparing, (16) and (17) we get,

$$\lim_{n\to\infty}\|x_{n+1}-u_{n+1}\|=0 \Rightarrow \lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}u_{n+1} \Rightarrow \lim_{n\to\infty}x_{n+1}=p. \text{ [By equation (9)]}$$

From, which we can say that, $\{x_n\}_{n=0}^{\infty}$ converges to the fixed point p, i.e., the Ishikawa iterative scheme converges to p.

This completes the step-1 of our proof.

Step-2: In this step we first prove that $(c) \Rightarrow (b)$.

Suppose that the Noor iteration scheme converges to p. Then it is clear that this p is a fixed point of T, i.e., $Tp = p$. By setting, $c_n = 0 \ \forall \ n \in \mathrm{N}$ (set of all natural numbers), in (viii) we obtain the convergence of Ishikawa iterative scheme (vi).

Conversely, we prove that $(b) \Rightarrow (c)$ i.e. the convergence of Ishikawa iterative scheme to the fixed point p implies the convergence of Noor iterative scheme to the fixed point p.

Since $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iterative scheme defined by (vi) and $\{p_n\}_{n=0}^{\infty}$ be the Noor iterative scheme defined by (viii), therefore, we get

$$\begin{aligned}\|p_{n+1}-x_{n+1}\| &= \|(1-a_n)p_n+a_nTq_n-(1-a_n)x_n-a_nTy_n\| \\ &= \|(1-a_n)(p_n-x_n)+a_n(Tq_n-Ty_n)\| \\ &\le (1-a_n)\|p_n-x_n\|+a_n\|Tq_n-Ty_n\| \\ &\le (1-a_n)\|p_n-x_n\|+a_n a\|q_n-y_n\| \end{aligned}$$

[Since T is a Zamfirescu operator]

$$= (1-a_n)\|p_n - x_n\| + a_n a\|(1-b_n)p_n + b_n Tr_n - (1-b_n)x_n - b_n Tx_n\|$$
$$\leq (1-a_n)\|p_n - x_n\| + a_n a(1-b_n)\|p_n - x_n\| + aa_n b_n\|Tr_n - Tx_n\|$$
$$= \{(1-a_n) + a_n a(1-b_n)\}\|p_n - x_n\| + aa_n b_n\|Tr_n - Tx_n\| \quad (18)$$

Now, according to the supposition, Ishikawa iterative scheme $\{x_n\}_{n=0}^{\infty}$ converge to the fixed point p, i.e., $\lim_{n\to\infty} x_n = p$.

$$\therefore \lim_{n\to\infty}[Tx_n - x_n] = Tp - p = 0$$
$$\Rightarrow \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_n = p \quad (19)$$
$$\Rightarrow \lim_{n\to\infty} Tx_n = p. \quad (20)$$

Again, by Ishikawa iterative scheme we have,

$$x_{n+1} = (1-a_n)x_n + a_n T[(1-b_n)x_n + b_n Tx_n]$$
$$\Rightarrow x_{n+1} = (1-a_n)x_n + a_n T[x_n - b_n(x_n - Tx_n)]$$
$$\Rightarrow x_{n+1} - x_n = -a_n x_n + a_n T[x_n - b_n(x_n - Tx_n)]$$
$$\Rightarrow \lim_{n\to\infty}(x_{n+1} - x_n) = 0$$
$$\Rightarrow \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = p. \quad (21)$$

Combining, (18), (19) and (20) we get,

$$\|p_{n+1} - x_{n+1}\| \leq \{(1-a_n) + aa_n(1-b_n)\}\|p_n - p\| + aa_n b_n\|Tr_n - p\| \quad (22)$$

Now, by setting $x = p$ & $y = r_n$ in (4) we get,

$$\|Tr_n - p\| \leq \lambda\|r_n - p\| \quad \text{[Where, } \lambda \text{ is given by (3)]} \quad (23)$$

We have,

$$\|r_n - p\| = \|(1-c_n)p_n + c_n Tp_n - (1-c_n+c_n)p\|$$
$$= \|(1-c_n)(p_n - p) + c_n(Tp_n - p)\| \quad (24)$$
$$\leq (1-c_n)\|p_n - p\| + c_n\|Tp_n - p\|$$

By, setting $x = p$ & $y = p_n$ in (4) we obtain,

$$\|Tp_n - p\| \leq \lambda\|p_n - p\| \quad \text{[Where, } \lambda \text{ is given by (3)]} \quad (25)$$

From, (24) and (25) we get,

$$\|r_n - p\| \leq (1-c_n)\|p_n - p\| + c_n\lambda\|p_n - p\| \quad (26)$$

Combining, (22), (23) and (26) we get,

$$\|p_{n+1} - x_{n+1}\| \leq \{(1-a_n) + aa_n(1-b_n)\}\|p_n - p\| + \lambda aa_n b_n(1-c_n)\|p_n - p\| + \lambda^2 aa_n b_n c_n\|p_n - p\|$$

$$= [1 - \{1 - a + (1-\lambda)(ab_n + \lambda ab_n c_n)\}a_n]\|p_n - p\|$$
$$= [1 - \{1 - a + (1-\lambda)ab_n(1 + \lambda c_n)\}a_n]\|p_n - p\|$$
$$= [1 - (1-a)a_n - (1-\lambda)aa_n b_n(1 + \lambda c_n)]\|p_n - p\|$$
$$\leq [1 - (1-a)a_n]\|p_n - p\| \quad (27)$$

By (27) we inductively obtain,

$$\|p_{n+1} - x_{n+1}\| \leq \prod_{k=0}^{n}[1 - (1-a)a_k]\|p_0 - p\|, \text{ where, } n = 0, 1, 2, \ldots \quad (27.a)$$

Now, using the fact that $0 < a < 1$, $a_k \in [0, 1]$ & $\sum_{n=0}^{\infty} a_n = \infty$ we obtain,

$$\lim_{n\to\infty} \prod_{k=0}^{n}[1 - (1-a)a_k] = 0 \quad (28)$$

Comparing, (27.a) and (28) we get,

$$\lim_{n\to\infty}\|p_{n+1} - x_{n+1}\| = 0 \Rightarrow \lim_{n\to\infty} p_{n+1} = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = p \Rightarrow \lim_{n\to\infty} p_{n+1} = p$$

[By equation (21)]

From, which we can say that, $\{p_n\}_{n=0}^{\infty}$ converges to the fixed point p. i.e., the Noor iterative scheme converges to p.

This completes the step-2 of our proof.

Step-3: In this step we first prove that $(c) \Rightarrow (a)$.

Suppose that the Noor iteration scheme converges to p. Then it is clear that this p is a fixed point of T. i.e., $Tp = p$.

By setting, $b_n = 0$ & $c_n = 0$ $\forall n \in \mathrm{N}$ (set of all natural numbers), in (vii) we obtain the convergence of Mann iterative scheme (v).

Conversely, we prove that $(a) \Rightarrow (c)$. i.e., the convergence of Mann iterative scheme to the fixed point p implies the convergence of Noor iterative scheme to the fixed point p.

Since $\{u_n\}_{n=0}^{\infty}$ be the Mann iterative scheme defined by (v), we get

$$\|p_{n+1} - u_{n+1}\| = \|(1-a_n)p_n + a_n Tq_n - (1-a_n)u_n - a_n Tu_n\| \quad (29)$$
$$\leq (1-a_n)\|p_n - u_n\| + a_n\|Tq_n - Tu_n\|$$

Now, combining (6), (7) and (29) we obtain,

$$\|p_{n+1} - u_{n+1}\| \leq (1-a_n)\|p_n - p\| + a_n\|Tq_n - y_n\| \quad (30)$$

By setting, $x = p$ & $y = q_n$ in (4) we get,

$$\|Tq_n - p\| \leq \lambda\|q_n - p\| \quad \text{[Where, } \lambda \text{ is given by (3)]} \tag{31}$$

We have,

$$\begin{aligned}\|q_n - p\| &= \|(1-b_n)p_n + b_n Tr_n - (1-b_n+b_n)p\| \\ &= \|(1-b_n)(p_n - p) + b_n(Tr_n - p)\| \\ &\leq (1-b_n)\|p_n - p\| + b_n\|Tr_n - p\|\end{aligned} \tag{32}$$

Again, by setting $x = p$ & $y = r_n$ in (4) we get,

$$\|Tr_n - p\| \leq \lambda\|r_n - p\| \tag{33}$$

We have,

$$\begin{aligned}\|r_n - p\| &= \|(1-c_n)p_n + c_n Tp_n - (1-c_n-c_n)p\| \\ &= \|(1-c_n)(p_n - p) + c_n(Tp_n - p)\| \\ &\leq (1-c_n)\|p_n - p\| + c_n\|Tp_n - p\|\end{aligned} \tag{34}$$

Again, by setting $x = p$ & $y = p_n$ in (4) we get,

$$\|Tp_n - p\| \leq \lambda\|p_n - p\| \tag{35}$$

From (34) and (35) we obtain,

$$\|r_n - p\| \leq (1-c_n)\|p_n - p\| + c_n\lambda\|p_n - p\| \tag{36}$$

From (33) and (36) we get,

$$\|Tr_n - p\| \leq \lambda[(1-c_n)\|p_n - p\| + c_n\lambda\|p_n - p\|] \tag{37}$$

From (32) and (37) we get,

$$\|q_n - p\| \leq (1-b_n)\|p_n - p\| + b_n\lambda[(1-c_n)\|p_n - p\| + c_n\lambda\|p_n - p\|] \tag{38}$$

From (31) and (38) we get,

$$\|Tq_n - p\| \leq \lambda[(1-b_n)\|p_n - p\| + b_n\lambda[(1-c_n)\|p_n - p\| + c_n\lambda\|p_n - p\|]] \tag{39}$$

Combining, (30) and (39) we get,

$$\begin{aligned}\|p_{n+1} - u_{n+1}\| &\leq (1-a_n)\|p_n - p\| + a_n[\lambda[(1-b_n)\|p_n - p\| \\ &\quad + b_n\lambda[(1-c_n)\|p_n - p\| + c_n\lambda\|p_n - p\|]]] \\ &= (1-a_n)\|p_n - p\| + a_n\lambda(1-b_n)\|p_n - p\| \\ &\quad + a_n b_n\lambda^2(1-c_n)\|p_n - p\| + a_n b_n c_n\lambda^3\|p_n - p\| \\ &= [1 - a_n(1-\lambda) - a_n b_n\lambda(1-\lambda) - a_n b_n c_n\lambda^2(1-\lambda)]\|p_n - p\| \\ &= [1 - a_n(1-\lambda)(1 + b_n\lambda + b_n c_n\lambda^2)]\|p_n - p\| \\ &\leq [1 - a_n(1-\lambda)]\|p_n - p\|\end{aligned} \tag{40}$$

By (40), we inductively obtain,

$$\|p_{n+1} - u_{n+1}\| \leq \prod_{k=0}^{n} [1-(1-\lambda)a_k] \|p_0 - p\|, \text{ where, } n = 0, 1, 2,... \tag{41}$$

Using the fact that $0 \leq \lambda < 1$, $a_k \in [0, 1]$ & $\sum_{n=0}^{\infty} a_n = \infty$ we obtain,

$$\lim_{n\to\infty} \prod_{k=0}^{n} [1-(1-\lambda)a_k] = 0 \tag{42}$$

Comparing, (41) and (42) we get,

$$\lim_{n\to\infty} \|p_{n+1} - u_{n+1}\| = 0 \Rightarrow \lim_{n\to\infty} p_{n+1} = \lim_{n\to\infty} u_{n+1} \Rightarrow \lim_{n\to\infty} p_{n+1} = p \qquad \text{[By equation (9)]}$$

From, which we can say that, $\{p_n\}_{n=0}^{\infty}$ converges to the fixed point p. i.e., the Noor iterative scheme converges to p. This completes the step-3 of our proof. ■

Now, we give an example, which proves numerically our theorem 4.4.1.

Example 4.5.2. Let $X = \mathrm{R}$ (set of all real numbers), $B = [0, 2]$ and $T : B \to B$ be a Zamfirescu operator defined by $Tx = \dfrac{x+1}{2}$. T has a fixed point $p = 1 \in B$ Now, let us choice the sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ such that $a_n = \dfrac{1}{n+1}$, $b_n = \dfrac{1}{n+2}$ and $c_n = \dfrac{1}{n+3}$ respectively and $x_0 = 0.1 \in B$. Then, all conditions of our theorem 4.4.1 are satisfied.

4.6 Equivalence of Mann and Multi step iterative schemes

We will generalize the above Theorem 4.3.1, (see also [19]), by proving that the Mann iterative scheme and Multi step iterative scheme are equivalent.

Theorem 4.6.1. *Let X be an arbitrary Banach space, B be a nonempty closed convex subset of X and $T : B \to B$ be an operator satisfying condition **Z** i.e., $T : B \to B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T where $F(T)$ denotes the set of fixed points of T. Let $\{u_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (v) *and $\{x_n\}_{n=0}^{\infty}$ be the Multi step iteration defined by* (viii). *If $x_0 = u_0 \in B$, then the following assertions are equivalent:*

(a) *The Mann iterative scheme converges to $p \in F(T)$;*

(b) *The Multi step iterative scheme* (viii) *converges to $p \in F(T)$.*

Proof. We shall use the equation (4) of the theorem 4.3.1:

$$\|Tx - Ty\| \le \delta\|x - y\| + 2\delta\|x - Tx\|, \ \forall x, y \forall B.$$

First we will prove the implication $(a) \Rightarrow (b)$. Suppose that $\lim_{n\to\infty} u_n = p$. Using $\lim_{n\to\infty}\|x_n - u_n\| = 0$, and $0 \le \|p - x_n\| \le \|u_n - p\| + \|x_n - u_n\|$ we get $\lim_{n\to\infty} x_n = p$.

Using now the definition of Mann and Multi-step iterative schemes and the equation (4) of the theorem 4.3.1 with $x = u_n$ *and* $y = y_n^1$, we have

$$\begin{aligned}\|u_{n+1} - x_{n+1}\| &\le \|(1-a_n)(u_n - x_n) + a_n(Tu_n - Ty_n^1)\| \\ &\le (1-a_n)\|u_n - x_n\| + a_n\|Tu_n - Ty_n^1\| \\ &\le (1-a_n)\|u_n - x_n\| + a_n\delta\|u_n - y_n^1\| + 2a_n\delta\|u_n - Tu_n\|\end{aligned} \qquad (1)$$

Using (1) with $x = u_n$ *and* $y = y_n^1$, we have

$$\begin{aligned}\|u_n - y_n^1\| &\le \|(1-b_n^1)(u_n - x_n) + b_n^1(u_n - Tx_n)\| \\ &\le (1-b_n^1)\|u_n - x_n\| + b_n^1\|u_n - Tx_n\| \\ &\le (1-b_n^1)\|u_n - x_n\| + b_n^1\|u_n - Tu_n\| + b_n^1\|Tu_n - Tx_n\| \\ &\le (1-b_n^1)\|u_n - x_n\| + b_n^1\|u_n - Tu_n\| + b_n^1\delta\|u_n - x_n\| + 2\delta b_n^1\|u_n - Tu_n\| \\ &= (1 - b_n^1(1-\delta))\|u_n - x_n\| + b_n^1\|u_n - Tu_n\|(1+2\delta).\end{aligned} \qquad (2)$$

Relations (1) and (2) lead to

$$\begin{aligned}\|u_{n+1} - x_{n+1}\| \le (1-a_n)\|u_n - x_n\| + a_n\delta(1 - b_n^1(1-\delta))\|u_n - x_n\| + \\ a_n b_n^1\delta\|u_n - Tu_n\|(1+2\delta) + a_n\delta\|u_n - y_n\| \\ = (1 - a_n(1 - \delta(1 - b_n^1(1-\delta))))\|u_n - x_n\| + a_n\delta\|u_n - Tu_n\|(b_n^1(1+2\delta) + 2\delta).\end{aligned} \qquad (3)$$

Denote by

$$\begin{aligned}\alpha_n &= \|u_n - x_n\|, \\ \lambda_n &= a_n(1 - \delta(1 - b_n^1(1-\delta))) \subset (0, 1), \\ \sigma_n &= a_n\delta\|u_n - Tu_n\|(b_n^1(1+2\delta) + 2\delta).\end{aligned}$$

Since, $\lim_{n\to\infty}\|u_n - p\| = 0$, T satisfies condition ***Z***, and $p \in F(T)$, from the equation (4) of the theorem 4.3.1 we obtain

$$\begin{aligned}0 &\le \|u_n - Tu_n\| \\ &\le \|u_n - p\| + \|p - Tu_n\| \\ &\le (\delta + 1)\|u_n - p\| \to 0 \ \text{as } n \to \infty.\end{aligned}$$

Hence $\lim_{n\to\infty}\|u_n - Tu_n\| = 0$, that is $\sigma_n = 0(\lambda_n)$. Lemma 4.4.1 leads to $\lim_{n\to\infty}\|u_n - x_n\| = 0$.

We will prove now that if Multi-step iteration converges then Mann iteration does, i.e., $(b)\Rightarrow(a)$. Using the equation (4) of the theorem 4.3.1 with $x=y_n^1$ and $y=u_n$, we obtain

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le \|(1-a_n)(x_n-u_n)+a_n(Ty_n^1-Tu_n)\|\\ &\le (1-a_n)\|x_n-u_n\|+a_n\|Ty_n^1-Tu_n\|\\ &\le (1-a_n)\|x_n-u_n\|+a_n\delta\|y_n^1-u_n\|+2a_n\delta\|y_n^1-Ty_n^1\|.\end{aligned} \tag{4}$$

The following relation holds

$$\begin{aligned}\|y_n^1-u_n\| &\le \|(1-b_n^1)(x_n-u_n)+b_n^1(Tx_n-u_n)\|\\ &\le (1-b_n^1)\|x_n-u_n\|+b_n^1\|Tx_n-u_n\|\\ &\le (1-b_n^1)\|x_n-u_n\|+b_n^1\|Tx_n-x_n\|+b_n^1\|x_n-u_n\|\\ &\le \|x_n-u_n\|+b_n^1\|Tx_n-x_n\|.\end{aligned} \tag{5}$$

Substituting (4) in (5), we obtain

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| &\le (1-a_n)\|x_n-u_n\|+a_n\delta(\|x_n-u_n\|+b_n^1\|Tx_n-x_n\|)+2a_n\delta\|y_n^1-Ty_n^1\|\\ &\le (1-(1-\delta)a_n)\|x_n-u_n\|+a_nb_n^1\delta\|Tx_n-x_n\|+2a_n\delta\|y_n^1-Ty_n^1\|.\end{aligned} \tag{6}$$

Denote by

$$\begin{aligned}\alpha_n &= \|x_n-u_n\|,\\ \lambda_n &= a_n(1-\delta)\subset(0,1),\\ \sigma_n &= a_nb_n^1\delta\|Tx_n-x_n\|+2a_n\delta\|y_n^1-Ty_n^1\|.\end{aligned}$$

Since, $\lim_{n\to\infty}\|x_n-p\|=0$, T satisfies condition ***Z***, and $p\in F(T)$, from the equation (4) of the theorem 4.3.1 we obtain

$$\begin{aligned}0 &\le \|x_n-Tx_n\|\\ &\le \|x_n-p\|+\|p-Tx_n\|\\ &\le (\delta+1)\|x_n-p\|\to 0 \ \text{as } n\to\infty.\end{aligned}$$

Note that $b_n^i\in[0,1),\ \forall n\ge 1,\ 1\le i\le p-1$, and the equation (4) of the theorem 4.3.1 to obtain

$$\begin{aligned}0 &\le \|y_n^1-Ty_n^1\|\\ &\le \|y_n^1-p\|+\|p-Ty_n^1\|\\ &\le (\delta+1)\|y_n^1-p\| \le (\delta+1)[(1-b_n^1)\|x_n-p\|+b_n^1\|Ty_n^2-p\|]\\ &\le (\delta+1)[\|x_n-p\|+\delta\|y_n^2-p\|]\end{aligned}$$

$$\leq (\delta+1)[\|x_n - p\| + \|y_n^2 - p\|]$$
$$\leq (\delta+1)[\|x_n - p\| + (1-b_n^2)\|x_n - p\| + b_n^2\|Ty_n^3 - p\|]$$
$$\leq (\delta+1)[\|x_n - p\| + \|x_n - p\| + \|Ty_n^3 - p\|]$$
$$\leq (\delta+1)[2\|x_n - p\| + \delta\|y_n^3 - p\|]$$
$$\leq (\delta+1)[2\|x_n - p\| + \|y_n^3 - p\|] \,.........$$
$$\leq (\delta+1)[(q-2)\|x_n - p\| + \|y_n^{p-1} - p\|]$$
$$\leq (\delta+1)[(q-2)\|x_n - p\| + (1-b_n^{q-1})\|x_n - p\| + b_n^{q-1}\|Tx_n - p\|]$$
$$\leq (\delta+1)[(q-1)\|x_n - p\| + \|Tx_n - p\|]$$
$$\leq (\delta+1)[(p-1)\|x_n - p\| + \delta\|x_n - p\|] \tag{7}$$
$$= (\delta+1)\|x_n - p\|[(q-1)+\delta] \to 0 \text{ as } n \to \infty,$$

Hence $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty}\|y_n^1 - Ty_n^1\| = 0$ that is $\sigma_n = o(\lambda_n)$. Lemma 4.4.1 and (4) lead to $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Thus, we get $\|p - u_n\| \leq \|x_n - u_n\| + \|x_n - p\| \to 0$.

This completes our theorem. ■

Theorem 4.6.1 and Corollary4.4.3 lead to the following result.

Corollary 4.6.2. *Let X be an arbitrary Banach space, B be a nonempty closed convex subset of X and $T : B \to B$ be an operator satisfying condition **Z** i.e., $T : B \to B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T where $F(T)$ denotes the set of fixed points of T. If the initial point is the same for all iterations, then the following are equivalent:*

(a) *the Mann iteration* (v) *converges to $p \in F(T)$;*

(b) *the Ishikawa iteration* (vi) *converges to $p \in F(T)$;*

(c) *the Multi-step iteration* (viii) *converges to $p \in F(T)$;*

(d) *the Noor iteration* (vii) *converges to $p \in F(T)$;*

(e) *the Krasnoselskii's iteration* (iii) *converges to $p \in F(T)$.* ■

4.7 T – Stability of the Equivalence of Mann and Ishikawa iterative schemes

Definition 4.7.1. Let $F(T) = \{p \in B : Tp = p\}$, $p \in F(T)$ .Consider

$$\eta_n = \|x_{n+1} - (1-a_n)x_n - a_n Tx_n\|$$

$$\mu_n = \|x_{n+1} - (1-a_n)x_n - a_n Ty_n\|$$

If $\lim_{n\to\infty}\eta_n = 0$ & $\lim_{n\to\infty}\mu_n = 0$ then the iterative schemes (v) and (vi) respectively is said to be T-stable.

Lemma 4.7.2. Let X be a Banach space, B be a nonempty, convex subset of X and $T: B \to B$ be a Zamfirescu operator. Now, if the Mann and Ishikawa iterative scheme converges, then $\lim_{n\to\infty}\eta_n = 0$ & $\lim_{n\to\infty}\mu_n = 0$ respectively. ■

Theorem 4.7.3. *Let X be an arbitrary Banach space, B be a nonempty closed convex subset of X and $T: B \to B$ be an operator satisfying condition* ***Z*** *i.e., $T: B \to B$ be a Zamfirescu operator. Let $p \in F(T)$ be a fixed point of T where $F(T)$ denotes the set of fixed points of T. Let $\{u_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (v) *and $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by* (vi). *Then the following assertions are equivalent:*

(a) *The Mann iterative scheme (v) is T-stable;*

(b) *The Ishikawa iterative scheme (vi) is T-stable.*

Proof. We show that $(a) \Leftrightarrow (b)$. From definition 4.7.1, $(a) \Leftrightarrow (b)$ means that $\lim_{n\to\infty}\eta_n = 0 \Leftrightarrow \lim_{n\to\infty}\mu_n = 0$. Now, $\lim_{n\to\infty}\mu_n = 0 \Rightarrow \lim_{n\to\infty}\eta_n = 0$ is obvious by setting, $b_n = 0 \ \forall n \in \mathrm{N}$(set of all natural numbers), in Ishikawa iterative scheme (vi). i.e., $(b) \Rightarrow (a)$.

Conversely, suppose that Mann iterative scheme (v) is T-stable. Using definition 4.7.1, we get

$$\lim_{n\to\infty}\eta_n = 0 \Rightarrow \lim_{n\to\infty}u_n = p$$

Now, by theorem 4.3.1we get,

$$\lim_{n\to\infty}u_n = p \Rightarrow \lim_{n\to\infty}x_n = p.$$

Using lemma 4.7.2 we have,

$$\lim_{n\to\infty}\mu_n = 0.$$

Thus, we get $\lim_{n\to\infty}\eta_n = 0 \Rightarrow \lim_{n\to\infty}\mu_n = 0$. i.e., $(a) \Rightarrow (b)$.

This completes the proof. ■

4.8 T – Stability of the Equivalence of Mann, Ishikawa and Noor iterative schemes

Theorem 4.8.1. *Let X be a Banach space, B be a nonempty closed convex subset of X and $T: B \to B$ be an operator satisfying condition* ***Z*** *i.e., $T: B \to B$ be a Zamfirescu* *operator. Let $p \in F(T)$ be a fixed point of T where $F(T)$ denotes the set of fixed points of T. Let $\{u_n\}_{n=0}^{\infty}$ be the Mann iteration defined by* (v), *$\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by* (vi) *and $\{p_n\}_{n=0}^{\infty}$ be the Noor iteration defined by* (vii). *Then the following assertions are equivalent:*

(a) *The Mann iterative scheme* (v) *is T-stable;*

(b) *The Ishikawa iterative scheme* (vi) *is T-stable;*

(c) *The Noor iterative scheme* (vii) *is T-stable.*

Proof. We prove our theorem in the following three steps: step-1: $(a) \Leftrightarrow (b)$ step-2: $(b) \Leftrightarrow (c)$ and step-3: $(a) \Leftrightarrow (c)$.

Step-1: From definition 2.3, $(a) \Leftrightarrow (b)$ means that $\lim_{n\to\infty} \eta_n = 0 \Leftrightarrow \lim_{n\to\infty} \mu_n = 0$ and so, $\lim_{n\to\infty} \mu_n = 0 \Rightarrow \lim_{n\to\infty} \eta_n = 0$ is obvious by setting, $b_n = 0\ \forall n \in \mathbf{N}$ (set of all natural numbers), in (vi).

Conversely, suppose that Mann iterative scheme (v) is *T*-stable. Using definition 4.7.1, we get

$$\lim_{n\to\infty} \eta_n = 0 \Rightarrow \lim_{n\to\infty} u_n = p\ .$$

Now, by theorem 4.3.1we get,

$$\lim_{n\to\infty} u_n = p \Rightarrow \lim_{n\to\infty} x_n = p\ .$$

Using lemma 4.7.2 we have,

$$\lim_{n\to\infty} \mu_n = 0\ .$$

Thus, we get $\lim_{n\to\infty} \eta_n = 0 \Rightarrow \lim_{n\to\infty} \mu_n = p$.

This completes the step-1 of our theorem.

Step-2: From definition 4.7.1, $(b) \Leftrightarrow (c)$ means that $\lim_{n\to\infty} \mu_n = 0 \Leftrightarrow \lim_{n\to\infty} \xi_n = 0$ and so, $\lim_{n\to\infty} \xi_n = 0 \Rightarrow \lim_{n\to\infty} \mu_n = 0$ is obvious by setting, $c_n = 0\ \forall n \in \mathbf{N}$ (set of all natural numbers), in (vii).

Conversely, suppose that Ishikawa iterative scheme (vi) is *T*-stable. Using definition 4.7.1, we get $\lim_{n\to\infty} \mu_n = 0 \Rightarrow \lim_{n\to\infty} x_n = p$.

Now, by theorem 4.3.1we get,

$$\lim_{n\to\infty} x_n = p \Rightarrow \lim_{n\to\infty} p_n = p\ .$$

Using lemma 4.7.2 we have,

$$\lim_{n\to\infty} \xi_n = 0 \ .$$

Thus, we get $\lim_{n\to\infty} \mu_n = 0 \Rightarrow \lim_{n\to\infty} \xi_n = 0$.

This completes the step-2 of our theorem.

Step-3: From definition 4.7.1, $(c) \Leftrightarrow (a)$ means that $\lim_{n\to\infty} \xi_n = 0 \Leftrightarrow \lim_{n\to\infty} \eta_n = 0$ and so, $\lim_{n\to\infty} \xi_n = 0 \Rightarrow \lim_{n\to\infty} \eta_n = 0$ is obvious by setting, b_n & $c_n = 0 \ \forall n \in \mathbb{N}$ (set of all natural numbers), in (vii).

Conversely, suppose that Mann iterative scheme (v) is T-stable. Using definition 4.7.1, we get

$$\lim_{n\to\infty} \eta_n = 0 \Rightarrow \lim_{n\to\infty} u_n = p .$$

Now, by theorem 4.3.1we get,

$$\lim_{n\to\infty} u_n = p \Rightarrow \lim_{n\to\infty} p_n = p .$$

Using lemma 4.7.2 we have,

$$\lim_{n\to\infty} \xi_n = 0 .$$

Thus, we get $\lim_{n\to\infty} \eta_n = 0 \Rightarrow \lim_{n\to\infty} \xi_n = 0$.

This completes the step-3 of our theorem. ■

CHAPTER-5

NUMERICAL COMPARISON OF FIXED POINT ITERATIVE SCHEMES

CHAPTER-5

NUMERICAL COMPARISON OF FIXED POINT ITERATIVE SCHEMES

5.1 Introduction

In the last three decades many numerous papers have been published on the iterative approximation of fixed points for certain classes of operators, by using Picard, Krasnoselskij, Mann, Ishikawa and Noor iterative methods. In those papers there are given various fixed point theorems. The importance of metrical fixed point theory consists mainly in the fact that for most functional equations $y = f(x)$ we can equivalently transform them in a fixed point problem $Tx = x$ and then apply a fixed point theorem to get information on the existence or existence and uniqueness of the fixed point, that is, of a solution for the original equation. Moreover, fixed point theorems usually provide a method for constructing such a solution.

Main applications of fixed point theorems: to obtain existence or existence and uniqueness theorems for various classes of operator equations (differential equations, integral equations, integro-differential equations, variational inequalities etc.)

In this chapter, first we will state some existence theorems and some existence and uniqueness theorems, which are based on some fixed point theorems. Secondly, we will give some examples to realized when a fixed point theorem is valuable and provide a suitable fixed point iterative scheme. Finally, we will study on the rate of convergence of different fixed point iterative schemes and compare their rate of convergence at fixed point.

The main aim of this chapter is to illustrate how, in the absence of theoretical results, we can perform an empirical study of the rate of convergence of fixed point iterative methods, by using the MATLAB-7 software package. The empirical approach of the rate of convergence of fixed point iterative schemes was firstly considered by B.E. Rhoades [3], [6].

5.2 Existence theorems obtained by fixed point theorems

Theorem 5.2.1, (see, [A. Constantin, Annali di Mathematica 184(2005), 131-138])

This theorem is based on Schauder fixed point theorem 1.5.3.

Statement. *Assume that* $|f(t,u)| \le g(t, |u|),\ t \ge 0,\ u \in R$ *(set of real numbers), where* $g \in C(R^+ \times R^+, R^+)$ *is such that the map* $r \mapsto g(t, r)$ *is non-decreasing on* R^+ *for every fixed* $t \ge 0$. *Then for every* $c > 0$ *for which* $\int_0^\infty g(t, 2ct)dt < c$, *the equation* $u'' + f(t, u) = 0,\ t > 0$, *where* $f \in C(R^+ \times R, R)$ *has a global solution* $u_c(t) > 0$ *for* $t > 0$, *and* $\lim_{t\to\infty} \frac{u_c(t)}{t} = c$. ■

Theorem 5.2.2, (see, [R.P.Agarwal, D. O' Regan, Proc. AMS 128 (2000), No.7, 2085-2094])

This theorem is based on Krasnoselskij's fixed point theorem 1.4.3.

Statement. *Consider the singular* $(n;\ p)$ *problem*

$$y^{(n)}(t) + \phi(t)[g(y(t)) + h(y(t))] = 0,\ \ 0 < t < 1$$
$$y^{(i)}(0) = 0,\ \ 0 \le i \le n$$
$$y^{(p)}(1) = 0$$

under several assumptions on $g, h\ \&\ \phi$, *the problem has a solution*

$y \in C^{n-1}[0, 1] \cap C^n(0,1]$ *with* $y > 0$. ■

5.3 Existence and uniqueness theorems obtained by fixed point theorems

Theorem 5.3.1, (see [2]). This theorem is based on a continuation theorem of Leray-Schauder type and Mawhin's coincidence degree theory.

Statement. *Let the domain* $\Omega \subset \boldsymbol{R}^2$ *is assumed to be bounded, with a boundary* $\Gamma = \partial\Omega$ *and* H *be a homeomorphism which can be lifted to a continuous map* $h_1 : \boldsymbol{R} \to \boldsymbol{R}$ *, which is an increasing function onto* $\boldsymbol{R}$ *such that* $0 \le h_1(0) < l$ *and* $h_1(s + l) = h_1(s) + l;\ s \in \boldsymbol{R}$, *and* $S(H(P)) = h_1(S(P))(\text{mod}\, l);\ P \in \Gamma$, *for each point* $P \in \Gamma$ *we denote its coordinate by* $S(P) \in [0, l[$. *Let the number* $\alpha(H)$ *is the winding number or rotation number of* H *such that* $\alpha(H)$ *is an irrational number, and* H^k *has no fixed point on* Γ *for any* $k \in \boldsymbol{N}$. *Also suppose that* $\alpha(H)$ *has a bounded sequence of partial* *quotients. Then there exists* $\varepsilon > 0$ *such that the Dirichlet problem for the semilinear equation of the vibrating string*

$$\begin{cases} u_{xx} - u_{yy} + h(x, y, u) = 0, & (x, y) \in \Omega \\ u|_{\partial\Omega} = 0, \end{cases}$$

has a unique weak solution for each $h \in F$ *when the condition* $\left|\frac{h(u) - h(v)}{u - v}\right| \le \varepsilon$, *holds for all* $u, v \in R$, $u \ne v$. *Here* $F = L^2(]0, \pi[\times]0, \frac{2\pi}{\beta}[)$, $\beta \in R$ *and* L *be the abstract realisation in* F *of the wave operator with Dirichlet boundary conditions on* $]0, \pi[\times]0, \frac{2\pi}{\beta}[$.

■

Theorem 5.3.2, (Classical, see [18])

This theorem is based on Banach fixed point theorem 1.4.2.

Statement. *Consider the integral equation*

$$y(x) = f(x) + \lambda \int_a^x K(x, s, y(s))\, ds, \quad x \in [0, T] \tag{1}$$

Using appropriately the contraction mapping principle, we get for equation (1) the following conclusions:

a) Existence and uniqueness of the solution;

b) A method for constructing the solution $\{y_n\}$,

$$y_{n+1}(x) = f(x) + \int_a^x K(x, s, y_n(s))\, ds,$$

c) Error estimate, rate of convergence;

d) Stability results.

■

Remark 5.3.3. In order to prove all the previous theorems, the key tool is to equivalently write the problem (equation) as a fixed point problem $x = Tx$ and then apply a certain fixed point theorem.

5.4 Some numerical concept of fixed point theorems

From a practical point of view, that is, from a numerical point of view, a fixed point theorem is valuable if, apart from ensuring the existence (and possible, uniqueness) of the fixed point, it also satisfies some minimal numerical requirements, (see, e.g., [5]) amongst which we mention:

(a) it provides a method (generally, iterative) for constructing the fixed point;

(b) it is able to provide information on the error estimate (rate of convergence) of the iterative process used to approximate the fixed point; and

(c) it can give concrete information on the stability of this iterative method, that is, on the data dependence of the fixed point.

Only a few fixed point theorems in literature do fulfill all three requirements above. Moreover, the error estimate and stablity of fixed points appear to have been given systematically, mainly for the Picard iteration (sequence of successive approximations), in conjunction with various contractions.

Now, we give two examples to illustrate our above discussion.

Example 5.4.1. If $T: X \to X$ is an α-contraction on a complete metric space (X, d), that is, there exists a constant $0 \le \alpha < 1$ such that $d(Tx, Ty) \le \alpha\, d(x, y), \forall\ x, y \in X$, then by Banach fixed point theorem (Theorem 1.5.2) we know that

(a) $F(T) = \{x^*\}$, Where $F(T)$ denotes the set of fixed point of T.

(b) $x_n = T^n(x_0)$ (Picard iteration) converges to x^* for all $x_0 \in X$.

(c) Both the a priori and a posteriori error estimates

$$d(x_n, x^*) \le \frac{\alpha^n}{1-\alpha} d(x_0, x_1),\ n = 0, 1, 2, \ldots \tag{1}$$

$$d(x_n, x^*) \le \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n),\ n = 1, 2, \ldots \tag{2}$$

hold.

Remark 5.4.2. The errors $d(x_n, x^*)$ are decreasing as rapidly as the term of geometric progression with ratio α, that is $\{x_n\}_{n=0}^{\infty}$ converges to x^* at least as rapidly as the geometric series. The convergence is however linear,

$$d(x_n, x^*) \le d(x_{n-1}, x^*),\ n = 1, 2, \ldots$$

If T satisfies a weaker contractive condition, e.g., T is non-expensive, then Picard iteration does not converges generally or even if it converges, its limit is not fixed point of T. More general iterative procedures are needed.

Example 5.4.3. Let $X = \mathrm{R}$ with the usual norm, $K = [\frac{1}{2}, 2]$ and $T: K \to K$ be a function given by $T(x) = \frac{1}{x}, \forall\ x \in K$. Then

(a) T is Lipschitzian with constant $L = 4$;

(b) T is strictly pseudocontractive;

(c) $F(T) = \{1\}$, where $F(T) = \{x \in K : Tx = x\}$;

(d) The Picard iteration associated to T does not converge to the fixed point of T, for any $x_0 \in K - \{1\}$;

(e) The Krasnoselskii's iteration associated to T converges to the fixed point $p = 1$, for any $x_0 \in K$ and $\lambda \notin (0, 1/16)$;

(f) The Mann iteration associated to T with $a_n = \frac{n}{2n+1}$, $n \geq 0$ and $x_0 = 2$ converges to 1, the unique fixed point of T.

5.5 Rate of convergence of fixed point iterative schemes

In numerical analysis, the speed at which a convergent sequence approaches its limit is called the ***rate of convergence.*** Although strictly speaking, a limit does not give information about any finite first part of the sequence, this concept is of practical importance if we deal with a sequence of successive approximations for an iterative method, as then typically fewer iterations are needed to yield a useful approximation if the rate of convergence is higher. This may even make the difference between needing ten or a million iterations.

Basic definition

Suppose that the sequence $\{x_k\}$ converges to the number ξ. We say that this sequence converges linearly to ξ, if there exists a number $\mu \in (0, 1)$ such that

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|} = \mu .$$

The number μ is called the ***rate of convergence.*** If the above holds with $\mu = 0$, then the sequence is said to ***converge superlinearly.*** One says that the sequence ***converges sublinearly*** if it converges, but $\mu = 1$.

The next definition is used to distinguish superlinear rates of convergence. We say that the sequence ***converges with order*** q for $q > 1$ to ξ if

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = \mu \text{ with } \mu > 0 .$$

In particular, convergence with order 2 is called ***quadratic convergence***, and convergence with order 3 is called ***cubic convergence.***

This is sometimes called ***Q-linear convergence, Q-quadratic convergence***, etc., to distinguish it from the definition below. The Q stands for "quotient," because the definition uses the quotient between two successive terms.

Extended definition. The drawback of the above definitions is that these do not catch some sequences which still converge reasonably fast, but whose "speed" is variable, such as the sequence $\{b_k\}$ below. Therefore, the definition of rate of convergence is sometimes extended as follows.

Under the new definition, the sequence $\{x_k\}$ converges with at least order q if there exists a sequence $\{\varepsilon_k\}$ such that $|x_k - \xi| \le \varepsilon_k \quad \forall\ k$, and the sequence $\{\varepsilon_k\}$ converges to zero with order q according to the above "simple" definition. To distinguish it from that definition, this is sometimes called ***R-linear convergence, R-quadratic convergence***, etc. (with the R standing for "root").

Example. Consider the following sequences:

$$a_0 = 1,\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{4},\ a_3 = \frac{1}{8},\ a_4 = \frac{1}{16},\ a_5 = \frac{1}{32},\ldots,a_k = \frac{1}{2^k},\ldots$$

$$b_0 = 1,\ b_1 = 1,\ b_2 = \frac{1}{4},\ b_3 = \frac{1}{4},\ b_4 = \frac{1}{16},\ b_5 = \frac{1}{16},\ldots,b_k = \frac{1}{4^{floor(k/2)}},\ldots$$

$$c_0 = \frac{1}{2},\ c_1 = \frac{1}{4},\ c_2 = \frac{1}{16},\ c_3 = \frac{1}{256},\ c_4 = \frac{1}{65536},\ \ldots..,c_k = \frac{1}{2^{2^k}},\ldots$$

$$d_0 = 1,\ d_1 = \frac{1}{2},\ d_2 = \frac{1}{3},\ d_3 = \frac{1}{4},\ d_4 = \frac{1}{5},\ d_5 = \frac{1}{6},\ldots,d_k = \frac{1}{k+1},\ldots$$

The sequence $\{a_k\}$ converges linearly to 0 with rate 1/2. The sequence $\{b_k\}$ also converges linearly to 0 with rate 1/2 under the extended definition, but not under the simple definition. The sequence $\{c_k\}$ converges superlinearly. In fact, it is quadratically convergent. Finally, the sequence $\{d_k\}$ converges sublinearly.

The problem of studying the rate of convergence of fixed point iterative schemes arises in two different contexts:

(a) For large classes of operator (quasi-contractive type operators) not only Picard iteration, but also the Mann, Ishikawa and Noor iterations can be used to approximate the fixed points.

In such situation, it is of theoretical importance to compare these methods in order to establish, if possible which one converges faster.

(b) For a certain fixed point iterative method (Picard, Kranoselskij, Mann, Ishikawa, Noor etc.) we do not know an analytical error estimate of the form (1) and (2) of example 5.4.1.

In this case we can try an empirical study of the rate of convergence. These two cases we will describe in the art 5.6 and art 5.7. Now, we give a theorem, which have been stated and proof by the help of Banach fixed point theorem. By this theorem we are able to provide some useful information about the rate of convergence of fixed point iterative schemes towards the fixed point.

Theorem 5.5.1. *Let T be a contraction mapping on a complete metric space M, with contraction constant λ and fixed point a. For any $x_0 \in M$, with $T-$iterates $\{x_n\}$, we have the error estimates*

$$d(x_n, a) \le \frac{\lambda^n}{1-\lambda} d(x_0, T(x_0)), \tag{1}$$

$$d(x_n, a) \le \lambda d(x_{n-1}, a), \tag{2}$$

$$\text{and } d(x_n, a) \le \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n), \tag{3}$$

Proof. From the proof of the Banach fixed point theorem 1.4.2, for $m \ge n$ we have

$$d(x_n, x_m) \le \frac{\lambda^n}{1-\lambda} d(x_0, x_1) = \frac{\lambda^n}{1-\lambda} d(x_0, T(x_0)).$$

The right side is independent of m. Let $m \to \infty$ to get

$$d(x_n, a) \le \frac{\lambda^n}{1-\lambda} d(x_0, T(x_0)).\text{This shows (1)}.$$

To show (2) from the contraction property and a being a fixed point we get

$$d(x_n,\ a)=d(T(x_{n-1}),\ T(a))\leq \lambda d(x_{n-1},\ a). \tag{4}$$

Applying the triangle inequality to $d(x_{n-1},\ a)$ on the right side of (4) using the three points x_{n-1}, x_n and a,

$$d(x_n,\ a)\leq \lambda(d(x_{n-1},\ x_n)+d(x_n,\ a)),$$

and solving for $d(x_n,\ a)$ shows

$$d(x_n,\ a)\leq \frac{\lambda}{1-\lambda}d(x_{n-1},\ x_n).$$

This completes our theorem. ■

5.5.2 Information about the rate of convergence

The inequality (1) of the theorem 5.5.1 tells us, in terms of x_0, how far we have to iterate T starting from x_0 to be certain that we are within a specified distance from the fixed point. Inequality (2) of the theorem 5.5.1 shows that the x_n's are always moving closer to the fixed point. The inequality (3) of the theorem 5.5.1 tells us, after each computation, how much closer we are to the fixed point in terms of the previous two iterations. This can be a much more useful estimate than (1), because two successive iterations which are nearly equal will tell us that we are very close to the fixed point.

Example 5.5.3. Returning to Example of the definition 1.4.1, how many iterates of cosine are needed to be sure we have found the solution to $\cos a = a$ in $[0, 1]$ accurately to 3 decimal places? On $[0, 1]$, cosine is a contraction mapping with contraction constant $\lambda = 0.8415$. Taking $x_0 = 0$, by (1) of the theorem 5.5.1we know $|\cos^n(0)-a|\leq(\lambda^n/(1-\lambda))|1-\cos(1)|$, where $\cos^n$ means the n-fold iterate of cosine, not its nth power. For which n is $(\lambda^n/(1-\lambda))|1-\cos(1)|<1/1000$? The first such n is $n=47$, so after 47 iterations of cosine starting at $x_0 = 0$ we guaranteed to have found the solution to $\cos a = a$ accurate 3 decimal places. This recovers the approximation $a\approx 0.739$ referred to in Example of the definition 1.4.1, where we did ignored error analysis.

Actually, the numerical evidence suggests the 3-digit approximation to a is already achieved (and remains this way) starting from the 21st iterate. While (1) of the theorem

5.5.1 tells us that the 47th iterate would be provably sufficient, if we use (2) of the theorem 5.5.1 we can essentially prove that the 21st iterate is accurate to 3 digits: $|x_{21}-a| \le \frac{0.8415}{1-0.8415}|x_{20}-x_{21}|$.

Since $x_{20} \approx 0.739184$ and $x_{21} \approx 0.739018$, we obtain $|x_{21}-a| \le 0.0013$, which is just slightly more than $1/1000$, so we have not proved $|x_{21}-a| \le 1/1000$. However, at the next step we have provable 3-digit accuracy: $x_{22} \approx 0.739018$ and

$$|x_{22}-a| \le \frac{0.8415}{1-0.8415}|x_{21}-x_{22}| \approx 0.00088 < 1/1000.$$

Of course, the iterative schemes should converge faster if we began our iteration closer to the fixed point. A comparison of the graphs of $y=\cos x$ and $y=x$ show the intersection point occurs where $x \approx 0.7$, so let's start with $x_0 = 0.7$ instead of starting at 0. The iterations appear to stabilize in the first 3 digits starting at x_{15}, and we can use inequality (3) of the theorem 5.5.1 to prove there is 3-digit accuracy at x_{16}: $|x_{16}-a| \le \frac{0.8415}{1-0.8415}|x_{16}-x_{15}| \approx 0.0009$.

If we had used inequality (1) of the theorem 5.5.1 instead of inequality (3) of the theorem 5.5.1 with $x_0 = 0.7$, we could guarantee x_n is within $1/1000$ of a for $n \ge 35$. This reinforces why it's good to have estimates of convergence using points computed along the way and not only in terms of the initial point x_0.

5.6 Theoretical approchement of the rate of convergence

Definition 5.6.1. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers that converge to a and b respectively. Assume there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} \tag{3}$$

(i) If $l = 0$, then it is said that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a faster than the sequence $\{b_n\}_{n=0}^{\infty}$ converges to b.

(ii) If $0 < l < \infty$ then we say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Remarks on the definition 5.6.1. (a) If $l = \infty$, then the sequence $\{b_n\}_{n=0}^{\infty}$ converges faster than the sequence $\{a_n\}_{n=0}^{\infty}$, that is $b_n - b = o(a_n - a)$.The concept introduced by definition 5.6.1 allows us to compare the rate of convergence of two sequences, and will be useful in the sequel.

(b) The concept of rate of convergence given by definition 5.6.1is a relative one, while in literature; there exists concepts of absolute rate of convergence. However, in the presence of an error estimate of the form (1) and (2) of example 5.4.1, the concept given by definition 5.6.1 is much more suitable. Indeed, the estimate (1) shows that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to x^* faster than any sequence $\{\theta_n\}$ to zero, where $0 < \theta < a$.

Suppose that for two fixed point iterations $\{x_n\}_{n=0}^{\infty}$, and $\{y_n\}_{n=0}^{\infty}$, converging to the same fixed point x^*, the following a priori error estimates

$$d(x_n, x^*) \le a_n, \; n = 0, 1, 2,\ldots\ldots\ldots \tag{4}$$

and

$$d(y_n, x^*) \le b_n, \; n = 0, 1, 2,\ldots\ldots\ldots \tag{5}$$

are available, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive real numbers(converging to zero). Then, in view of definition 5.6.1, the following definition appears to be very natural.

Definition 5.6.2. If $\{a_n\}_{n=0}^{\infty}$ converges faster then $\{b_n\}_{n=0}^{\infty}$, then we shall say that the fixed point iteration $\{x_n\}_{n=0}^{\infty}$ converges faster to x^* than the fixed point iteration $\{y_n\}_{n=0}^{\infty}$ or, simply, that $\{x_n\}_{n=0}^{\infty}$ is better than $\{y_n\}_{n=0}^{\infty}$.

Theorem 5.6.3, (see [71, 78]). *Let B be an arbitrary Banach space, K a closed convex subset of B, and $T : K \to K$ an operator satisfying Zamfirescu's conditions, i.e., there exist the real numbers $a, b \,\&\, c$ satisfying $0 \le a < 1,\ 0 \le b < 1,\ c < 1/2$ such that for each pair x, y in X, at least one of the following is true:*

(z_1) $\|Tx - Ty\| \le a\|x - y\|$;

(z_2) $\|Tx - Ty\| \le b[\|x - Tx\| + \|y - Ty\|]$;

(z_3) $\|Tx - Ty\| \le c[\|x - Ty\| + \|y - Tx\|]$.

Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration associated with T, starting from $x_0 \in K$, and $\{y_n\}_{n=0}^{\infty}$ be the Mann iteration, where $\{a_n\}_{n=0}^{\infty}$ is a sequence satisfying (a) $a_0 = 1$ (b) $0 \le a_n < 1$ and *(c)* $\sum_{n=0}^{\infty} a_n$, *i.e.,* $\{a_n\}$ *is divergent. Then, 1)* T *has a unique fixed point* $p \in B$*; 2) Picard iteration* $\{x_n\}_{n=0}^{\infty}$ *converges to* p *for any* $x_0 \in K$*; 3) Mann iteration* $\{y_n\}_{n=0}^{\infty}$ *converges to* p *for any* $y_0 \in K$ *and* $\{a_n\}_{n=0}^{\infty}$ *satisfying(v); 4) Picard iteration is faster than Mann iteration.*

■

The same result we established in our theorem 3.9.2 for Noor iterative scheme. This result we also established in another theorem 3.7.2, but there we used another contractive definition.

However, in most of practical problems, we do not know error estimates like those in (1) and therefore there is no possibility of getting information on the rate of convergence of the corresponding iterative processes as in Theorem 5.6.3. There are several results in literature concerning Picard and Krasnoselskii's iterative schemes, see [4], but there is no systematic study of the numerical aspects related to other fixed point iteration procedures like Mann, Ishikawa, Noor, Mann type, Ishikawa type and Noor type etc. For classes of real functions and fixed point methods, it is possible to perform empirical studies, by using the software package MATLAB-7 and test functions and then by inferring conclusions for the entire class of mappings and a certain fixed point iterative method.

5.7 Empirical approachment of the rate of convergence

Some empirical studies on different numerical iterative schemes by using the software package MATLAB-7 follows:

1) The Krasnoselskii's iteration converges to the fixed point $p = 1$ for any $\lambda \in (0, 1)$ and initial guess x_0 (recall that the Picard iteration does not converge for any initial value $x \in [1/2, 2]$ different from the fixed point). The convergence is slow for λ close enough to 0 (that is, for Krasnoselskii's iteration close enough to the Picard iteration) or close enough to 1. If λ closer to **1/2** = the middle point of the interval (0, 1), then the Krasnoselskii's iteration converges very fast to the fixed point $p = 1$. In the Example-5.4.3 if we put $\lambda = 0.5$, then the Krasnoselskii's iteration converges very fast to the fixed point $p = 1$, which is the unique fixed point of T. For example, starting with $x_0 = 1.5$, only four iterations are needed in order to obtain p with 6 decimal places:

$$x_1 = 1.08335,\ x_2 = 1.00325,\ x_3 = 1.000053,\ x_4 = 1$$

For the same value of λ and $x_0 = 2$ again only four iterations are needed to obtain p with the same precision, even though the initial guess is for away from the fixed point: $x_1 = 1.25$, $x_2 = 1.025$, $x_3 = 1.0003$, $x_4 = 1$

2) The speed of Mann, Ishikawa and Noor iterative schemes also depends on the position of the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in the interval $(0, 1)$. If we take $x_0 = 1.5$, $a_n = 1/(n+1)$, $b_n = 1/(n+2)$ and $c_n = 1/(n+3)$ then the Mann, Ishikawa and Noor iterative schemes converge (slowly) to $p = 1$: after $n = 35$ iterations and we get $x_{35} = 1.000155$ for Mann iterative scheme, $x_{35} = 1.0007$ for Ishikawa iterative scheme and $x_{35} = 1.43369034273599$ for Noor iterative scheme.

For $a_n = 1/\sqrt[3]{(n+1)}$, $b_n = 1/\sqrt[4]{(n+2)}$ and $c_n = 1/\sqrt[5]{(n+3)}$ we obtain the fixed point with 6 exact digits performing 7 iterations for the Mann iterative scheme whereas 9 iterations are necessary for the Ishikawa iterative scheme and 30 iterations are necessary for the Noor iterative scheme to obtain the same result. Notice that in this case Mann, Ishikawa and Noor iterative schemes converge not monotonically to $p = 1$. Conditions like that $a_n \to 0\,(\text{as}\, n \to \infty)$ or/and $b_n \to 0\,(\text{as}\, n \to \infty)$ or/and $c_n \to \infty\,(\text{as}\, n \to \infty)$ are usually involved in many convergence theorems. The next results show that these conditions are in general not necessary for convergence of Mann, Ishikawa and Noor iterations.

If we taking $x_0 = 2$, $a_n = \frac{n}{2n+1} \to \frac{1}{2}$, $b_n = \frac{n+1}{2n} \to \frac{1}{2}$, $c_n = \frac{n+2}{2n-1} \to \frac{1}{2}$, then we obtain the following results:

For the Mann iteration:

$$x_1 = 1.50000000000000,\ x_2 = 1.16666666666667,\ x_3 = 1.03401360544218$$

$$x_4 = 1.00427656442694,\ x_5 = 1.00039705636711, \ldots\ldots\ldots\ldots, x_{10} = 1.$$

For the Ishikawa iteration:

$$x_1 = 1.50000000000000,\ x_2 = 1.25000000000000,\ x_3 = 1.12142857142857$$

$$x_4 = 1.05762056414923,\ x_5 = 1.02704732308074, \ldots\ldots\ldots\ldots, x_{16} = 1.$$

For the Noor iteration:

$$x_1 = 0.50000000000000,\ x_2 = 1.10000000000000,\ x_3 = 1.04545454545455$$

$x_4 = 1.02428006775833$, $x_5 = 1.01410151875570$,......................, $x_{32} = 1$.

For all combinations of x_0, λ, a_n *and* b_n we notice the following decreasing (with respect to their speed of convergence) chain of iterative processes: Krasnoselskii's, Mann, Ishikawa, Noor. Consequently, if for a certain operator in the same class, all this schemes converge, then we shall use faster one (empirically deduced).

5.8 Numerical examples and conclusions

Now, we give two examples which represent a function with two repulsive fixed points with respect to the Picard iterative scheme. By these examples we compare the rate of convergence of different iterative schemes.

Example 5.8.1. Let $K = [0, 1]$ and let $T : K \to K$ given by $T(x) = 4x(1-x)$ be the famous logistic function. Then T has $p_1 = 0$ and $p_2 = 0.75$ as fixed points. It is clear from the following figure 5.1. Both of them are repulsive fixed points with respect to the Picard iteration, since $T'(p_1) = 4$ and $T'(p_2) = -2$. The numerical tests show p_2 is attractive with respect to Krasnoselskii's, Mann, Ishikawa, Noor and Newton-Raphson iterative schemes while p_1 stays repulsive. Information regarding the rate of convergence of the convergent methods is illustrated by the following numerical results obtained by running the new version of the software package MATLAB-7.

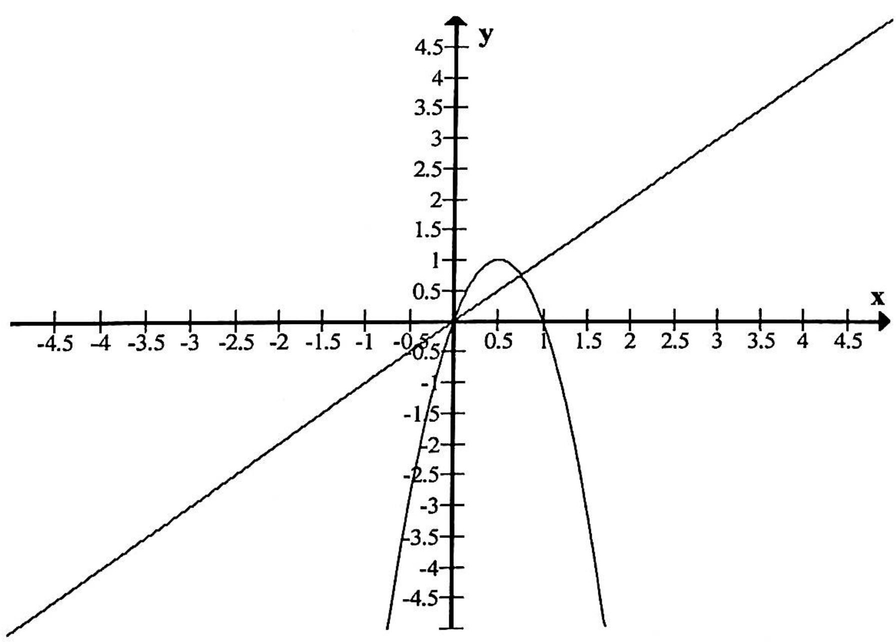

Figure - 5.1

For the Krasnoselskii's iterative scheme:

If we start from $x_0 = 0.3$ and the parameter that defines the iteration is $\lambda = 0.5$, then we obtain following results:

Iteration Number(n)	Approximated value obtained by Krasnoselskii's iterative scheme (x_n)	Iteration Number(n)	Approximated value obtained by Krasnoselskii's iterative scheme (x_n)
$n=1$	$x_1 = 0.57000000000000$	....	....
$n=2$	$x_2 = 0.77520000000000$	$n=10$	$x_{10} = 0.75010404295762$
$n=3$	$x_3 = 0.73612992000000$	....	....
$n=4$	$x_4 = 0.75655028176159$	$n=30$	$x_{30} = 0.75000000009925$
$n=5$	$x_5 = 0.74663904673689$	....	....
$n=6$	$x_6 = 0.75165788461788$	$n=45$	$x_{45} = 0.75000000000000$
$n=7$	$x_7 = 0.74916556052825$	....	....

For the Mann iterative scheme:

If we start from $x_0 = 0.3$ and the parameter sequence is $a_n = 1/(n+1)$, then we obtain following results:

Iteration Number(n)	Approximated value obtained by Mann iterative scheme (x_n)	Iteration Number(n)	Approximated value obtained by Mann iterative scheme (x_n)
$n=1$	$x_1 = 0.57000000000000$	....	....
$n=2$	$x_2 = 0.70680000000000$	$n=10$	$x_{10} = 0.74968138766394$
$n=3$	$x_3 = 0.73733376000000$	....	....
$n=4$	$x_4 = 0.74480515709141$	$n=30$	$x_{30} = 0.74998829850963$
$n=5$	$x_5 = 0.74738458761714$	....	....
$n=6$	$x_6 = 0.74850156984869$	$n=45$	$x_{45} = 0.74999653499154$
$n=7$	$x_7 = 0.74906235850897$	....	....

For the Ishikawa iterative scheme:

If we start from $x_0 = 0.3$ and the parameter sequences are $a_n = 1/(n+1)$ and $b_n = 1/(n+2)$, then we obtain following results:

Iteration Number(n)	Approximated value obtained by Ishikawa iterative scheme (x_n)	Iteration Number(n)	Approximated value obtained by Ishikawa iterative scheme (x_n)
$n=1$	$x_1 = 0.39000000000000$	....	....
$n=2$	$x_2 = 0.43680000000000$	$n=10$	$x_{10} = 0.52354622661835$
$n=3$	$x_3 = 0.46416115200000$	....	....
$n=4$	$x_4 = 0.48185119052987$	$n=30$	$x_{30} = 0.54750839727341$
$n=5$	$x_5 = 0.49415669749009$	....	....
$n=6$	$x_6 = 0.50318717473604$	$n=45$	$x_{45} = 0.55190280649804$
$n=7$	$x_7 = 0.51008678852677$	....	....

For the Noor iterative scheme:

If we start from $x_0 = 0.3$ and the parameter sequences are $a_n = 1/(n+1)$, $b_n = 1/(n+2)$, and $c_n = 1/(n+3)$ then we obtain following results:

Iteration Number(n)	Approximated value obtained by Noor iterative scheme (x_n)	Iteration Number(n)	Approximated value obtained by Noor iterative scheme (x_n)
$n=1$	$x_1 = 0.32250000000000$	....	....
$n=2$	$x_2 = 0.33169125000000$	$n=10$	$x_{10} = 0.34381918676870$
$n=3$	$x_3 = 0.33631622840578$	....	....
$n=4$	$x_4 = 0.33896629632607$	$n=30$	$x_{30} = 0.34534557972094$
$n=5$	$x_5 = 0.34062494599511$	....	....
$n=6$	$x_6 = 0.34173163929402$	$n=45$	$x_{45} = 0.34548636537893$
$n=7$	$x_7 = 0.34173163929402$	....	....

For the Newton-Raphson iterative scheme:

If we start from $x_0 = 0.3$, then we obtain following results:

Iteration Number(n)	**Approximated value obtained by Newton-Raphson iterative scheme (x_n)**	**Iteration Number(n)**	**Approximated value obtained by Newton-Raphson iterative scheme (x_n)**
$n=1$	x_1 = - 0.22500000000000	....	....
$n=2$	x_2 = -0.03491379310345	$n=10$	$x_{10} = 0$
$n=3$	x_3 = -0.00113941065326	....	....
$n=4$	x_4 = -0.00000129530487	$n=30$	$x_{30} = 0$
$n=5$	x_5 = -0.00000000000168	....	....
$n=6$	x_6 = - 0.00000000000000	$n=45$	$x_{45} = 0$
$n=7$	$x_7 = 0$	....	....

The previous numerical results suggest that Krasnoselskii's iterative scheme converges to the fixed point faster than Mann, Ishikawa and Noor iterative schemes. Although, the Newton-Raphson iterative scheme converge to the fixed point $p_1 = 0$ very fast but this fixed point not attractive. The same fact is illustrated for all values of x_0 we tested. We may infer that, for the function above and, possibly, for all functions possessing similar properties (i.e., Lipschitzian), one can expect that Krasnoselskii's iterative scheme always converges to the fixed point faster than Mann or Ishikawa or Noor iterative scheme. The next step would be of course to try to prove (or disprove) this assertion, if possible, but certainly this is not an easy task. Sometimes this approach could be successful. It is perhaps important to stress on the fact that the conclusions of Theorem 5.6.3 was reached in this way: we first observed empirically the behavior of Picard iterative scheme, Mann iterative scheme, Ishikawa iterative scheme and Noor iterative scheme iterative scheme for many different sets of initial data and parameters and then succeeded to prove analytically the observed property.

Example 5.8.2. Let $K = [0, 1]$ and let $T : K \to K$ be given by $T(x) = (1-x)^6$, then T has $p_1 \approx 0.2219$ and $p_2 \approx 2.1347$ as fixed points. It is clear from the following figure.

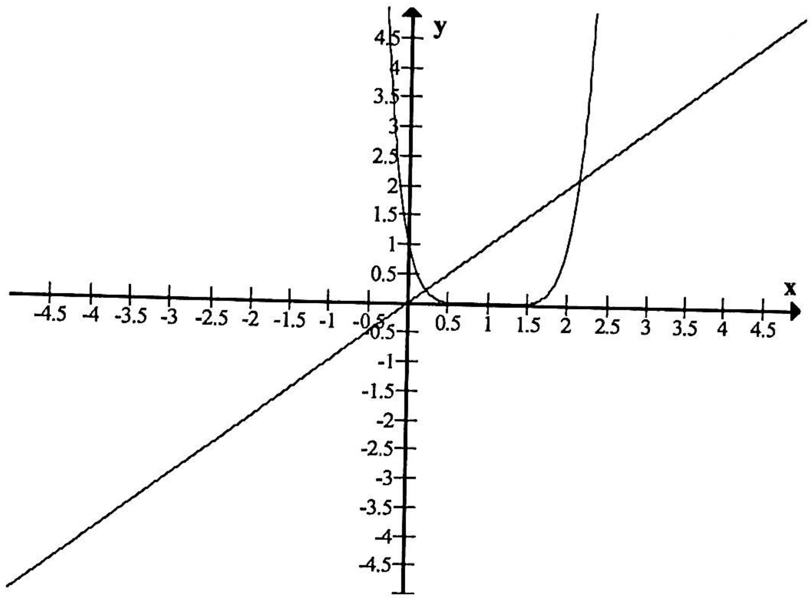

Figure - 5.2

Now, we compute some numerical results from example 5.8.1 for different iterative schemes by using the software package MATLAB-7.

For the Picard iterative scheme:

For $x_0 = 2$, we obtain $x_1 = 1,\ x_2 = 0, x_3 = 1,\ x_4 = 0, x_5 = 1,\ x_6 = 0,\ x_7 = 1$, i.e., the Picard iteration is repulsive for the function T.

For the Krasnoselskii's iterative scheme:

For $x_0 = 2\ \&\ \lambda = 0.5$, we obtain following results:

Iteration Number(n)	**Approximated value obtained by Krasnoselskii's iterative scheme (x_n)**	**Iteration Number(n)**	**Approximated value obtained by Krasnoselskii's iterative scheme (x_n)**
$n=1$	$x_1 = 1.500000$	....	....
$n=2$	$x_2 = 0.757813$	$n=9$	$x_9 = 0.221932$
$n=3$	$x_3 = 0.379007$	....	....
$n=4$	$x_4 = 0.218178$	$n=100$	x_{100}= 0.22191040
$n=5$	$x_5 = 0.223276$	....	....
$n=6$	$x_6 = 0.221430$	$n = 500$	x_{500} = 0.21910400

For the Mann iterative scheme:

For $x_0 = 2\ \&\ a_n = 1/(n+1)$, we obtain following results:

Iteration Number(n)	Approximated value obtained by Mann iterative scheme (x_n)	Iteration Number(n)	Approximated value obtained by Mann iterative scheme (x_n)
$n=1$	$x_1 = 1.50000000$	....	....
$n=2$	$x_2 = 1.00520833$	$n=100$	$x_{100} = 0.22209929$
$n=3$	$x_3 = 0.75390625$	....	....
$n=4$	$x_4 = 0.60316942$	$n=132$	$x_{132} = 0.2219994$
$n=5$	$x_5 = 0.50329203$	....	....
$n=6$	$x_6 = 0.43353857$	$n=500$	$x_{500} = 0.22191281$

For the Ishikawa iterative scheme:

For $x_0 = 2,\ a_n = 1/(n+1) \& b_n = 1/(n+2)$, we obtain following results:

Iteration Number(n)	Approximated value obtained by Ishikawa iterative scheme (x_n)	Iteration Number(n)	Approximated value obtained by Ishikawa iterative scheme (x_n)
$n=1$	$x_1 = 1.0439$	....	....
$n=2$	$x_2 = 0.69596537$	$n=100$	$x_{100} = 0.22209929$
$n=3$	$x_3 = 0.52386538$	....	....
$n=4$	$x_4 = 0.42536054$	$n=111$	$x_{111} = 0.22199995$
$n=5$	$x_5 = 0.36491368$	....	....
$n=6$	$x_6 = 0.325997668$	$n=500$	$x_{500} = 0.22191196$

For the Noor iterative scheme:

For $x_0 = 2,\ a_n = 1/(n+1),\ b_n = 1/(n+2)\ \&\ c_n = 1/(n+3)$ we obtain following results:

Iteration Number(n)	Approximated value obtained by Noor iterative scheme (x_n)	Iteration Number(n)	Approximated value obtained by Noor iterative scheme (x_n)
$n=1$	$x_1 = 7.1538e^{-009}$	$n=6$	$x_6 = 0.23177046$
$n=2$	$x_2 = 0.34642111$	....	....

$n=3$	$x_3=0.27324844$	$n=100$	$x_{100}=0.22191615$
$n=4$	$x_4=0.224818052$	....	....
$n=5$	$x_5=0.23729487$	$n=500$	$x_{500}=0.221904837$

For the Newton-Raphson iterative scheme:

For $x_0=2$, we obtain following results:

Iteration Number(n)	Approximated value obtained by Newton-Raphson iterative scheme (x_n)	Iteration Number(n)	Approximated value obtained by Newton-Raphson iterative scheme (x_n)
$n=1$	$x_1=1.83333333333333$	$n=6$	$x_6=1.33489797668038$
$n=2$	$x_2=1.69444444444444$	....	....
$n=3$	$x_3=1.57870370370370$	$n=100$	$x_{100}=1.00000001207467$
$n=4$	$x_4=1.48225308641975$	....	....
$n=5$	$x_5=1.40187757201646$	$n=500$	$x_{500}=1.00000000000000$

From, the previous numerical results we can see that Krasnoselskii's iterative scheme converge to the fixed point 0.2219 after 9 iterations, Mann iterative scheme converge to the fixed point 0.2219 after 132 iterations and Ishikawa iterative scheme converge to the fixed point 0.2219 after 111 iterations, where as Noor iterative scheme converge to the fixed point 0.2219 very slowly and Newton-Raphson iterative scheme never converge to the fixed point 0.2219. So, we can suggest that Krasnoselskii's iterative scheme converges faster than Mann, Ishikawa, Noor and Newton-Raphson iterative schemes. This fact is more clearly illustrated if we choose $x_0=p_2\approx 2.1347$: after 17 iterations, Krasnoselskii's iteration gives $x_{17}=0.22190721$, while Mann, Ishikawa and Noor iterative schemes give $x_{17}=1.03862188$, $x_{17}=2.13397981$ and $x_{17}=2.13467288$ respectively. The convergence rate of Mann, Ishikawa and Noor iterative schemes is indeed very slow in this case: after 500 iterations we get for Mann iterative scheme $x_{500}=0.22233185$, for Ishikawa iterative scheme $x_{500}=0.22191196$ and for Noor iterative scheme $x_{500}=0.221904837$, where as Newton-Raphson iteration never converges to the fixed point 0.2219.

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