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Quantum Gravity in Curved Space-Time of General Relativity

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" Symmetry , as wide or as narrow as you may difine it , is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection . "

— Hermann Weyl

" Beauty is the proper conformity of the parts to one another and to the whole . "

— Werner Heisenberg .

QUANTUM GRAVITY IN CURVED SPACE-TIME OF GENERAL RELATIVITY

BY

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DEDICATED TO MY PARENTS

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NOTATIONS AND CONVENTIONS

Bar indicates complex conjugation. Dot denotes differentiation with respect to the argument.

Units are choosen such that $G = c = h = 1$, where G is the Newtonian gravitational constant, c is the speed of light and h is the Planck's constant.

We shall adopt the following abbreviations :

Newman - Unti - Tamburino	NUT
NUT - Kerr - Newman	NUTKN
NUT - Kerr - Newman - de Sitter	HNUTKN
NUT - Kerr - Newman - Kasuya	NUTKNK
Kerr - Newman - Kasuya - de Sitter	HKNK
Kerr - Newman - de Sitter	HKN

ABSTRACT

The main contents of this thesis may be briefly summarized as follows :

Chapter 1 contains a brief account of the Plebanski spacetime . We specify there the various special class of spacetimes covered by the Plebanski spacetime. Among the various special cases of the Plebanski spacetime a new spacetime discovered there is the NUT - Kerr - Newman - Kasuya - de Sitter spacetime . This chapter also includes a discussion on the modification of the Plebanski spacetime and presents the modified form of the Plebanski spacetime .

Chapter 2 deals with the separation of the variables of the Dirac equation in an arbitrary curved background spacetime . We discuss there some of the special cases of the separated Dirac equation. The pertinent equation of the separated Dirac equation will be used to derive the radial decoupled Dirac equations for the concerned background spacetimes which will be useful in studying the problems of Hawking radiation .

Since our efforts are concentrated on quantum field theory in some interesting spacetimes of general relativity which are not the black hole spacetimes but include the black hole spacetimes as special cases , we review in Chapter 3 , Hawking's quantum effects near the event horizon of NUT-Kerr-Newman spacetime containing flat black

hole spacetimes as special cases .

Chapter 4 also reviews and extends the result obtained in Chapter 3. The Hawking radiation of Dirac particles is studied in NUT-Kerr-Newman de Sitter spacetime containing black hole spacetimes which are asymptotically flat as well as asymptotically de Sitter as special cases.

Chapter 5 includes the investigation of Dirac particles in Kasner- type spacetime .

Chapter 6 presents Hawking's thermal radiation by black hole near the horizons of the Plebanski spacetime containing a large number of spacetimes of which some are important from the physical point of view. The result obtained in this chapter not only encompasses the known results but also includes some new results . This chapter ends with a concluding remarks .

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INTRODUCTION

The problem of reconciling the description of the gravitational force embodied in Einstein's general theory of relativity with quantum theory is a central issue in theoretical physics . There are two approaches in which quantum idea can be brought into relativity. In the dynamic approach one requires the investigation of quantum equivalent of Einstein's equation . Unfortunately there are some difficulties in this approach to quantize gravity itself . However, there is a great deal of work going in this area and some progress has been made.

In the kinematic approach one tries to study the quantized matter fields in the presence of the gravitational field. For example, writing the Dirac equation in curved spacetime, one studies how Dirac particles behave in a gravitational field such as that of a black hole. The limitation of this approach is that it does not help us to quantize the gravitational field itself. In spite of the limitations, one could hope that the results obtained in this semi-classical approach may be an integral part of the full quantum gravity and the gravitational field will be quantized.

Actually, the investigation of matter fields in curved spacetime (Friedman - Robertson - Walker) was initiated by Schrodinger [1] in 1939, although Rosenfeld [2] seems to be the first to treat the quantum mechanical interaction of matter and gravity. Following the original lead of Schrodinger , in the late sixties Parker [3] and Sexl and Urbantke [4] began discussion of the creation of particles during the initial rapidly expanding phase of the universe . However, the investigation of matter fields in curved spacetime caught fire when Hawking [5] in 1974, demonstrated how a black hole can in fact lose mass by a process of quantum particle creation. Hawking's quantum effects interpreted as the emission of a thermal spectrum of particles near black hole event horizon , opened a new and existing era in quantum gravity research.

Hawking [6] used the technique of quantum field theory on a given background spacetime and showed that black holes create and emit particles at a steady rate and that the predicted rate is just that of the thermal emission of a body with the temperature $\frac{K}{2\pi}$, where K is the surface gravity [7] of the black hole .

Gibbons and Hawking [8] extended Hawking's [9] results on thermal radiation by black holes near black hole event horizon to the cosmological event horizon and deduced the thermodynamic properties of black holes using the arguments of Bekenstein [10]. Subsequently, several studies that attempt to relate the thermal properties of black holes to quantum gravity have appeared [11, 12].

Hawking's result on thermal radiation by black holes have been studied by different authors in different types of spacetimes such as Kerr [13] , Kerr-Newman [14] , Kerr-Newman-de Sitter [15, 16] spacetimes . Recently Ahmed [17,18] studied Hawking radiation of Dirac particles near the horizons of NUT-Kerr-Newman and NUT-Kerr-Newman - de Sitter spacetimes . Ahmed's works are interesting in that Hawking's and Gibbons and Hawking's results could also be obtained in the case of NUT and NUT - de Sitter spacetimes which have very peculiar properties . Most recently Ahmed and Mondal [19] have shown that Hawking's result on thermal radiation by black holes also occurs in the case of Kasner - type spacetime . In this work Ahmed and Mondal observed that particle is creating in Kasner spacetime which is contraction phase of the Schwarzschild spacetime . So we have seen that the particles are creating in the contraction phase of the universe which is in contradiction to the fact that particles are created in the expanding phase of the universe . Particles should be disappeared in the contraction phase of the universe whereas they are being created . To overcome the awkward situation we have interpreted the result as saying that souls of the particles are creating in the contraction phase of the universe .

From the works of Ahmed [20] and Ahmed and Mondal [21] it is clear that Hawking's result could be obtained in the case of spacetimes which are not black hole spacetimes . The aim of this thesis is to show that Hawking's and Gibbons and Hawking's results could also be obtained in a more general background space-time viz . Plebanski spacetime having horizons .

In Chapter 1 , the various special cases of the Plebanski space-time have been described. It has been shown that the Plebanski space-time includes a large number of solutions of Einstein or Einstein-Maxwell equations with or without cosmological constant . Besides the black hole spacetimes (asymptotically flat as well as asymptotically de Sitter) , the Plebanski spacetime includes many interesting space-times which are not black hole spacetimes but are important from the physical point of view . While most of the various special cases of the Plebanski spacetime have been discussed in the literature [22], a number of new spacetimes (such as Kerr - de Sitter , NUT - Kerr - de Sitter , NUT - Kerr - Newman - Kasuya - de Sitter spacetimes and so forth) do not appear to have been stated explicitly in the litera-ture previously . It is shown that with appropriate specialization of the parameters and coordinate transformation the NUT - Kerr - Newman - Kasuya - de Sitter spacetime can be derived from the Plebanski space-time . It is also shown that if the Plebanski spacetime is modified then the modified form of the Plebanski spacetime yields the various special cases of the Plebanski spacetime in an elegant way .

As we would like to study Hawking radiation of Dirac particles in curved spacetime , it is necessary to separate the Dirac equation in the background spacetime considered . Kamran and McLenaghan [23] separated Dirac equation in an arbitrary curved background spacetime and obtained the decoupled quantum field equations . The second chap-ter is devoted to study of the separation of variables of Dirac equation and to show that the earlier results of the separated Dirac equation of Chandrasekhar[24,25] and Page [26] are included in the present general result.

Taking the advantages of the separation of Dirac equation in an arbitrary curved background spacetime we derive the radial decoupled Dirac equations in the NUT-Kerr - Newman, NUT-Kerr - Newman-de Sitter, Kasner - type and Plebanski background spacetimes in a limiting procedure and then study Hawking radiation in Chapters 3, 4, 5 and 6.

Chapter 3 reviews the Hawking radiation of Dirac particles near the event horizon of NUT - Kerr- Newman spacetime which includes the asymptotically flat black hole spacetimes as special cases. From this result it is clear that Hawking's result could be obtained in the case of NUT spacetime.

Chapter 4 also reviews the Hawking radiation of Dirac particles near the event horizon and cosmological horizon of NUT - Kerr - Newman - de Sitter spacetime which includes the black hole spacetimes which are asymptotically flat as well as asymptotically de Sitter as special cases. Here it is observed that Hawking's result is also valid in the case of NUT - de Sitter spacetime.

Chapter 5 includes our study of Dirac particles in Kasner - type spacetime which gives very interesting results that particles are created in the contraction phase of the universe.

Chapter 6 presents the Hawking radiation of Dirac particles near the horizons of the Plebanski spacetime . The result obtained in this chapter includes the results of Chapters 3 , 4 and 5 . This result not only encompasses the results obtained in [27] but also includes some new results . The last section of this chapter ends with conclusive remarks .

CHAPTER - 1

PLEBANSKI SPACETIME

Plebanski [28] studied a class of solutions of Einstein-Maxwell equations. In Boyer coordinates (p, σ, q, τ) the Plebanski spacetime is given by

$$ds^2 = \frac{p^2+q^2}{X} dp^2 + \frac{X}{p^2+q^2}(d\tau + q^2 d\sigma)^2 + \frac{p^2+q^2}{Y} dq^2$$

$$- \frac{Y}{p^2+q^2}(d\tau - p^2 d\sigma)^2 \qquad \ldots \quad (1.1a)$$

where

$$X = X(p) = b - g^2 + 2np - \varepsilon p^2 - (\Lambda/3)\, p^4 \qquad \ldots \quad (1.1b)$$

$$Y = Y(q) = b + e^2 - 2Mq + \varepsilon q^2 - (\Lambda/3)\, q^4 \qquad \ldots \quad (1.1c)$$

with electric potential

$$A_\mu dx^\mu = \frac{eq}{p^2+q^2}(d\tau - p^2 d\sigma) \qquad \ldots \quad (1.2)$$

Besides the cosmological constant Λ, the metric (1.1) contains six parameters $b, e, g, M, n, \varepsilon$. Under the proper coordinate transformation along with the suitable adjustment of the kinetical parameters b and ε , the metric (1.1) gives many physically interesting solutions of Einstein or Einstein - Maxwell equations with or without cosmological constant . The surfaces $Y = 0$ have been interpreted as the horizon of the metric.

1.1 Some Properties of the Plebanski Spacetime

The electromagnetic field associated with the metric is given by

$$W = - d \left\{ \frac{e + ig}{q + ip} (d\tau - i p q \, d\sigma) \right\} \qquad \ldots (1.3)$$

The components of the Weyl tensor $C^{(a)}$ are

$$C^{(1)} = C^{(2)} = C^{(4)} = C^{(5)} = 0$$

$$\ldots (1.4)$$

$$C^{(3)} = \frac{1}{6} \frac{\ddot{X} + \ddot{Y}}{p^2 + q^2} - \frac{1}{p^2 + q^2} \left\{ \frac{\dot{Y} + i\dot{X}}{q + ip} - \frac{2(Y - X)}{(q + ip)^2} \right\}$$

The components of Ricci tensor R_{ab} are

$$R_{11} = R_{22} = R_{13} = R_{33} = R_{14} = R_{44} = R_{23} = R_{24} = 0$$

$$R_{12} = \frac{1}{2} \frac{\ddot{X}}{p^2 + q^2} - \frac{1}{p^2 + q^2} \{ (p\dot{X} - q\dot{Y}) - (X - Y) \} \qquad \ldots (1.5)$$

$$R_{34} = \frac{1}{2} \frac{\ddot{Y}}{p^2 + q^2} + \frac{1}{p^2 + q^2} \{ (p\dot{X} - q\dot{Y}) - (X - Y) \}$$

The scalar curvature R is

$$R = \frac{\ddot{X} + \ddot{Y}}{p^2 + q^2} = 2 (R_{12} + R_{34}) \qquad \ldots (1.6)$$

where here and above dots denote differentiation with respect to the argument.

Now we turn to the equation (1.4). This equation can be simplified further for $C^{(3)}$. Using equations (1.1b) and (1.1c) we obtain from (1.4)

$$C^{(3)} = \frac{-2}{(q^2 + p^2)(q + ip)^2} [Mq + np - e^2 - g^2 - i (Mp - nq)] \qquad (1.7)$$

Consequently if any of the constants M, n, e, g is not zero, the metric (1.1) is of Petrov type D. In the case $C^{(3)} = 0$, the metric is conformally flat. In addition to $C^{(a)} = 0$, if also $R_{ab} = 0$, then the metric becomes flat. The metric is asymptotically de Sitter when $\Lambda \neq 0$ but asymptotically flat when $\Lambda = 0$.

A large number of solutions can be obtained by contracting the metric (1.1) by appropriate limiting procedures. In the course of this discussion, we give the interpretation of the parameters involved in the metric (1.1) and their correspondence to some well- known solutions.

1.2 Contractions of the Plebanski Spacetime

(a) We consider the coordinate transformation from $x^{\mu} = (p, \sigma, q, \tau)$ to $x'^{\mu} = (p', \sigma', q', \tau')$ defined by

$$p = p_o + \varepsilon_o p', \quad \sigma = \sigma'/\varepsilon_o, \quad q = q', \quad \tau = \tau' + (p_o^2 \sigma')/\varepsilon_o \qquad \ldots (1.8)$$

where p_o is an arbitrary constant and ε_o denotes the contraction parameter.

Under the coordinate transformation given by (1.8) the metric (1.1) reduces to the form

$$ds^2 = \frac{\{(p_o + \varepsilon_o p')^2 + q'^2\}}{\varepsilon_o^{-2} X(p_o + \varepsilon_o p')} dp'^2$$

$$+ \frac{\varepsilon_o^{-2} X(p_o + \varepsilon_o p')}{\{(p_o + \varepsilon_o p')^2 + q'^2\}} \{\varepsilon_o d\tau' + (p_o^2 + q'^2) d\sigma'\}^2$$

$$+ \frac{1}{\{(p_o + \varepsilon_o p')^2 + q'^2\}^{-1} Y(q')} dq'^2$$

$$- \{(p_o + \varepsilon_o p')^2 + q'^2\}^{-1} Y(q') \{d\tau' - (2p_o p' + \varepsilon_o p'^2) d\sigma'\}^2 \quad (1.9a)$$

where

$$\varepsilon_o^{-2} X(p_o + \varepsilon_o p') = \varepsilon_o^{-2} \{b - g^2 + 2n(p_o + \varepsilon_o p')$$

$$- \varepsilon(p_o + \varepsilon_o p')^2 - (\Lambda/3)(p_o + \varepsilon_o p')^4\} \quad \ldots (1.9b)$$

$$\{(p_o + \varepsilon_o p')^2 + q'^2\}^{-1} \; Y(q')$$

$$= \{(p_o + \varepsilon_o p')^2 + q'\}^{-1} (b + e^2 - 2Mq' + \varepsilon q'^2 - \frac{\wedge}{3} q'^4) \qquad (1.9c)$$

Simplifying we can write equation (1.9) in the form

$$ds^2 = \frac{\{(p_o + \varepsilon_o p')^2 + q'^2\}}{X_1} dp'^2 + \frac{X_1}{\{(p_o + \varepsilon_o p')^2 + q'^2\}} \{\varepsilon_o d\tau' + (p_o^2 + q'^2) d\sigma'\}^2$$

$$+ \frac{dq'^2}{Y_1} Y_1 \{d\tau' - (2p_o p' + \varepsilon_o p'^2) d\sigma'\}^2 \qquad \ldots (1.10a)$$

where

$$X_1 = \varepsilon_o^{-2} X(p_o + \varepsilon_o p') = \alpha_o + 2\beta_o p' - \gamma_o p'^2 - \frac{4}{3} \wedge p_o \varepsilon_o p'^3 - \frac{\wedge}{3} \varepsilon_o^2 p'^4 \quad (1.10b)$$

$$Y_1 = \{(p_o + \varepsilon_o p')^2 + q'^2\}^{-1} \; Y(q')$$

$$= \{(p_o + \varepsilon_o p')^2 + q'^2\} (\alpha_o \varepsilon_o^2 - 2\beta_o p_o \varepsilon_o - \gamma_o p_o^2 + g^2 + \wedge p_o^4$$

$$+ e^2 - 2Mq' + \varepsilon q'^2 - \frac{\wedge}{3} q'^4) \qquad \ldots (1.10c)$$

$$\alpha_o = \varepsilon_o^{-2} (b - g^2 + 2np_o - \varepsilon p_o - \frac{\wedge}{3} p_o^4) \qquad \ldots (1.10d)$$

$$\beta_o = \varepsilon_o^{-2} (n - \varepsilon p_o - \frac{2\wedge}{3} p_o^3) \qquad \ldots (1.10e)$$

$$\gamma_o = \varepsilon + 2\wedge p_o^2 \qquad \ldots (1.10f)$$

with α_o, β_o and γ_o being constants independent on ε_o.

Now if $\varepsilon_o \rightarrow 0$, then we have from (1.10)

$$ds^2 = (p_o^2 + q'^2)\left(\frac{dp'^2}{X_1} + X_1\, d\sigma'^2\right) + \frac{dq'^2}{Y_1} - Y_1 (d\tau' - 2\, p_o p' d\sigma')^2 \qquad \ldots \; (1.11a)$$

where

$$X_1 = \alpha_o + 2\beta_o p' - \gamma_o p'^2 \qquad \ldots \; (1.11b)$$

$$Y_1 = (p_o^2 + q'^2)(g^2 - \gamma_o p_o^2 + \Lambda p_o^4 + e^2 - 2Mq' + \varepsilon q'^2 - \frac{\Lambda}{3} q'^4) \qquad \ldots \; (1.11c)$$

The metric (1.11) can ultimately be put in the form

$$ds^2 = (p_o^2 + q'^2)\left(\frac{dp'^2}{X_1} + X_1\, d\sigma'^2\right) + \frac{dq'^2}{Y_1} - Y_1 (d\tau' - 2\, p_o p' d\sigma')^2 \qquad \ldots \; (1.12a)$$

where

$$X_1 = \alpha_o + 2\beta_o p' - \gamma_o p'^2 \qquad \ldots \; (1.12b)$$

$$Y_1 = \gamma_0 - \frac{\wedge}{3}(q'^2 + 5p_0^2) - 2\,\mathrm{Re}\left(\frac{M + in_0}{q' + ip_0}\right) + \left|\frac{e + ig}{q' + ip_0}\right|^2 \qquad (1.12c)$$

$$n_0 = \gamma_0 p_0 - \frac{4\wedge}{3} p_0^3 \qquad \ldots \quad (1.12d)$$

The contracted solution given by equation (1.12) in the coordinates $x'^{\mu} = (p', \sigma', q', \tau')$ is the generalized NUT solution.

(b) We now consider the coordinate transformation from $x^{\mu} = (p, \sigma, q, \tau)$ to $x'^{\mu} = (p', \sigma', q', \tau')$ defined by

$$p = p', \; \sigma = \sigma'/\varepsilon_0, \; q = q_0 + \varepsilon_0 q', \; \tau = \tau' - (q_0^2\, \sigma')/\varepsilon_0 \qquad \ldots \quad (1.13)$$

where q_0 is arbitrary constant and ε_0 is the contraction parameter.

Under the coordinate transformation (1.13) the metric (1.1) reduces to the form

$$ds^2 = \frac{1}{\{p'^2 + (q_0 + \varepsilon_0 q')^2\}^{-1} X(p')}\, dp'^2$$

$$+ \{p'^2 + (q_0 + \varepsilon_0 q')^2\}^{-1} X(p') \{d\tau' + (2q_0 q' + \varepsilon_0 q'^2) d\sigma'\}^2$$

$$+ \frac{\{p'^2 + (q_0 + \varepsilon_0 q')^2\}}{\varepsilon_0^{-2} Y(q_0 + \varepsilon_0 q')}\, dq'^2$$

$$- \frac{\varepsilon_0^{-2} Y(q_0 + \varepsilon_0 q')}{\{p'^2 + (q_0 + \varepsilon_0 q')^2\}} \{\varepsilon_0 d\tau' - (q_0^2 + p'^2)\, d\sigma'\}^2 \qquad \ldots \quad (1.14a)$$

where

$$\{ p'^2 + (q_o + \varepsilon_o q')^2 \}^{-1} X(p')$$

$$= \{p'^2 + (q_o + \varepsilon_o q')^2 \}^{-1} (b - g^2 + 2n\, p' - \varepsilon\, p'^2 - \frac{\Lambda}{3} p'^4) \qquad (1.14b)$$

$$\varepsilon_o^{-2}\ Y(q_o + \varepsilon_o q')$$

$$= \varepsilon_o^{-2} \{b + e^2 - 2M(q_o + \varepsilon_o q') + \varepsilon\,(q_o + \varepsilon_o q')^2 - \frac{\Lambda}{3}(q_o + \varepsilon_o q')^4\} \quad (1.14c)$$

Simplifying we can write equation (1.14) in the form

$$ds^2 = \frac{d\,p'^2}{X_2} + X_2 \{ d\tau' + (2\, q_o q' + \varepsilon_o q'^2)\, d\sigma' \}^2$$

$$+ \frac{p'^2 + (q_o + \varepsilon_o q')^2}{Y_2}\, d\,q'^2$$

$$- \frac{Y_2}{p'^2 + (q_o + \varepsilon_o\, q')^2} \{ \varepsilon_o\, d\tau' - (q_o^2 + p'^2)\, d\sigma' \}^2 \quad \ldots \quad (1.15a)$$

where

$$Y_2 = \varepsilon_o^{-2}\, Y(q_o + \varepsilon_o q')$$

$$= \alpha_1 + 2\beta_1 q' + \gamma_1 q'^2 - \frac{4}{3}\Lambda\, q_o q' \varepsilon_o - \frac{1}{3}\Lambda\, q'^4 \varepsilon_o^2 \quad \ldots \quad (1.15b)$$

$$X_2 = \{ p'^2 + (q_o + \varepsilon_o q')^2 \}^{-1} X(p')$$

$$= \{p'^2 + (q_o + \varepsilon_o q')^2\}^{-1} \{\alpha_1 \varepsilon_o^2 - 2\beta_1 q_o \varepsilon_o + \gamma_o q_o^2 + \Lambda q_o^4$$

$$- e^2 - g^2 + 2np' - \varepsilon p'^2 - \frac{1}{3} \Lambda p'^4\} \quad \ldots \quad (1.15c)$$

$$\alpha_1 = \varepsilon_o^{-2} (b + e^2 - 2Mq_o + \varepsilon q_o^2 - \frac{1}{3} \Lambda q_o^4) \quad \ldots \quad (1.15d)$$

$$\beta_1 = \varepsilon_o^{-1} (\varepsilon q_o - M - \frac{2}{3} \Lambda q_o^3) \quad \ldots \quad (1.15e)$$

$$\gamma_1 = \varepsilon - 2 \Lambda q_o^2 \quad \ldots \quad (1.15f)$$

with α_1 , β_1 and γ_1 being constants independent on ε_o .

Now if $\varepsilon_o \rightarrow 0$, then from (1.15) we have

$$ds^2 = \frac{dp'^2}{X_2} + X_2 (d\tau' + 2 q_o q' d\sigma')^2$$

$$+ (q_o^2 + p'^2) \left(\frac{dq'^2}{Y_2} - Y_2 d\sigma'^2 \right) \quad \ldots \quad (1.16a)$$

where

$$Y_2 = \alpha_1 + 2\beta_1 q' + \gamma_1 q'^2 \quad \ldots \quad (1.16b)$$

$$X_2 = (p'^2 + q_o^2)^{-1} \{\gamma_o q_o^2 + \Lambda q_o^4 - e^2 - g^2 + 2np' + \varepsilon p'^2 - \frac{1}{3} \Lambda p'^4\} \quad (1.16c)$$

The metric (1.16) can ultimately be put into the form

$$ds^2 = \frac{d\,p'^2}{X_2} + X_2 (d\,\tau' + 2\,q_o q'\, d\sigma')^2$$

$$+ (q_o^2 + p'^2) \left(\frac{d\,q'^2}{Y_2} - Y_2\, d\,\sigma'^2 \right) \qquad \ldots \quad (1.17a)$$

where

$$Y_2 = \alpha_1 + 2\,\beta_1 q' + \gamma_1 q'^2 \qquad \ldots \quad (1.17b)$$

$$X_2 = -\gamma_1 - \frac{1}{3} \wedge (p'^2 + 5\,q_o^2) + 2\,\mathrm{Re}\left(\frac{m_o + in}{q_o + ip'} \right) - \left| \frac{e + ig}{q_o + ip'} \right|^2 \qquad (1.17c)$$

$$m_o = \gamma_1 q_o + \frac{4}{3} \wedge q_o^3 \qquad \ldots \quad (1.17d)$$

This contracted solution given by (1.17) is the generalized anti - NUT solution .

1.3 Canonical forms of the Contracted Solutions

(a) Let us consider the generalized NUT solution given by (1.12) and restrict the parameter γ_o associated with it to the discrete values

$$\gamma_o = 1 \,, \quad 0 \quad , \quad -1$$

Case I : $\gamma_o = 1$

If we put $\alpha_o = 1$, $\beta_o = 0$ and consider coordinate transformation from $x'^{\mu} = (p', \sigma', q', \tau')$ to $x''^{\mu} = (\theta , \phi , q', t')$ defined by

$$p' = \cos\theta , \quad \sigma' = \phi , \quad q' = q', \quad \tau' = t' \qquad \ldots (1.18)$$

then the metric given by (1.12) takes the form

$$ds^2 = (p_o^2 + q'^2)(d\theta^2 + \sin^2\theta \, d\phi^2) + \frac{dq'^2}{Y_1}$$

$$- Y_1 (dt' - 2 p_o \cos\theta \, d\phi)^2 \qquad \ldots (1.19a)$$

where

$$Y_1 = 1 - \frac{\Lambda}{3}(q'^2 + 5 p_o^2) - 2 \, \mathrm{Re}\left(\frac{M + i n_o}{q' + i p_o}\right) + \left|\frac{e + i g}{q' + i p_o}\right|^2 \qquad \ldots (1.19b)$$

$$n_o = p_o - \frac{4}{3} \Lambda \, p_o^3 \qquad \ldots (1.19c)$$

Case II : $\gamma = 0$

If we set $\alpha_o = 1$, $\beta_o = 0$ and introduce new coordinates

$$p' = \theta \cos\phi , \quad \sigma' = \theta \sin\phi , \quad \tau' = t' + p_o p' \sigma' \qquad \ldots (1.20)$$

then the metric given by (1.12) reduces to the form

$$ds^2 = (p_o^2 + q'^2)(d\theta^2 + \theta^2 d\phi^2) + \frac{d q'^2}{Y_1}$$

$$- Y_1 (dt' + p_o \theta^2 d\phi)^2 \quad \ldots \quad (1.21a)$$

where

$$Y_1 = - \frac{\Lambda}{3} (q'^2 + 5 p_o^2) - 2 \operatorname{Re} \left(\frac{M + in_o}{q' + ip_o} \right) + \left| \frac{e + ig}{q' + ip_o} \right|^2 \quad \ldots \quad (1.21b)$$

$$n_o = - \frac{4}{3} \Lambda p_o^3 \quad \ldots \quad (1.21c)$$

<u>Case III:</u> $\gamma_o = -1$

If we set $\alpha_o = -1$, $\beta_o = 0$ and introduce new coordinates

$$p' = \cosh\theta , \sigma' = \phi , \tau' = t' \quad \ldots \quad (1.22)$$

then we have (1.12) as

$$ds^2 = (p_o^2 + q'^2)(d\theta^2 + \sinh^2\theta \, d\phi^2) + \frac{d q'^2}{Y_1}$$

$$- Y_1 (dt' - 2 p_o \cosh\theta \, d\phi)^2 \quad \ldots \quad (1.23a)$$

where

$$Y_1 = - 1 - \frac{\Lambda}{3} (q'^2 + 5 p_o^2) - 2 \operatorname{Re} \left(\frac{M + in_o}{q' + ip_o} \right) + \left| \frac{e + ig}{q' + ip_o} \right|^2 \quad \ldots \quad (1.23b)$$

$$n_o = - p_o - \frac{4}{3} \Lambda p_o^3 \quad \ldots \quad (1.23c)$$

The equations (1.19) , (1.21) and (1.23) are the canonical representations of the generalized NUT solutions.

Now we would like to provide some comments concerning the interpretation of the generalized NUT solution described in canonical forms.

If $p_o = 0$, then we have from (1.19)

$$ds^2 = q'^2 (d\theta^2 + \sin^2\theta \, d\phi^2) + \frac{d q'^2}{Y_1} - Y_1 \, dt'^2 \quad \ldots (1.24a)$$

$$Y_1 = 1 - \frac{\Lambda}{3} q'^2 - \frac{2M}{q'} + \frac{e^2 + g^2}{q'^2} \quad \ldots (1.24b)$$

If $g = 0$ we recognize (1.24) as the Reissner - Nordstrom solution with the cosmological constant Λ. The coordinate q' plays the role of the radial variable. The constant M and e are interpreted as mass and charge associated with this solution. If g is different from zero, then (1.24) represents a slight generalization of the cosmological Reissner - Nordstrom solution ; g is interpreted as magnetic charge. For $e = g = 0$ the solution given by (1.24) becomes the Schwarzschild solution generalized by the cosmological constant. Further with $e = g = \Lambda = 0$, we obtain from (1.24), the basic Schwarzschild solution.

If $p_o \neq 0$ but $e = g = 0$, then the solution given by (1.19) reduces to the form

$$ds^2 = (p_o^2 + q'^2)(d\theta^2 + \sin^2\theta \, d\phi^2) + \frac{dq'^2}{Y_1}$$

$$- Y_1 (dt' - 2p_o \cos\theta \, d\phi)^2 \qquad \ldots (1.25a)$$

where

$$Y_1 = 1 - \frac{\Lambda}{3}(q'^2 + 5p_o^2) - 2\,\mathrm{Re}\left(\frac{M + in_o}{q' + ip_o}\right) \qquad \ldots (1.25b)$$

$$n_o = p_o - \frac{4}{3}\Lambda \, p_o^3 \qquad \ldots (1.25c)$$

The solution given by (1.25) is the NUT solution generalized by the presence of the cosmological constant. The parameter n_o coincides with p_o when $\Lambda = 0$. This parameter is the NUT parameter or magnetic mass parameter.

It is clear that the generalized family of NUT solutions described in the canonical form by (1.19) and obtained by a contraction from Plebanski spacetime (1.1) represents the combined NUT- Reissner-Nordstrom solution with the cosmological constant, additionally generalized by the possible presence of magnetic monopole. The parameters e and g have interpretation of the electric and magnetic charges; M and n_o have interpretation of the mass and the NUT parameter.

If we put $p_o = 0$ in (1.21) then we have

$$ds^2 = q'^2 (d\theta^2 + \theta^2 d\phi^2) + \frac{dq'^2}{Y_1} - Y_1 dt'^2 \qquad \ldots (1.26a)$$

where

$$Y_1 = -\frac{\Lambda}{3} q'^2 - \frac{2M}{q'} + \frac{e^2 + g^2}{q'^2} \quad \ldots \quad (1.26b)$$

Equation (1.26) is the Kasner-type spacetime, an anisotropic universe. Setting $\Lambda = e = g = 0$, equation (1.26) can be transformed to Kasner form.

Now if we set $p_o = 0$ in (1.23) then we have

$$ds^2 = q'^2 (d\theta^2 + \sinh^2 \theta \, d\phi^2) + \frac{dq'^2}{Y_1} - Y_1 \, dt'^2 \quad \ldots \quad (1.27a)$$

where

$$Y_1 = -1 - \frac{\Lambda}{3} q'^2 - \frac{2M}{q'} + \frac{e^2 + g^2}{q'^2} \quad \ldots \quad (1.27b)$$

The solution (1.27) presents Levi-Civita's type of generalization of the cosmological Reissner-Nordstrom solution with charges of both types. The Levi-Civita's metric is sometimes interpreted as the metric of a heavy tachyon.

(b) Now we consider " generalized anti-NUT solutions " given by (1.17) and restrict the parameter γ_1 to the discrete values

$\gamma_1 = 1, \quad 0, \quad -1$

<u>Case I :</u> $\gamma_1 = 1$

If we put $\alpha_1 = 1$, $\beta_1 = 0$ and introduce new coordinates $x''^\mu = (p', \phi, \theta, \tau')$ in place of $x'^\mu = (p', \sigma', q', \tau')$ defined by

$$p' = p' , \quad \sigma' = \phi , \quad q' = \sinh \theta , \quad \tau' = \tau' \qquad \ldots (1.28)$$

then (1.17) reduces to the form

$$ds^2 = \frac{d\,p'^2}{X_2} + X_2 (d\tau' + 2q_o \sinh \theta \, d\phi)^2$$

$$+ (q_o^2 + p'^2)(d\theta^2 - \cosh \theta \, d\phi^2) \qquad \ldots (1.29a)$$

where

$$X_2 = -1 - \frac{\Lambda}{3}(p'^2 + 5q_o^2) + 2 \, \mathrm{Re} \left(\frac{m_o + in}{q_o + ip'} \right) - \left| \frac{e + ig}{q_o + ip'} \right|^2 \qquad \ldots (1.29b)$$

$$m_o = q_o + \frac{4}{3} \Lambda \, q_o^3 \qquad \ldots (1.29c)$$

<u>Case II :</u> $\gamma_1 = 0$

If we now take $\alpha_1 = 1$, $\beta_1 = 0$ and introduce new coordinates $x''^{\mu} = (p', y, x, \tau')$ for $x'^{\mu} = (p', \sigma', q', \tau')$ defined by

$$p' = p' , \quad \sigma' = y , \quad q' = x , \quad \tau' = \tau' \qquad \ldots (1.30)$$

then (1.17) reduces to the form

$$ds^2 = \frac{d\,p'^2}{X_2} + X_2 (d\tau' + 2 q_o x dy)^2$$

$$+ (q_o^2 + p'^2)(dx^2 - dy^2) \qquad \ldots (1.31a)$$

where

$$X_2 = -\frac{\Lambda}{3}(p'^2 + 5q_o^2) + 2\,\mathrm{Re}\left(\frac{m_o + in}{q_o + ip'}\right) - \left|\frac{e + ig}{q_o + ip'}\right|^2 \quad \ldots \quad (1.31b)$$

$$m_o = \frac{4}{3}\Lambda q_o^3 \quad \ldots \quad (1.31c)$$

<u>Case III :</u> $\gamma_1 = -1$

In this case if we set $\alpha_1 = -1$, $\beta_1 = 0$ and consider the transformation defined by

$$p' = p', \quad \sigma' = \phi, \quad q' = \sinh\theta, \quad \tau' = \tau' \quad \ldots \quad (1.32)$$

then we have (1.17) as

$$ds^2 = \frac{dp'^2}{X_2} + X_2(d\tau' + 2q_o \sinh\theta \; d\phi)^2$$

$$- (q_o^2 + p'^2)(d\theta^2 - \cosh^2\theta \; d\phi^2) \quad \ldots \quad (1.33a)$$

where

$$X_2 = 1 - \frac{\Lambda}{3}(p'^2 + 5q_o^2) + 2\,\mathrm{Re}\left(\frac{m_o + in}{q_o + ip'}\right) - \left|\frac{e + ig}{q_o + ip'}\right|^2 \quad \ldots \quad (1.33b)$$

$$m_o = -q_o + \frac{4}{3}\Lambda q_o^3 \quad \ldots \quad (1.33c)$$

Equations (1.29), (1.31) and (1.33) are the canonical representations of the anti - NUT solution.

If we put $e = g = q_o = \Lambda = 0$, then the equation (1.33) takes the form

$$ds^2 = \frac{dp'^2}{X_2} + X_2\, d\tau'^2 - p'^2 (d\theta^2 - \cosh^2\theta\, d\phi^2) \qquad \ldots (1.34a)$$

where

$$X_2 = 1 + \frac{2n}{p'} \qquad \ldots (1.34b)$$

which is a solution of Einstein's equation in vacuum.

1.4 The Combined NUT - Kerr - Newman - Kasuya Spacetime

If we set

$$\varepsilon = 1 , \Lambda = 0 , b = a^2 - n^2 + g^2 \qquad \ldots (1.35)$$

then the metric (1.1) reduces to the form

$$ds^2 = \frac{p^2 + q^2}{X} dp^2 + \frac{X}{p^2 + q^2} (d\tau + q^2 d\sigma)^2$$

$$+ \frac{p^2 + q^2}{Y} dq^2 - \frac{Y}{p^2 + q^2} (d\tau - p^2 d\sigma)^2 \qquad \ldots (1.36a)$$

where

$$X = a^2 - (n - p)^2 \qquad \ldots (1.36b)$$

$$Y = q^2 - 2Mq + a^2 - n^2 + e^2 + g^2 \qquad \ldots \quad (1.36c)$$

and the parameter a has the interpretation of angular momentum per unit mass. The above equation represents the combined NUT - Kerr - Newman - Kasuya spacetime in Boyer coordinates. For $n = e = g = 0$ the equation (1.36) reduces to the Kerr spacetime [29].

If we perform the coordinate transformation

$$p = n + a\cos\theta, \qquad \sigma = -\phi/a$$
$$q = r, \qquad \tau = t - \frac{(n^2 + a^2)}{a}\phi \qquad \ldots \quad (1.37)$$

then we obtain from (1.36)

$$ds^2 = \Sigma\, d\theta^2 + \frac{\Sigma}{Y} dr^2 + \frac{\sin^2\theta}{\Sigma}(a\,dt - \rho\, d\phi)^2$$
$$- \frac{Y}{\Sigma}(dt - A\, d\phi)^2 \qquad \ldots \quad (1.38a)$$

where

$$\Sigma = r^2 + (n + a\cos\theta)^2$$
$$Y = r^2 - 2Mr + a^2 - n^2 + e^2 + g^2$$
$$\rho = r^2 + a^2 + n^2 \qquad \ldots \quad (1.38b)$$
$$A = a\sin^2\theta - 2n\cos\theta .$$

Equation (1.38) represents the NUT - Kerr - Newman - Kasuya (NUTKNK) spacetime in Boyer - Lindquist coordinates. The NUTKNK spacetime includes

(i) Kerr - Newman - Kasuya spacetime [30] when $n = 0$

(ii) NUT - Kerr - Newman spacetime for $g = 0$

(iii) NUT - Kerr spacetime [31] with $e = g = 0$

(iv) Kerr - Newman spacetime [32] if $n = g = 0$

(v) Kerr spacetime [33] provided $n = g = e = 0$

(vi) Reissner - Nordstrom spacetime [34,35] for $n = g = a = 0$

(vii) Schwarzschild spacetime [36] when $n = g = a = e = 0$

(viii) charged NUT spacetime [37] if $a = g = 0$

(ix) NUT spacetime [38] for $a = e = g = 0$.

So we see that the NUTKNK spacetime includes all the black hole spacetimes (iv) - (vii), which are asymptotically flat. In particular the NUTKNK spacetime contains the NUT spacetime which has peculiar properties.

1.5 The Combined NUT - Kerr - Newman - Kasuya - de Sitter Spacetime

Setting

$$\varepsilon = 1 - \frac{\Lambda}{3}(a^2 + 6n^2)$$

$$b = a^2 - n^2 + g^2 - \frac{5}{3}\Lambda n^2 (a^2 + n^2) \qquad \ldots (1.39)$$

and replacing n by

$$n + \frac{\Lambda}{3}\left(\frac{3a^2n^2}{p} + \frac{2n^4}{p} + 2n^3 - 6n^2 p - a^2 n + 2np^2\right) \qquad \ldots (1.40)$$

the metric (1.1) can be brought to the form

$$ds^2 = \frac{p^2+q^2}{X}dp^2 + \frac{X}{p^2+q^2}(d\tau + q^2 d\sigma)^2 + \frac{p^2+q^2}{Y}dq^2$$

$$- \frac{Y}{p^2+q^2}(d\tau - p^2 d\sigma)^2 \qquad \ldots (1.41a)$$

where

$$X = [a^2 - (n-p)^2][1 + \frac{\Lambda}{3}(n-p)^2] \qquad \ldots (1.41b)$$

$$Y = (q^2 + a^2 + n^2)[1 - \frac{\Lambda}{3}(q^2 + 5n^2)]$$

$$- 2(Mq + n^2) + e^2 + g^2 \qquad \ldots (1.41c)$$

Equation (1.41) represents the combined NUT - Kerr- Newman - Kasuya- de Sitter spacetime [39] in Boyer coordinates.

If we now perform the coordinate transformation

$$p = n + a\cos\theta \quad , \qquad \sigma = - \Xi^{-1}(\phi / a) \qquad \ldots \quad (1.42)$$

$$q = r \quad , \qquad \tau = \Xi^{-1}\left[t - \frac{(n^2 + a^2)}{a} \phi \right]$$

where $\Xi = 1 + \dfrac{\Lambda}{3} a^2$

then the metric (1.41) can be transformed to the form

$$ds^2 = \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Sigma}{\Delta_r} dr^2 + \frac{\Xi^{-2} \Delta_\theta \sin^2\theta}{\Sigma} (a\,dt - \rho\, d\phi)^2$$

$$- \frac{\Xi^{-2} \Delta_r}{\Sigma} (dt - A\, d\phi)^2 \qquad \ldots \quad (1.43a)$$

where

$$\Sigma = r^2 + (n + a\cos\theta)^2$$

$$\Delta_\theta = 1 + \frac{\Lambda}{3} a^2 \cos^2\theta \qquad \ldots \quad (1.43b)$$

$$\Delta_r = (r^2 + a^2 + n^2)\left[1 - \frac{\Lambda}{3}(r^2 + 5n^2)\right] - 2(Mr + n^2) + e^2 + g^2$$

$$\rho = r^2 + a^2 + n^2$$

$$A = a\sin^2\theta - 2n\cos\theta$$

The equation (1.43) represents the combined NUT - Kerr - Newman - Kasuya - de Sitter spacetime [40] in Boyer - Lindquist coordinates . We call the metric (1.43) as hot NUT - Kerr - Newman-Kasuya (HNUTKNK) spacetime since the de Sitter spacetime has been interpreted as being hot [41] . The HNUTKNK spacetime includes :

(i) NUTKNK spacetime for $\Lambda = 0$

(ii) hot Kerr - Newman - Kasuya (HKNK) spacetime when $n = 0$

(iii) hot NUT - Kerr - Newman (HNUTKN) spacetime for $g = 0$

(iv) hot Kerr - Newman spacetime [42] with $n = g = 0$

(v) hot Kerr spacetime [43] if $n = g = e = 0$

(vi) hot Reissner - Nordstrom spacetime for $n = g = a = 0$

(vii) hot Schwarzschild spacetime [44] for $n = g = a = e = 0$

(viii) hot NUT spacetime [45] for $a = e = g = 0$.

So we see that the HNUTKNK spacetime includes the NUTKNK, HKNK , HNUTKN , hot NUT spacetimes as well as all the black hole spacetimes (iv) - (vii) which are asymptotically de Sitter . Further if we set $\Lambda = 0$, in the cases (ii) - (vii) we get the Kerr-Newman - Kasuya , NUT - Kerr - Newman spacetimes and all the black hole spacetimes which are asymptotically flat . In the limit $\Lambda = 0$, the case (viii) reduces to the NUT spacetime which is considered as homogeneous anisotropic cosmological model [46].

Thus the Plebanski spacetime (1.1) contains a large number of solutions of Einstein - Maxwell equations with or without cosmological constant which are important from the physical point of view. The metric (1.1) contains some spacetimes with cosmological parameter which may be found interesting from the point of view of its inflationary scenario of the early universe [47].

1.6 A Modified form of the Plebanski Spacetime

In this section we present a new form of the Plebanski spacetime. We show that a simple coordinate transformation brings the Plebanski spacetime to a form which includes an additional parameter other than the parameters present in the Plebanski spacetime. For this purpose if we introduce the new coordinate (p', σ', q', τ') in place of the coordinate (p, σ, q, τ) defined by

$$p' = p \;, \quad q' = q \;, \quad \tau' = \nu\tau \;, \quad \sigma' = \nu\sigma \qquad \ldots \quad (1.44)$$

where ν is a real constant, then the metric (1.1) takes the form

$$ds^2 = \frac{p^2+q^2}{X} dp^2 + \frac{p^2+q^2}{Y} dq^2 + \frac{X}{\nu^2(p^2+q^2)}(d\tau + q^2 d\sigma)^2$$

$$- \frac{Y}{\nu^2(p^2+q^2)}(d\tau - p^2 d\sigma)^2 \quad \ldots \quad (1.45a)$$

$$X = b - g^2 + 2np - \varepsilon p^2 - \frac{\wedge}{3} p^4 \quad \ldots \quad (1.45b)$$

$$Y = b + e^2 - 2Mq + \varepsilon q^2 - \frac{\wedge}{3} q^4 \quad \ldots \quad (1.45c)$$

where we have dropped the primes .

The equation (1.45) can be identified as the modified form of the Plebanski spacetime [48] in Boyer coordinates . The parameters M , n , e , g have the same interpretations as in (1.1) and the parameter ν is the adjustable parameter like the parameters b and ε in (1.1) . By the similar arguments as given in [49] one can show that the metric (1.45) represents a class of solutions of Einstein - Maxwell equations.

For $\nu = 1$, the metric (1.45) gives the original Plebanski metric (1.1).

Setting

$$\nu = 1 + \frac{\wedge}{3} a^2, \quad \varepsilon = 1 - \frac{\wedge}{3}(a^2 + 6n^2)$$
$$b = a^2 - n^2 + g^2 - \frac{5}{3} \wedge n^2 (a^2 + n^2) \quad \ldots \quad (1.46)$$

and replacing n by

$$n + \frac{\Lambda}{3}\left(\frac{3a^2n^2}{p} + \frac{2n^4}{p} + 2n^3 - 6n^2p - a^2n + 2np^2 \right) \quad \ldots \quad (1.47)$$

and then accomplishing the coordinate transformation

$$p = n + a\cos\theta \,, \qquad \sigma = -\phi/a$$
$$q = r \,, \qquad \tau = t - \frac{(n^2 + a^2)}{a}\phi \quad \ldots \quad (1.48)$$

the metric (1.45) can be brought to the form

$$ds^2 = \frac{\Sigma}{P} d\theta^2 + \frac{\Sigma}{Y} dr^2 + \frac{P \sin^2\theta}{(1 + \frac{\Lambda}{3}a^2)^2\Sigma}(a\,dt - \rho\,d\phi)^2$$

$$- \frac{Y}{(1 + \frac{\Lambda}{3}a^2)^2\Sigma}(dt - A\,d\phi)^2 \quad \ldots \quad (1.49a)$$

where

$$\Sigma = r^2 + (n + a\cos\theta)^2$$

$$P = 1 + \frac{\Lambda}{3}a^2\cos^2\theta$$

$$\ldots \quad (1.49b)$$

$$Y = (r^2 + a^2 + n^2)\left[1 - \frac{\Lambda}{3}(r^2 + 5n^2)\right] - 2(Mr + n^2) + e^2 + g^2$$

$$\rho = r^2 + a^2 + n^2$$

$$A = a\sin^2\theta - 2n\cos\theta \,.$$

CHAPTER - 2

SEPARATED DIRAC EQUATION IN AN ARBITRARY CURVED SPACETIME

The separability properties of Hamilton - Jacobi and Klein - Gordon equations have been studied by Carter [50,51] in the Kerr - Newman spacetime. Debever et al. [52] investigated the separability properties of the Hamilton - Jacobi and Klein - Gordon equations in an arbitrary curved background spacetime. The separability of Weyl neutrino equation, Maxwell's equations and the perturbed Einstein gravitational field equations has been established by Teukolsky [53] in the Kerr spacetime and using an analogous method, in the Plebanski - Demianski background [54] by Dudley and Finley [55]. Teukolsky's separation of the variables of the equations governing the electromagnetic, the gravitational and the two - component neutrino - field perturbations of Kerr black hole has been central to much of the later developments. Following Teukolsky's separation method, Chandrasekhar [56] separated Dirac's equation in the Kerr spacetime and showed that the solutions can be expressed in terms of certain radial and angular function (satisfying decoupled equations). Chandrasekhar's result was immediately extended by Page [57] to the Kerr - Newman spacetime and subsequently analyzed by Carter and McLenaghan [58,59]. Recently Kamran and McLenaghan [60] performed a systematic study of the separability properties of neutrino and massive charged Dirac equations and obtained a system of coupled first order ordinary differential equations and a system of decoupled second order ordinary differential equations.

In this chapter we study the separability properties of the massive charged Dirac equation in an arbitrary curved background spacetime [61] and obtain the separated Dirac equation as a system of decoupled second order ordinary differential equation from which the pertinent equation will be used to derive the radial equation for the metric concerned to study the Hawking radiation in the later chapters .

Finally , we show how the separated Dirac equation of Page [62] in the Kerr-Newman background and of Chandrasekhar [63] in the Kerr background emerge from the general expression of the separated Dirac equation in an arbitrary curved background spacetime as special cases .

2.1 Arbitrary Curved Spacetime and Some Preliminaries

The general structures for the spacetimes which are solutions of Einstein-Maxwell equations with cosmological constant have been established by Debever and McLenaghan [64 , 65] and Diaz and Salazar I [66] . But recently Debever el al. [67 , 68] and Diaz [69] determined , in a systematic and unified manner , the complete set of solutions of Petrov type D to the Einstein - Maxwell equations with cosmological constant . The obtained solutions contain , all known type D metrics: for instance Plebanski - Demianski solution [70] , Plebanski solution [71] , Kinnersley solution [72] Carter solutions [73 , 74 , 75] and so forth .

For our purpose we consider the spacetime [76] in the coordinates (u, v, w, x)

$$ds^2 = \frac{Z}{T^2}[fW^2 Z^{-2}(\varepsilon_1 du + m\,dv)^2 + (1-f^2)g^{-2} Z^{-1}(\varepsilon_1 du + m\,dv)dw$$

$$-f g^{-4} W^{-2} dw^2 - X^2 Z^{-2}(\varepsilon_2 du + p\,dv)^2 - X^{-2} dx^2] \quad \ldots \quad (2.1a)$$

with

$$A = A_\mu dx^\mu = \frac{1}{Z} [(\varepsilon_2 G + \varepsilon_1 H) du + (pG + mH) dv] \quad \ldots \quad (2.1b)$$

where

$$W = W(w) , X = X(x) , T = T(w, x) , G = G(x) , H = H(w) ,$$

$$p = p(w) , \quad m = m(x) , g = [(1 + f^2)/2]^{1/2} \quad \ldots \quad (2.1c)$$

$$Z = Z(w, x) = \varepsilon_1 p(w) - \varepsilon_2 m(x)$$

and ε_1, ε_2 and f are real constants satisfying $\varepsilon_1^2 + \varepsilon_2^2 \neq 0$ and all functions are real valued .

By specializing the constants and the functions appearing in (2.1), a particular class of spacetimes of the metric may be deduced . For example if we set in equation (2.1) the following substitutions :

$$\varepsilon_1 = \varepsilon_2 = f = g = T = 1 , Z = w^2 + x^2 , p = w^2 , m = -x^2, X = \sqrt{X}, W = \sqrt{Y}$$

where X and Y are given by (1.1b) and (1.1c) , we get the Plebanski spacetime .

The spin coefficients for the metric (2.1) are given by [77]

$$\rho = (\sqrt{2})^{-1} W T Z^{-1/2} [T^{-1} T_w - \frac{1}{2} Z^{-1} Z_w + \frac{i}{2} Z^{-1} \varepsilon_1 m_x]$$

$$\tau = (\sqrt{2})^{-1} X T Z^{-1/2} [\frac{1}{2} Z^{-1} \varepsilon_2 p_w + i (\frac{1}{2} Z^{-1} Z_x - T^{-1} T_x)]$$

$$\varepsilon = \frac{1}{2\sqrt{2}} W T Z^{-1/2} [W^{-1} W' - \frac{1}{2} Z^{-1} Z_w - T^{-1} T_w + \frac{i}{2} Z^{-1} \varepsilon_1 m_x] \qquad (2.2)$$

$$\alpha = \frac{1}{2\sqrt{2}} X T Z^{-1/2} [\frac{1}{2} Z^{-1} \varepsilon_2 p_w + i (T^{-1} T_x + \frac{1}{2} Z^{-1} Z_x - X^{-1} X')]$$

$$\mu = f\rho , \quad \gamma = f\varepsilon , \quad \pi = \tau , \quad \beta = \alpha , \quad \kappa = \sigma = \lambda = \nu = 0 , \quad f^2 = 1 .$$

The derivative operators are given by

$$D = \frac{1}{\sqrt{2}} T Z^{-1/2} [f g^{-2} W^{-1} (p \frac{\partial}{\partial u} - \varepsilon_2 \frac{\partial}{\partial v}) + W \frac{\partial}{\partial w}]$$

$$\Delta = \frac{1}{\sqrt{2}} T Z^{-1/2} [g^{-2} W^{-1} (p \frac{\partial}{\partial u} - \varepsilon_2 \frac{\partial}{\partial v}) - f W \frac{\partial}{\partial w}] \qquad (2.3)$$

$$\delta = \frac{1}{\sqrt{2}} T Z^{-1/2} [X^{-1} (\varepsilon_1 \frac{\partial}{\partial u} - m \frac{\partial}{\partial v}) - i X \frac{\partial}{\partial x}]$$

2.2 Separation of the Dirac Equation

In the Newman - Penrose formalism the Dirac equation can be written in the form , using the Weyl representation for ψ

$$H_D \psi = \begin{bmatrix} -i\mu_e & 0 & D+\bar{\epsilon}-\bar{\rho}-iQA_1 & \delta+\bar{\pi}-\bar{\alpha}-iQA_3 \\ 0 & -i\mu_e & \bar{\delta}+\bar{\beta}-\bar{\tau}-iQA_4 & \Delta+\bar{\mu}-\bar{\gamma}-iQA_2 \\ \Delta+\mu-\gamma-iQA_2 & -(\delta+\beta-\tau)+iQA_3 & -i\mu_e & 0 \\ -(\bar{\delta}+\pi-\alpha)+iQA_4 & D+\epsilon-\rho-iQA_1 & 0 & -i\mu_e \end{bmatrix} \begin{bmatrix} P_o \\ P_1 \\ \bar{Q}^{0'} \\ \bar{Q}^{1'} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (2.4)$$

where μ_e and Q are the mass and the electric charge of the Dirac particle respectively .

We have from (2.1b) , (2.2) and (2.3)

$$D + \varepsilon - \rho - iQA_1 = \frac{1}{\sqrt{2}} T Z^{-1/2} \Big[W \frac{\partial}{\partial w} + f g^{-2} \Big(p \frac{\partial}{\partial u} - \varepsilon_2 \frac{\partial}{\partial v} \Big)$$

$$- \frac{1}{4} W Z^{-1} \varepsilon_1 (- p' + i m') + \frac{1}{2} W' - \frac{3}{2} W T^{-1} T_w \Big]$$

$$- iQ Z^{-1} (\varepsilon_2 G + \varepsilon_1 H) \qquad (2.5a)$$

$$\Delta + \mu - \gamma - iQA_2 = \frac{1}{\sqrt{2}} T Z^{-1/2} \Big[- f W \frac{\partial}{\partial w} + g^{-2} W^{-1} \Big(p \frac{\partial}{\partial u} - \varepsilon_2 \frac{\partial}{\partial v} \Big)$$

$$+ \frac{1}{4} f W Z^{-1} \varepsilon_1 (- p' + i m') - \frac{1}{2} f W'$$

$$+ \frac{3}{2} f W T^{-1} T_w \Big] - iQ Z^{-1} (p G + m H) \qquad (2.5b)$$

$$\delta + \beta - \tau - iQA_3 = \frac{1}{\sqrt{2}} T Z^{-1/2} \Big[- i X \frac{\partial}{\partial x} + X^{-1} \Big(\varepsilon_1 \frac{\partial}{\partial u} - m \frac{\partial}{\partial v} \Big)$$

$$- \frac{1}{4} X Z^{-1} \varepsilon_2 (p' - i m') - \frac{i}{2} X' + \frac{3}{2} i X T^{-1} T_x \Big] \qquad (2.5c)$$

$$\bar{\delta} + \pi - \alpha - iQA_4 = \frac{1}{\sqrt{2}} T Z^{-1/2} \Big[i X \frac{\partial}{\partial x} + X^{-1} \Big(\varepsilon_1 \frac{\partial}{\partial u} - m \frac{\partial}{\partial v} \Big)$$

$$+ \frac{1}{4} X Z^{-1} \varepsilon_2 (p' - i m') + \frac{i}{2} X' - \frac{3}{2} i X T^{-1} T_x \Big] \qquad (2.5d)$$

Let us perform the transformation

$$\psi \quad = \quad S\,\psi' \qquad \ldots \qquad (2.6a)$$

where

$$S = T^{3/2}\, Z^{-1/4}\, \mathrm{diag}\;(\, e^{iB},\, e^{iB},\, e^{-iB},\, e^{-iB}\,) \qquad \ldots \qquad (2.6b)$$

$$dB = (4\,Z)^{-1}\;(\,\varepsilon_1\, m'\, dw + \varepsilon_2\, p'\, dx\,) \qquad \ldots \qquad (2.6c)$$

and separating matrix defined by

$$U = Z^{1/2}\; T^{-1}\; \mathrm{diag}\;(e^{2iB},\, -\, e^{2iB},\, -\, e^{-2iB},\, e^{-2iB}) \qquad (2.7)$$

We obtain from (2.4) , (2.6b) and (2.7)

$$W_D = U\, S^{-1}\; H_D\, S$$

$$= \begin{bmatrix} -i\mu_e\, Z^{1/2}\, T^{-1}\, e^{2iB} & 0 & D_w^{+} & D_x^{-} \\ 0 & i\mu_e Z^{1/2}\, T^{-1}\, e^{2iB} & -D_x^{-} & -D_w^{-} \\ -\, D_w^{-} & D_x^{-} & i\mu_e\, Z^{1/2}\, T^{-1}\, e^{-2iB} & 0 \\ -\, D_x^{+} & D_w^{+} & 0 & -i\mu_e Z^{1/2}\, T^{-1}\, e^{-2iB} \end{bmatrix}$$

$$\ldots \qquad (2.8)$$

where the operators

$$D_x^+ = \frac{1}{\sqrt{2}} \left[i \left(X \frac{\partial}{\partial x} + \frac{X'}{2} - \frac{QG}{X} \right) + \frac{1}{X} \left(\varepsilon_1 \frac{\partial}{\partial v} - m \frac{\partial}{\partial u} \right) \right] \quad \ldots \qquad (2.9a)$$

$$D_x^- = \frac{1}{\sqrt{2}} \left[- i \left(X \frac{\partial}{\partial x} + \frac{X'}{2} + \frac{QG}{X} \right) + \frac{1}{X} \left(\varepsilon_1 \frac{\partial}{\partial v} - m \frac{\partial}{\partial u} \right) \right] \quad \ldots \qquad (2.9b)$$

$$D_w^+ = \frac{1}{\sqrt{2}} \left[W \frac{\partial}{\partial w} + \frac{W'}{2} - \frac{i\, f Q H}{g^2 W} + \frac{f}{g^2 W} \left(p \frac{\partial}{\partial u} - \varepsilon_2 \frac{\partial}{\partial v} \right) \right] \quad \ldots \qquad (2.9c)$$

$$D_w^- = \frac{1}{\sqrt{2}} \left[- f W \frac{\partial}{\partial w} - \frac{f W'}{2} - \frac{i\, Q H}{g^2 W} + \frac{1}{g^2 W} \left(p \frac{\partial}{\partial u} - \varepsilon_2 \frac{\partial}{\partial v} \right) \right] \quad \ldots \qquad (2.9d)$$

The separability condition for the transformed equation $W_D \psi' = 0$ is that

$$Z^{1/2} \; T^{-1} \; e^{2iB} = [\, f(x) + i g(x) \,] + [\, h(w) + i k(w) \,] \qquad (2.10)$$

where $f(x)$, $g(x)$, $h(w)$ and $k(w)$ are real valued functions. But this condition is not sufficient for separability with ψ' given by

$$\psi'(u,v,w,x) = e^{i(\alpha u + \beta v)} \begin{bmatrix} H_1(x) & K_2(w) \\ H_2(x) & K_1(w) \\ H_1(x) & K_1(w) \\ H_2(x) & K_2(w) \end{bmatrix} \quad \ldots \qquad (2.11)$$

where α and β are arbitrary constants.

Substituting (2.10) and (2.11) into $W_D \psi' = 0$ with W_D as given by (2.8) we obtain the following equations :

$$\overset{o}{D}{}^{+}_{w} K_1(w) H_1(x) - i\mu_e [h(w) + ik(w)] K_2(w) H_1(x) + \overset{o}{D}{}^{-}_{x} H_2(x) K_2(w) - i\mu_e [f(x) + ig(x)] H_1(x) K_2(w) = 0 \quad (2.12a)$$

$$\overset{o}{D}{}^{-}_{w} K_2(w) H_2(x) - i\mu_e [h(w) + ik(w)] K_1(w) H_2(x) + \overset{o}{D}{}^{+}_{x} H_1(x) K_1(w) - i\mu_e [f(x) + ig(x)] H_2(x) K_1(w) = 0 \quad (2.12b)$$

$$\overset{o}{D}{}^{-}_{w} K_2(w) H_1(x) - i\mu_e [h(w) - ik(w)] K_1(w) H_1(x) - \overset{o}{D}{}^{-}_{x} H_2(x) K_1(w) - i\mu_e [f(x) - ig(x)] H_1(x) K_1(w) = 0 \quad (2.12c)$$

$$\overset{o}{D}{}^{+}_{w} K_1(w) H_2(x) - i\mu_e [h(w) - ik(w)] K_2(w) H_2(x) - \overset{o}{D}{}^{+}_{x} H_1(x) K_2(w) - i\mu_e [f(x) - ig(x)] H_2(x) K_2(w) = 0 \quad (2.12d)$$

where the operators

$$\overset{o}{D}{}^{+}_{x} = \frac{i}{\sqrt{2}}\left[X \frac{\partial}{\partial x} + \frac{X'}{2} - \frac{QG}{X} + \frac{1}{X}(\beta \varepsilon_1 - \alpha m)\right] \quad \ldots \quad (2.12e)$$

$$\overset{o}{D}{}^{-}_{x} = \frac{-i}{\sqrt{2}}\left[X \frac{\partial}{\partial x} + \frac{X'}{2} + \frac{QG}{X} - \frac{1}{X}(\beta \varepsilon_1 - \alpha m)\right] \quad \ldots \quad (2.12f)$$

$$\overset{o}{D}{}^{+}_{w} = \frac{1}{\sqrt{2}}\left[W \frac{\partial}{\partial w} + \frac{W'}{2} - \frac{if\,QH}{g^2 W} + \frac{if}{g^2 W}(\alpha p - \beta \varepsilon_2)\right] \quad \ldots \quad (2.12g)$$

$$\overset{o}{D}{}^{-}_{w} = \frac{1}{\sqrt{2}}\left[-fW \frac{\partial}{\partial w} - \frac{f\,W'}{2} - \frac{i\,QH}{g^2 W} + \frac{i}{g^2 W}(\alpha p - \beta \varepsilon_2)\right] \quad (2.12h)$$

The relations (2.12a) - (2.12d) give then immediately

$$\overset{o}{D}{}^{+}_{w} K_1(w) - i \mu_e [h(w) + ik(w)] K_2(w) = \mu_1 K_2(w) \quad \ldots \qquad (2.13a)$$

$$\overset{o}{D}{}^{-}_{x} H_2(x) - i \mu_e [f(x) + i g(x)] H_1(x) = - \mu_1 H_1(x) \quad \ldots \qquad (2.13b)$$

$$\overset{o}{D}{}^{-}_{w} K_2(w) - i \mu_e [h(w) + i k(w)] K_1(w) = \mu_2 K_1(w) \quad \ldots \qquad (2.13c)$$

$$\overset{o}{D}{}^{+}_{x} H_1(x) - i \mu_e [f(x) + i g(x)] H_2(x) = - \mu_2 H_2(x) \quad \ldots \qquad (2.13d)$$

$$\overset{o}{D}{}^{-}_{w} K_2(w) - i \mu_e [h(w) - i k(w)] K_1(w) = \mu_3 K_1(w) \quad \ldots \qquad (2.13e)$$

$$-\overset{o}{D}{}^{-}_{x} H_2(x) - i \mu_e [f(x) - i g(x)] H_1(x) = - \mu_3 H_1(x) \quad \ldots \qquad (2.13f)$$

$$\overset{o}{D}{}^{+}_{w} K_1(w) - i \mu_e [h(w) - i k(w)] K_2(w) = \mu_4 K_2(w) \quad \ldots \qquad (2.13g)$$

$$-\overset{o}{D}{}^{+}_{x} H_1(x) - i \mu_e [f(x) - i g(x)] H_2(x) = - \mu_4 H_2(x) \quad \ldots \qquad (2.13h)$$

Subtraction of (2.13g) from (2.13a) yields

$$k(w) = C_1 \quad \ldots \qquad (2.14a)$$

and the addition of (2.13h) to (2.13d) yields

$$f(x) = C_2 \quad \ldots \qquad (2.14b)$$

where C_1 and C_2 are real constants.

If we now define $g_1(x)$ and $h_1(w)$ by

$$g_1(x) = g(x) + C_1 \qquad \ldots \qquad (2.15a)$$

and

$$h_1(w) = h(w) + C_2 \qquad \ldots \qquad (2.15b)$$

then the condition (2.10) can be reduced , after dropping the suffixes , to the form

$$Z^{1/2}\, T^{-1}\, e^{2iB} = h(w) + i\, g(x) \qquad \ldots \qquad (2.16)$$

which is the separability condition for the existence of a separable solution .

Substituting (2.16) into $W_D \psi' = 0$ with W_D as given by (2.8) we obtain

$$\begin{bmatrix} -i\mu_e [h(w) + i g(x)] & 0 & \overset{o}{D}{}_w^+ & \overset{o}{D}{}_x^+ \\ 0 & i\mu_e [h(w) + i g(x)] & -\overset{o}{D}{}_x^- & -\overset{o}{D}{}_w^- \\ -\overset{o}{D}{}_w^- & \overset{o}{D}{}_x^- & i\mu_e [h(w) + i g(x)] & 0 \\ -\overset{o}{D}{}_x^+ & \overset{o}{D}{}_w^+ & 0 & -i\mu_e [h(w) + i g(x)] \end{bmatrix} \begin{bmatrix} H_1(x)\, K_2(w) \\ H_2(x)\, K_1(w) \\ H_1(x)\, K_1(w) \\ H_2(x)\, K_2(w) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or,

$$\begin{bmatrix} -i\mu_e h(w) + \mu_e g(x) & 0 & \overset{o}{D}{}_w^+ & \overset{o}{D}{}_x^- \\ 0 & i\mu_e h(w) - \mu_e g(x) & -\overset{o}{D}{}_x^+ & -\overset{o}{D}{}_w^- \\ -\overset{o}{D}{}_w^- & \overset{o}{D}{}_x^- & i\mu_e h(w) - \mu_e g(x) & 0 \\ -\overset{o}{D}{}_x^+ & \overset{o}{D}{}_w^+ & 0 & -i\mu_e h(w) + \mu_e g(x) \end{bmatrix} \begin{bmatrix} H_1(x)\,K_2(w) \\ H_2(x)\,K_1(w) \\ H_1(x)\,K_1(w) \\ H_2(x)\,K_2(w) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

from which we obtain

$$\frac{1}{K_2(w)}\,\overset{o}{D}{}_w^+\,K_1(w) + \frac{1}{H_1(x)}\,\overset{o}{D}{}_x^-\,H_2(x) = i\,\mu_e h(w) + \mu_e g(x)\ldots \qquad (2.17a)$$

$$\frac{1}{H_2(x)}\,\overset{o}{D}{}_x^+\,H_1(x) + \frac{1}{K_1(w)}\,\overset{o}{D}{}_w^-\,K_2(w) = i\mu_e h(w) - \mu_e g(x)\ldots \qquad (2.17b)$$

$$\frac{1}{K_1(w)}\,\overset{o}{D}{}_w^-\,K_2(w) - \frac{1}{H_1(x)}\,\overset{o}{D}{}_x^-\,H_2(x) = i\mu_e h(w) - \mu_e g(x)\ldots \qquad (2.17c)$$

$$\frac{1}{H_2(x)}\,\overset{o}{D}{}_x^+\,H_1(x) - \frac{1}{K_2(w)}\,\overset{o}{D}{}_w^+\,K_1(w) = -i\,\mu_e h(w) - \mu_e g(x) \qquad (2.17d)$$

These relations imply that

$$\overset{o}{D}{}^{+}_{w}\, K_1(w) - i\mu_e\, h(w)\, K_2(w) = \lambda_1 K_2(w)$$

$$\overset{o}{D}{}^{-}_{x}\, H_2(x) - \mu_e\, g(x)\, H_1(x) = -\lambda_1 H_1(x) \quad \ldots \quad (2.18a)$$

$$\overset{o}{D}{}^{+}_{x}\, H_1(x) + \mu_e\, g(x)\, H_2(x) = \lambda_2 H_2(x)$$

$$\overset{o}{D}{}^{-}_{w}\, K_2(w) - i\,\mu_e\, h(w)\, K_1(w) = -\lambda_2 K_1(w) \quad \ldots \quad (2.18b)$$

$$\overset{o}{D}{}^{-}_{w}\, K_2(w) - i\,\mu_e\, h(w)\, K_1(w) = \lambda_3 K_1(w)$$

$$\overset{o}{D}{}^{-}_{x}\, H_2(x) - \mu_e\, g(x)\, H_1(x) = \lambda_3 H_1(x) \quad \ldots \quad (2.18c)$$

$$\overset{o}{D}{}^{+}_{x}\, H_1(x) + \mu_e\, g(x)\, H_2(x) = \lambda_4 H_2(x)$$

$$\overset{o}{D}{}^{+}_{w}\, K_1(w) - i\,\mu_e\, h(w)\, K_2(w) = \lambda_4 K_2(w) \quad \ldots \quad (2.18d)$$

where λ_1, λ_2, λ_3 and λ_4 are the separation constants. However, it is manifest that the consistency of the foregoing equations requires that

$$\lambda_1 = \lambda_2 = -\lambda_3 = \lambda_4 = \lambda . \quad \ldots \quad (2.19)$$

We are thus left with the following equations :

$$\overset{o}{D}{}^{+}_{w}\, K_1(w) - i\,\mu_e\, h(w)\, K_2(w) = \lambda K_2(w)$$

$$\overset{o}{D}{}^{-}_{w}\, K_2(w) - i\,\mu_e\, h(w)\, K_1(w) = -\lambda K_1(w) \quad \ldots \quad (2.20a)$$

$$\overset{o}{D}{}^{-}_{x}\, H_2(x) + \mu_e\, g(x)\, H_1(x) = -\lambda H_1(x)$$

$$\overset{o}{D}{}^{+}_{x}\, H_1(x) + \mu_e\, g(x)\, H_2(x) = \lambda H_2(x) \quad \ldots \quad (2.20b)$$

The system of coupled first - order ordinary differential equations (2.20) can be written as a system of decoupled second- order ordinary differential equations in the following forms :

$$\overset{o}{D}{}_w^- \overset{o}{D}{}_w^+ K_1(w) - \frac{i \mu_e [\overset{o}{D}{}_w^- h(w)]}{\lambda + i \mu_e h(w)} \overset{o}{D}{}_w^+ K_1(w)$$

$$+ [\lambda^2 + \{\mu_e h(w)\}^2] K_1(w) = 0 \quad \ldots \quad (2.21a)$$

$$\overset{o}{D}{}_w^+ \overset{o}{D}{}_w^- K_2(w) - \frac{i \mu_e [\overset{o}{D}{}_w^+ h(w)]}{-\lambda + i \mu_e h(w)} \overset{o}{D}{}_w^- K_2(w)$$

$$+ [\lambda^2 + \{\mu_e h(w)\}^2] K_2(w) = 0 \quad \ldots \quad (2.21b)$$

$$\overset{o}{D}{}_x^- \overset{o}{D}{}_x^+ H_1(x) - \frac{\mu_e [\overset{o}{D}{}_x^- g(x)]}{-\lambda + \mu_e g(x)} \overset{o}{D}{}_x^+ H_1(x)$$

$$- [\lambda^2 - \{\mu_e g(x)\}^2] H_1(x) = 0 \quad \ldots \quad (2.21c)$$

$$\overset{o}{D}{}_x^+ \overset{o}{D}{}_x^- H_2(x) - \frac{\mu_e [\overset{o}{D}{}_x^+ g(x)]}{\lambda + \mu_e g(x)} \overset{o}{D}{}_x^- H_2(x)$$

$$+ [\lambda^2 - \{\mu_e g(x)\}^2] H_2(x) = 0 \quad \ldots \quad (2.21d)$$

The equations(2.20) and (2.21) reduce to the separated equations of Chandrasekhar [78] , Page [79], Guven [80] when the constants and the functions appearing in (2.1) are appropriately specialized.

2.3 Reduction to Special Cases

Let us now discuss some of the special cases of equations (2.20) and (2.21).

For the Kerr metric we obtain from equation (2.1),

$$\begin{aligned}
&\varepsilon_1 = \varepsilon_2 = f = g = 1, \quad m(x) = -x^2 \quad p(w) = w^2, \\
&T(w,x) = 1, \quad H(w) = G(x) = 0, \quad h(w) = w, \quad Z = w^2 + x^2, \\
&\tilde{g}(x) = x, \quad W(w) = \sqrt{Y}, \quad X(x) = \sqrt{X}, \quad X = a^2 - x^2, \\
&Y = w^2 - 2Mw + a^2, \quad \alpha = i\sigma, \quad \beta = -i(am_o + \sigma a^2)
\end{aligned} \tag{2.22}$$

where M is the mass parameter, a is the angular momentum per unit mass parameter, σ is the energy of the Dirac particle and m_o is any arbitrary constant.

From equations (2.12 g,h), (2.20 a) and (2.22) we obtain

$$\begin{aligned}
&(\sqrt{2})^{-1}\sqrt{Y}\left\{\frac{\partial}{\partial w} + \frac{iK}{Y} - \frac{w-M}{2Y}\right\} K_1(w) = (\lambda + i\mu_e w) K_2(w) \\
&(\sqrt{2})^{-1}\sqrt{Y}\left\{\frac{\partial}{\partial w} - \frac{iK}{Y} + \frac{w-M}{2Y}\right\} K_2(w) = (\lambda - i\mu_e w) K_1(w)
\end{aligned} \tag{2.23}$$

where

$Y = w^2 - 2Mw + a^2$,

$K = (w^2 + a^2)\sigma + a m_o$.

If we now allow the coordinate w to play the role of the new radial variable r, then replacing w by r and letting

$$K_1(r) = R_2 Y^{-1/4} \quad , \quad K_2(r) = (\sqrt{2})^{-1} R_1 Y^{1/4}$$

we obtain (2.23), after simplification, as

$$\left(\frac{\partial}{\partial r} + i K \right) R_2 = (\lambda + i \mu_e r) R_1$$

$$Y \left(\frac{\partial}{\partial r} - \frac{i K}{Y} + \frac{r - M}{Y} \right) R_1 = 2 (\lambda - i \mu_e r) R_2 \qquad \ldots \qquad (2.24)$$

where

$$Y = r^2 - 2Mr + a^2 ,$$

$$K = (r^2 + a^2)\sigma + a m_o .$$

The equations (2.24) correspond to Chandrasekhar's equations (40) [81] in the context of Kerr geometry.

Now from equations (2.12e,f), (2.20b) and (2.22) we obtain

$$(\sqrt{2})^{-1}\sqrt{X}\left[\frac{\partial}{\partial x} + \frac{a m_o + \sigma(a^2 - x^2)}{X} - \frac{x}{2X}\right] i\, H_2(x) = (\lambda + \mu_e x) H_1(x)$$

$$(\sqrt{2})^{-1}\sqrt{X}\left[\frac{\partial}{\partial x} - \frac{a m_o + \sigma(a^2 - x^2)}{X} - \frac{x}{2X}\right] H_1(x) = (\lambda - \mu_e x)(-i H_2(x)) \qquad (2.25)$$

If $- i H_2(x) = S_1(\theta)$, $H_1(x) = S_2(\theta)$

and $x = a \cos\theta$ so that $X = a^2 \sin^2\theta$

then we obtain (2.25) as

$$\left(\frac{\partial}{\partial\theta} - m_o \operatorname{cosec}\theta - a\sigma \sin\theta + \frac{1}{2}\cot\theta\right) S_1(\theta) = \sqrt{2}\,(\lambda + a\mu_e \cos\theta) S_2(\theta)$$

$$\left(\frac{\partial}{\partial\theta} + m_o \operatorname{cosec}\theta + a\sigma \sin\theta + \frac{1}{2}\cot\theta\right) S_2(\theta) = -\sqrt{2}\,(\lambda - a\mu_e \cos\theta) S_1(\theta) \qquad (2.26)$$

The above equations correspond to those first derived by Chandrasekhar [82] in the context of Kerr geometry.

For the Kerr - Newman metric we set

$$\varepsilon_1 = \varepsilon_2 = f = g = 1 , \quad m(x) = -x^2 , \quad p(w) = w^2 ,$$

$$T(w, x) = 1 , \quad H(w) = ew , \quad G(x) = 0 , \quad h(w) = w ,$$

$$Z = w^2 + x^2 , \quad g(x) = x , \quad W(w) = \sqrt{Y} , \quad X(x) = \sqrt{X} , \quad \ldots \qquad (2.27)$$

$$X = a^2 - x^2 , \quad Y = w^2 - 2Mw + a^2 + e^2 , \quad \alpha = i\sigma ,$$

$$\beta = -i(a m_o + \sigma a^2)$$

where M is the mass parameter , a is the angular momentum per unit mass parameter , e is the electric charge parameter, σ is the energy of the Dirac particle and m_o is any constant.

Then from equations (2.12 g,h) (2.21 a) and (2.27) we obtain the following equation :

$$\sqrt{Y}\,[-\frac{\partial}{\partial w}+\frac{iK}{Y}-\frac{Y'}{4Y}]\sqrt{Y}\,[\frac{\partial}{\partial w}+\frac{iK}{Y}+\frac{Y'}{4Y}]K_1(w)$$

$$+\frac{i\mu_e Y}{\lambda+i\mu_e w}[\frac{\partial}{\partial w}+\frac{iK}{Y}+\frac{Y'}{4Y}]K_1(w)$$

$$+2(\lambda^2+\mu_e^2 w^2)K_1(w)=0 \quad \ldots \qquad (2.28)$$

where

$$Y = w^2 - 2Mw + a^2 + e^2$$

$$K = (w^2+a^2)\sigma - eQw + am_o$$

and the prime denotes differentiation with respect to the argument.

Now if w plays the role of the radial variable r, then replacing w by r and letting $K_1(r) = RY^{-1/4}$, we find that the equation (2.28) reduces to an exceedingly compact form

$$\sqrt{Y}\frac{d}{dr}(\sqrt{Y}\frac{dR}{dr})-\frac{i\mu_e Y}{\lambda+i\mu_e r}\frac{dR}{dr}+[\frac{K^2-i(r-M)K}{Y}-ieQ$$

$$+2i\sigma r+\frac{\mu_e K}{\lambda+i\mu_e r}-2(\lambda^2+\mu_e^2 r^2)]R=0 \qquad (2.29)$$

where

$$Y = r^2 - 2Mr + a^2 + e^2,$$

$$K = (r^2+a^2)\sigma - eQr + am_o.$$

If $\sqrt{2}\lambda \to \lambda$, $\sqrt{2}\mu_e \to \mu_e$, then the equation (2.29) corresponds to Page's equation (20) [83] in the context of Kerr - Newman spacetime .

For $e = 0$, the equation (2.29) reduces to the Chandrasekhar's equation (45) [84] in the case of Kerr metric .

When both $e = 0$ and $\mu_e = 0$, the equation (2.29) corresponds to Teukolsky's equation (4.10) [85] for neutrinos.

CHAPTER - 3

HAWKING RADIATION OF DIRAC PARTICLES IN NUT-KERR-NEWMAN SPACETIME

Hawking's [86] thermal radiation by black holes near black hole event horizon have been studied by different authors in different types of asymptotically flat background spacetimes . Chandrasekhar's [87] work provided the possibility of attacking the problem of Hawking evaporation of Dirac particles to the Kerr black hole spacetime . Liu Liao and Xu Dianyan [88] based on Chandrasekhar's work studied the Hawking evaporation of Dirac particles in the Kerr background . Page [89] extended Chandrasekhar's work to the Kerr - Newman black hole . Zhao Zheng et al.[90] based on Page's work extended the work of Liu Liao and Xu Dianyan to the spacetime describing the Kerr - Newman black hole . Ahmed [91] taking the advantage of the separability of the Dirac equation in an arbitrary curved background spacetime performed by Kamran and McLenaghan [92] extended the work of Zhao Zheng et al. to the NUT - Kerr - Newman (NUTKN) spacetime which includes all the black hole spactimes which are asymtotically flat . Ahmed's work are interesting in that Hawking's thermal radiation by black holes can also be obtained in the case of NUT spacetime which has peculiar properties . In this chapter we review Ahmed's work which serves a useful purpose in providing with a firm basis for the work on Hawking's thermal radiation near the event horizon of asymptotically flat background spacetimes . We use equation (2.21a) and derive the radial decoupled Dirac equation in the NUTKN limit and study Hawking radiation near the event horizon of NUTKN spacetime.

3.1 The NUTKN Spacetime

In the coordinates (u , v , w , x) the NUTKN spacetime is

$$ds^2 = \frac{x^2 + w^2}{X} dx^2 + \frac{x^2 + w^2}{Y} dw^2 + \frac{X}{x^2 + w^2} (du + w^2 dv)^2$$

$$- \frac{Y}{x^2 + w^2} (du - x^2 dv)^2 \qquad \ldots \quad (3.1a)$$

where

$$X = a^2 - (n - x)^2 \qquad \ldots \quad (3.1b)$$

$$Y = w^2 - 2M w + a^2 - n^2 + e^2 \qquad \ldots \quad (3.1c)$$

with electric potential

$$A_\mu \, d Z^\mu = \frac{e w}{x^2 + w^2} (du - x^2 dv) \qquad \ldots \quad (3.2)$$

The NUTKN spacetime contains four real parameters : the mass parameter M , the NUT (magnetic mass) parameter n , the angular momentum per unit mass parameter a , the electric charge parameter e . The NUTKN spacetime includes

(i) Kerr - Newman spacetime when $n = 0$

(ii) Kerr spacetime with $n = e = 0$

(iii) Reissner - Nordstrom spacetime if $n = a = 0$

(iv) Schwarzschild spacetime for $n = a = e = 0$

(v) NUT Kerr spacetime provided $e = 0$

(vi) NUT spacetime when $a = e = 0$.

Thus we see that the NUTKN spacetime contains all the black hole spacetimes (Kerr - Newman , Kerr , Reissner - Nordstrom , Schwarzschild) which are asymptotically flat.

3.2 Radial Wave Equation

To obtain the NUTKN spacetime and the radial decoupled Dirac equation in this background spacetime it suffices to set in equations (2.1), (2.21 a), (2.12 g) and (2.12 h) the following substitutions :

$$\varepsilon_1 = \varepsilon_2 = f = g = 1 , \quad m(x) = -x^2 , \quad p(w) = w^2 ,$$

$$T(w, x) = 1 , \quad H(w) = ew , \quad G(x) = 0 . \quad h(w) = w ,$$

$$g(x) = x , \quad Z = w^2 + x^2 , \quad \alpha = i\sigma , \quad \beta = -iA , \qquad \ldots \qquad (3.3)$$

$$A = a m_0 + \sigma (a + n)^2 , \quad W(w) = \sqrt{Y} , \quad X(x) = \sqrt{X} ,$$

$$Y = w^2 - 2Mw + a^2 - n^2 + e^2 , \quad X = a^2 - (n - x)^2$$

where σ is the energy of the Dirac particle and m_0 is any arbitrary constant.

From equations (2.21a), (2.12g,h) and (3.3) we obtain the following equation

$$\sqrt{Y}\left[-\frac{\partial}{\partial w}+\frac{iK}{Y}-\frac{Y'}{4Y}\right]\sqrt{Y}\left[\frac{\partial}{\partial w}+\frac{iK}{Y}+\frac{Y'}{4Y}\right]K_1(w)$$

$$+\frac{i\mu_e Y}{\lambda+i\mu_e w}\left[\frac{\partial}{\partial w}+\frac{iK}{Y}+\frac{Y'}{4Y}\right]K_1(w)$$

$$+2(\lambda^2+\mu_e^2 w^2)K_1(w)=0 \qquad \ldots \quad (3.4a)$$

where

$$K = w^2\sigma - A - eQw$$

$$A = am_0 + \sigma(a+n)^2 \qquad \ldots \quad (3.4b)$$

$$Y = w^2 - 2Mw + a^2 - n^2 + e^2$$

and prime denotes differentiation with respect to the argument.

If w plays the role of the radial variable r, then replacing w by r and letting $K_1(r) = Y^{-1/4} R(r)$ we find, after some further simplifications, that equation (3.4) becomes

$$\sqrt{Y}\frac{d}{dr}\left(\sqrt{Y}\frac{dR}{dr}\right) - \frac{i\mu_e Y}{\lambda + i\mu_e r}\frac{dR}{dr} + \left[\frac{K^2}{Y} - 2(\lambda^2 + \mu_e^2 r^2)\right.$$

$$\left. + \frac{K\mu_e}{\lambda + i\mu_e r} + i\sqrt{Y}\frac{d}{dr}\left(\frac{K}{\sqrt{Y}}\right)\right] R = 0 \qquad \ldots (3.5a)$$

$$K = r^2\sigma - A - eQr$$
$$Y = r^2 - 2Mr + a^2 - n^2 + e^2 \qquad \ldots (3.5b)$$

The equation (3.5) reduces to the radial decoupled Dirac equation obtained by

(i) Page [93] for the Kerr - Newman black hole when $n = 0$, $\sqrt{2}\lambda \rightarrow \lambda$, $\sqrt{2}\mu_e \rightarrow \mu_e$

(ii) Chandrasekhar [94] for the Kerr black hole provided $e = n = 0$.

Introducing the coordinate

$$\frac{d}{d\hat{r}} = \frac{Y}{r^2 + (a+n)^2}\frac{d}{dr} \qquad \ldots (3.6)$$

the equation (3.5), near the horizon, reduces to

$$\frac{d^2R}{d\hat{r}^2} + (\sigma - \sigma_o)^2 R = 0 \qquad \ldots (3.7)$$

where

$$\sigma_o = \frac{(a+n)^2\sigma + A}{r_+^2 + (a+n)^2} + \frac{e\,Q\,r_+}{r_+^2 + (a+n)^2} \qquad \ldots (3.8)$$

and r_+ called the event horizon is the positive root of $Y = 0$, provided $M^2 > a^2 - n^2 + e^2$. The other root of $Y = 0$, denoted by r_- is called the Cauchy horizon.

If $n = 0$, the equation (3.7) reduces to the wave equation obtained by Zhao Zheng et al. [95] for the Kerr - Newman black hole. For $n = e = 0$, the equation (3.7) reduces to the wave equation obtained by Liu Liao and Xu Dianyan [96] for the Kerr black hole.

3.3 Hawking Thermal Spectrum

Now we turn to the wave equation (3.7). The solution of this equation can easily be found to be

$$R \sim \exp\left[\pm\, i\,(\sigma - \sigma_o)\,\hat{r}\,\right] \qquad \ldots (3.9)$$

We now write the radial wave function

$$\psi_r = \exp\left[-\,i\sigma\,(\,t \pm \hat{r}_1)\,\right] \qquad \ldots (3.10)$$

where

$$\hat{r}_1 = \frac{(\sigma - \sigma_0)\,\hat{r}}{\sigma} \qquad \ldots (3.11)$$

We resolve ψ_r into ingoing and outgoing waves as

$$\psi_r^{in} \sim \exp\left[- i\sigma(t + \hat{r}_1) \right] \qquad \ldots (3.12)$$

$$\psi_r^{out} \sim \exp\left[- i\sigma(t - \hat{r}_1) \right] \qquad \ldots (3.13)$$

Introducing the Eddington - Finkelstein coordinates

$$v = t + \hat{r}_1 \qquad \ldots (3.14)$$

we obtain

$$\psi_r^{in} \sim \exp(- i \sigma v) \qquad \ldots (3.15)$$

$$\psi_r^{out} \sim \exp\left[- i\sigma v + 2i(\sigma - \sigma_0)\hat{r} \right] \qquad \ldots (3.16)$$

Near $r = r_+$, equation (3.6) can be integrated to give

$$\hat{r} = \frac{1}{2\kappa_+} \ln(r - r_+) \qquad \ldots (3.17)$$

where

$$\kappa_+ = \frac{1}{2} \frac{r_+ - r_-}{r_+^2 + (a + n)^2} \qquad \ldots (3.18)$$

is the surface gravity [97] near the event horizon of the NUTKN spacetime.

Just ouside the event horizon

$$\psi_r^{out} \sim e^{-i\sigma v} (r - r_+)^{i(\sigma - \sigma_o)/K_+} \qquad \ldots (3.19)$$

We now extend the outgoing wave outside the horizon to the region inside. Since on the event horizon the outgoing wave function is not analytic and can not be straightforwardly extended to the region inside, it can be continued analytically to the complex plane by going round the event horizon.

Hence inside the event horizon

$$\psi_r^{out} \sim e^{-i\sigma v} (r_+ - r)^{i(\sigma - \sigma_o)/K_+} \; e^{\pi(\sigma - \sigma_o)/K_+} \qquad \ldots (3.20)$$

Introducing the step function

$$y(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \qquad \ldots (3.21)$$

the outgoing wave function can generally be written as

$$\phi_r^{out} = N_r [\, y(r - r_+)\, \psi_r^{out}(r - r_+)$$

$$+ y(r_+ - r)\, \psi_r^{out}(r_+ - r) \exp\{\frac{\pi}{K_+}(\sigma - \sigma_o)\}\,] \qquad (3.22)$$

where ψ_r^{out} is the normalized Dirac wave function. Expression (3.22) describes the splitting of ϕ_r^{out} into two components :

(a) a flow of positive energy particles of strength N_r^2 outgoing from the event horizon and

(b) a flow of positive energy particles propagating in the reverse time , since inside the event horizon , r represents the time axis due to the interchange of time and space . This can be interpreted as a flow in time of negative energy particles ingoing towards the singularity region .

This shows that a wave function near the event horizon gives rise to the creation of Dirac particle-antiparticle pair [98 , 99] .

Obviously, from the normalization condition , we obtain

$$< \phi_r^{out} , \phi_r^{out} > = N_r^2 \{ \exp [\frac{2\pi}{K_+} (\sigma - \sigma_o)] + 1 \} = 1 \qquad \ldots (3.23)$$

or ,

$$N_r^2 = \{ \exp [\frac{2\pi}{K_+} (\sigma - \sigma_o)] + 1 \}^{-1}$$

$$= \{ \exp [\frac{1}{K_b T_+} (\sigma - \sigma_o)] + 1 \}^{-1} \qquad \ldots (3.24)$$

where

$$T_+ = \frac{K_+}{2\pi K_b} \qquad \ldots (3.25)$$

is the temperature near the event horizon of the NUTKN spacetime.

Equation (3.24) is the Hawking thermal spectrum formula near the event horizon of the NUTKN spacetime [100].

In the proper limits the relation (3.24) reduces to the Hawking thermal spectrum of Dirac particles near the event horizon of

(i) Kerr - Newman black hole [101] for $n = 0$

(ii) Kerr black hole [102] if $n = e = 0$

(iii) Reissner - Nordstrom black hole when $n = a = 0$

(iv) Schwarzschild black hole provided $n = a = e = 0$

(v) NUT Kerr spacetime with $e = 0$

(vi) NUT spacetime if $a = e = 0$.

Thus we observe that Hawking's thermal radiation occurs not only in black hole spacetimes but also in NUT Kerr and NUT spacetimes.

3.4 Discussion

We observe particle emission near the event horizon r_+ of the NUTKN spacetime which **is not** black hole spacetime but includes all the black hole spacetimes which are asymptotically flat. The particle flux has thermal character which is evident from the formula given by (3.25). The surface gravity κ_+ is the limit of the product of two quantities : the magnitude of proper acceleration of a stationary observer and the redshift factor. Thus particle emission can be interpreted as the escape of particles to infinity that an observer just outside the horizon sees on account of his acceleration. This fact manifests that the thermal emission is closely related to the thermal properties of the vacuum state. Finally, we conclude that Hawking's thermal radiation by black holes holds good to the spacetimes which are not black hole spacetimes but include black hole spacetimes as special cases.

CHAPTER - 4

HAWKING RADIATION OF DIRAC PARTICLES IN NUT-KERR-NEWMAN-DE SITTER SPACETIME

Hawking's [103] work on thermal radiation by black holes near black hole event horizon has been extended by Gibbons and Hawking [104] to the spacetime of cosmological event horizon. Xu Dianyan and Wang Huiya [105] and Shen You Gen [106] extended Gibbons and Hawking's work to the Kerr-Newman de Sitter spacetime. Recently Ahmed [107] carried on this work [108] to the NUT-Kerr-Newman de Sitter spacetime. Ahmed's works [109] are interesting in that Hawking's and Gibbons and Hawking's thermal radiation could also be obtained in the case of NUT de Sitter and NUT spacetimes which have peculiar properties. In this chapter we study Ahmed's work on Hawking radiation of Dirac particles near the horizons of NUT-Kerr-Newman-de Sitter spacetime which includes all the black hole spacetimes (asymptotically flat or asymptotically de Sitter). We call the NUT-Kerr-Newman-de Sitter spacetime as hot NUT-Kerr-Newman (HNUTKN) spacetime since the de Sitter spacetime has been interpreted as being hot [110]. This work also provides a basis for the extension of the work on Hawking's thermal radiation by black holes near the horizons of asymptotically de Sitter background spacetimes. Using equation (2.21a) we derive the radial decoupled Dirac equation in the HNUTKN limit and then study Hawking radiation near the horizons of the HNUTKN spacetime.

4.1 The HNUTKN Spacetime

In the coordinates (u, v, w, x) the HNUTKN spacetime is given by

$$ds^2 = \frac{x^2 + w^2}{X} dx^2 + \frac{x^2 + w^2}{Y} dw^2 + \frac{X}{x^2 + w^2} (du + w^2 dv)^2$$

$$- \frac{Y}{x^2 + w^2} (du - x^2 dv)^2 \qquad \ldots \qquad (4.1a)$$

where

$$X = [a^2 - (n - x)^2] [1 + \frac{\Lambda}{3} (n - x^2)] \qquad \ldots \qquad (4.1b)$$

$$Y = (w^2 + a^2 + n^2) [1 - \frac{\Lambda}{3}(w^2 + 5n^2)] - 2 (Mw + n^2) + e^2 \qquad (4.1c)$$

with electric potential

$$A_\mu \, d Z^\mu = \frac{e w}{x^2 + w^2} (du - x^2 dv) \qquad \ldots \qquad (4.2)$$

Besides the cosmological constant Λ, the HNUTKN spacetime contains : the mass parameter M, the NUT (magnetic mass) parameter n, the angular momentum per unit mass parameter a and the electric charge parameter e. The HNUTKN spacetime includes

(i) hot Kerr - Newman spacetime for $n = 0$

(ii) hot Kerr spacetime when $n = e = 0$

(iii) hot Reissner - Nordstrom spacetime for $n = a = 0$

(iv) hot Schwarzschild spacetime with $n = a = e = 0$

(v) hot NUT spacetime if $a = e = 0$.

Thus we see that the HNUTKN spacetime includes all the black hole spacetimes (i) - (iv) which are asymptotically de Sitter. Further if we set $\Lambda = 0$, in the cases (i) - (iv), we get all the black hole spacetimes which are asymptotically flat. In the limit $\Lambda = 0$, the case (v) reduces to the NUT spacetime.

4.2 Radial Wave Equation

To obtain the HNUTKN spacetime and the radial decoupled Dirac equation we set in equations (2.1), (2.21a), (2.12g) and (2.12h) the following substitutions :

$$\varepsilon_1 = \varepsilon_2 = f = g = 1 , \quad m(x) = -x^2 , \; p(w) = w^2 , \; T(w, x) = 1 ,$$

$$H(w) = ew , \; G(x) = 0 , \; h(w) = w , \; g(x) = x , \quad Z = w^2 + x^2 ,$$

$$\alpha = i\sigma , \; \beta = -iA , \; A = a m_0 + \sigma (a + n)^2 , \quad X(x) = \sqrt{X} , \qquad (4.3)$$

$$W(w) = \sqrt{Y} , \quad X = [a^2 - (n - x)^2] [1 + \frac{\Lambda}{3} (n - x)^2] ,$$

$$Y = (w^2 + a^2 + n^2) [1 - \frac{\Lambda}{3} (w^2 + 5n^2)] - 2(Mw + n^2) + e^2$$

where σ is the energy of the Dirac particle and m_o is any constant.

From equations (2.21a), (2.12g,h) and (4.3) we obtain after cumbersome calculations

$$\sqrt{Y}\left[-\frac{\partial}{\partial w}+\frac{iK}{Y}-\frac{Y'}{4Y}\right]\sqrt{Y}\left[\frac{\partial}{\partial w}+\frac{iK}{Y}+\frac{Y'}{4Y}\right]K_1(w)$$

$$+\frac{i\mu_e Y}{\lambda+i\mu_e w}\left[\frac{\partial}{\partial w}+\frac{iK}{Y}+\frac{Y'}{4Y}\right]K_1(w)$$

$$+2(\lambda^2+\mu_e^2 w^2)K_1(w)=0 \qquad \ldots \qquad (4.4a)$$

where

$$K = w^2 - A - \frac{eQw}{\Xi}$$

$$\ldots \qquad (4.4b)$$

$$Y = (w^2+a^2+n^2)\left[1-\frac{\Lambda}{3}(w^2+5n^2)\right]-2(Mw+n^2)+e^2$$

and prime denotes differentiation with respect to the argument.

If w plays the role of the radial variable r, then replacing w by r and letting $K_1(r) = R\,Y^{-1/4}$ we obtain equation (4.4) after some algebraic calculations and rearranging the terms in the following form

$$\sqrt{Y}\frac{d}{dr}\left(\sqrt{Y}\frac{dR}{dr}\right)-\frac{i\mu_e Y}{\lambda+i\mu_e r}\frac{dR}{dr}+\left[\frac{\Xi^2K^2}{Y}-2(\lambda^2+\mu_e^2r^2)\right.$$

$$\left.+\frac{\Xi K\mu_e}{\lambda+i\mu_e r}+i\Xi\sqrt{Y}\frac{d}{dr}\left(\frac{K}{\sqrt{Y}}\right)\right]R=0 \quad \ldots \quad (4.5a)$$

$$K = r^2\sigma - A - \frac{eQr}{\Xi} \quad \ldots \quad (4.5b)$$

$$Y = (r^2+a^2+n^2)\left[1-\frac{\Lambda}{3}(r^2+5n^2)\right]-2(Mr+n^2)+e^2$$

which represents the radial decoupled Dirac equation for the HNUTKN spacetime.

The equation (4.5) reduces to the radial decoupled Dirac equation obtained by

(i) Xu Dianyan and Wang Huiya [111] and Shen You-Gen [112] for the Kerr - Newman de Sitter spacetime when $n = 0$

(ii) Khanal [113] for the Kerr de Sitter spacetime if $n = e = 0$

(iii) Page [114] for the Kerr - Newman spacetime provided $\Lambda = n = 0$, $\sqrt{2}\lambda \to \lambda$, $\sqrt{2}\mu_e \to \mu_e$

(iv) Chandrasekhar [115] for the Kerr spacetime when $\Lambda = e = n = 0$.

Introducing the coordinate

$$\frac{d}{d\hat{r}} = \frac{Y}{r^2 + (a+n)^2} \frac{d}{dr} \qquad \ldots \qquad (4.6)$$

the equation (4.5), near the horizon, reduces to

$$\frac{d^2 R}{d\hat{r}^2} + \Xi^2 (\sigma - \sigma_0)^2 R = 0 \qquad \ldots \qquad (4.7)$$

where

$$\sigma_0 = \frac{(a+n)^2 \sigma + A}{r_+^2 + (a+n)^2} + \frac{e Q r_+}{\Xi [r_+^2 + (a+n)^2]}$$

and r_+, called the event horizon, is the smaller of the two positive values of r for which $Y = 0$, provided the roots are real (i. e. $\frac{1}{\Lambda} > M^2 > a^2 - n^2 + e^2$). The larger positive value of $Y = 0$ denoted by r_{++} represents the cosmological horizon.

If $n = 0$, the equation (4.7) reduces to the wave equation obtained by Xu Dianyan and Wang Huiya [116] and Shen You Gen [117] for the Kerr - Newman de Sitter spacetime. For $\Lambda = n = 0$, the equation (4.7) reduces to the wave equation obtained by Zhao Zheng et al. [118] for the Kerr - Newman black hole. When $\Lambda = n = e = 0$, the equation (4.7) reduces to the wave equation obtained by Liu Liao and Xu Dianyan [119] for the Kerr blak hole.

4.3 Hawking Thermal Spectrum

The solution of the equation (4.7) can easily be found to be

$$R \sim \exp [\pm i \exists (\sigma - \sigma_o)] \hat{r} \quad \ldots \quad (4.8)$$

Now we can write the radial wave function as

$$\psi_r = \exp [- i \sigma (t \pm \hat{r}_1)] \quad \ldots \quad (4.9)$$

where

$$\hat{r}_1 = \frac{\exists (\sigma - \sigma_o) \hat{r}}{\sigma}$$

We resolve ψ_r into ingoing and outoing waves as

$$\psi_r^{in} \sim \exp [- i \sigma (t + \hat{r}_1)] \quad \ldots \quad (4.10)$$

$$\psi_r^{out} \sim \exp [- i \sigma (t - \hat{r}_1)] \quad \ldots \quad (4.11)$$

Introducing the Eddington - Finkelstein coordinates

$$v = t + \hat{r}_1 \quad \ldots \quad (4.12)$$

we obtain

$$\psi_r^{in} \sim \exp (- i \sigma v) \quad \ldots \quad (4.13)$$

$$\psi_r^{out} \sim \exp [- i \sigma v + 2 i \exists (\sigma - \sigma_o) \hat{r}] \quad \ldots \quad (4.14)$$

Integrating equation (4.6) , near $r = r_+$, we obtain

$$\hat{r} = \frac{1}{2 \Xi K_+} \ln (r - r_+) \qquad \ldots (4.15)$$

where

$$K_+ = - \frac{\Lambda}{6 \Xi [r_+^2 + (a + n)^2]} (r_+ - r_{++})(r_+ - r_-)(r_+ - r_{--})$$

is the surface gravity [120] of the event horizon of the HNUTKN spacetime . Here r_- is the inner black hole horizon and r_{--} is another cosmological horizon .

Just outside the event horizon we have

$$\psi_r^{out} \sim e^{-i\sigma v} (r - r_+)^{(i/K_+)(\sigma - \sigma_0)} \qquad \ldots (4.16)$$

We now extend the outgoing wave outside the horizon to the region inside . Since on the event horizon , the outgoing wave function is not analytic and can not be straightforwardly extended to the region inside , it can be continued analytically to the complex plane by going round the event horizon .

Hence inside the event horizon

$$\psi_r^{out} \sim e^{-i\sigma v} (r_+ - r)^{(i/K_+)(\sigma - \sigma_0)} e^{(\pi/K_+)(\sigma - \sigma_0)} \qquad \ldots (4.17)$$

Introducing the step function

$$y(x) = \begin{cases} 1 & , \quad x \geqslant 0 \\ 0 & , \quad x < 0 \end{cases} \quad \ldots \quad (4.18)$$

the outgoing wave function can generally be written as

$$\phi_r^{out} = N_r [y(r - r_+) \psi_r^{out} (r - r_+)$$

$$+ y(r_+ - r) \psi_r^{out} (r_+ - r) \exp \{ \frac{\pi}{K_+} (\sigma - \sigma_o) \}] \quad \ldots \quad (4.19)$$

where ψ_r^{out} is the normalized Dirac wave function.

Expression (4.19) describes the splitting of ϕ_r^{out} into two components :

(a) a flow of positive energy particles of strength N_r^2 outgoing from the event horizon and

(b) a flow of positive energy particles propagating in the HNUTKN background gravitational field in the reverse time, since inside the event horizon, r represents the time axis due to the interchange of time and space. This can be interpreted as a flow in time of negative energy antiparticles ingoing towards the singularity region. This shows that a wave function near the event horizon gives rise to the creation of Dirac particle antiparticle pair [121].

From the normalization condition , we have

$$< \phi_r^{out} , \phi_r^{out} > = N_r^2 \{ \exp [\frac{2\pi}{K_+} (\sigma - \sigma_0)] + 1 \} = 1 \qquad \ldots \qquad (4.20)$$

or $$N_r^2 = \{ \exp [\frac{2\pi}{K_+} (\sigma - \sigma_0)] + 1 \}^{-1}$$

$$= \{ \exp [\frac{1}{K_b T_+} (\sigma - \sigma_0)] + 1 \}^{-1} \qquad \ldots \qquad (4.21)$$

where

$$T_+ = \frac{K_+}{2\pi K_b} \qquad \ldots \qquad (4.22)$$

T_+ being the temperature of the region inside the event horizon, K_b is the Boltzmann's constant . Equation (4.21) is the formula for the Hawking thermal spectrum of Dirac particles in the HNUTKN spacetime [122].

In the limits

(i) $n = 0$, the relation (4.21) will give the Hawking thermal spectrum of Dirac particles in HKN spacetime [123]

(ii) $n = e = 0$, we get from (4.21) the Hawking thermal spectrum of Dirac particles in the hot Kerr spacetime

(iii) $n = a = 0$, the relation (4.21) gives the Hawking thermal spectrum in hot Reissner - Nordstrom spacetime

(iv) $n = a = e = 0$, the relation (4.21) is the Hawking thermal spectrum in the hot Schwarzschild spacetime .

Further in the limits $\Lambda = 0$, in the cases (i) - (vi) we will get the Hawking thermal spectrum formula for

(i) Kerr - Newman spacetime [124]

(ii) Kerr spacetime [125]

(iii) Reissner Nordstrom spacetime

(iv) Schwarzschild spacetime .

In the limit $a = e = 0$, the relation (4.21) will give the formula for the Hawking thermal spectrum in the hot NUT spacetime. Further in the limit $\Lambda = 0$, we get the formula for the Hawking thermal spectrum in the NUT spacetime .

Following in the similar way we could have

$$T_{++} = \frac{K_{++}}{2\pi K_b} \qquad \ldots (4.23)$$

where

$$K_{++} = - \frac{\Lambda}{6 \Xi [r_{++}^2 + (a+n)^2]} (r_{++} - r_+)(r_{++} - r_-)(r_{++} - r_{--}) \qquad (4.24)$$

is the surface gravity of the cosmological horizon . In the proper limits (4.24) goes for the results obtained in [126] .

4.4 Discussion

We observe particle emission near the horizons r_+ and r_{++} of the HNUTKN spacetime which is not a black hole spacetime but includes all the black hole spacetimes which are asymptotically flat as well as asymptotically de Sitter as special cases . In the next chapter we shall show that Gibbons and Hawking's thermal radiation by black holes occurs in the case of Kasner - type spacetime which describes an anisotropic model of the universe.

CHAPTER - 5

HAWKING RADIATION OF DIRAC PARTICLES IN KASNER- TYPE SPACETIME

Hawking's [127] investigations of quantum effects interpreted as the emission of a thermal spectrum of particles near a black hole event horizon has been extended by Gibbons and Hawking [128] to the spacetime of cosmological event horizons including the de Sitter spacetime , which has attracted renewed interest as a model of the early universe . In Chapter 3, we have observed that Hawking's [129] result on thermal radiation by black holes holds good in the case of NUT - Kerr - Newman spacetime which includes all the black hole spacetimes (asymptotically flat) as well as NUT spacetime as special cases. Whereas in Chapter 4, we have observed that Gibbons and Hawking's [130] result holds good in the case of NUT - Kerr - Newman - de Sitter spacetime which includes black hole spacetimes(asymptotically flat as well as asymptotically de Sitter) and NUT de Sitter spacetime as special cases . In the present chapter we have attempted to show that Hawking's [131] and Gibbons and Hawking's [132] results on the thermal radiation by black holes that have been found to occur in the case of NUT - Kerr - Newman and NUT - Kerr - Newman - de Sitter spacetimes also occur in the case of Kasner - type spacetime , an anisotropic universe .

5.1 The Kasner - Type Spacetime

We consider the spacetime

$$d s^2 = r^2 (d\theta^2 + \theta^2 d\phi^2) + \frac{d r^2}{\Delta} - \Delta \, d t^2 \qquad \ldots (5.1a)$$

where

$$\Delta = - \frac{1}{3} \Lambda r^2 - \frac{2M}{r} + \frac{e^2 + g^2}{r^2} \qquad \ldots (5.1b)$$

Besides the cosmological constant Λ , the spacetime given by (5.1) contains three real parameters : the mass M , the electric charge e and the magnetic charge g .

The spacetime (5.1) can be transformed to the Kasner form when we put $\Lambda = e = g = 0$. Equation (5.1) can be written in the form

$$d s^2 = r^2 (d\theta^2 + \theta^2 d\phi^2) + \frac{r^2}{Y} d r^2 - \frac{Y}{r^2} d t^2 \qquad \ldots (5.2a)$$

where

$$Y = - \frac{\Lambda}{3} r^4 - 2Mr + e^2 + g^2 \qquad \ldots (5.2b)$$

The surfaces $Y = 0$ are the horizons of the metric (5.2).

5.2 Radial Wave Equation

From equations (2.21a), (2.12g,h) in the limit, we obtain the radial ($w \sim r$) decoupled Dirac equation in the case of Kasner - type spacetime as follows :

$$Y \frac{d^2R}{dr^2} + [\sqrt{Y} \frac{d}{dr} (\sqrt{Y}) - \frac{i \mu_e Y}{\lambda + i \mu_e r}] \frac{dR}{dr} + [\frac{K^2}{Y} - 2(\lambda^2 + \mu_e^2 r^2)$$

$$+ \frac{\mu_e K}{\lambda + i \mu_e r} + i \sqrt{Y} \frac{d}{dr} (\frac{K}{\sqrt{Y}})] R = 0 \qquad \ldots (5.3)$$

$$K = r^2 \sigma - e Q r$$

$$Y = -\frac{\Lambda}{3} r^4 - 2 M r + e^2 + g^2$$

where σ is the energy of the Dirac particle, λ is the separation constant, μ_e and Q are the mass and the electric charge of the Dirac particle respectively.

With the coordinate transformation

$$\frac{d}{d\hat{r}} = \frac{Y}{r^2} \frac{d}{dr} \qquad \ldots (5.4)$$

equation (5.3) reduces near the horizon to the form

$$\frac{d^2 R}{d\hat{r}^2} + (\sigma - \sigma_o)^2 R = 0 \qquad \ldots (5.5)$$

where

$$\sigma_o = \frac{e Q}{r_+} \qquad \ldots (5.6)$$

and r_+ , called the event horizon , is the smaller of the two positive values of r at which $Y = 0$, provided the roots are real . The larger positive value of $Y = 0$, denoted by r_{++} , represents the cosmological horizon .

5.3 Hawking Thermal Spectrum

The solution of equation (5.5) can easily be found to be

$$R \sim \exp \left[\pm i (\sigma - \sigma_o) \right] \hat{r} \qquad \ldots (5.7)$$

Now we can write the radial wave function as

$$\psi_r = \exp \left[- i \sigma (t \pm \hat{r}_1) \right] \qquad \ldots (5.8)$$

where

$$\hat{r}_1 = \frac{\sigma - \sigma_o}{\sigma} \hat{r} \qquad \ldots (5.9)$$

We resolve ψ_r into ingoing and outgoing waves as

$$\psi_r^{in} \sim \exp\left[- i \sigma (t + \hat{r}_1) \right] \qquad \ldots \quad (5.10)$$

$$\psi_r^{out} \sim \exp\left[- i \sigma (t - \hat{r}_1) \right] \qquad \ldots \quad (5.11)$$

Introducing the Eddington - Finkelstein coordinates

$$v = t + \hat{r}_1 \qquad \ldots \quad (5.12)$$

we obtain

$$\psi_r^{in} \sim \exp(- i \sigma v) \qquad \ldots \quad (5.13)$$

$$\psi_r^{out} \sim \exp\left[- i \sigma v + 2i (\sigma - \sigma_o) \hat{r} \right] \qquad \ldots \quad (5.14)$$

Near $r = r_+$, equation (5.4) can be integrated to give

$$\hat{r} = \frac{1}{2 \kappa_+} \ln (r - r_+) \qquad \ldots \quad (5.15)$$

where

$$\kappa_+ = - \frac{\Lambda}{6 r_+^2} (r_+ - r_{++})(r_+ - r_-)(r_+ - r_{--}) \qquad \ldots \quad (5.16)$$

is the surface gravity of the event horizon of the Kasner-type spacetime . Here r_- is the inner black hole horizon and r_{--} is another cosmological horizon.

Just outside the event horizon we have

$$\psi_r^{out} \sim e^{-i\sigma v}(r - r_+)^{(i/K_+)(\sigma - \sigma_0)} \quad \ldots \quad (5.17)$$

We now extend the outgoing wave outside the horizon to the region inside . Since on the event horizon the outgoing wave function is not analytic and cannot be straightforwardly extended to the region inside , it can be continued analytically to the complex plane by going around the event horizon.

Hence inside the event horizon

$$\psi_r^{out} \sim e^{-i\sigma v}(r_+ - r)^{(i/K_+)(\sigma - \sigma_0)}\, e^{(\pi/K_+)(\sigma - \sigma_0)} \qquad (5.18)$$

Introducing the step function

$$y(x) = \begin{cases} 1 & \text{for} \quad x \geqslant 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \quad \ldots \quad (5.19)$$

we can generally write the outgoing wave function as

$$\phi_r^{out} = N_r \{ y(r - r_+)\, \psi_r^{out}(r - r_+)$$

$$+ y(r_+ - r)\, \psi_r^{out}(r_+ - r) \exp\left[\frac{\pi}{K_+}(\sigma - \sigma_0)\right]\} \quad \ldots \quad (5.20)$$

where ψ_r^{out} is the normalized Dirac wave function.

Expression (5.20) describes the splitting of ϕ_r^{out} into two components :

(a) A flow of positive - energy particles of strength N_r^2 outgoing from the event horizon .

(b) A flow of positive - energy particles in the Kasner- type background in the reverse time , since inside the event horizon , r represents the time axis due to the interchange of time and space . This can be interpreted as a flow in time of negative - energy particles ingoing toward the singularity region. This shows that a wave function near the event horizon gives rise to the creation of a Dirac particle - antiparticle pair [133] .

Obviously, from the normalization condition we have

$$< \phi_r^{out} , \phi_r^{out} > = N_r^2 \{ \exp [\frac{2\pi}{K_+} (\sigma - \sigma_o)] + 1 \} = 1 \qquad \ldots (5.21)$$

$$\text{or,} \quad N_r^2 = \{ \exp [\frac{2\pi}{K_+} (\sigma - \sigma_o)] + 1 \}^{-1}$$

$$= \{\exp [\frac{1}{K_b T_+} (\sigma - \sigma_o)] + 1 \}^{-1} \qquad \ldots (5.22)$$

where

$$T_+ = \frac{K_+}{2 \pi K_b} \qquad \ldots (5.23)$$

T_+ is the temperature of the region inside the event horizon and K_b is Boltzmann's constant. Equation (5.22) is the formula for the Hawking thermal spectrum of Dirac particles in the Kasner - type spacetime [134].

Following in a similar way, we have

$$T_{++} = \frac{K_{++}}{2\pi K_b} \qquad \ldots \quad (5.24)$$

where

$$K_{++} = -\frac{\Lambda}{6r_{++}^2}(r_{++} - r_+)(r_{++} - r_-)(r_{++} - r_{--}) \qquad \ldots \quad (5.25)$$

is the surface gravity of the cosmological event horizon.

5.4 Remarks

From this work it has appeared that an anisotropic model like Kasner- type spacetime gives rise to particle creation. The Kasner-type spacetime is due to the contraction of the Schwarzschild space-time generalized with the cosmological constant and electric and magnetic monopole charge. This result of particle creation in the Kasner - type spacetime goes beyond the idea that in the contraction phase it is necessary that the matter should disappear. To avoid this situation , it could be said that particles do not disappear in the process of contraction but become immaterial . It will be more interesting to say that " immaterial souls " of particles are created during contraction .

CHAPTER - 6

HAWKING RADIATION OF DIRAC PARTICLES IN PLEBANSKI SPACETIME

In the preceding three chapters we have seen that Hawking's and Gibbons and Hawking's thermal spectrum of Dirac particles near the event horizon and the cosmological horizon of black hole spacetimes holds good near the event horizon and the cosmological horizon of NUTKN , HNUTKN and Kasner- type spacetimes . In the present chapter we shall generalize these results and show that Hawking's and Gibbons and Hawking's thermal radiation by black holes holds good also in a more general background spacetime viz. Plebanski spacetime . Using equation (2.21a) we derive the radial decoupled Dirac equation in the Plebanski limit and then study Hawking radiation near the horizons of the metric .

6.1 The Plebanski Spacetime

In the coordinates (u, v, w, x) the Plebanski spacetime is given by

$$ds^2 = \frac{x^2 + w^2}{X} dx^2 + \frac{x^2 + w^2}{Y} dw^2 + \frac{X}{x^2 + w^2} (du + w^2 dv)^2 - \frac{Y}{x^2 + w^2} (du - x^2 dv)^2 \quad \ldots \quad (6.1a)$$

where

$$X = b - g^2 + 2n\,x - \varepsilon\,x^2 - \frac{\Lambda}{3}\,x^4 \qquad \ldots \qquad (6.1b)$$

$$Y = b + e^2 - 2M\,w + \varepsilon\,w^2 - \frac{\Lambda}{3}\,w^4 \qquad \ldots \qquad (6.1c)$$

with electric potential

$$A_\mu\,dx^\mu = \frac{e\,w}{x^2 + w^2}\,(\,du - x^2\,dv\,) \qquad \ldots \qquad (6.2)$$

Besides the cosmological constant Λ, the metric given by (6.1) includes : the mass parameter M, the NUT (magnetic mass) parameter n, the electric charge parameter e, the magnetic charge parameter g and the kinetical parameters b and ε. The metric is of Petrov type - D and the surfaces $Y = 0$ are the horizons of the metric [135] .

With suitable adjustment of the parameters and appropriate coordinate transformation the metric (6.1) gives many of the physically interesting solutions of Einstein or Einstein - Maxwell equation which has already been discussed in Chapter 1.

6.2 Radial Wave Equation

To obtain the Plebanski spacetime and the radial decoupled Dirac equation in this background spacetime , we set in equations (2.1) , (2.21 a),(2.12g) and (2.12h) the following substitutions:

$$\varepsilon_1 = \varepsilon_2 = f = g = 1 , \quad m(x) = -x^2 , \quad p(w) = w^2 , \quad T(w, x) = 1 ,$$

$$H(w) = ew , \quad G(x) = 0 , \quad h(w) = w , \quad g(x) = x , \quad Z = w^2 + x^2 ,$$

$$W(w) = \sqrt{Y} , \quad X(x) = \sqrt{X} , \quad \alpha = i\sigma , \quad \beta = -iA , \tag{6.3}$$

$$Y = b + e^2 - 2Mw + \varepsilon w^2 - \frac{\Lambda}{3} w^4 ,$$

$$X = b - g^2 + 2nx - \varepsilon x^2 - \frac{\Lambda}{3} x^4$$

where σ is the energy of the Dirac particle and A is the arbitrary constant .

From equations (2.21a) , (2.12g, h) and (6.3) we obtain the following equation

$$\sqrt{Y} \left[-\frac{\partial}{\partial w} + \frac{iK}{Y} - \frac{Y'}{4Y} \right] \sqrt{Y} \left[\frac{\partial}{\partial w} + \frac{iK}{Y} + \frac{Y'}{4Y} \right] K_1(w)$$

$$+ \frac{i\mu_e Y}{\lambda + i\mu_e w} \left[\frac{\partial}{\partial w} + \frac{iK}{Y} + \frac{Y'}{4Y} \right] K_1(w)$$

$$+ 2(\lambda^2 + \mu_e^2 w^2) K_1(w) = 0 \qquad \ldots \tag{6.4}$$

where

$$K = w^2 \sigma - A - eQw$$

$$Y = b + e^2 - 2Mw + \varepsilon w^2 - \frac{\Lambda}{3} w^4$$

If w plays the role of the radial variable r, then replacing w by r and letting $K_1(r) = R\,Y^{-1/4}$ we obtain (6.4) as

$$\sqrt{Y}\,\frac{d}{dr}\left(\sqrt{Y}\,\frac{dR}{dr}\right) - \frac{i\,\mu_e\,Y}{\lambda + i\mu_e\,r}\,\frac{dR}{dr} + \Big[\frac{K^2}{Y} - 2\,(\lambda^2 + \mu_e^2\,r^2)$$

$$+ \frac{K\,\mu_e}{\lambda + i\,\mu_e\,r} + i\sqrt{Y}\,\frac{d}{dr}\left(\frac{K}{\sqrt{Y}}\right)\Big]\,R = 0 \qquad \ldots \qquad (6.5)$$

$$K = r^2\sigma - A - eQr$$

$$Y = b + e^2 - 2Mr + \varepsilon r^2 - \frac{\Lambda}{3}\,r^4$$

Under the coordinate transformation (1.42) along with the adjustment of the parameters (1.39) and (1.40) the equation (6.5) transforms to the radial decoupled Dirac equation for the NUT-Kerr-Newman-Kasuya-de Sitter spacetime which reduces to the radial decoupled Dirac equation obtained by

(i) Ahmed [136] for the HNUTKN spacetime when $g = 0$

(ii) Xu Dianyan and Wang Huiya [137] for the Kerr-Newman-de Sitter spacetime if $n = g = 0$

(iii) Khanal [138] for the Kerr-de Sitter spacetime provided $n = e = g = 0$

(iv) Page [139] for the Kerr - Newman spacetime when $\sqrt{2}\,\lambda \to \lambda$, $\sqrt{2}\,\mu_e \to \mu_e$, $\Lambda = n = g = 0$

(v) Chandrasekhar [140] for the Kerr spacetime if $\Lambda = e = n = g = 0$.

Introducing the coordinate $\hat{r}$ defined by

$$\frac{d}{d\hat{r}} = \frac{Y}{r^2 + x^2}\,\frac{d}{dr} \qquad \ldots \qquad (6.6)$$

the equation (6.5) near the horizon $Y = 0$, reduces to the following form

$$\frac{d^2 R}{d\hat{r}^2} + (\sigma - \sigma_o)^2 R = 0 \qquad \ldots \qquad (6.7)$$

where

$$\sigma_o = \left[\frac{x^2 \sigma + A}{r^2 + x^2} + \frac{e\,Q\,r}{r^2 + x^2} \right]_{r = r_+} \qquad \ldots \qquad (6.8)$$

and r_+, called the event horizon, is one of the positive values of r for which $Y = 0$, provided the roots of $Y = 0$ are real.

Under the coordinate transformation (1.42) accompanied by the adjustment of the parameters (1.39) and (1.40), the equation (6.7) will be transformed to the wave equation for the HNUTKNK spacetime.

6.3 Hawking Thermal Spectrum

The solution of equation (6.7) is evidently

$$R \approx \exp\left[\pm i (\sigma - \sigma_0) \right] \hat{r} \qquad \ldots \quad (6.9)$$

Now we write the radial wave function

$$\psi_r = \exp\left[- i\sigma (t \pm \hat{r}_1) \right] \qquad \ldots \quad (6.10)$$

where

$$\hat{r}_1 = \frac{(\sigma - \sigma_0)\hat{r}}{\sigma} \qquad \ldots \quad (6.11)$$

We resolve ψ_r into ingoing and outgoing waves as

$$\psi_r^{in} \sim \exp\left[- i \sigma (t + \hat{r}_1) \right] \qquad \ldots \quad (6.12)$$

$$\psi_r^{out} \sim \exp\left[- i \sigma (t - \hat{r}_1) \right] \qquad \ldots \quad (6.13)$$

Introducing the Eddington - Finkelstein coordinates

$$v = t + \hat{r}_1 \qquad \ldots \quad (6.14)$$

we obtain

$$\psi_r^{in} \sim \exp (- i \sigma v) \qquad \ldots \quad (6.15)$$

$$\psi_r^{out} \sim \exp\left[- i \sigma v + 2i (\sigma - \sigma_0) \hat{r} \right] \qquad \ldots \quad (6.16)$$

Now on integration of equation (6.6), we find near $r = r_+$ that the equation takes the following form

$$\hat{r} = \frac{1}{2K_+} \ln (r - r_+) \quad \ldots \quad (6.17)$$

where

$$K_+ = \frac{1}{2} \left[\frac{1}{r^2 + x^2} \frac{\partial Y}{\partial r} \right]_{r = r_+} \quad \ldots \quad (6.18)$$

is the surface gravity [141] near the event horizon of the Plebanski spacetime.

Just outside the event horizon

$$\psi_r^{out} \sim e^{-i\sigma v} (r - r_+)^{(i/K_+)(\sigma - \sigma_0)} \quad \ldots \quad (6.19)$$

By analytically continuing the outgoing wave outside the horizon to the region inside we obtain inside the event horizon

$$\psi_r^{out} \sim e^{-i\sigma v} (r_+ - r)^{(i/K_+)(\sigma - \sigma_0)} \, e^{(\pi/K_+)(\sigma - \sigma_0)} \quad \ldots \quad (6.20)$$

Introducing the step function

$$y(x) = \begin{cases} 1 & \text{for} \quad x \geqslant 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \quad \ldots \quad (6.21)$$

the outgoing wave function can generally be written as

$$\phi_r^{out} = N_r [y(r - r_+) \psi_r^{out} (r - r_+)$$

$$+ y(r_+ - r) \psi_r^{out} (r_+ - r) \exp \{ \frac{\pi}{K_+} (\sigma - \sigma_o) \}] \quad \ldots \quad (6.22)$$

where ψ_r^{out} is the normalized Dirac wave function. The above expression shows that a wave function near the event horizon gives rise to the creation of Dirac particle - antiparticle pair [142].

From the normalization condition, we obtain

$$< \phi_r^{out} , \phi_r^{out} > = N_r^2 \{ \exp [\frac{2\pi}{K_+} (\sigma - \sigma_o)] + 1 \} = 1 \quad \ldots \quad (6.23)$$

$$\text{or,} \quad N_r^2 = \{ \exp [\frac{2\pi}{K_+} (\sigma - \sigma_o)] + 1 \}^{-1}$$

$$= \{ \exp [\frac{1}{K_b T_+} (\sigma - \sigma_o)] + 1 \}^{-1} \quad \ldots \quad (6.24)$$

where

$$T_+ = \frac{K_+}{2 \pi K_b} \quad \ldots \quad (6.25)$$

being the temperature of the region inside the event horizon and K_b is the Boltzmann's constant. Equation (6.24) is the formula for the Hawking thermal spactrum of Dirac particles near the event horizon of the Plebanski spacetime .

Under the coordinate transformation (1.42) accompanied by the adjustment of the parameters (1.39) and (1.40) , the equations (6.5) - (6.25) , will be transformed to the corresponding equations of HNUTKNK spacetime [143] . The Hawking thermal spectrum formula thus obtained will be reduced to the Hawking thermal spectrum formula of Dirac particles near the event horizons of

(i) HKNK spacetime when $n = 0$

(ii) HNUTKN spacetime [144] for $g = 0$

(iii) hot Kerr - Newman spacetime [145] for $n = g = 0$

(iv) hot Kerr - spacetime if $n = g = e = 0$

(v) hot Reissner - Nordstrom spacetime provided $n = g = a = 0$

(vi) hot Schwarzschild spacetime for $n = g = a = e = 0$

(vii) hot NUT spacetime when $a = e = g = 0$.

Further in the limits $\Lambda = 0$ in the cases (i) - (vii) , we will get the Hawking thermal spectrum for

(i) Kerr - Newman-Kasuya [146] spacetime

(ii) NUT - Kerr - Newman [147] spacetime

(iii) Kerr - Newman [148] spacetime

(iv) Kerr [149] spacetime

(v) Reissner - Nordstrom spacetime

(vi) Schwarzschild spacetime

(vii) NUT spacetime .

Following the similar way, we could have

$$T_{++} = \frac{\kappa_{++}}{2\pi \kappa_b} \qquad \ldots \qquad (6.26)$$

where

$$\kappa_{++} = \frac{1}{2} \left[\frac{1}{r^2 + x^2} \frac{\partial Y}{\partial r} \right]_{r = r_{++}} \qquad \ldots \qquad (6.27)$$

and r_{++} , called the cosmological horizon , is the larger positive root of $Y = 0$ and κ_{++} represents the surface gravity [150] of the cosmological horizon of the Plebanski spacetime .

Under the coordinate transformation (1.42) accompanied by the adjustment of the parameters (1.39) and (1.40) , the equation (6.27) will be transformed to the corresponding equation for the HNUTKNK spacetime [151] which with proper limits goes for the results obtained in [152].

Thus we observe that our result is a more general one and encompasses all the results obtained in [153].

6.4 Discussion

We observe that Hawking- Gibbon's thermal spectrum formula of Dirac particles near the event horizon and the cosmological horizon of the black hole spacetimes , can also be obtained near the event horizon and the cosmological horizon of the more general spacetime such as Plebanski spacetime which includes many interesting spacetimes besides the black hole spacetimes as special cases. So we see that the spacetimes which are not black hole spacetimes but includes black hole spacetimes having horizons can produce thermal particles near the horizons. The Hawking thermal spectrum formula obtained in the Plebanski background spacetime will be reduced to all the particular cases of the Plebanski spacetime . From the observation of the particular cases of the Plebanski spacetime which has already been discussed in Chapter 1, we see that the Hawking radiation is valid for all types of spacetimes having horizons . The result goes most of the way in confirming the theory obtained recently by Kay and Wald [154] that Hawking radiation is valid for the arbitrary spacetime having killing horizon.

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