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# Study of Sums and Products in Some Branches of Mathematics

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# **Study of Sums and Products in Some Branches of Mathematics**

**THESIS SUBMITTED FOR THE DEGREE OF  
Doctor of Philosophy  
in  
Mathematics**

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No. ....

Dated: 26.12.06.

Certified that the thesis entitled “Study of Sums and Products in Some Branches of Mathematics” submitted by Kalyan Kumar Dey in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi, Bangladesh has been completed under my supervision. I believe that this research work is an original one and that it has not been submitted elsewhere for any degree.

Subrata Majumdar  
(Subrata Majumdar)  
Supervisor

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*Kalyan Kumar Dey*  
(Kalyan Kumar Dey)



## STATEMENT OF ORIGINALITY

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University, and to the best of my knowledge and belief, does not contain any material previously published or written by another person except where due reference is made in the text.

*Kalyan Kumar Dey*  
(Kalyan Kumar Dey)

# Synopsis

In many branches of Mathematics a strategy for study of the mathematical entities is to view them as being made up of simpler entities in one or more ways. The task is then divided into two parts : (i) identifying and studying simple entities, (ii) investigating various manners of amalgamating these simple building blocks together. When two mathematical objects of the same nature are glued together to obtain a third such object, the latter is often called a sum or a product. In this thesis our objects of study are sums and products and their similar counterparts in various areas of Mathematics.

The first chapter gives a survey of various known sums and products in many mathematical branches including Algebra, Topology and Graph Theory.

In the second chapter direct product and wreath products of transformation semigroups have been defined and studied. Their nature and applicability have been investigated and their associativity and mutual distributivity have been established.

In the third chapter two kinds of 'product's of partially ordered sets have been studied. In particular such products of lattices have been considered.

The fourth chapter introduces two kinds of 'sum's for topological spaces. One of them is a generalisation of 'connected sum' of surfaces, while the other is constructed after the pattern of 'amalgamated free product' for groups. Some properties of these products have been studied.

In the fifth and last chapter a number of 'sums' and 'products' have been defined and studied for two objects in a category. These have a maximal or a minimal property in some sense.

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# Chapter 1

## A Survey of Well-known Sums and Products

### Introduction

We shall describe here how in different branches of Mathematics various procedures have been used to amalgamate two or more mathematical objects of the same nature to yield a new such object. The resulting object is usually called a sum or a product. The method of forming such sums and products has often proved very useful for building complex mathematical entities out of a number of simpler objects.

In this chapter we shall give an account of forming sums and products or some such separations in a number of branches of Mathematics and hint at their possible applications. This will illustrate the significance and importance of these processes. The branches of Mathematics considered here include Algebra, Topology and Graph Theory.

### 1. Algebra

#### (a) Groups:

##### (i) External Direct Product

Let  $A$  and  $B$  be groups. Then the external direct product of  $A$  and  $B$ , written  $A \times B$ , is defined as the set  $A \times B$  together with multiplication  $(a, b) \cdot (a', b') = (aa', bb')$ . As an example,  $C_m \times C_n \cong C_{mn}$ , if  $(m, n) = 1$  and  $C_r$  is the cyclic group of order  $r$ .

In general, if  $\{A_\alpha\}_{\alpha \in I}$  is any non-empty collection of groups, then the external direct product  $\prod_{\alpha \in I} A_\alpha$  is the group is the Cartesian product  $\times_{\alpha \in I} A_\alpha$  together with multiplication  $\{a_\alpha\} \cdot \{a_{\alpha'}\} = \{a_\alpha a_{\alpha'}\}$ .

Sometimes it is called the **unrestricted direct product**.

**The restricted direct product** is the subgroup  $S$  of the external direct product  $\prod_{\alpha \in I} A_\alpha$  consisting of all those elements  $\{a_\alpha\}$  for which  $a_\alpha \neq e_\alpha$ , for only a finite number of  $\alpha$ 's.

If  $I$  is finite, the restricted and the unrestricted direct product coincide.

If  $\mathfrak{R}$  denotes the additive group of real numbers,  $\mathfrak{R}^n$  is the external direct product of  $n$  copies of  $\mathfrak{R}$  of with

$$\mathfrak{R}^n = \mathfrak{R} \times \mathfrak{R} \times \dots \times \mathfrak{R} \text{ (n copies) and}$$

$$(a_1, a_2, \dots, a_n) \cdot (b_1, b_2, \dots, b_n) = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n).$$

For abelian groups i.e. commutative groups, written additively, the restricted direct product is called the direct sum (external).

Thus  $\mathfrak{R}^n$  is the direct sum of  $n$  copies of  $(\mathfrak{R}, +)$ .

## (ii) Internal Direct Product

Let  $G$  be a group and let  $\{N_\alpha\}_{\alpha \in I}$  be a nonempty collection of normal subgroups of  $G$  such that (i) for each  $\alpha \in I$ ,  $N_\alpha \cap \bar{N}_\alpha = \{e\}$  where  $\bar{N}_\alpha$  is the subgroup of  $G$  generated by  $\bigcup_{\beta \in I, \beta \neq \alpha} N_\beta$ , (ii)  $G$  is generated by

$$\bigcup_{\alpha \in I} N_\alpha.$$

In this case, every element of  $G$  can be expressed uniquely as a product  $n_{\alpha_1} n_{\alpha_2} \dots n_{\alpha_r}$  for some  $r$  where  $\alpha_1, \alpha_2, \dots, \alpha_r$  are distinct and the uniqueness is upto the order of  $n_{\alpha}$ 's.

Also  $n_{\alpha} n_{\alpha'} = n_{\alpha'} n_{\alpha}$  for each distinct pair  $(\alpha, \alpha')$ .

It can be shown that ([], p.34) that the internal direct product  $\{N_{\alpha}\}$  is isomorphic to the restricted external direct product of  $\{N_{\alpha}\}_{\alpha \in I}$ .

### Direct Sum

If the group is additive abelian, the above product is called the direct sum (internal). The direct sum is denoted by  $\oplus$ . Thus

$$Z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\} \cong \{\bar{0}, \bar{2}, \bar{4}\} \oplus \{\bar{0}, \bar{3}\} = Z_3 \oplus Z_2.$$

### Application

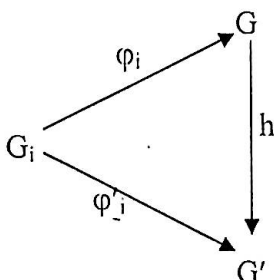
Direct sums are used to describe the structures of certain classes of abelian groups, e.g. free abelian groups, divisible groups and finitely generated abelian groups.

**Direct Sum (external and internal) of vector spaces, modules is similarly defined.**

We have the following uniqueness proposition about free products.

### Proposition ([61], p.97)

Let  $G$  and  $G'$  are free products of a collection  $\{G_i: i \in I\}$  of groups (with respect to homomorphism  $\varphi_i: G_i \rightarrow G$  and  $\varphi'_i: G_i \rightarrow G'$ , respectively). Then there exists a unique isomorphism  $h: G \rightarrow G'$  such that the following diagram is commutative for any  $i \in I$ :



### Theorem ([61], p. 98)

Given any collection  $\{G_i: i \in I\}$  of groups, their free products exists.

We denote the free product of groups  $G_1, G_2, \dots, G_n$  by  $G_1 * G_2 * \dots * G_n$  or  $\prod_{1 \leq i \leq n}^* G_i$ .

The free product of the family of groups  $\{G_i: i \in I\}$  is denoted by  $\prod_{i \in I}^* G_i$ .

### Examples

1. A free group on  $n$  generators is a free product of  $n$  infinite cyclic groups.
2. If  $G = \langle x, y \mid x^m = e, y^n = e \rangle$ , then  $G$  is a free product of cyclic group of order  $m$  and  $n$ , i.e.,  $G = C_m * C_n$ .

### Application

Free product is used to describe the structure of fundamental groups of path-connected topological spaces.

If  $X$  and  $Y$  are two path-connected topological spaces such that  $X \cap Y$  is a singleton, then  $\pi(X \cup Y) = \pi(X) * \pi(Y)$ .

### (iii) Semidirect Product

Let  $G$  be a group,  $N$  a normal subgroup and  $H$  a subgroup of  $G$  such that every element of  $G$  can be expressed uniquely a product  $hn$ ,



$h \in H, n \in N$ . Then  $G$  is called the semidirect product of  $N$  by  $H$ , denoted by  $H \ltimes N$ . The element  $hn$  is denoted by  $[h, n]$ ,  $h \in H, n \in N$ .

**Theorem ([61], p. 89)**

If  $G$  is the semidirect product of  $N$  by  $H$  then the elements of  $[h, 1]$  of  $G$  form a subgroup isomorphic to  $H$  and the elements  $[1, n]$  form a normal subgroup isomorphic to  $N$ . Moreover, the automorphism of  $N$  as a subgroup of  $G$  is induced by transformation by the element  $h=[h, 1]$  of  $H$  as a subgroup of  $G$ . Moreover,  $G=H \cup N=HN$ .

**Theorem ([61], p. 89)**

$G$  is the semidirect product of  $N$  by  $H$  if and only if  $N$  is a normal subgroup of  $G$  and  $H$  is a subgroup of  $G$  whose elements may be taken as the coset representation of  $N$ . Otherwise expressed

- i.  $N$  is a normal subgroup of  $G$
- ii.  $H$  is a subgroup of  $G$
- iii.  $H \cap N = 1$
- iv.  $H \cup N = G$ .

**Examples**

$S_3$  is the semidirect product of  $C_3$  with  $C_2$ .

$S_3$  is the semidirect product of  $A_3$  by  $S_2=C_2$ .

**(iv) Wreath Products of Permutation Groups**

Let  $G$  and  $H$  be permutation groups on sets  $A$  and  $B$ , respectively. We define **wreath product** of  $G$  by  $H$ , written  $G \wr H$  in the following way:

$G \wr H$  is the group of all permutations  $\theta$  on  $A \times B$  of the following kind  $\theta(a,b) = (a\gamma_b, b\eta)$ ,  $a \in A$ ,  $b \in B$  where for each  $b \in B$ ,  $\gamma_b$  is a permutation of  $G$  on  $A$ , but for different  $b$ 's the choices of the permutations  $\gamma_b$  are independent. The permutation  $\eta$  is a permutation of  $H$  on  $B$ . The permutation  $\theta$  with  $\eta=1$  form a normal subgroup  $G^*$  isomorphic to the direct product of  $n$  copies of  $G$ , where  $n$  is the number of **letters** in the  $B$ . The factor group  $G \wr H / G^*$  is isomorphic to  $H$ , and the permutation  $\theta$  with all  $\gamma_b = 1$  form a subgroup isomorphic to  $H$ , whose elements may be taken as coset representation of  $G^*$  in  $G$ . Let  $X$  and  $Y$  be finite set with  $X \cap Y = \emptyset$ . Let  $G$  be a permutation group on  $X$  and  $H$  be a permutation group on  $Y$ . Then the wreath product  $G \wr H$  is permutation group on  $X \times Y$  and the direct product  $G \times H$  is permutation group on  $X \cup Y$ .

### (v) Free Product

A group  $G$  is said to be the **free product** of its subgroups  $A_\alpha$  ( $\alpha$  ranges over some index set) if the subgroups  $A_\alpha$  generate  $G$ , that is, if every element  $g$  of  $G$  is the product of a finite number of the elements of the  $A_\alpha$ ,  $g = a_1 a_2 \dots a_n$ ,  $a_i \in A_{\alpha_i}$ ,  $i=1, 2, \dots, n$ , (1)

and if every element  $g$  of  $G$ ,  $g \neq 1$ , has a *unique* representation in the form (1) subject to the condition that all the elements  $a_i$  are different from the unit element and that in (1) no two adjacent elements are in the same subgroup  $A_\alpha$ —although the product (1) may, in general, contain several factors from one and the same subgroup.

The free product is denoted by the symbol.

$$G = \prod_{\alpha}^* A_{\alpha} \quad (2)$$

and if  $G$  is the free product of a finite number of subgroups  $A_1, A_2, \dots, A_k$ , by the symbol

$$G = A_1 * A_2 * \dots * A_k.$$

The subgroups  $A_a$  are called the **free factors** of the free decomposition (2) of  $G$ . The expression (1) (under the restrictions imposed on it) is called the **normal** form (or irreducible representation) of the element  $g$  in the decomposition (2), and the number  $n$  the length of  $g$  in this decomposition; we write  $n=l(g)$ .

If a group  $G$  is generated by subgroups  $A_a$  (where  $a$  ranges over an index set), then  $G$  is the free product of these subgroups if and only if for every group  $H$  and every set of homomorphic mappings  $\Phi_a$  of each  $A_a$  into  $H$  there exists a homomorphic mapping  $\varphi$ , of  $G$  into  $H$  that coincides with  $\varphi_a$  on each  $A_a$ .

### Free Product with an Amalgamated Subgroup

In some connections an even more general construction than that of the free product turns out to be useful. Let  $A_a$  be groups, where  $a$  ranges over a set of indices, and let a proper subgroup  $B_a$  be chosen in every  $A_a$  such that all these subgroups are isomorphic to a fixed group  $B$ . By  $\varphi_a$  we denote a specific isomorphic mapping of  $B_a$  onto  $B$ ; then  $\phi_{\alpha\beta} = \varphi_{\alpha}\varphi_{\beta}^{-1}$  is an isomorphic mapping of  $B_{\alpha}$  onto  $B_{\beta}$ .

The free product of the groups  $A_{\alpha}$  with the amalgamated subgroup  $B$  is defined as the factor group  $G$  of the free product of the groups  $A_{\alpha}$  with respect to the normal subgroup generated by all elements of the form  $b_{\alpha}b_{\beta}^{-1}$ , where  $b_{\beta} = b_{\alpha}\psi_{\alpha\beta}$ , where  $b_{\alpha}$  ranges over the whole subgroup  $B_{\alpha}$ , and where  $\alpha$  and  $\beta$  are all possible index pairs. In other words, if every

group  $A_\alpha$  is given by a system of generators  $M_\alpha$  and a system of defining relations  $\Phi_\alpha$  between these generators, then  $G$  has as a system of generators the union of all sets  $M_\alpha$ , as a system of defining relations the union of the sets  $\Phi_\alpha$  and, in addition, all relations obtained by identifying those elements of different subgroups  $B_\alpha$  and  $B_\beta$  which are mapped by the isomorphisms  $\varphi_\alpha$  and  $\varphi_\beta$  onto one and the same element of  $B$ . The subgroups  $B_\alpha$  are “amalgamated”, as it were, in accordance with the isomorphisms  $\Psi_{\alpha\beta}$ .

We note two important results involving free products.

**( $\alpha$ ) Kurosh's Subgroup Theorem [19]**

$$\text{If } G = \prod_{\alpha}^* A_{\alpha} \quad (1)$$

and if  $H$  is an arbitrary subgroup of  $G$ , then there exists a free decomposition of  $H$ .

$$G = F * \prod_{\alpha}^* A_{\alpha} \quad (2)$$

where  $F$  is a free group and every  $B$  is conjugate in  $G$  to a subgroup of one of the free factors  $A_{\alpha}$ .

**( $\beta$ ) Theorem (Baer and Levi) [19]**

No group can be decomposable both into a free product and into a direct product.

**Torsion Products of Abelian Groups**

For abelian groups  $A$  and  $G$  we define the torsion product  $\text{Tor}(G, A)$  as that abelian group which has generators all symbols  $\langle g, m, a \rangle$ ,

with  $m \in \mathbb{Z}$ ,  $mg = 0$  in  $G$ , and  $ma=0$  in  $A$ , subject to the relations (“additivity” and “slide” rules for factors  $m, n$ )

$$\langle g_1+g_2, m, a \rangle = \langle g_1, m, a \rangle + \langle g_2, m, a \rangle, \quad mg_i = 0 = ma$$

$$\langle g, m a_1+a_2 \rangle = \langle g, m, a_1 \rangle + \langle g, m, a_2 \rangle, \quad mg = 0 = ma_i$$

$$\langle g, mn, a \rangle = \langle gm, n, a \rangle, \quad mng = 0 = na$$

$$\langle g, mn, a \rangle = \langle g, m, na \rangle, \quad mg = 0 = mna$$

Each relation is imposed whenever both sides are defined; in each case this amounts to the requirement that the symbols on the right hand side be defined. The additivity relations imply that  $\langle 0, m, a \rangle = 0 = \langle g, m, 0 \rangle$ . Hence  $\text{Tor}(G, A) = 0$  when  $A$  has no elements (except 0) of finite order. Also  $\text{Tor}(A, G) \cong \text{Tor}(G, A)$ .

In particular,  $\text{Tor}(\mathbb{Z}, A) = \text{Tor}(Q, A) = \text{Tor}(\mathbb{R}, A) = 0$ , for each abelian group  $A$ , and  $\text{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m, n)}$ .

## Modules and Vector Spaces

Let  $R$  be a commutative ring with 1. Let  $A$  and  $B$  be two  $R$ -modules. The **tensor product** of  $A$  and  $B$ , written  $A \otimes_R B$  is the  $R$ -module generated by all symbols  $a \otimes b$ ,  $a \in A$ ,  $b \in B$  with defining relations.

$$(i) (a_1+a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$$

$$(ii) a \otimes (b_1+b_2) = a \otimes b_1 + a \otimes b_2$$

$$(iii) ar \otimes b = a \otimes rb,$$

for each  $a_1, a_2, a \in A$ ,  $b_1, b_2, b \in B$ ,  $r \in R$ .

As particular cases,

$$R \otimes_R A \cong A, \quad \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_m \cong \mathbb{Z}_m, \quad \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n \cong \mathbb{Z}_{(m, n)}.$$

The tensor product of two vector spaces  $U$  and  $V$  over a field  $F$  is defined similarly. If  $\dim U=m$  and  $\dim V=n$ , then  $\dim (U \otimes_F V) = mn$ .

Tensor product of modules and vector spaces are associative in the sense:

$$\begin{cases} (A \otimes_R B) \otimes_R C \cong A \otimes_R (B \otimes_R C), \\ (U \otimes_F V) \otimes_F W \cong U \otimes_F (V \otimes_F W). \end{cases}$$

Also tensor product is distributive over direct sum in the following sense:

$$\begin{cases} A \otimes_R (B \oplus C) \cong (A \otimes_R B) \oplus (A \otimes_R C), \\ (A \oplus B) \otimes_R C \cong (A \otimes_R C) \oplus (B \otimes_R C). \end{cases}$$

$$\begin{cases} U \otimes_F (V \oplus W) \cong (U \otimes_F V) \oplus (U \otimes_F W), \\ (U \oplus V) \otimes_F W \cong (U \otimes_F W) \oplus (V \otimes_F W). \end{cases}$$

### Cayley-Dickson Construction of Division Algebras

An **algebra**  $A$  is a vector space equipped with a multiplication of vectors and a multiplicative identity. A **division algebra** is an algebra  $A$  such that if  $a, b \in A$  and  $ab=0$  then either  $a = 0$  or  $b=0$ , i.e., every non-zero element of  $A$  has a multiplicative inverse.

There are only 4 normed division algebras:  $\mathbf{R}$ , the real numbers,  $\mathbf{C}$ , the complex number,  $\mathbf{H}$ , the quaternions, and  $\mathbf{O}$ , the octonions.  $\mathbf{C}$  has a basis  $\{1, i\}$  over  $\mathbf{R}$  with  $i^2 = -1$ .  $\mathbf{H}$  has a basis  $\{1, i, j, k\}$  over  $\mathbf{R}$  with  $i^2 = j^2 = k^2 = ijk = -1$ .  $\mathbf{O}$  has a basis  $\{1, e_1, e_2, \dots, e_7\}$  over  $\mathbf{R}$  with

	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$e_1$	$-1$	$e_4$	$e_7$	$-e_2$	$e_6$	$-e_5$	$-e_3$
$e_2$	$e_4$	$-1$	$e_5$	$e_1$	$-e_3$	$e_7$	$-e_6$
$e_3$	$-e_7$	$-e_5$	$-1$	$e_6$	$e_2$	$-e_4$	$e_1$
$e_4$	$e_2$	$-e_1$	$-e_6$	$-1$	$e_7$	$e_3$	$-e_5$
$e_5$	$-e_6$	$e_3$	$-e_2$	$-e_7$	$-1$	$e_1$	$e_4$
$e_6$	$e_5$	$-e_7$	$e_4$	$-e_3$	$-e_1$	$-1$	$e_2$
$e_7$	$e_3$	$e_6$	$-e_1$	$e_5$	$-e_4$	$-e_2$	$-1$

There is an interesting method of constructing these algebras step by step.

A algebra is an algebra equipped with a conjugation, i.e., real linear map  $*$  :  $A \rightarrow A$  with  $a^{**} = a$ ,  $(ab)^* = b^* a^*$ .

A complex number can be thought of as an order  $(a, b)$  or real numbers with component-wise addition and multiplication given by  $(a, b)(c, d) = (ac - db, ad + cb)$ .

We can also define conjugate of a complex number by  $(a, b)^* = (a, -b)$ .

We can then define a quaternion in a similar way. A quaternion is a pair of complex numbers. Addition is component-wise and multiplication is given by  $(a, b)(c, d) = (ac - db^*, a^*d + cb)$  (2)

We can also define the conjugate of a quaternion by (3)  $(a, b)^* = (a^*, -b)$ .

We can now define an octonion by an ordered pair of quaternions. As before we can define addition and multiplication using (2) and (3).

This method can be continued to obtain a new algebra from an old one. This method is called **Cayley-Dickson construction**.

Let  $G$  and  $H$  be two groups given by the following generators and relations:  $G = \langle x_1, \dots; r_1, \dots \rangle$  and  $H = \langle y_1, \dots; s_1, \dots \rangle$ . Let  $A$  and  $B$  be subgroups of  $G$  and  $H$  respectively such that there exists an isomorphism  $\varphi : A \rightarrow B$ . Then the free product of  $G$  and  $H$ , amalgamating the subgroups  $A$  and  $B$  by their isomorphism  $\varphi$  is the group  $\langle x_1, \dots; y_1, \dots; r_1, \dots, s_1, \dots; a = \varphi(a), a \in A \rangle$ .

An important concept very closely related to the free product of two groups with an amalgamated subgroup is the Higman-Neumann-Neuman extension or HNN extension of a group relative to two of its subgroups  $A, B$  and an isomorphism  $\varphi : A \rightarrow B$ .

### HNN Extension

Let  $G$  be a group and let  $A$  and  $B$  be subgroups of  $G$  with  $\varphi : A \rightarrow B$  an isomorphism. **The HNN extension of  $G$  related to  $A, B$  and  $\varphi$  is the group  $G^* = \langle G, t; t^{-1}at = \varphi(a), a \in A \rangle$ .**

Free products with amalgamation and HNN extensions are often motivated by consideration of fundamental groups of topological spaces.

All topological spaces are assumed to be path connected. If  $X$  is a topological space  $\pi_1(X)$  will denote the fundamental group of  $X$ . Let  $X$  and  $Y$  be spaces, and let  $U$  and  $V$  be open path connected subspaces of  $X$  and  $Y$  respectively such that there is a homeomorphism,  $h: U \rightarrow V$ . Choose a base point  $\mu \in U$  for the fundamental groups of  $U$  and  $X$ . Similarly, choose as base point  $h(\mu) = v \in V$ . There is a homomorphism  $\eta: \pi_1(U) \rightarrow \pi_1(X)$  defined by simply considering a loop in  $U$  as a loop in  $X$ . Suppose that  $\eta$  and the similarly defined homomorphism  $\delta: \pi_1(V) \rightarrow \pi_1(Y)$  are both injections. The homeomorphism  $h$  induces an



isomorphism  $h^*: \pi_1(U) \rightarrow \pi_1(V)$ . Suppose we identify  $U$  and  $V$  by the homeomorphism  $h$  to obtain a new space  $Z$ . Under the assumptions made, the Seifert-van Kampen Theorem (Massey [61]) says that

$$\pi_1(Z) = (\pi_1(X) * \pi_1(Y); \pi_1(V), h^*)$$

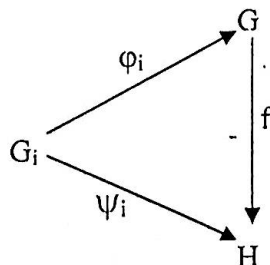
The HNN extension has a similar topological interpretation. Suppose that  $U$  and  $V$  are both subspaces of the arcwise connected space  $X$ . Assume the same hypothesis on  $U$  and  $V$  as above. Let  $I$  be the unit interval, and let  $C = U \times I$ . Identify  $U \times \{0\}$  with  $U$  and identify  $U \times \{1\}$  with  $V$  by the homeomorphism  $h$ . Let  $Z$  be the resulting space. (What we have done is to attach a handle to  $X$ .) The Seifert-van Kampen Theorem can be used to show that  $\pi_1(Z) = \langle \pi_1(X), t; t^{-1}ut = h(u), u \in U \rangle$

### Free Products of Groups

Let  $\{G_i\}_{i \in I}$  be a collection of groups, and assume there is given for each index  $i$  a homomorphism  $\varphi_i$  of  $G_i$  into a fixed group  $G$ . We say that  $G$  is the **free product** of the groups  $G_i$  (with respect to the homomorphism  $\varphi_i$ ) if and only if the following condition holds:

For any group  $H$  and any homomorphism  $\psi_i : G_i \rightarrow H, i \in I$

there exists a unique homomorphism  $f: G \rightarrow H$  such that for any  $i \in I$ , the following diagram is commutative



## (b) Rings :

### (i) Complete Direct Sum and Subdirect Sum

Let  $\{A_i : i \in I\}$  be a class of rings.

Consider the set  $S$  of symbols  $(a_1, a_2, \dots)$  where  $a_i$  belongs to  $A_i$ . These are infinite vectors. Addition and multiplication are defined coordinatewise.

$$(a_1, a_2, \dots) + (b_1, b_2, \dots) = (a_1 + b_1, a_2 + b_2, \dots)$$

$$(a_1, a_2, \dots) \cdot (b_1, b_2, \dots) = (a_1 b_1, a_2 b_2, \dots)$$

Then  $S$  is a ring and it is called the **complete direct sum** of the rings  $A_i$ .

The set of all elements of the form  $(0, 0, \dots, 0, a_i, 0, \dots)$  is a subring  $A'_i$  which is isomorphic to  $A_i$  and the mapping

$(a_1, a_2, \dots, a_i, \dots) \rightarrow (0, 0, \dots, 0, a_i, 0, \dots)$  is homomorphism of  $S$  onto  $A'_i$ . The subring which consists of all elements of  $S$  which have only a finite number of nonzero entries is called the **weak direct sum** of the  $A_i$ . The weak direct sum is also called the **direct sum**.

Thus the natural homomorphisms of the weak direct sum to  $A'_i$  are onto for every  $i$ . We say that a subring  $S^*$  of the complete direct sum  $S$  is a **subdirect sum** of the rings  $A_i$  if natural homomorphism of  $S^*$  to  $A'_i$

$(a_1, a_2, \dots, a_i, \dots) \rightarrow (0, 0, \dots, 0, a_i, 0, \dots)$  is an onto mapping for every  $i$ .

The subdirect sum has a very useful characterisation in the following result.

### Theorem ([34],p 64)

A ring  $R$  is isomorphic to a subdirect sum of the rings  $A_i$  if and only if  $R$  contains a class of ideals  $\{B_i\}$  such that  $\bigcap_i B_i = 0$  and  $R/B_i \cong A_i$ .

### (c) Radicals

A non-empty class  $R$  of rings is called a radical or a radical class if

( $\alpha$ )  $R$  is homomorphically closed;

( $\beta$ ) every ring  $A$  has an  $R$ -ideal  $R(A)$  which contains every  $R$ -ideal of  $A$ ;

( $\gamma$ ) for every ring  $A$ ,  $\frac{A}{R(A)}$  has no non-zero  $R$ -ideals.

If  $R(A) = A$ ,  $A$  is called an  **$R$ -radical** ring and if  $R(A) = 0$ ,  $A$  is called an  **$R$ -semisimple** ring.

### Examples

(1) The set of nil rings is a radical  $N$ . Here a nil ring is a ring  $A$  such that for each  $x$  in  $A$ ,  $x^n = 0$  for some positive integer  $n$ .

(2) The set of all right quasi-regular rings is a radical  $J$ . A ring  $A$  is called *right quasi regular* if for each  $x \in A$ , there is an element  $y \in A$  such that  $x + y + xy = 0$

Given a non-empty class of rings  $C$ , there is a smallest radical containing  $C$ . It is called the *lower radical determined by  $C$* . and is denoted by  $L_C$ . If  $C$  satisfies the condition: *every non-zero ideal of a non-zero ring in  $C$  can be mapped homomorphically onto a non-zero ring in  $C$* , then the class  $U_C$  of rings  $A$  which cannot be mapped onto a non-zero

ring in  $C$  is radical and is the largest radical with respect to which  $A$  is semisimple.

We now describe a number of methods of forming a new radicals out of two given radicals.

### (i) Join (Sum) and Meet (Product) of two Radicals

Let  $R_1$  and  $R_2$  be two radicals, then the **Join** or the **sum** of  $R_1$  and  $R_2$  is the radical  $L_{R_1 \cup R_2}$  (see Leavitt [60], Majumdar and Dey [49]).

For two radicals  $R_1$  and  $R_2$ , the meet or **the product** of  $R_1$  and  $R_2$  is the radical  $R_1 \cap R_2$  ([55], [43]). This is denoted by  $R_1 \wedge R_2$  or  $R_1 R_2$  and is the largest radical contained in both  $R_1$  and  $R_2$ .

### (ii) Radical Pairs and their likes

For two radicals  $R_1$  and  $R_2$ -Snider [43] defined a radical  $(R_1 : R_2)$ , called the **radical pair**, defined as

(1)  $(R_1 : R_2)$  = The class of all rings  $A$  such that for each ideal  $I$  of  $A$ ,  $R_1(\frac{A}{I}) \supseteq R_2(\frac{A}{I})$ .

Majumdar and Paul [46] generalised this construction and obtained three radicals and called them **radicals similar to radical pairs**. These are :

(2)  $(R_1 ; R_2)$  = The class of all rings  $A$  such that for each ideal  $I$  of  $A$  and each ideal  $J$  of  $I$ ,  $R_1(\frac{I}{J}) \supseteq R_2(\frac{I}{J})$ .

(3)  $(R_1 || R_2)$  = The class of all rings  $A$  such that for each ideal  $I$  of  $A$ ,  $R_1(\frac{A}{I}) = R_2(\frac{A}{I})$ .

- (4)  $(R_1 \ominus R_2) =$  The class of all rings  $A$  such that for each ideal  $I$  of  $A$  and for each ideal  $J$  of  $I$ ,  $R_1(\frac{I}{J}) = R_2(\frac{I}{J})$ .

The classes of rings (1) –(4) are not always radicals. Conditions for which these classes of rings are radicals have been stated and proved in [43] and [46].

#### (d) Lattices

Majumdar and Sultana [ ] defined two kinds of internal and external chain products and also internal and external tag products of lattices. Bae and Lee [ ] define truncated product and Bennet [ ] defined rectangular product of lattices.

We describe :

There are a number of products of lattices. We shall describe here some of them.

#### (i) Direct Product

Let  $L$  and  $K$  be lattices. Consider  $L \times K$ , the Cartesian product of the sets  $L$  and  $K$ . Define two binary operations  $\wedge$  and  $\vee$  on  $L \times K$  by

$$(a, b) \wedge (a', b') = (a \wedge a', b \wedge b'),$$

$$(a, b) \vee (a', b') = (a \vee a', b \vee b').$$

Then  $L \times K$  becomes a lattice under these operations. This lattice is called the direct product of  $L$  and  $K$ . The definition of direct product can be extended to an arbitrary family of lattices.

### Example

The set of all real numbers is a lattice under the usual ordering. If for  $z = a + ib$ ,  $z' = a' + ib'$  ( $a, b, a', b' \in \mathfrak{R}$ ), we define  $z \leq z'$  if either  $a \leq a'$ , or ( $a = a'$  and  $b \leq b'$ ) then  $\mathbb{C}$ , the set of complex numbers is a lattice under this ordering and is isomorphic to the direct product  $\mathfrak{R} \times \mathfrak{R}$ .

### (ii) Free Distributive Product [13]

Let  $L_1$  and  $L_2$  be two disjoint distributive lattices. Then  $Q = L_1 \cup L_2$  is a partially ordered set. A free lattice generated by  $Q$  over the class  $\mathcal{D}$  of all distributive lattices is called a free distributive product  $L_1$  and  $L_2$ .

### (iii) Tensor Product of Distributive Lattices

Let  $A, B$  and  $C$  be distributive lattices. A Function  $f : A \times B \rightarrow C$  is called a bihomomorphism if the functions  $g_a : B \rightarrow C$  defined by  $g_a(b) = f(a, b)$  and  $h_b : A \rightarrow C$  defined by  $h_b(a) = f(a, b)$  are lattice homomorphism for each  $a \in A$  and  $b \in B$ .

Let  $A$  and  $B$  be distributive lattices. A distributive lattice  $C$  is a tensor product of  $A$  and  $B$  if there exists a canonical bihomomorphism  $f : A \times B \rightarrow C$  such that  $C$  is generated by  $f(A \times B)$  and for any distributive and any bihomomorphism  $g : A \times B \rightarrow D$ , there is a homomorphism  $h : C \rightarrow D$  satisfying  $g = hf$ .

### (i) Internal Chain Products [55]

Let  $(L, \leq)$  be a lattice and  $A, B$  be two sublattices of  $L$  such that  $A \not\subset B$ ,  $B \not\subset A$ .

$A \cup B$  will be called the **internal chain product** (first kind) of  $A$  with  $B$  if

- (i)  $A \cap B = \emptyset$ ,
- (ii) for each  $x \in A$  and  $y \in B$ ,  $y \leq x$ .

We denote this product by  $A \cdot B$ . Clearly  $A \cdot B$  is a sublattice of  $L$ .

$A \cup B$  will be called the **internal chain product** ( second kind ) of  $A$  with  $B$  if

- (i)  $A \cap B = C$  (say),  $C \neq \emptyset$ ,
- (ii) for each  $x \in A - B$ ,  $y \in C$  and  $z \in B - A$ ,  $z \leq y \leq x$ .

We denote this product by  $A \odot B$ . Then  $A \odot B$  is a sublattice of  $L$ .

## (ii) External Chain products [55]

Let  $(A, \leq_1)$  and  $(B, \leq_2)$  be two disjoint lattices. Let  $X = A \cup B$  and define a relation  $\leq$  on  $X$  by

- (i) for each  $x, y \in A$ ,  $x \leq y$  if  $x \leq_1 y$ ,
- (ii) for each  $x, y \in B$ ,  $x \leq y$  if  $x \leq_2 y$ ,
- (iii) for each  $x \in B$ ,  $y \in A$ ,  $x \leq y$ .

Then  $\leq$  is a partial ordering of  $X$  and  $(X, \leq)$  is a lattice. We call it the external chain product (First kind) of  $A$  with  $B$  and denoted by  $A * B$ .

Let  $(A, \leq_1)$  and  $(B, \leq_2)$  be two disjoint lattices such that there exists sublattices  $C$  of  $A$  and  $D$  of  $B$  with the properties:

- (i) for each  $x \in A - C$  and for each  $y \in C$ ,  $y \leq_1 x$ ;
- (ii) for each  $u \in B - D$  and for each  $v \in D$ ,  $u \leq_2 v$ ;
- (iii) there is an isomorphism  $f$  from  $C$  onto  $D$ .

Define an equivalence relation  $\sim$  on  $A \cup B$  by the partition

$$\Pi = \{\{x\} \mid x \in (A - C) \cup (B - D)\} \cup \{\{x, f(x)\} \mid x \in C\}.$$

Thus, for each  $x \in (A - C) \cup (B - D)$ ,  $\text{cls } x = \{x\}$ , and for each  $x \in C$ ,  $\text{cls } x = \text{cls } f(x) = \{x, f(x)\}$ . Let  $X = \frac{A \cup B}{\sim}$ . Define a relation  $\leq$  on  $X$  by

(a) for each  $x, y \in A$ ,  $\text{cls } x \leq \text{cls } y$  if  $x \leq_1 y$ ,

(b) for each  $x, y \in B$ ,  $\text{cls } x \leq \text{cls } y$  if  $x \leq_2 y$ ,

(c) for each  $x \in B - D$  and for each  $y \in A - C$ ,  $\text{cls } x \leq \text{cls } y$ .

Then  $\leq$  is a partial ordering on  $X$  and  $(X, \leq)$  is a lattice.

We call  $X$  the **external chain product** (second kind) of  $A$  and  $B$ , and denote it by  $A \odot B$ .

#### (iv) Internal Tag Product [55]

Let  $(L, \leq_L)$  be a lattice and let  $A$  and  $B$  be two distinct bounded sublattices of  $L$  such that

$$(1) A \cap B = \{1_A, 1_B, 0_A, 0_B\}$$

$$(2) 1_A = 1_B, 0_A = 0_B.$$

Define the relation  $\leq$  on  $A \cup B$  by the following:

(i) for each  $a_1, a_2 \in A$ ,  $a_1 \leq a_2$  if  $a_1 \leq_L a_2$ ,

(ii) for each  $b_1, b_2 \in B$ ,  $b_1 \leq b_2$  if  $b_1 \leq_L b_2$

(iii) for each  $a \in A$ ,  $b \in B$ ,  $a \neq 1_A, 0_A$  and  $b \neq 1_B, 0_B \Rightarrow a \leq b, b \leq a$ .

Then  $\leq$  is a partial ordering and  $A \cup B$  is a lattice under  $\leq$ . This will be called the **internal tag product** of  $A$  and  $B$  and will be denoted by  $A \circ B$ . Clearly,  $A \circ B$  is bounded, and  $1_{A \circ B} = 1_A = 1_B, 0_{A \circ B} = 0_A = 0_B$ .



### (v) External Tag Product [55]

Let  $(A, \leq_1)$  and  $(B, \leq_2)$  be two disjoint bounded lattices. Let  $X = A \cup B$ , and let  $\sim$  be the equivalence relation of  $X$  defined by the partition:  $\Pi = \{\{x\} \mid x \neq 1_A, 1_B, 0_A, 0_B\} \cup \{\{1_A, 1_B\}, \{0_A, 0_B\}\}$ . Thus,  $\text{cls } 1_A = \text{cls } 1_B = \{1_A, 1_B\}$ ,  $\text{cls } 0_A = \text{cls } 0_B = \{0_A, 0_B\}$ ,  $\text{cls } x = \{x\}$ ,  $x \neq 1_A, 1_B, 0_A, 0_B$ .

Define a relation  $\leq$  on  $\frac{X}{\sim}$  by the following:

(i) for each  $x, y \in A$ ,  $\text{cls } x \leq \text{cls } y$  if  $x \leq_1 y$

(ii) for each  $x, y \in B$ ,  $\text{cls } x \leq \text{cls } y$  if  $x \leq_2 y$

(iii) for each  $x \in A$ ,  $y \in B$ ,  $x \neq 1_A, 0_A$ , and  $y \neq 1_B, 0_B \Rightarrow \text{cls } x \leq \text{cls } y$ ,  $\text{cls } y \leq \text{cls } x$ .

Then  $\leq$  is a partial ordering and  $\frac{X}{\sim}$  is a lattice under  $\leq$ . It will be called the external tag product of  $A$  and  $B$ .

We denote it by  $A \square B$ .

All lattices  $L$  with 1 to 6 elements are expressed in terms of chain products and tag products as follow:

Number of elements	Lattices
1	$S^1$
2	$S^2$
3	$S^3$
4	$S^4, S^3 \square S^3$
5	$S^5, S^1, (S^3 \square S^3), (S^3 \square S^3), S^1, S^3 \square S^3 \square S^3, S^4 \square S^3, S^6, S^1, (S^4 \square S^3), (S^4 \square S^3), S^1, S^1, (S^3 \square S^3 \square S^3), (S^3 \square S^3 \square S^3), S^1$
6	$S^6, S^1, (S^4 \square S^3), (S^4 \square S^3), S^1, S^1, (S^3 \square S^3 \square S^3), (S^3 \square S^3 \square S^3), S^1$

Majumdar and Sultana [55] defined two kinds of internal and external chain products and also internal and external tag products of lattices. Bae and Lee [63] define truncated product and Bennet [64] defined rectangular product of lattices.

We describe :

There are a number of products of lattices. We shall describe here some of them.

### (i) Direct Product [55]

Let  $L$  and  $K$  be lattices. Consider  $L \times K$ , the Cartesian product of the sets  $L$  and  $K$ . Define two binary operations  $\wedge$  and  $\vee$  on  $L \times K$  by

$$(a, b) \wedge (a', b') = (a \wedge a', b \wedge b'),$$

$$(a, b) \vee (a', b') = (a \vee a', b \vee b').$$

Then  $L \times K$  becomes a lattice under these operations. This lattice is called the direct product of  $L$  and  $K$ . The definition of direct product can be extended to an arbitrary family of lattices.

### Example

The set of all real numbers is a lattice under the usual ordering. If for  $z = a + ib$ ,  $z' = a' + ib'$  ( $a, b, a', b' \in \mathbb{R}$ ), we define  $z \leq z'$  if either  $a \leq a'$ , or ( $a = a'$  and  $b \leq b'$ ) then  $\mathbb{C}$ , the set of complex numbers is a lattice under this ordering and is isomorphic to the direct product  $\mathbb{R} \times \mathbb{R}$ .

### (ii) Free Distributive Product [13]

Let  $L_1$  and  $L_2$  be two disjoint distributive lattices. Then  $Q = L_1 \cup L_2$  is a partially ordered set. A free lattice generated by  $Q$  over the class  $\mathcal{D}$

### (iii) Tensor Product of Distributive Lattices

Let  $A$ ,  $B$  and  $C$  be distributive lattices. A Function  $f : A \times B \rightarrow C$  is called a bihomomorphism if the functions  $g_a : B \rightarrow C$  defined by  $g_a(b) = f(a, b)$  and  $h_b : A \rightarrow C$  defined by  $h_b(a) = f(a, b)$  are lattice homomorphism for each  $a \in A$  and  $b \in B$ .

Let  $A$  and  $B$  be distributive lattices. A distributive lattices  $C$  is a tensor product of  $A$  and  $B$  if there exists a canonical bihomomorphism  $f : A \times B \rightarrow C$  such that  $C$  is generated by  $f(A \times B)$  and for any distributive and any bihomomorphism  $g : A \times B \rightarrow D$ , there is a homomorphism  $h : C \rightarrow D$  satisfying  $g = hf$ .

## 2. Topology

### (i) Product space

Let  $X$  and  $Y$  be two topological space. Consider the Cartesian product  $X \times Y$  together with the topology  $\mathbf{T}$  of generated by all sets of the form  $U \times Y$  and  $X \times V$ , where  $U$  and  $V$  are open sets in  $X$  and  $Y$  respectively. This topological space  $X \times Y$  is called the **product of  $X$  with  $Y$** .

### Example

The Euclidean plane with the usual metric topology is the product of the real line with itself.

### (ii) Sum

Let  $X$  and  $Y$  be two topology spaces such either  $X \cap Y = \phi$  or the topology on  $X \cap Y$  as subspaces of  $X$  and  $Y$  are identical. Then  $X \cup Y$

together with the topology  $\mathbf{T}$  consisting of all subsets of  $X \cup Y$  which are of the form  $G \cup H$ , where  $G$  and  $H$  are open in  $X$  and  $Y$  respectively, has been defined as the sum of  $X$  and  $Y$  and has been denoted by  $X+Y$  in [ ]. If  $X \cap Y = \phi$ , the sum is called the **direct sum** and  $X+Y$  is written  $X \oplus Y$ .

### (iii) Fibre Space

Let  $X, B$  be a topological space, a map  $p: X \rightarrow B$  is called (Serre) fibre map if for each polyhedron  $|K|$ , map  $f_0: |K| \rightarrow X$  and homotopy  $g_t: |K| \rightarrow B$  with  $g_0 = f_0 p$  there is a homotopy  $f_t: |K| \rightarrow X$  with  $g_t = f_t p$ . If we pick a base point  $b_0 \in B$  then  $p^{-1}b_0$  called the fibre (over  $b_0$ ),  $X$  is called the **fibre space** or **total space** and  $B$  the base-space of the fibration.

### Connected Sum

Let  $S_1$  and  $S_2$  be disjoint surfaces. The **connected sum**, denoted by  $S_1 \# S_2$ , is formed by cutting a small circular hole in each surface, and then gluing the two surfaces together along the boundaries of the holes. We choose the subsets  $D_1 \subset S_1$  and  $D_2 \subset S_2$  such that  $D_1$  and  $D_2$  are closed disc (i.e., homeomorphic to  $E^2$ ). Let  $S'_i$  denote the complement of the interior of  $D_i$  in  $S_i$  for  $i = 1$  and  $2$ . Choose a homeomorphism  $h$  of the boundary circle of  $D_1$  onto the boundary of  $D_2$ . Then  $S_1 \# S_2$  is the quotient space of  $S'_1 \cup S'_2$  obtained by identifying the points  $x$  and  $h(x)$  for all points  $x$  in the boundary of  $D_1$ . It is clear that  $S_1 \# S_2$  is a surface.

### (iv) Suspension

For a space  $X$ , the suspension  $SX$  is the quotient of  $X \times I$  obtained by collapsing  $X \times \{0\}$  to one point and  $X \times \{1\}$  to another point. The

motivating example is  $X = S^n$ , when  $SX = S^{n+1}$  with the two 'suspension points' at the north and south poles of  $S^{n+1}$ , the points  $(0, \dots, 0, \pm 1)$ . One can regard  $SX$  as a double cone on  $X$ , the union of two copies of the cone  $CX = (X \times I) / (X \times \{0\})$ . If  $X$  is a CW complex, so are  $SX$  and  $CX$  as quotients of  $X \times I$  with its product cell structure,  $I$  being given the standard cell structure of two 0-cells joined by a 1-cell.

### (v) Join

The cone  $CX$  is the union of all line segments joining points of  $X$  to an external vertex, and similarly the suspension  $SX$  is the union of all line segments joining points of  $X$  to two external vertices. More generally, given  $X$  and a second space  $Y$ , one can define the space of all line segments joining points in  $X$  to points in  $Y$ . This is the join  $X * Y$ , the quotient space of  $X \times Y \times I$  under the identifications  $(x, y_1, 0) \sim (x, y_2, 0)$  and  $(x_1, y, 1) \sim (x_2, y, 1)$ . Thus we are collapsing the subspace  $X \times Y \times \{0\}$  to  $X$  and  $X \times Y \times \{1\}$  to  $Y$ .

### (vi) Wedge Sum

Given spaces  $X$  and  $Y$  with chosen points  $x_0 \in X$  and  $y_0 \in Y$ , then the **wedge sum**  $X \vee Y$  is the quotient of the disjoint union  $X \cup Y$  obtained by identifying  $x_0$  and  $y_0$  to a single point. For example,  $S^1 \vee S^2$  is homeomorphic to the figure '8', two circles touching at a point. More generally one could form the wedge sum  $\bigvee_{\alpha} X_{\alpha}$  of an arbitrary collection of spaces  $X_{\alpha}$  by starting with the disjoint union  $\coprod_{\alpha} X_{\alpha}$  and identifying points  $x_{\alpha} \in X_{\alpha}$  to a single point. In case the spaces  $X_{\alpha}$  are cell complexes and the points  $x_{\alpha}$  are 0-cells, then  $\bigvee_{\alpha} X_{\alpha}$  is a cell complex since it is

obtained from the cell complex  $\Pi_{\alpha} X_{\alpha}$  by collapsing a subcomplex to a point.

### (vii) Smash Product

Inside a product space  $X \times Y$  there are copies of  $X$  and  $Y$ ; namely  $X \times \{y_0\}$  and  $\{x_0\} \times Y$  for points  $x_0 \in X$  and  $y_0 \in Y$ . These two copies of  $X$  and  $Y$  in  $X \times Y$  intersect only at the point  $(x_0, y_0)$ , so their union can be identified with wedge sum  $X \vee Y$ . The **smash product**  $X \wedge Y$  is then defined to be the quotient  $X \times Y / X \vee Y$ . One can think of  $X \wedge Y$  as a reduced version of  $X \times Y$  obtained by collapsing away the parts that are not genuinely a product, the separate factors  $X$  and  $Y$ .

### (vii) CW- complex [1]

An important and useful method of building up complex topological out of simple one is provided by the following procedure. The constituents are called  $n$ -cells and the constituted structure is called a **cell complex** or **CW-complex**.

We describe below the procedure :

- (1) Start with a discrete set  $X^0$ , whose points are regarded as 0-cells.
- (2) Inductively, form the  $n$ -skeleton  $X^n$  from  $X^{n-1}$  by attaching  $n$ -cells  $e_{\alpha}''$  via maps  $\varphi_{\alpha} : S^{n-1} \rightarrow X^{n-1}$ . This means that  $X^n$  is the quotient space of the disjoint union  $X^{n-1} \cup_{\alpha} e_{\alpha}''$  of  $X^{n-1}$  with a collection of  $n$ -disks  $D_{\alpha}''$  under the identifications  $x \sim \varphi_{\alpha}(x)$  for  $x \in \partial D_{\alpha}''$ . Thus a set  $X^n = X^{n-1} \cup_{\alpha} e_{\alpha}''$  where each  $e_{\alpha}''$  is an open  $n$ -disk.

- (3) One can either stop this inductive process at a finite stage, setting  $X = X^n$  for some  $n < \infty$ , or one can continue indefinitely, setting  $X = \bigcup_n X^n$ . In the latter case  $X$  is given the weak topology : A set  $A \subset X$  is open (or closed) if and only if  $A \cap X^n$  is open (or closed) in  $X^n$  for each  $n$ .

### 3. Graphs

Let  $G_1$  and  $G_2$  be graphs with disjoint points sets  $V_1$  and  $V_2$  and live sets  $X_1$  and  $X_2$  respectively.

#### (i) Union

The union  $G = G_1 \cup G_2$  is a graph with  $V = V_1 \cup V_2$  and  $X = X_1 \cup X_2$ .

#### (iii) Join

The join of  $G_1$  and  $G_2$  is  $G_1 + G_2$  is the graph of consisting  $G_1 \cup G_2$  and all lines joining  $V_1$  with  $V_2$ .

#### (iii) Product

The product  $G_1 \times G_2$  has  $V = V_1 \times V_2$  and any two pts  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$  are adjacent in  $G_1 \times G_2$  whenever  $(u_1 = v_1 \text{ and } u_2 \text{ adj } v_2)$  or  $(u_2 = v_2 \text{ and } u_1 \text{ adj } v_1)$ .

#### (iv) Composition

The composition  $G = G_1 [G_2]$  has  $V = V_1 \times V_2$  and  $u = (u_1, u_2)$  is adjacent with  $v = (v_1, v_2)$  wherever  $(u_1 \text{ adj } v_1 \text{ or } u_1 = v_1 \text{ and } u_2 \text{ adj } v_2)$ .

### 4. Categories

A category is a class  $\mathbf{A}$ , together with a class  $M$  which is a disjoint union of the form  $M = \bigcup_{A, B \in \mathbf{A}} [A, B]_{\mathbf{A}}$  (To avoid logical difficulties we

postulate that each  $[A, B]_A$  is a set). For each triple  $A, B, C \in \mathbf{A}$ , we are to have a function from  $[B, C] \times [A, B]$  into  $[A, C]$ . The image of the pair  $(\beta, \alpha)$  under this function will be called the composition of  $\beta$  by  $\alpha$ , and will be denoted by  $\beta\alpha$ . The composition function are subject to axioms.

(i) Associativity : When ever the compositions make sense are have  $(\gamma\beta)\alpha = \gamma(\beta\alpha)$ .

(ii) Existence of identities : For each  $A \in \mathbf{A}$ , we have an element  $1_A \in [A, A]$  such that  $1_A\alpha = \alpha$  and  $\beta 1_A = \beta$  whenever the compositions make sense.

The members of  $\mathbf{A}$  are called objects and the members of  $M$  are called morphisms. If  $\alpha \in [A, B]$  we shall call  $A$  the domains and  $B$  the codomain of  $\alpha$ .  $\alpha$  is called a morphism from  $A$  to  $B$  and this is represented by ' $\alpha : A \rightarrow B$ ' or ' $A \xrightarrow{\alpha} B$ '.

If  $\mathbf{A}$  is a set the category is called small.

### The Nonobjective Approach

A category can also be defined as a class  $M$ , together with a binary operation on  $M$ , called composition, which is not always defined (that is a function from a subclass of  $M \times M$  to  $M$ ). The image of the pair  $(\beta, \alpha)$  under this operation is denoted by  $\beta\alpha$  (if defined). An element  $e \in M$  is called an identity if  $e\alpha = \alpha$  and  $\beta e = \beta$  whenever the compositions make sense. We assume the following axioms:

(i) If either  $(\gamma\beta)\alpha$  or  $\gamma(\beta\alpha)$  is defined, the other is defined, and they are equal.



- (ii) If  $\gamma\beta$  and  $\beta\alpha$  are defined and  $\beta$  is an identity, then  $\gamma\beta$  is defined.
- (iii) Given  $\alpha \in M$ , there are identities and  $e_R$  in  $M$  such that  $e_L\alpha$  and  $\alpha e_R$  are defined (and hence equal  $\alpha$ ).
- (iv) For any pair of identities  $e_L$  and  $e_R$ , the class  $\{\alpha \in M \mid (e_L\alpha)e_R \text{ is defined}\}$  is a set.

## Examples

1. The category  $S$  whose class of objects is the class of all sets, where  $[A, B]_S$  is the class of all functions from  $A$  to  $B$ , is called the category of sets.  $S$  is not small.

2. A similar definition applies to the category  $T$  of all topological spaces where the morphisms from space  $A$  to space  $B$  are the continuous functions from  $A$  to  $B$ .

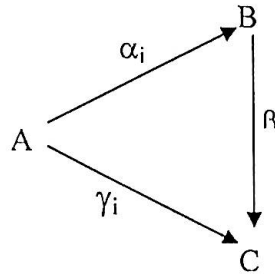
## Special Morphisms

$\theta : A \rightarrow B$  is a coretraction if there exists  $\theta' : B \rightarrow A$  such that  $\theta'\theta = 1_A$ .  $A$  is called a retract of  $B$ . Dually,  $\theta$  is a retraction if there is  $\theta'' : B \rightarrow A$  such that,  $\theta\theta'' = 1_B$ . If  $\theta$  is both a retraction and coretraction then  $\theta$  is called an isomorphism or equivalence. If  $\alpha : A' \rightarrow A$  is a monomorphism,  $A'$  is called a **subobject** of  $A$ .

Dually if  $\alpha : A \rightarrow A'$  is an epimorphism,  $A'$  is called a quotient object of  $A$ .

## Equivalence

A diagram



is called **commutative** if  $\beta\alpha = \gamma$  and we say in this case that  $\gamma$  **factors through**  $\beta$ .

## Pull backs and Push out

Given two morphisms  $\alpha_1 : A_1 \rightarrow A$  and  $\alpha_2 : A_2 \rightarrow A$  with a common codomain, a commutative diagram is called a pull back for  $\alpha_1, \alpha_2$  if for every pair of morphisms  $\beta'_1 : P \rightarrow A_1, \beta'_2 : P \rightarrow A_2$  such that  $\alpha_1 \beta'_1 = \alpha_2 \beta'_2$  there exists a morphism  $\gamma : P \rightarrow P$  such that  $\beta'_1 = \beta_1 \gamma, \beta'_2 = \beta_2 \gamma$ .

## Intersections

If the intersection exists for every set of subobjects of every object in  $\mathbf{A}$  we say that  $\mathbf{A}$  has intersections. If intersection exists only for every finite set of subobjects we say that  $\mathbf{A}$  has a finite intersections.

## Proposition

If  $A_1 \rightarrow A$  and  $A_2 \rightarrow A$  are monomorphisms in a category  $\mathbf{A}$ , then the diagram  $P \rightarrow A_2 \rightarrow A = P \rightarrow A_1 \rightarrow A$  is a pullback if and only if

$P \rightarrow A_2 \rightarrow A = P \rightarrow A_1 \rightarrow A$  is the intersection of  $A_1$  and  $A_2$ . Hence if  $\mathbf{A}$  has pull backs then  $\mathbf{A}$  has finite intersections.

The dual of intersection is cointersection

## Product and Coproduct

Let  $\{A_i\}_{i \in I}$  be a set of objects in an arbitrary category  $A$ . A **product** for the family is a object  $A$  of morphisms  $\{p_i : A \rightarrow A_i\}_{i \in I}$  with the property that for any family  $\{\alpha_i : A' \rightarrow A_i\}_{i \in I}$  there is a unique morphism  $\alpha : A' \rightarrow A$  such that  $p_i \alpha = \alpha_i$ , for all  $i \in I$ . The object  $A$  is denoted by  $\prod_{i \in I} A_i$ . The morphisms  $p_i$  are called the projection morphism from the product. Each  $p_i$  is a retraction. In particular, this will always be true when  $A$  has a zero object. In this case we can define morphisms  $u_i : A_i \rightarrow A$  is called the injection morphisms into the product such that  $p_j u_i = \delta_{ij}$  where  $\delta_{ij} = 0$  for  $i \neq j$  and  $\delta_{ij} = 1_A$ .

The **coproduct** of the family  $\{A_i\}_{i \in I}$  is defined dually to the product. Thus the coproduct is a family of morphisms  $\{u_i : A_i \rightarrow A\}$  called injections such that for each family of morphisms  $\{\alpha_i : A_i \rightarrow A'\}_{i \in I}$  we have a unique morphisms  $\alpha : A \rightarrow A'$  with  $\alpha u_i = \alpha_i$ , for all  $i \in I$ . The object  $A$  is denoted by  $\bigoplus_{i \in I} A_i$ . If  $A$  has a zero object then we can define projections  $p_i : A \rightarrow A_i$  such that  $p_j u_i = \delta_{ij}$ .

Suppose that for all  $i \in I$  we have  $A_i = A$ . In this case we denote the product of the family by  $A^I$  and the coproduct by  ${}^I A$ . We have the diagonal morphism  $\Delta : A \rightarrow A^I$  defined by  $P_i \Delta = 1_A$  for all  $i \in I$ , and dually, the **codiagonal** morphism  $\nabla : {}^I A \rightarrow A$  defined by  $\nabla u_i = 1_A$  for all  $i \in I$ . Then  $\Delta$  is necessarily a monomorphism and  $\nabla$  is necessarily an epimorphism.

In a general category  $A$ , a morphism  $f$  from the coproduct  $\bigoplus_{j \in I} A_j$  to a product  $\prod_{i \in I} B_i$  is completely determined by its coordinate morphisms

$f_{ij} = P_i f \mu_j$ , where  $\mu_j$  is the  $j$ -th injection into the coproduct and  $P_i$  is the  $i$ th projection from the product.

For all  $A' \in A$  the set of morphisms  $[A', A]$  is in 1-1 correspondence with the cartesian product of sets  $\prod_{i \in I} [A', A_i] \left[ \bigoplus_{j \in J} A_j, \prod_{i \in I} B_i \right]$  is in 1-1 correspondence with the set of all  $I \times J$  matrices of the form  $(f_{ij})$  where  $f_{ij} \in [A_j, B_i]$ . We shall frequently denote such a morphism by its corresponding matrix.

In particular, when  $A$  has a zero object, we have the morphism  $\delta = (\delta_{ij})$  :  $\left[ \bigoplus_{j \in I} A_j \rightarrow \prod_{i \in I} A_i \right]$ . If  $\delta$  is an isomorphism then  $\left[ \bigoplus_{j \in I} A_j \right]$  is called a **biproduct**.

## Chapter 2

# Direct Product and Wreath Product of Transformation Semigroups

### Introduction

Here we have defined and studied two kinds of amalgamating two transformation semigroups to obtain a third transformation semigroup. These have been called direct product and wreath product and have been defined as generalisations of the corresponding concepts for transformation groups. Uses of such products have been illustrated, and the associability of both the products as well as the distributivity of wreath product over direct product have been established.

### Definition

Let  $S$  be a semigroup and  $X$  a non-empty set.  $S$  will be called a **transformation semigroup** on  $X$  if there is a mapping  $\phi: S \times X \rightarrow X$ , for which we write  $\phi(s, x) = s(x)$  and which satisfies the condition  $(s_1 s_2)(x) = s_2(s_1(x))$ , for each  $x \in X$  and for each  $s_1, s_2 \in S$ .

If  $S$  is a monoid, i.e., if  $S$  has an identity element  $1$ , then the mapping  $\phi$  is further assumed to satisfy  $1(x) = x$ , for each  $x \in X$ .

Clearly, the semigroup  $E(X)$  of all endomappings of  $X$ , is a transformation semigroup on  $X$ . For every transformation semigroup  $S$  on  $X$ , is a homomorphism  $\psi: S \rightarrow E(X)$  given by  $\psi(s) = f$ , where  $f(x) = s(x)$ . If  $\psi(s_1) = f_1$ ,  $\psi(s_2) = f_2$ , then  $(s_1 s_2)(x) = s_1(s_2(x)) = f_1(f_2(x)) = (f_1 f_2)(x)$ , and so,  $\psi(s_1 s_2) = f_1 f_2$ . Thus,  $\psi$  is indeed a homomorphism.

$E(X)$  will be called the **full transformation semigroup** on  $X$ . If  $X$  is finite, so is  $E(X)$ ; and if the number of elements of  $X$  is  $n$ , the number of elements of  $E(X)$  is  $n^n$ . If  $S$  is a semigroup, then  $\text{End } S$ , the set of all endomorphisms of  $S$ , i.e. the set of all homomorphisms of  $S$  into itself, is a subsemigroup of  $E(S)$  under the composition of maps.

### **Direct Product and Wreath Product of Transformation Semigroups**

Direct product and wreath product of transformation groups are well known and useful concepts (see [ ] ). Using Majumdar's ideas we generalize these concepts of transformation semigroups and give some characterizations and study their properties.

We consider two transformation semigroups  $S_1$  and  $S_2$  on non-empty sets  $X_1$  and  $X_2$  respectively, we shall see how  $S_1$  and  $S_2$  naturally yield transformation semigroups on  $X_1 \cup X_2$  and  $X_1 \times X_2$  respectively.

#### **Definition**

The **direct product** of  $S_1$  and  $S_2$ , written  $S_1 \times S_2$ , is defined as a transformation semigroup on  $X_1 \cup X_2$ , the elements of  $S_1 \times S_2$  being the ordered pairs  $(s_1, s_2)$ ,  $s_1 \in S_1$ ,  $s_2 \in S_2$ , with  $(s_1, s_2)(x_1) = s_1(x_1)$ ,  $(s_1, s_2)(x_2) = s_2(x_2)$ , for each  $x_1 \in X_1$ ,  $x_2 \in X_2$ . The multiplication in  $S_1 \times S_2$  is component-wise. It is easily seen that  $S_1 \times S_2$  is indeed a transformation semigroup. If  $S_1$  and  $S_2$  are finite, the number of elements of  $S_1 \times S_2$  is obviously the product of the numbers of elements of  $S_1$  and  $S_2$ .

#### **Theorem**

If  $S_1, S_2, S_3$  are transformation semigroups on  $X_1, X_2, X_3$ , then  $(S_1 \times S_2) \times S_3 \cong S_1 \times (S_2 \times S_3)$  as transformation semigroup on  $X_1 \cup X_2 \cup X_3$ .

### Proof

Obviously, the map  $((s_1, s_2), s_3) \rightarrow (s_1, (s_2, s_3))$  is an isomorphism of semigroups  $(S_1 \times S_2) \times S_3$  and  $S_1 \times (S_2 \times S_3)$ . To see that is also so as transformation semigroups, we note as a typical case,  $((s_1, s_2), s_3)(x_1) = (s_1, s_2)(x_1) = s_1(x_1)$ , and also,  $(s_1, (s_2, s_3))(x_1) = s_1(x_1)$ .

### Definition

The **wreath product** of  $S_1$  with  $S_2$ , written  $S_1 \wr S_2$  is the transformation semigroup on  $X_1 \times X_2$  consisting of elements  $\theta$  on  $X_1 \times X_2$  which are given by  $\theta : X_1 \times X_2 \rightarrow X_1 \times X_2$  such that,  $\theta(x_1, x_2) = (s_{1,x_2}(x_1), s_2(x_2))$ , with  $s_2$  in  $S_2$  and for each  $s_{1,x_2}$  in  $S_1$ ,  $s_{1,x_2}$  being determined by  $x_2$ .

It follows from the definition that if  $S_1, S_2, X_1, X_2$  are finite, then  $|S_1 \wr S_2| = |S_1|^{|X_2|} \times |S_2|$ , where  $|S_i|$  and  $|X_2|$  denote the numbers of elements of  $S_i$  ( $i=1,2$ ) and  $X_2$  respectively.

An alternative description of the wreath product in terms of direct products is given below:

### Theorem

If  $(S_1, X_1)$  and  $(S_2, X_2)$  are transformation semigroups, then  $(S_1 \wr S_2, X_1 \times X_2) \cong \left( \left( \prod_{x_2 \in X_2} S_{1,x_2} \right) \times S_2, \left( \bigcup_{x_2 \in X_2} X_{1,x_2} \right) \cup X_2 \right)$  where each  $S_{1,x_2} \cong S_1$  and  $X_{1,x_2} = X_1$

### Proof

Define  $\phi : (S_1 \wr S_2, X_1 \times X_2) \rightarrow \left( \prod_{x_2 \in X_2} S_{1,x_2} \times S_2, \bigcup_{x_2 \in X_2} X_{1,x_2} \right)$  by

$\phi(\theta_{(\sigma,\tau)}) = (\{\sigma_{x_2}\}, \tau)$  and define

$$\Psi \left( \times_{x_2 \in X_2} S_1, x_2 \times S_2, \bigcup_{x \in X_2} X_1, x_2 \right) \rightarrow (S_1 \varsigma S_2, X_1 \times X_2) \text{ by}$$

$$\Psi (\{\sigma_{x_2}\}, \tau) = \theta_{(\sigma, \tau)}.$$

It follows from the definitions  $\Phi$  and  $\psi$  are inverses of each other, so that both  $\phi$  and  $\psi$  are 1-1 and onto.

$$\text{Now, } (\theta_{(\sigma, \tau)} \theta'_{(\sigma', \tau')})(x_1, x_2)$$

$$= \theta_{(\sigma, \tau)} (\sigma'_{x_2}(x_1), \tau'(x_2))$$

$$= (\sigma_{\tau'(x_2)} \sigma'_{x_2}(x_1), \tau(\tau'(x_2)))$$

$$= (\sigma_{\tau'(x_2)} \sigma'_{x_2})(x_1), (\tau \tau')(x_2))$$

$$= (\sigma''_{x_2}(x_1), \tau \tau'(x_2))$$

$$= \theta_{(\sigma'', \tau \tau')}(x_1, x_2)$$

$$\text{where } \sigma''_{x_2}(x_1) = (\sigma_{\tau'(x_2)} \sigma'_{x_2})(x_1)$$

Hence

$$(\phi(\theta_{(\sigma, \tau)} \theta'_{(\sigma', \tau')}))(x_1, x_2)$$

$$= \phi(\theta_{(\sigma'', \tau \tau')})(x_1, x_2)$$

$$= (\{\sigma''_{x_2}\}, \tau \tau')(x_1, x_2)$$

$$= \sigma''_{x_2}(x_1), \tau \tau'(x_2)$$

$$= ((\sigma_{\tau'(x_2)} \sigma'_{x_2})(x_1), \tau \tau'(x_2))$$

(1)

Also

$$(\phi(\theta_{(\sigma, \tau)}), \phi(\theta'_{(\sigma', \tau')}))(x_1, x_2)$$



$$\begin{aligned}
&= (\phi(\theta_{(\sigma, \tau)}), (\sigma'_{x_2}(x_1), \tau'(x_2))) \\
&= (\sigma_{\tau'(x_2)}(\sigma'_{x_2}(x_1)), \tau(\tau'(x_2))) \\
&= ((\sigma_{\tau'(x_2)} \sigma'_{x_2}(x_1), \tau\tau'(x_2))) \quad (2)
\end{aligned}$$

Also,

$$\begin{aligned}
&= (\phi(\theta_{(\sigma, \tau)} \theta'_{(\sigma', \tau')}))(x_2) \\
&= (\phi(\theta_{\sigma', \tau'}))(x_2) \\
&= (\tau\tau')(x_2)
\end{aligned}$$

And

$$\begin{aligned}
&= (\phi(\theta_{(\sigma, \tau)} \theta'_{(\sigma', \tau')}))(x_2) \\
&= (\phi(\theta_{(\sigma, \tau)})(\tau'(x_2))) \\
&= \tau(\tau'(x_2)) \\
&= (\tau\tau')(x_2)
\end{aligned}$$

It follows from (1) and (2) that

$$(\phi(\theta_{(\sigma, \tau)} \theta'_{(\sigma', \tau')})) = (\phi(\theta_{(\sigma, \tau)}), \phi(\theta'_{(\sigma', \tau')})).$$

Thus  $\phi$  is a homomorphism.

Therefore  $\phi$  is an isomorphism and

$$(S_1 \wr S_2, X_1 \times X_2) \cong \left( \left( \times_{x_2 \in X_2} S_1, x_2 \times S_2, \bigcup_{x \in X_2} X_1, x_2 \right) \cup X_2 \right).$$

Clearly, this description also holds for wreath products of transformation groups.

Wreath products are very much useful when one has a situation of transformation groups or (resp. transformation semigroups) on a union of

collection of exactly similar classes, or sets with structures. Applications of wreath products of groups and semigroups appear in Majumdar and Sultana [55] in case of lattices and Majumdar [50], Hossain [52] and Majumdar, Hossain and Dey [53] in case of directed graphs specially, directed rooted trees.

We explain and illustrate applications of direct products and wreath products through the following example.

### Example

Let  $X = \{a, b, c, d, e, f\}$ . The full transformation semigroup  $E(X)$  consists of all mappings of  $X$  into itself, i.e., under  $\varphi \in E(X)$ , each of  $a, b, c, d, e, f$  may be mapped onto any one of  $a, b, c, d, e, f$ , of course,  $\varphi$  determines which one will be mapped onto which element. Thus the transformation semigroup  $(E(X), X)$  has  $5^5 = 46656$  such elements, i.e.,  $|E(X)| = 46656$  with  $|X| = 6$ .

Next, write  $X = X_1 \cup X_2$ , where  $X_1 = \{a, b\}$ ,  $X_2 = \{c, d, e, f\}$ . Then, if we consider the semigroups of all those endomappings of  $X$  which map  $X_1$  into  $X_1$  and  $X_2$  into  $X_2$ . Then  $S$  is precisely the direct product  $E(X_1) \times E(X_2)$  transforming  $X_1 \cup X_2$  into itself so each  $X_i$  is mapped into itself. Hence the transformation semigroup  $(S, X) = (E(X_1) \times E(X_2), X_1 \cup X_2)$  has  $2^2 \times 4^4 = 4 \times 256 = 1024$  elements. Thus,  $|E(X_1) \times E(X_2)| = 1024$ , with  $|X| = |X_1 \cup X_2| = 6$ .

Finally, write  $X = E_1 \cup E_2 \cup E_3$ , where  $E_1 = \{a, b\}$ ,  $E_2 = \{c, d\}$  and  $E_3 = \{e, f\}$ . Consider the set of all endomappings of  $X$  which maps all elements of an  $E_i$  into an  $E_j$   $i, j$  not necessarily distinct. They form a subsemigroup  $S'$  of  $E(X)$  is precisely of the form  $E(2) \wr E(3)$ . Here  $E(n)$  is the full transformation semigroup on a set with  $n$  elements. This is

explained as follows. We may write  $X_1 = \{x_{11}, x_{21}\}$ ,  $X_2 = \{x_{12}, x_{22}\}$ ,  $X_3 = \{x_{13}, x_{23}\}$ , i.e., we write  $a = x_{11}$ ,  $b = x_{21}$ ,  $c = x_{12}$ ,  $d = x_{22}$ ,  $e = x_{13}$ ,  $f = x_{23}$ . Again we may write  $Y = \{y_1, y_2\}$  and  $X = Y \times \{1, 2, 3\}$ , so that  $(y_{ij}) = x_{ij}$ ,  $i=1, 2$ ,  $j=1, 2, 3$ . Then  $S'$  is precisely  $E(Y) \subseteq E(\{1, 2, 3\})$ . Because, under the action of the transformations in  $S'$ , each of the  $Y \times \{1\}$ ,  $Y \times \{2\}$  and  $Y \times \{3\}$  is mapped arbitrarily into any of these yielding the semigroup  $E(\{1, 2, 3\})$ , and each such map is to be considered together with independent maps of each of the sets  $Y \times \{j\}$ ,  $j = 1, 2, 3$  into itself, i.e., with a copy of  $E(Y)$ , the total number of required endomappings in  $S'$  is  $|E(Y)|^3 \times |E(\{1, 2, 3\})| = (2^2)^3 \times 3^3 = 4^3 \times 3^3 = 64 \times 27 = 1728$ .

### Remarks

(i) If  $(S_1, X_1)$  and  $(S_2, X_2)$  are transformation semigroups with  $S_2 \{1_{x_2}\}$ , then both  $(S_1 \times S_2, X_1 \cup X_2)$  and  $(S_1 \wr S_2, X_1 \times X_2)$  may be identified with  $(S_1, X_1)$  and  $\prod_{x_2 \in X_2} S_{1, x_2}, X_1$  respectively, ignoring the trivial action of  $S_2$  on  $X_2$ . Thus, in this case,  $S_1 \times S_2 \cong S_1$  and  $S_1 \wr S_2 \cong \left( \prod_{x_2 \in X_2} S_{1, x_2} \right)$  (direct product) as semigroups.

(ii) If  $S_1 \{1_{x_1}\}$ , then both  $(S_1 \times S_2, X_1 \cup X_2)$  and  $(S_1 \wr S_2, X_1 \times X_2)$  may be identified with  $(S_2, X_2)$  since  $s_{1, x_2} = s_{1, x'_2} = 1_{x_2}$ , for each pair of elements  $x_2, x'_2 \in X_2$ .

(iii) If  $S$  and  $S'$  are transformation semigroups on the same set  $X$ , then  $(S_1 \wr S', X \times X)$  may be identified with  $\left( \left( \prod_{x \in X} S_x \right) \times S', \bigcup_{x \in X} X_x \cup X \right)$ . As semigroups,  $S \wr S' \cong \left( \prod_{x \in X} S_x \right) \times S'$ .

If, in particular,  $S=S'$  and  $X$  is finite with  $|X|=n$ , then  $S \wr S \cong \underbrace{S \times S \times S \times \dots \times S \times S}_{(n+1 \text{ copies})}$ .

We shall now establish a result which will show that wreath product is distributive over direct product.

### Theorem

Let  $(S_1, X_1)$ ,  $(S_2, X_2)$  and  $(S_3, X_3)$  be three transformation semigroups. Then  $(S_1 \wr (S_2 \times S_3), X_1 \times (X_2 \cup X_3)) \cong ((S_1 \wr S_2) \times (S_1 \wr S_3), (X_1 \times X_2) \cup (X_1 \times X_3))$

### Proof

Define  $\varphi : (S_1 \wr (S_2 \times S_3), X_1 \times (X_2 \cup X_3)) \rightarrow ((S_1 \wr S_2) \times (S_1 \wr S_3), (X_1 \times X_2) \cup (X_1 \times X_3))$  by  $\varphi(\theta) = (\theta', \theta'')$  (1)

where, if  $\theta(x_1, x_2) = (\sigma_{1,x_2}(x_1), (\sigma_2, \sigma_3)(x_2))$

$$= (\sigma_{1,x_2}(x_1), \sigma_2(x_2))$$

$$\theta(x_1, x_3) = (\sigma_{1,x_3}(x_1), (\sigma_2, \sigma_3)(x_3))$$

$$= (\sigma_{1,x_3}(x_1), \sigma_3(x_3)) \quad (2)$$

$$\text{Then } (\theta', \theta'')(x_1, x_2) = \theta'(x_1, x_2)$$

$$= (\sigma_{1,x_2}(x_1), \sigma_2(x_2)) \quad (3)$$

$$(\theta', \theta'')(x_1, x_3) = \theta''(x_1, x_3)$$

$$= (\sigma_{1,x_3}(x_1), \sigma_3(x_3)) \quad (4)$$

Also define

$$\psi : (S_1 \wr S_2) \times (S_1 \wr S_3), (X_1 \times X_2) \cup (X_1 \times X_3) \rightarrow (S_1 \wr (S_2 \times S_3), X_1 \times (X_2 \cup X_3))$$

by  $\psi(\theta, \theta') = \theta''$

where if  $(\theta, \theta')(x_1, x_2) = \theta'(x_1, x_2)$

$$= (\sigma_{1,x_2}(x_1), \sigma_2(x_2))$$

$$(\theta, \theta')(x_1, x_3) = \theta'(x_1, x_3)$$

$$= (\sigma_{1,x_3}(x_1), \sigma_3(x_3)) \quad (8)$$

then

$$\theta''(x_1, x_2) = (\sigma_{1,x_2}(x_1), (\sigma_2, \sigma_3)(x_2))$$

$$= (\sigma_{1,x_2}(x_1), \sigma_2(x_2)) \quad (9)$$

$$\theta''(x_1, x_3) = (\sigma_{1,x_3}(x_1), (\sigma_2, \sigma_3)(x_3))$$

$$= (\sigma_{1,x_3}(x_1), \sigma_3(x_3)) \quad (10)$$

It follows from (1) --- (10) that

$$\varphi\psi = 1_{((S_1 \wr S_2) \times (S_1 \wr S_3), (X_1 \times X_2) \cup (X_1 \times X_3))}$$

$$\text{and } \psi\varphi = 1_{(S_1 \wr (S_2 \times S_3), X_1 \times (X_2 \cup X_3))}$$

Thus both  $\varphi$  and  $\psi$  are 1-1 and onto.

Now, Let  $\theta, \bar{\theta} \in (S_1 \wr (S_2 \times S_3), X_1 \times (X_2 \cup X_3))$  by given by

$$\theta(x_1, x_2) = \sigma_{1,x_2}(x_1), \sigma_2(x_2))$$

$$\bar{\theta}(x_1, x_3) = \sigma_{1,x_3}(x_1), \sigma_3(x_3))$$

where  $\sigma_2 \in S_2$ ,  $\sigma_3 \in S_3$ , and  $\sigma_{1,x_2}, \sigma_{1,x_3} \in S_1$

the former being determined by  $x_2$  and the latter by  $x_3$ .

and

$$\bar{\theta}(x_1, x_2) = (\overline{\sigma_{1,x_2}}(x_1), \sigma_2(x_2))$$

$$\bar{\theta}(x_1, x_3) = (\overline{\sigma_{1,x_3}}(x_1), \sigma_2(x_3))$$

where  $\bar{\sigma}_2 \in S_2$ ,  $\bar{\sigma}_3 \in S_3$ , and  $\overline{\sigma_{1,x_2}}, \overline{\sigma_{1,x_3}} \in S_1$ , the former being determined by  $x_2$  and the latter by  $x_3$ .

Then

$$(\theta, \bar{\theta})(x_1, x_2) = (\sigma_{1,x_2} \overline{\sigma_{1,x_2}}(x_1), \sigma_2 \bar{\sigma}_2(x_2)),$$

$$(\theta, \bar{\theta})(x_1, x_3) = (\sigma_{1,x_1} \overline{\sigma_{1,x_3}^{(x_1)}}, \sigma_3 \bar{\sigma}_3(x_3))$$

Since

$$\varphi(\theta) = (\theta'_1, \theta'_2) \text{ and } \varphi(\bar{\theta}) = (\bar{\theta}'_1, \bar{\theta}'_2),$$

Where

$$(\theta'_1, \theta'_2)(x_1, x_2) = (\sigma_{1,x_2}(x_1), \sigma_2(x_2)),$$

$$(\theta'_1, \theta'_2)(x_1, x_3) = (\sigma_{1,x_2}(x_1), \sigma_3(x_3)),$$

and

$$(\bar{\theta}'_1, \bar{\theta}'_2)(x_1, x_2) = (\overline{\sigma_{1,x_2}}(x_1), \bar{\sigma}_2(x_2))$$

$$(\bar{\theta}'_1, \bar{\theta}'_2)(x_1, x_3) = (\overline{\sigma_{1,x_3}}(x_1), \bar{\sigma}_3(x_3))$$

$$\text{we have } \varphi(\theta, \bar{\theta}) = \varphi(\theta) \varphi(\bar{\theta}),$$

Hence  $\varphi$  is a homomorphism. Therefore  $\varphi$  is an isomorphism. Thus

$$(S_1 \wr (S_2 \times S_3), X_1 \times (X_2 \cup X_3)) \cong ((S_1 \wr S_2) \times (S_1 \wr S_3), (X_1 \times X_2) \cup (X_1 \times X_3)).$$

## Chapter 3

# Maximal and Minimal Sums and Products

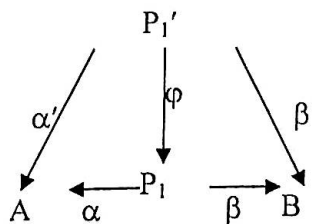
### Introduction

We shall construct from two objects in a category a third object which will be maximal or minimal in some sense. The construction will be done in a number of ways and the resulting object will be termed a kind of sum or product of the given two objects. We shall examine the existence and uniqueness of such products, and in possible cases, give examples in certain particular categories. A brief study of their properties will be made in a few cases. For groups two other kinds of constructions will also be considered.

#### 1. Let $C$ be a category.

##### (a) Definition

(a) Let  $A$  and  $B$  be two objects in  $C$ . An object  $P_1$  will be called a product of the first kind for  $A$  and  $B$  if (i) there exist epics  $P_1 \xrightarrow{\alpha} A$ ,  $P_1 \xrightarrow{\beta} B$  (ii) if  $P'_1$  is any object in  $C$  such that there epics  $P'_1 \xrightarrow{\alpha'} A$ ,  $P'_1 \xrightarrow{\beta'} B$ , then there is an epic  $\varphi: P'_1 \rightarrow P_1$  such that  $\alpha\varphi = \alpha'$ ,  $\beta\varphi = \beta'$ , i.e. the diagram



is commutative.

In general, let  $\{A_i\}$  be a class of objects in  $C$ . An object  $P$  in  $C$  will be called a **product of the first kind** for  $\{A_i\}$ , if (i) for each  $i$ , there exist an epic  $\alpha_i: P \rightarrow A_i$  and (ii) if there is an object  $P'$  and for each  $i$ , there is an epic  $\alpha'_i: P' \rightarrow A_i$ , then there exists an epic  $\varphi: P' \rightarrow P$  such that  $\alpha'_i = \alpha_i \varphi$  for each  $i$ .

If  $\varphi$  is unique, the product  $P$ , if it exists, is unique upto equivalence.

### Examples

- (i) If  $C = S$ , then the category of sets,  $P = \prod_{\alpha} A_{\alpha}$ , the cartesian product.
- (ii) If  $C = G$ , the category of groups, then  $P = \prod_i A_i$ , the restricted direct product of  $A_i$ 's.  $\{\{a_i\} \mid a_i \in A_i\}$  with component wise multiplication with  $\alpha_i(\{a_i\}) = a_i$ , i.e.,  $\alpha_i$  is the projection onto the  $i$ -th component.
- (iii) If  $C = M_R$ , the category of all  $R$ -modules,  $R$  being a ring with 1, then  $P =$  the direct product of the  $A_i$ 's  $= \{\{a_i\} \mid a_i \in A_i\}$ , with component-wise addition and  $R$ -multiplication and  $\alpha_i$  is the projection onto the  $i$ -th component.
- (iv) If  $V_F$  is the category of vector spaces over a field  $F$ , then  $P = \{\{v_i\} \mid v_i \in V_i\}$ , with component wise addition and scalar multiplication with the projection onto the  $i$ -th component.
- (v) If  $C = T =$  the category of all topological spaces, then  $P = \prod_i A_i$ , the product space of the  $A_i$ 's and  $\alpha_i$  is the projection onto the  $i$ -th component.



Let  $\{A_i\}$  and  $\{B_i\}$  be two class of objects in  $C$  with  $P_1$  and  $P'_1$  as their products of the first kind on  $C$ , and let  $f_i : A_i \rightarrow B_i$  be an epic, for each  $i$ . Then, for each  $i$ ,  $f_i \alpha_i : P \rightarrow B_i$  and  $f_i \alpha'_i : P' \rightarrow B_i$  are epics. Hence by the definition of the product there is an epic  $\phi' : P \rightarrow P'$  such that  $\phi' f_i \alpha_i = f_i \alpha'_i$ .

### (b) Definition

If  $C$  is such a category of finite sets/finite groups / finite topological spaces, then the product does not exist for infinite class of objects. If in the above definition we remove the restriction of uniqueness from  $\phi$ , then the product  $P_1$ , so defined will be called a **semiweak product** of the first kind.

### Examples

(i) Every product of the first kind is obviously semi-weak product of the first kind.

### (c) Definition

An object  $P_2$  will called a weak product of the first kind for a class of objects  $\{A_i\}$  if there exist epics  $\alpha_i : P_2 \rightarrow A_i$ , for each  $i$  and an epic  $\phi : P_2' \rightarrow P_2$ .

### Examples

(i) Clearly every product and every semi-weak product of the first kind are weak products of the first kind.

### Comments

(i) It remains to be decided whether the three concepts of a product, a semi weak product and a weak product are all distinct .We are yet to find an example of a semi-weak product which is not a product and an

example of a weak product which is not a semi-weak product. In fact we have not been able to provide an example of a weak product which is not even a product.

Also it remains to be decided whether a product defined here is different from the usual concept of product (an universal object) in categories, the given examples of our product are all products in the usual sense.

(ii) The products in certain categories like  $G$ ,  $M_R$ ,  $V$ ,  $T$  represent objects which are minimal object in same sense.

### Theorem

Let  $\{A_i\}$  be a class of objects. If a product of the first kind exists for  $\{A_i\}$ , then it is unique.

### Proof

Let  $P_1$  and  $P'_1$  be two product of the first kind for  $\{A_i\}$ . Then for each  $i$ , there exist epics  $P_1 \xrightarrow{\alpha_i} A_i$  and  $P'_1 \xrightarrow{\alpha'_i} A_i$  and also unique epics  $P_1 \xrightarrow{\varphi} P'_1$  and  $P'_1 \xrightarrow{\varphi'} P_1$  such that  $\alpha'_i \varphi = \alpha_i$  and  $\alpha_i \varphi' = \alpha'_i$  for each  $i$ . So  $\alpha_i \varphi \varphi' = \alpha_i$  and  $\alpha'_i \varphi' \varphi = \alpha'_i$ , for each  $i$ . It follows that the diagram

$$\begin{array}{ccc}
 P_1 & \xrightarrow[\quad 1_{S_1} \quad]{\varphi' \varphi} & P_1 \\
 \searrow \alpha_i & & \searrow \alpha_i \\
 & A_i &
 \end{array}
 \qquad
 \begin{array}{ccc}
 P'_1 & \xrightarrow[\quad 1_{S'_1} \quad]{\varphi \varphi'} & P'_1 \\
 \searrow \alpha'_i & & \searrow \alpha'_i \\
 & A_i &
 \end{array}$$

are commutative. By the definition of a product of the first kind  $\varphi' \varphi = 1_{P_1}$  and  $\varphi \varphi' = 1_{P'_1}$ . Hence both  $\varphi$  and  $\varphi'$  are equivalence.

## 2. Definition

For a category  $C$ , let  $\{A_i\}$  be a class of objects and let  $S_1$  denote the object in  $C$  such that (i) for each  $i$ , there exists a monic  $A_i \xrightarrow{\alpha_i} S_1$  and if  $S'$  is an object in  $C$  such that there are monics  $A_i \xrightarrow{\alpha'_i} S'$  and if  $S'$  is an object in  $C$  such that there are monics  $S_1 \xrightarrow{\varphi} S'$  such that  $\varphi\alpha_i = \alpha'_i$ .  $S_1$  will be called a **sum of the first kind** for the class  $\{A_i\}$ .

A semi-weak sum and a weak product of  $\{A_i\}$  are defined as the corresponding products as  $S_1$  with the arrows reversed and 'monic' replacing 'epic'.

## Theorem

Let  $\{A_i\}$  be a class of objects. If a sum of the first kind exists, then it is unique.

## Proof

Let  $S_1$  and  $S'_1$  be two products of the first kind for  $\{A_i\}$ . Then for each  $i$ , there exist monics  $A_i \xrightarrow{\alpha_i} S_1$  and  $A_i \xrightarrow{\alpha'_i} S'_1$  also unique monics  $S_1 \xrightarrow{\varphi} S'_1$  and  $S_1 \xrightarrow{\varphi'} S'_1$  such that  $\alpha'_i\varphi = \alpha_i$  and  $\alpha_i\varphi' = \alpha'_i$ . Hence  $\alpha_i\varphi\varphi' = \alpha_i$  and  $\alpha'_i\varphi'\varphi = \alpha'_i$ , for each  $i$ , thus the diagram

$$\begin{array}{ccc}
 S_1 & \xrightarrow{\varphi'\varphi} & S_1 \\
 \alpha_i \searrow & \downarrow S_1 & \swarrow \alpha_i \\
 & A_i &
 \end{array}
 \qquad
 \begin{array}{ccc}
 S'_1 & \xrightarrow{\varphi\varphi'} & S'_1 \\
 \alpha'_i \searrow & \downarrow S'_1 & \swarrow \alpha'_i \\
 & A_i &
 \end{array}$$

are commutative. By the definition of a sum of the first kind  $\varphi'\varphi = 1_{S_1}$  and  $\varphi\varphi' = 1_{S'_1}$ . Hence both  $\varphi$  and  $\varphi'$  are equivalences.

## Examples

- (i) For  $S$ ,  $S_1 = \prod_i A_i$ , the Cartesian product.
- (ii) For  $G$ ,  $S_1 = \sum A_i$ , the restricted direct product of the  $A_i$ 's, i.e.  $\{\{a_i\} \mid a_i \in A_i\}$  with  $A_i$  not equal to the identity element only for a finite number of  $i$ 's and with multiplication defined componentwise.
- (iii) For both  $M_R$  and  $V_F$ ,  $S_1$  is the direct sum of  $\{A_i\}$ .
- (iv) For  $T$ ,  $S_1$  is equal to the product  $P_1$  which is the product space  $\prod_i A_i$ .

3. Let  $C$  be a category. Let  $\{A_i\}$  be a class of objects in  $C$ . An object  $P_2$  of  $C$  will be called a **product of the second kind** for  $\{A_i\}$  if (i) there exist monics  $P_2 \xrightarrow{\alpha_i} A_i$ , and (ii) if there is an object  $P'_2$  and there are monics  $P'_2 \xrightarrow{\alpha'_i} A_i$  then there exists a monic  $P'_2 \xrightarrow{\varphi} P_2$  such that  $\alpha_i \varphi = \alpha'_i$  for each  $i$ .

## Remark

If we replace 'monic' by 'morphism' and demand  $\varphi$  to be unique then, 'the product of the second kind' becomes 'the product' in the usual sense.

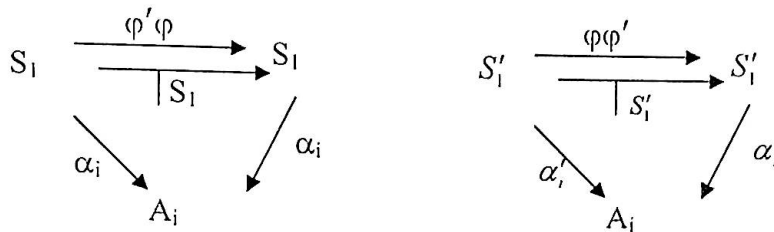
## Theorem

If  $\{A_i\}$  is a class of objects of  $C$  for which a product of the second kind exists, then this product is unique upto equivalence.

## Proof

Let  $P_2$  and  $P'_2$  be two products of the first kind for  $\{A_i\}$ . Then for each  $i$ , there exist monics  $A_i \xrightarrow{\alpha_i} P_2$  and  $A_i \xrightarrow{\alpha'_i} P'_2$  also unique monics

$S_i \xrightarrow{\varphi} S'_i$  and  $S_i \xrightarrow{\varphi'} S'_i$  such that  $\alpha'_i \varphi = \alpha_i$  and  $\alpha_i \varphi' = \alpha'_i$ . Hence  $\alpha_i \varphi \varphi' = \alpha_i$  and  $\alpha'_i \varphi' \varphi = \alpha'_i$ , for each  $i$ , thus the diagram



are commutative. By the definition of  $P_2$  and  $P'_2$  it following that  $\varphi \varphi' = 1_{P_2}$  and  $\varphi' \varphi = 1_{P'_2}$  i.e.  $\varphi$  and  $\varphi'$  are equivalences, since  $\alpha_i$  and  $\alpha'_i$  are monics. Hence both  $\varphi$  and  $\varphi'$  are monics.

### Examples

- (i) For  $S$ ,  $S_1 = \times_i A_i$ , the cartesian product.
- (ii) For  $G$ ,  $S_1 = \sum A_i$ , the restricted direct product of the  $A_i$ 's, i.e.  $\{\{a_i\} \mid a_i \in A_i\}$  with  $A_i$  not equal to the identity element only for a finite number of  $i$ 's and with multiplication defined component wise.
- (iii) For both  $M_R$  and  $V_F$ ,  $S_1$  is the direct sum of  $\{A_i\}$ .
- (iv) For  $T$ ,  $S_1$  is equal to the product  $P_1$  which is the product space  $\prod_i A_i$ .

### Comments

- (i) The product of the second kind does not always exist. For example  $Z_2$  and  $Z$  both are  $Z$ -modules or abelian groups. and there is no product of this second kind for them in  $M_Z = A$ .

- (ii) As known by example (i) above, in certain categories the product of the second kind is a maximal object (a set) which is contained in each of the given objects.
- (iii) The products of the two kinds introduced here are thus dual to each other in some sense.

4. We now consider a second kind of sum in a category. Let  $C$  be a category and  $\{A_i\}$  a class of objects in  $C$ . An object  $S_2$  will be called a product of the second kind for  $\{A_i\}$  if (i) there exists an epic  $A_i \xrightarrow{\alpha_i} S_2$  for each  $i$ , and (ii) if  $S'_2$  is an object such that there is an epic  $A_i \xrightarrow{\alpha'_i} S'_2$ , then there is an epic  $S_1 \xrightarrow{\varphi} S'_2$ , such that  $\alpha_i \varphi = \alpha'_i$ .

### Remark

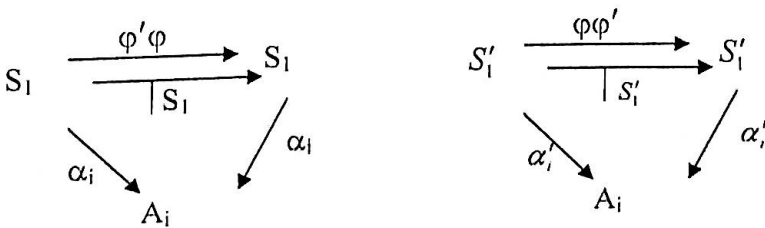
If epic is replaced by morphism and  $\varphi$  is assumed to be unique, the sum of the second kind becomes the usual coproduct.

### Theorem

If  $\{A_i\}$  be a class of objects in a category. If a sum of the second kind exists for  $\{A_i\}$ , then it is unique upto equivalence.

### Proof

Let  $S_2$  and  $S'_2$  be two sums of the second kind for  $\{A_i\}$ . Then for each  $i$ , there exist epics  $A_i \xrightarrow{\alpha_i} S_2$  and  $A_i \xrightarrow{\alpha'_i} S'_2$ , and also there exist epics  $A_i \xrightarrow{\varphi} S'_2$  and  $A_i \xrightarrow{\varphi'} S'_2$  such that  $\varphi \alpha'_i = \alpha_i$  and  $\varphi' \alpha_i = \alpha'_i$  for each  $i$ . Hence  $\varphi' \varphi \alpha_i = \alpha_i = 1_{S_2}$  and  $\varphi \varphi' \alpha'_i = \alpha'_i = 1_{S'_2}$  for each  $i$ , i.e., the diagram



are commutative. By the definition of  $S_2$  and  $S'_2$   $\varphi'\varphi = 1_{S_2}$  and  $\varphi\varphi' = 1_{S'_2}$ , since  $\alpha_i$  and  $\alpha'_i$  are epics. Hence both  $\varphi$  and  $\varphi'$  are equivalences.

### Examples

- (i) If  $A_1 = C_6$  and  $A_2 = S_3$ , then the product of the second kind for  $\{A_1, A_2\}$  is  $C_2$ .
- (ii) If  $C_{m_1}, C_{m_2} \dots C_{m_r}$  are cyclic groups of order  $m_1, m_2, \dots m_r$  with the h.c.f of  $m_1, m_2, \dots m_r = p$  a prime, then  $C_p$  is the sum of the second kind for  $C_{m_1}, C_{m_2} \dots C_{m_r}$ .

**5. (a) We now consider only  $G$ , the category of groups.**

### Definition

For two groups  $A$  and  $B$  we call a **sum of the third kind** for  $A$  and  $B$  if (i)  $Q$  contains  $A$  and  $B$  as subgroups and (ii) no proper subgroup of  $Q$  contains both  $A$  and  $B$ .  $Q$  is thus a minimal group containing  $A, B$  as subgroup.

The definition is similar for an arbitrary class  $\{A_i\}$  replacing  $A$  and  $B$ .

### Comments

- (i) It is easily seen that  $Q$  need not be unique. For if  $A=C_3$  and  $B=C_2$ , the cyclic groups of order 3 and 2 respectively, then both  $S_3$ , the semidirect product of  $A$  and  $B$ , and  $C_6$ , the direct product of  $C_3$  and  $C_2$ , are sums of the third kind for  $A$  and  $B$ .
- (ii) In general, for any two groups  $A$  and  $B$ , their direct product and semidirect product both are sums of the third kind for  $A$  and  $B$ .

(a) The previous examples also show that the product of the second kind for  $A$  and  $B$  need not be abelian even though both  $A$  and  $B$  are abelian.

(iii) If we impose  $A \trianglelefteq Q$  in the definition of the product  $Q$ , the totality of such  $Q$ 's include as particular cases the extensions  $1 \rightarrow A \rightarrow Q \rightarrow B \rightarrow 1$ , and in these cases, all these extensions build up the second cohomology group  $H^2(G, Z)$  with integral coefficients. In such situations the further restriction is  $B \cong \frac{Q}{A}$ .

If  $A, B$  are finite, consideration of the orders of  $A, B$  and  $Q$  proves that such  $Q$  is induced a sum of the third kind for  $A$  and  $B$ .

(iv) The free product of two groups  $A$  and  $B$ , two is a product of the second kind for  $A$  and  $B$ . We thus see that if  $A$  and  $B$  both are finite, a product of the second kind for  $A$  and  $B$  need not be finite.

(b) The corresponding sum in the category of sets is the union.  $A \cup B$  is the sum of second kind for sets  $A$  and  $B$ . For an arbitrary class of sets  $\{A_i\}$ , this sum  $Q$  is  $\bigcup_i A_i$ .

If in  $G$ ,  $A$  and  $B$  are subgroups of a group  $G$ , then the sum  $Q$  of the third kind for  $A$  and  $B$  is the group theoretic union of  $A$  and  $B$ . This means that  $Q$  is the subgroup of  $G$  generated by  $A \cup B$ .  $Q$  for  $\{A_i\}$  where are subgroups of a group, is the subgroup generated by  $\bigcup_i A_i$ .

(c) In the category  $M_R$  and  $V_F$ , the sum of the third kind for  $A$  and  $B$ , is their direct product  $A \oplus B$ . If  $A, B$  are submodules of a module or



subspace of a vector space  $V$ , then their sum  $Q$  is  $A+B$  in  $M_R$  or  $V_F$  respectively.

(d) In the category  $T$ , sum  $Q$  for two disjoint topological spaces  $A$  and  $B$  is their direct sum and if  $A$  and  $B$  are two intersecting topological spaces for which the intersection has the same topology as subspaces of  $A$  and  $B$ , then the sum  $Q$  is the sum  $A+B$  defined in **Chapter 2**. We have discussed its properties there.

(e) In the category of lattices a product of the third kind for a class of lattices  $\{A_i\}$  is a lattice  $S_3$  such that (i) each  $A_i$  is a sublattice of  $S_3$ , and (ii)  $S'_3$  is a lattice containing each  $A_i$  as a sublattice, then  $S_3$  is a sublattice of  $S'_3$ .

### Comment

(i) A sum of the third kind for a class of lattices need not always be unique.

For if  $A_1$  and  $A_2$  both are chains of length 3, then  $A_1$ ,  $A_2$  themselves as well as the lattices are sums of the third kind for  $A_1$  and  $A_2$  and these are non isomorphic lattices.

(ii) If the  $A_i$ 's are sublattices of a given lattice  $L$ , then  $S_3$  for  $\{A_i\}$  is the sublattice of  $L$  generated by  $\bigcup_i A_i$ .

6. (a) Again we consider only the category of groups. Let  $A$  and  $B$  be two groups. Let  $R$  denotes a group which has both  $A$  and  $B$  as factor groups but no proper factor group of  $R$  has this property. We call  $R$  a **product of the third kind** for  $A$  and  $B$ .

The definition for a product of the third kind for an arbitrary class  $\{A_i\}$  of groups is similar.

## Examples

(i) If  $A=C_4$ ,  $B=C_6$ , cyclic groups of order 4 and 6 respectively, then  $C_{12}$  and  $C_4 \times C_6$  both are products of the third kind for  $\{A, B\}$ . Thus a product of the third kind may not always be unique.

(ii) Let  $A_i = C_{p_i}$ ,  $i=1,2, \dots$  then a product of the third kind for  $\{A_i\}$  is the quasi-cyclic group  $(Z_p)$ , the group with generators,  $x_1, x_2, x_3 \dots$   
 $=\{x_n \mid n \in \mathbb{N}\}$  and defining relations.

$$x_1^p = 1, x_1 = x_2^p, x_2 = x_3^p, \dots, x_n = x_{n+1}^p = \{x_1^p = 1, x_n = x_{n+1}^p \mid n \in \mathbb{N}$$

(b) In  $V_F$ , the corresponding product of  $V_1$  and  $V_2$  is  $V_1$ , if  $V_1$  and  $V_2$  are finite dimensional with  $\dim V_1 \geq \dim V_2$ .

(c) (i) We now consider a product of the third kind in  $T$ , defined as in (a). If  $X$  and  $Y$  are topological space none of which is homeomorphic to a quotient space of the other, then  $X \times Y$  is a product of the third kind for  $X$  and  $Y$ .

(ii) If  $X$  is a topological space, then a product of the third kind for  $X^m$  and  $X^n$  is  $X^m$ , if  $m \geq n$ . In particular, for  $A=\mathbb{R}^m$ ,  $B=\mathbb{R}^n$  ( $m \geq n$ ),  $A$  is a product of the third kind for  $A$  and  $B$ .

## Chapter 4

# Semi-periodic Product

### Introduction

Two kinds of products of partially ordered sets have been defined. These have been called semiperiodic product and periodic product. These products for lattices have been studied and a few properties have been established.

### Definition

A partially ordered set  $(X, \leq)$  will be called a **semi-periodic product** of a partially ordered set  $(A, \leq_A)$  with a collection of partially ordered sets  $B = \{B_a, \leq_a\} \mid a \in A$  if (i) there exists an onto order homomorphism  $p: X \rightarrow A$ , such that  $p^{-1}(a) \cong B_a$ , for each  $a \in A$ . (ii) for each  $a \in A$ , there exist order homeomorphisms  $i_a: B_a \rightarrow X$  and  $\pi_a: X \rightarrow B_a$  such that  $\pi_a i_a = 1_{B_a}$  (iii) for each pair  $a, a' \in A$ ,  $a \leq a'$  and  $a \neq a'$  together implies  $x \leq x'$  whenever  $x \in p^{-1}(a)$ ,  $x' \in p^{-1}(a')$  and if  $a = a'$ , then  $x \leq x'$  if  $f_a(x) \leq f_{a'}(x')$   $X$  is denoted by  $A \square B$ .  $X$  will be said to be **semi-periodically ordered**.

### Periodic Product

If each  $B_a$  is order-isomorphic to a partially ordered set  $(B, \leq_B)$ , then  $X$  is said to be periodically ordered. In this case,  $X$  is called the **periodic product** of  $A$  with  $B$  and is denoted by  $A \square B$ .

## Examples

1. Let  $(A, \leq_A)$  be a partially ordered set and  $B = \{B_a\}$  where  $(B_a, \leq_a)$  is a bounded partially ordered set. Let  $X = \{(a, b_a), a \in A, b_a \in B_a\}$  and define  $\leq$  on  $X$  by  $(a, b_a) < (a', b'_{a'})$  if either  $a \leq_A a'$  or  $a = a'$  and  $b_a <_a b'_{a'}$ . Then  $X = A \square B$ . Here  $p: X \rightarrow A$  is given by  $p(a, b_a) = a$ , and  $i(a) = (a, 0_{B_a})$ , for each  $a \in A, b_a \in B_a$  while  $\pi_a: X \rightarrow B_a$  and  $i_a: B_a \rightarrow X$ , are given by  $\pi_a(a, b_a) = b_a, i_a(b_a) = (b_a, a)$  for each  $a \in A, b_a \in B_a$ .

In particular, if each  $B_a$  is order-isomorphic to a partially ordered set  $B$ , then  $A \square B_a = A \square B$  is order isomorphic to the direct product  $A \times B$ . In particular if  $A = [0, 1] = B$ , then  $A \times B$ , the closed unit square in the Euclidean plane is periodically ordered.

Also if  $A = S^1$ , the open unit circle  $= \{(1, \varphi, 0) \mid 0 \leq \varphi < 2\pi\}$  in cylindrical polar coordinates, and  $B = [0, 1]$ , then the  $X = A \times B$  is the vertically half-open right cylindrical surface with a circular base  $S^1$ . In cylindrical polar coordinates  $X = \{(1, \varphi, z) \mid 0 \leq \varphi < 2\pi, 0 \leq z \leq 1\}$ .  $(1, \varphi, z) \leq (1, \varphi', z')$ , if and only if  $z \leq z'$  or  $z = z', \varphi \leq \varphi'$ . After a regular period of  $2\pi$  in the variation of  $\varphi$ ; each point on the cylinder returns to the same vertical line, a fact which motivates the name of the term periodic.

2. Let  $A$  be a totally ordered set, and for each  $a \in A$ , let  $B_a$  be a bounded partially ordered set and let  $B = \{B_a\}_{a \in A}$ . Let  $A$  be such that each  $a \in A$  has an immediate successor  $a'$ , i.e., there is an element  $a' \in A$  such that  $a \neq a', a \leq a'$  and if  $a \leq \bar{a}$ , for some  $\bar{a} \in A$ , then either  $a = \bar{a}$  or  $\bar{a} = a'$ . Let  $X = (\bigcup_{a \in A} B_a)$ , with  $0_{B_a} = 1_{B_a} = a$ , for each  $a \in A$ . Then  $X = A \square B$

,where  $p: X \rightarrow A$ ,  $i: A \rightarrow X$ ,  $\pi_a: X \rightarrow B_a$  and  $i_a: B_a \rightarrow X$  are given by  $p(b_a) = a$ ,  $i(a) = 1_{B_a}$ ,  $\pi(b_a) = b_a$ ,  $i(a) = b_a$ .

As a particular case, we have the following semiperiodic product  $A \square B$  as shown in the figure

Here  $B = \{B_{a_1}, B_{a_1}, B_{a_1}, B_{a_1}, B_{a_1}\}$ ,  $A = \{a_1, a_2, a_3, a_4, a_5\}$ . This example is easily seen to be a lattice obtained from a chain product of  $B_{a_1}, B_{a_1}, B_{a_1}, B_{a_1}, B_{a_1}$  by identifying  $0_{a_i}$  with  $1_{a_{i+1}}$   $i=1,2,\dots,4$ .

3. Let  $X$  be the right circular cone with height  $h$ , semi-vertical angle  $\alpha$  and base the circle  $x^2+y^2=1$ ,  $z=0$  and axis the  $z$ -axis. Then, in Cartesian coordinates,

$$X = \left\{ \left( \frac{h-u}{h} \cos \nu, \frac{h-u}{h} \sin \nu, u \right) \mid 0 \leq u \leq h, 0 \leq \nu \leq 2\pi \right\}.$$

For two points  $P, Q$  with the parametric values  $(u_1, \nu_1), (u_2, \nu_2)$ ,  $P \leq Q$  if  $u_1 \leq u_2$  or if  $u_1 = u_2$  and  $\nu_1 \leq \nu_2$ . Then  $X = A \square B$ . Where  $A$  is the segment of the line  $\frac{x}{1} = \frac{z}{h}$ ,  $y=0$ , between the points  $(1, 0, 0)$  and  $(0, 0, h)$  and  $B$  is the circle  $x^2+y^2=1$ ,  $z=0$ , i.e.,  $B = \{(\cos \nu, \sin \nu, 0) \mid 0 \leq \nu < 2\pi\}$  with the natural ordering  $i: A \rightarrow X$   $p: X \rightarrow A$  are given by  $i(x, y, z) = (x, y, z)$ ,  $p\left(\frac{h-u}{h} \cos \nu, \frac{h-u}{h} \sin \nu, u\right) = (\cos \nu, \sin \nu, 0)$ .

3. Let  $X$  be the segment of a circular helix  $X = \{(\cos t, \sin t, t) \mid 0 \leq t < n, 0 \leq t \leq 2n\pi\}$  winding  $n$  times round the right circular cylinder with base  $x^2+y^2=1$ ,  $z=0$  and axis the  $z$ -axis. Then  $X = A \square B$ , where  $A = \{0, 1, 2, \dots, n-1\}$  (with one point removed) i.e.,  $B = \{(\cos \theta, \sin \theta, 0) \mid 0 \leq \theta < 2\pi\}$  with ordering on  $B$  induced by that on  $\theta$  and the order of  $A$  is natural.

Here  $p(\cos t, \sin t, t) = k$ , if  $t \in [2k\pi, (2k+1)\pi)$  and for  $0 \leq k \leq n-1$  and  $0 \leq t < 2\pi$ ,  $\pi_k(\cos t, \sin t, t) = (\cos t, \sin t, 0)$   $i_k(\cos t, \sin t, t) = (\cos t, \sin t, (2k-1)\pi+t)$ . Hence  $p^{-1}(k) = \{(\cos t, \sin t, t) \mid t \in [2k\pi, (2k+1)\pi)\} \cong \{(\cos t, \sin t, 0) \mid t \in [0, 2\pi)\}$  (order isomorphic)  $= B$ . Also  $\pi_k i_k(\cos t, \sin t, 0) = \pi_k(\cos t, \sin t, (2k-1)\pi+t) = (\cos t, \sin t, 0)$ .  $\therefore \pi_k i_k = 1_B$ .

4. Let  $B = \{a, b, c, d, e, f, g\}$  be the set of seven musical notes, viz., do, re, mi, pha, sol, la, ti in an octave and  $A = \{1, 2, 3, \dots\}$  is the set of numbers representing the audible octaves in the ascending order of the pitch. Then the frequencies of the notes in all the octaves has a natural ascending order. Also  $B$  has a natural ascending order in accordance with their pitches. If we write the note  $x$  in the  $n$ -th octave as  $(x, n)$  then all the audible  $X = \{(x, n) \mid x \in B, n \in A\}$ . Let  $\overline{(x, n)}$  denote the frequency of the note  $(x, n)$ . If we write  $(x, n) \leq (x', n')$  if  $\overline{(x, n)} < \overline{(x', n')}$ ; i.e., if either  $n \leq n'$  or  $n = n'$  and  $n \leq n'$ , then  $X = A \square B$ .

Here  $p(x, n) = n$ , while  $\pi(x, n) = x$ . We may take  $i(x) = (x, n_0)$  for a fixed  $n_0 \in A$ . In particular, we may write  $i(x) = (x, 1)$ . Then  $p^{-1}(n) = \{(a, n), (b, n) \dots (g, n)\} \cong \{a, b, c \dots g\} = B$ . Also  $i(x) = x$ , so that  $\pi i = 1_B$ .

In this example, the periodicity of the product is reflected in the repeated return of the same tunes in the same order as one passes from one octave to another. This is an audio property which does not have a counter part in the visual case, where as the frequency of the light wave goes on increasing or decreasing, the corresponding nature of the colour goes on changing in violet  $\rightarrow$  red direction or red-violet direction without any kind of repetition occurring.

## Properties of Periodic Product

We shall now prove a few results concerning the properties of a semiperiodic and periodic products.

### Theorem 1

If  $(A \leq_A)$  and each  $(B_a, \leq_a)$  are totally ordered,  $A \square B$  is totally ordered.

#### Proof

Let  $x, x' \in X$ . Let  $p(x) = a$  and  $p(x') = a'$ . First suppose  $a \neq a'$ . Since  $A$  is totally ordered either  $a \leq a'$  or  $a' \leq a$ . Suppose  $a \leq a'$ . Then by the definition of a semiperiodic product,  $x \leq x'$ . Next let  $a = a'$ . Then  $f_a(x)$  and  $f_a(x') = f_{a'}(x')$  both belong to  $B_a$ . The latter being totally ordered  $f_a(x) \leq f_a(x')$  or  $f_a(x') \leq f_a(x)$ ,  $f_a$  being an order isomorphism,  $x \leq x'$  or  $x' \leq x$ .

### Corollary 1

If  $A$  and  $B$  are totally ordered sets, so is  $A \square B$

### Theorem 2

If  $(A \leq_A)$  and each  $(B_a, \leq_a)$  are complete lattices, then  $A \square B$  is a complete lattice.

#### Proof

Let  $x, x' \in X$ . Let  $u = p(x) \vee p(x')$ ,  $l = p(x) \wedge p(x')$  and  $u' = \pi(x) \vee \pi(x')$ ,  $l' = \pi(x) \wedge \pi(x')$ . If  $p(x) = p(x')$ , define  $x \vee x' = \inf \pi^{-1}(u')$ ,  $x \wedge x' = \sup \pi^{-1}(l')$ .

Let  $x, x' \in X$ . If  $p(x) = p(x') = a$ , say then  $x, x' \in p^{-1}(a)$ . Hence  $x \vee x'$  and  $x \wedge x'$  are  $f_a^{-1}(f_a(x) \vee f_a(x'))$  and  $f_a^{-1}(f_a(x) \wedge f_a(x'))$ , the latter are

well defined elements of  $X$  and are indeed equal to  $x \vee x'$  and  $x \wedge x'$  respectively since  $B_a$  is a lattice and  $f_a$  is an order-isomorphism. Next let  $p(x) \neq p(x')$ . Then  $x \vee x' = \inf p^{-1}(p(x) \vee p(x'))$  and  $x \wedge x' = \sup p^{-1}(p(x) \wedge p(x'))$ . The existence of the elements on the right hand side follows from the facts that (i)  $A$  is a lattice and that each  $B_a$  is a complete lattice.

It is clear from the nature of  $x \vee x'$  and  $x \wedge x'$  in both the above cases that  $A \square B$  is a complete.

### Corollary 2

If  $A$  and  $B$  are complete lattices so is  $A \square B$ .

### Theorem 3

Let  $A$  and  $B$  each  $B_a$  be bounded complete lattices. Then  $A \square B$  is a bounded complete lattice.

### Proof

We are only to show that the lattice  $A \square B$  is a bounded. Let  $1_A$ ,  $0_A$ ,  $1_{B_a}$  and  $0_{B_a}$  be the greatest and lowest elements in  $A$  and  $B_a$  respectively. Then  $1_{B_{1,A}} = 1_{A \triangleleft B}$  and  $0_{B_{0,A}} = 0_{A \triangleleft B}$ . The truth of this statement follows from the definition of  $A \square B$ .

### Corollary 3

If  $A$  and  $B$  are bounded complete lattices, then so is  $A \square B$ .

### Theorem 4

If  $A$ ,  $B_a$  are complete lattices from each  $a \in A$  and if each  $B_a$  is order isomorphic to  $B$ . Regard  $A$ ,  $B$  and  $A \square B$  as topological spaces with order topology. Then  $A \square B$  is a covering space of  $B$ .



**Proof:**

By Theorem 2,  $A \sqcup B$  is a complete lattice. We recall that the order topology on a partially ordered set  $X$  is the topology generated by all intervals  $(a, b)$  on  $X$ . Here  $a, b \in X$  and  $(a, b) = \{ x \in X \mid a \leq x \leq b, x \neq a, x \neq b \}$ .

We first prove that each of  $A$ ,  $B$  and  $A \sqcup B$  is path-connected. Let  $a_1, a_2 \in A$ . Define  $f: I \rightarrow A$  by  $f(x) = a_1$ , if  $0 \leq x < \frac{1}{2}$

$$= a_1 \vee a_2, \text{ if } x = \frac{1}{2}$$

$$= a_2, \text{ if } \frac{1}{2} < x \leq 1$$

Then  $f$  is continuous, to see this let  $U$  be an open interval in  $A$ . If  $a_1, a_1 \vee a_2 \notin U$ , then  $f^{-1}(U) = \emptyset$ , which is open. If  $a_1 \in U$ ,  $a_1 \vee a_2 \notin U$ , then  $f^{-1}(U) = [0, x)$ , for some  $x < \frac{1}{2}$ . If  $a_2 \in U$ ,  $a_1 \vee a_2 \notin U$ , then  $f^{-1}(U) = (x, 1]$ , for some  $x > \frac{1}{2}$ . If  $a_1, a_1 \vee a_2 \in U$ ,  $a_2 \notin U$  then  $f^{-1}(U) = [0, x]$ , for some  $x$ , such that  $\frac{1}{2} < x < 1$ . If

If  $a_2 \notin U$ ,  $a_1 \vee a_2 \in U$ , then  $f^{-1}(U) = (x, 1]$ , for some  $x$  such that  $0 < x < \frac{1}{2}$ . If  $a_1, a_1 \vee a_2 \in U$ , then  $f^{-1}(U) = [0, 1]$ . Thus, in each case,  $f^{-1}(U)$  is open. Hence  $f$  is continuous. Also  $f(0) = a_1, f(1) = a_2$ . So  $f$  is indeed a path-connected.

Next, for each  $a \in A$ , Let  $f_a: B_a \rightarrow B$  be an order isomorphism. Define  $g: A \sqcup B \rightarrow B$  by  $g(x) = f_a \pi_a(x)$ , for some  $a$ , arbitrary but fixed. Let  $b \in B$  and let  $U_b$  be an open interval in  $B$  such that  $b \in U_b$ . Then  $f_a^{-1}(U_b)$  is an open set  $V_{ab}$  in  $B_a$  which is order-isomorphism to  $U_b$ . Then  $\pi_a^{-1} f_a^{-1}(U) = \pi_a^{-1}(V_{ab})$  is an open set in  $A \sqcup B$ . It is clear that the arc-components of  $\pi_a^{-1}(V_{ab})$  are open intervals in  $A$  and each of these is order isomorphic to  $V_{ab}$ , and hence, to  $U$ . Therefore,  $A \sqcup B$  is a covering space of  $B$ .

## Chapter 5

# On Connected Sums and External Sums of Topological Spaces

### Introduction

Connected sums of surfaces were introduced and applied for classification of compact surfaces by Möbius. Here a similar concept has been introduced for more general situations like metric spaces and topological spaces. Investigation has been made regarding the conservation of various properties of a topological space under connected sum .

1. Given two topological spaces  $X$  and  $Y$  , if is sometimes possible to define a topology on  $X \cup Y$  . Such a space  $X \cup Y$  may be called a sum of  $X$  and  $Y$ . It is obvious that the existence of a sum requires the intersection  $X \cap Y$  to be either  $\emptyset$  or to have the same topology as subspaces of  $X$  and  $Y$  . This may be called the compatibility condition.

Majumdar and Asaduzzaman [50] have considered two spaces  $X$  and  $Y$  satisfying the above condition and called  $X \cup Y$  the **sum** of  $X$  and  $Y$  when the topology on  $X \cup Y$  is precisely  $\{G \cup H \mid G, H \text{ open in } X \text{ and } Y \text{ respectively}\}$  . They denoted it by  $X + Y$  . When  $X \cap Y = \emptyset$  , they called the sum direct sum and denoted it by  $X \oplus Y$  . In the case the expression of an open set in  $X \oplus Y$  as a union of open sets in  $X$  and  $Y$  is unique. They studied some properties of  $X+Y$ .

A concept some what similar to a sum of two spaces occur in Bourbaki [31] and the idea of the direct sum occurs in a different name in Dugunji [61].

When  $X$  and  $Y$  are subspace of a topological space  $Z$ , the natural way of considering  $X$  as a topological space such that  $X$  and  $Y$  are its subspaces would be to regard  $X \cup Y$  as a subspace of  $Z$  i.e., to consider the subspace topology on  $X \cup Y$  with this topology on  $X \cup Y$  we call  $X \cup Y$  the usual or the normal sum of  $X$  and  $Y$

Here we shall make use of the concepts of sum, direct sum and normal sum to define new types of amalgamations of topological spaces. These will be called the **connected sum** and the **external sum**.

2. We shall make the union of two metric spaces into a metric space by defining a suitable metric on the union. This is done in the following theorem.

### Theorem (A)

Let  $(X, d_1)$  and  $(Y, d_2)$  be two metric space such that  $X \cap Y \neq \emptyset$  and let

$$d_1|_{(X \cap Y) \times (X \cap Y)} = d_2|_{(X \cap Y) \times (X \cap Y)}$$

Let us define

$$d : (X \cup Y) \times (X \cup Y) \rightarrow \mathbb{R}$$

$$\text{be given by } d(z, z') = d_1(z, z') \text{ if } z, z' \in X - Y$$

$$= d_2(z, z') \text{ if } z, z' \in X - X$$

$$= \inf \{d_1(z, c) + d_2(c, z') \mid c \in X \cap Y\}, \text{ if } z \in X - Y, z' \in Y - X$$

Then  $d$  is a well-defined metric on  $X \cup Y$ . Since

$$d_1(z, z') = d_2(z, z') \text{ for all } (z, z') \in (X \cap Y) \times (X \cap Y).$$

(i) Now  $d \geq 0$  since  $d_1 \geq 0$  and  $d_2 \geq 0$ .

(ii) From the definition of  $d$ , it is clear that  $d(t, t) = 0$  for all  $t \in X \cup Y$ .

(iii) For any  $t_1, t_2$  in  $X \cup Y$ ,

$$d(t_1, t_2) = d_1(t_1, t_2) = d_1(t_2, t_1), \text{ if } t_1, t_2 \in X - Y,$$

$$= d_2(t_1, t_2) = d_2(t_2, t_1), \text{ if } t_1, t_2 \in Y - X,$$

$$= \inf \{d_1(t_1, c) + d_2(c, t_2) \mid c \in X \cap Y\}, \text{ if } t_1 \in X - Y, t_2 \in Y - X,$$

$$= d(t_2, t_1) \text{ when any two of the } t\text{'s are in } X \text{ one is in } Y.$$

Let  $t_1, t_2 \in X$  and  $t_3 \in Y$  then  $d(t_1, t_2) + d(t_2, t_3) = d_1(t_1, t_2) + \inf \{d_1(t_2, c) + d_2(c, t_3), c \in X \cap Y\} = \inf \{d_1(t_1, t_2) + d_1(t_2, c) + d_2(c, t_3), c \in X \cap Y\} \geq \inf_{c \in X \cap Y} \{d_1(t_1, c) + d_2(c, t_3)\} = d(t_1, t_3).$

i.e.  $d(t_1, t_2) + d(t_2, t_3) \geq d(t_1, t_3).$

Therefore  $d$  is a metric on  $X \cup Y$ .

### 3. Connected Sum of Metric Spaces

#### Definition

Let  $(X, d), (Y, d')$  be two disjoint metric spaces such that the closed sphere

$S = S_r(x_0) = \{x \in X \mid d(x, x_0) \leq r\}$  and  $S' = S_{r'}(y_0) = \{y \in Y \mid d'(y, y_0) \leq r'\}$  are homeomorphic. Let  $f : S \rightarrow S'$  be a homeomorphism. If  $C$  and  $C'$

denote the boundary of  $S$  and  $S'$  respectively, then  $C = \{x \in X \mid d(x, x_0) = r\}$  and  $C' = \{y \in Y \mid d'(y, y_0) = r'\}$ . Also suppose that  $f$  restricted to  $C$  is not only a homomorphism but also an isometry [and so  $d(x_1, x_2) = d(f(x_1), f(x_2))$  for every pair of points  $x_1, x_2$  on  $C$ ] of  $C$  into  $C'$ . Let  $Z = (X - \text{int}(S)) \cup (Y - \text{int}(S'))$ . Define a relation  $\sim$  on  $Z$  as follows :

- (i) for each  $z \in Z - (C \cup C')$ ,  $z \sim z$
- (i) for each  $z \in C$ ,  $z \sim z$  and  $z \sim f(z)$
- (i) for each  $z' \in C'$ ,  $z' \sim z'$  and  $z' \sim f^{-1}(z')$ .

Then  $\sim$  is an equivalence relation on  $Z$ .

Under this identification topology  $\frac{Z}{\sim}$  is termed as the **connected sum** of  $X$  and  $Y$ . Then  $\bar{Z} = \frac{Z}{\sim}$  is thus a topological space. We can regard  $\bar{Z}$  as  $(X \cup Y - (\text{int}(S) \cap \text{int}(S')))$  under the identification  $x$  with  $f(x)$  for all  $x \in C$ . So here we regard  $C = C' = (X - \text{int}(S)) \cap (Y - \text{int}(S'))$ . We shall investigate whether  $\bar{Z}$  inherits a metric from the metric  $d$  and  $d'$  or not.

The connected sum of  $X$  and  $Y$  is denoted by  $X \#_f Y$ .

Let us define a function  $\bar{d}$  on  $\bar{Z}$

$$\begin{aligned} \bar{d}(z_1, z_2) &= d(z_1, z_2) \text{ if } z_1, z_2 \in (X - \text{int } S) - (Y - \text{int } S) \\ &= d'(z_1, z_2) \text{ if } z_1, z_2 \in (Y - \text{int } S) - (X - \text{int } S) \\ &= \inf \{d(z_1, c) + d'(c, z_2) \mid c \in C\}, \text{ if } z_1 \in (X - \text{int } S) - (Y - \text{int } S), \\ &\quad z_2 \in (Y - \text{int } S) - (X - \text{int } S). \end{aligned}$$

Then  $d$  is well-defined, since  $d|_{C \times C} = d'|_{C \times C}$ . Now by Theorem (A)

$d$  is a metric on  $\bar{Z}$ . Thus  $\bar{Z}$  is also a metric space  $(\bar{Z}, d)$  and  $d$  induces the required identification topology on  $\bar{Z}$ .

### Theorem (B)

Let  $(X, d_1)$  and  $(Y, d_2)$  be compact, then  $(\bar{Z}, d) = X \# Y$ , the connected sum of  $X$  and  $Y$  is compact.

### Proof

To prove that  $(\bar{Z}, d)$  is compact, it suffices to show that  $(\bar{Z}, d)$  is sequentially compact. Let  $\{z_n\}$  be a sequence in  $\bar{Z}$ . Then there exist subsequences  $\{Z_{n_i}\}$  and  $\{Z_{n_j}\}$  of  $\{z_n\}$  such that  $\{Z_{n_i}\}$  and  $\{Z_{n_j}\}$  are sequences in  $X$  and  $Y$  respectively. Here at least one of these subsequences, say  $\{Z_{n_i}\}$ , must contain an infinite number of terms of the sequence  $\{z_n\}$ . Since  $X$  is compact,  $\{Z_{n_i}\}$  has a convergent subsequence of  $\{Z_{n_{i_k}}\}$  which is also a convergent subsequence of  $\{z_n\}$ . Thus  $\bar{Z}$  is compact.

## 4. Connected Sum of Topological Spaces

### Definition

Let  $(X, \mathfrak{T})$ ,  $(Y, \mathfrak{T}')$  be two topological spaces such that  $X \cap Y = \emptyset$ . Suppose that there exist nonempty closed sets  $F$  and  $F'$  of  $X$  and  $Y$  respectively such that  $F$  is homeomorphic to  $F'$ . Let  $f : F \rightarrow F'$  be a homeomorphism. Let  $\bar{f} = f|_b(F)$ . Then  $\bar{f}$  is a homeomorphism. Let  $\bar{f} : b(F) (=B) \rightarrow b(F') = B'$ . Let  $Z = (X - \text{int}(F)) \cup (Y - \text{int}(F'))$ . We define a relation  $\sim$  on  $Z$  as follows :

- (i) for each  $z \in Z - (B \cup B')$ ,  $z \sim z$

(ii) for each  $z \in B$ ,  $z \sim z$  and  $z \sim f(z)$

(iii) for each  $z' \in B'$ ,  $z' \sim z'$  and  $z' \sim f^{-1}(z')$ .

Then  $\sim$  is an equivalence relation on  $Z$ .

We write  $\bar{Z} = \frac{Z}{\sim}$  Then  $\bar{Z} = X_1 \cup B \cup Y_1$  where  $X_1 = X - F$ ,  $Y_1 = Y - F'$ , and

$B$  has been identified with  $B'$  via the homeomorphism  $\bar{f}$ . We regard  $\bar{Z} = X_1 \oplus B \oplus Y_1$ , where  $X_1$ ,  $B$  and  $Y_1$  have subspace topologies from  $X$  and  $Y$ .  $\bar{Z}$  will be called **connected sum** of  $X$  and  $Y$  and will be denoted by  $X \#_F Y$ .

If  $X$  and  $Y$  are disjoint subspaces of a topological  $Z$  and the topology on the union  $(X - \text{int } S) \cup (Y - \text{int } S')$  is the subspace topology, then the resulting connected sum will be called the **usual** or the **normal** connected sum. As is well known for the both compact orientable and non-orientable subspaces,

- (i) The connected sum of two spheres is homeomorphic to a sphere.
- (ii) The connected sum of two tori is a sphere with two handles.  
In general the connected sum of  $n$  tori is a sphere with  $n$  handles.

(iii) Any orientable compact surface is either homeomorphic to a sphere or to a connected sum of tori onto a connected sum of projective planes. If  $S_1$  and  $S_2$  are projective planes then  $S_1 \# S_2$  is a Klein Bottle.

Normal connected sum is useful for describing structures of compact surfaces. Also it is of great help for determination of



fundamental groups of surfaces .It is expected that general connected sum will also be of some use in these regards.

### Theorem 1

The connected sum  $X \#_F Y$  is connected if and only if both  $X$ -int  $(F)$  and  $Y$ - int  $(F')$  are connected.

Before proving the theorem, we recall the definition or sum of two topological spaces. Let  $X$  and  $Y$  be topological spaces such that either  $X \cap Y = \emptyset$  or the topology on induced by  $X$  and  $Y$  are the same. Then  $X \cup Y$  is a topological spaces where the topology is  $\{G \cup H \mid G \text{ is open in } X \text{ and } H \text{ is open in } Y\}$ . Here  $X \cup Y$  is called the **sum** of  $X$  and  $Y$  and denoted by  $X+Y$ . We first prove a result due to Majumdar and Asaduzzaman [50].

### Lemma 1

$X+Y$  is connected if and only if both  $X$  and  $Y$  are connected and  $X \cap Y \neq \emptyset$ .

### Proof

First, let  $X \cup Y \neq \emptyset$ . Suppose  $(X \cup Y, T)$  is not connected. Then there exist two nonempty  $T$ -open sets  $G_1$  and  $G_2$  such that  $G_1 \cap G_2 = \emptyset$  and  $X \cup Y = G_1 \cup G_2$ . Now  $X \cap G_1$  and  $X \cap G_2$  are  $T_1$ -open sets and  $(X \cap G_1) \cap (X \cap G_2) = X \cap (G_1 \cap G_2) = \emptyset$  and  $(X \cap G_1) \cup (X \cap G_2) = X \cap (G_1 \cup G_2) = X \cap (X \cup Y) = X$ . Since  $(X, T_1)$  is connected, one of the sets  $X \cap G_1$  and  $X \cap G_2$  must be empty. Let  $X \cap G_1 = \emptyset$  then  $X \cap G_2 = X \Rightarrow X \subseteq G_2$ . Similarly, if  $X \cap G_2 = \emptyset$  then  $X \cap G_1 = X \Rightarrow X \subseteq G_1$ . Thus  $X \subseteq G_1$  or  $X \subseteq G_2$ . Similarly the connectedness of  $(Y, T_2)$   $Y \subseteq G_1$  or  $Y \subseteq G_2$  then  $X \cap Y \subseteq G_1 \cap G_2 = \emptyset$ . Similarly,  $Y \subseteq G_1$  and  $Y \subseteq G_2$  implies  $X \cap Y = \emptyset$ . Hence both  $X$  and  $Y$  are in

$G_1$  or both are in  $G_2$ . If  $X, Y \subseteq G_1$  then  $G_1 \cup G_2 = X \cup Y \subseteq G_1 \Rightarrow G_2 = \Phi$ . Similarly  $X, Y \subseteq G_2$  then  $G_1 = \Phi$ . The contradictions in both the cases prove that  $(X \cup Y, T)$  is connected.

Conversely, if  $X \cap Y = \Phi$  then  $X \cup Y$  is obviously disconnected since  $X$  and  $Y$  are open in  $X \cup Y$ .

### Proof of Theorem 1

We know that  $X \#_F Y = (X - \text{int}(F)) + (Y - \text{int}(F))$  where  $F$  and  $F'$  have been identified via a homeomorphism of  $F$  onto  $F'$ . Since  $(X - \text{int}(F)) \cap (Y - \text{int}(F)) = b(F) \neq \Phi$ , lemma 1 shows that  $X \#_F Y$  is connected if and only if both  $(X - \text{int}(F))$  and  $(Y - \text{int}(F'))$  are connected i.e., if and only if  $(X - \text{int}(F))$  and  $(Y - \text{int}(F'))$  are both connected.

We now prove

### Lemma 2

If  $X$  and  $Y$  are locally connected then the sum  $X+Y$  is locally connected.

### Proof

Let  $z \in X+Y$ . If  $W$  is an open set in  $X+Y$  with  $z \in W$ , then  $W = U \cup V$  with  $U$  open in  $X$  and  $V$  open in  $Y$ . If  $z \in U$ , there exist connected open sets  $U'$  in  $X$  with  $z \in U'$  and  $U' \subseteq U$ . Since  $U$  and  $U'$  are open in  $X+Y$ , the latter is locally connected.

### Lemma 3

If  $X$  is locally connected and  $R$  is an equivalence relation on  $X$ , then the quotient space  $\frac{X}{R}$  is locally connected.

**Proof**

Let  $\pi: X \rightarrow \frac{X}{R}$  denote the mapping given by  $\pi(x) = \text{class } x = [x]$ .

Then  $\pi$  is continuous, open and onto. Let  $x \in X$  and let  $\bar{U}$  be an open set in  $\frac{X}{R}$  such that  $\text{class } x \in \bar{U}$ . Then  $\bar{U} = \pi(U)$ , for some open set  $U$  in  $X$  such that  $x \in U$ . Since  $X$  is locally connected there exists a connected open set  $U'$  in  $X$  such that  $x \in U'$  and  $U' \subseteq U$ . Then  $\pi(U')$  is connected and  $[x] \in \pi(U') \subseteq \bar{U}$ . Hence  $\frac{X}{R}$  is locally connected.

**Theorem 2**

Let  $X$  and  $Y$  be topological spaces and let  $F$  and  $F'$  be closed sets of  $X$  and  $Y$  respectively such that  $F$  is homeomorphic to  $F'$  are locally connected then  $X \#_F Y$  is locally connected if both  $X\text{-int}(F)$  and  $Y\text{-int}(F')$  are locally connected.

**Proof**

Let  $X\text{-int}(F)$  and  $Y\text{-int}(F')$  be locally connected. Then by Lemma 2,  $(X\text{-int}(F)) + (Y\text{-int}(F'))$  is locally connected. By Lemma 3 and the definition of  $X \#_F Y$  it follows that  $X \#_F Y$  is locally connected.

**Theorem 3**

If both  $X$  and  $Y$  are compact, then  $X \#_F Y$  is compact. If  $X \#_F Y$  is compact then both  $X\text{-int}(F)$  and  $Y\text{-int}(F')$  are compact.

**Proof**

Let both  $X$  and  $Y$  be compact. We regard  $X \#_F Y$  as sum  $(X\text{-int } F) + (Y\text{-int } F)$ .  $X\text{-int}(F)$  and  $Y\text{-int}(F)$  being closed subsets of compact spaces  $X$

and  $Y$  respectively, are themselves compact. If  $\{G_\alpha\}$  is an open cover of  $X \#_F Y$ , then  $\{G_\alpha\} = \{G_\beta\} \cup \{G_\gamma\}$  where  $\{G_\beta\}$  is an open cover of  $X\text{-int } F$  and  $\{G_\gamma\}$  is an open cover of  $Y\text{-int}(F)$ . Hence for some positive integers  $m$  and  $n$ , there exists  $G_{\beta_1} \dots G_{\beta_m}$  and  $G_{\gamma_1} \dots G_{\gamma_n}$  such that  $X\text{-int } F \subseteq G_{\beta_1} \dots G_{\beta_m}$  and  $G_{\gamma_1} \dots G_{\gamma_n}$  and  $Y\text{-int } F \subseteq G_{\beta_1} \dots G_{\beta_m}$  and  $G_{\gamma_1} \dots G_{\gamma_n}$ . Hence  $G_{\beta_1} \dots G_{\beta_m}$  and  $G_{\gamma_1} \dots G_{\gamma_n}$  is a finite subcover of  $\{G_\alpha\}$ . Therefore  $X \#_F Y$  is compact.

Now, let  $X \#_F Y$  be compact. Let  $\{G_\alpha\}, \{G'_\beta\}$  be open cover of  $(X\text{-int}(F))$  and  $(Y\text{-int}(F'))$  respectively. Let  $G_\alpha$  and  $G'_\beta$  be the images of  $G_\alpha$  and  $G'_\beta$  in  $X \#_F Y$  under the identification.

We next prove:

#### Lemma 4

Let  $X$  be a locally compact topological space and  $Y$  a closed subspace of  $X$ . Then  $Y$  is locally compact.

#### Proof

Let  $y \in Y$ . Since  $y \in X$  and  $X$  is locally compact there exists  $V$  open in  $X$  such that  $y \in V$ , and  $\bar{V}$  is compact in  $X$ . Then  $V \cap Y$  is open in  $Y$  and  $y \in V \cap Y$ . Let  $\{W_\alpha\}$  be an open cover of  $(V \cap Y)_Y$  in  $Y$ . Then  $\forall_\alpha W_\alpha = U_\alpha \cap Y$ , for some  $U_\alpha$  open in  $X$ . Thus  $\{U_\alpha\}$  is an open cover of  $\overline{(V \cap Y)}_X$  in  $X$ . Since  $\overline{(V \cap Y)}_X$  is a closed subset of  $\bar{V}$  in  $X$ ,  $\overline{(V \cap Y)}_X$  is compact in  $X$ . Hence  $\exists U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}$  such  $\overline{(V \cap Y)}_X \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \dots \cup U_{\alpha_n}$ . So  $\overline{(V \cap Y)}_Y \subseteq W_{\alpha_1} \cup W_{\alpha_2} \cup \dots \cup W_{\alpha_n}$ . Therefore,  $Y$  is locally compact.

We note that  $X \# Y = (X-Y) \cup (Y-X) \cup C$ , and that  $X, Y, C$ , and  $Y-X$  are locally compact by the above lemma, since these are closed subspaces of the locally compact spaces  $X$  and  $Y$ . Hence  $X \#_F Y$  is locally compact here  $C=b(F)$ .

### Lemma 5

If  $X$  is locally compact ,then every quotient space of  $X$  is locally compact .

#### Proof

Let  $X$  be locally compact and let  $R$  be an equivalence relation .Let  $z \in \frac{X}{R}$  and let  $z = \text{cls } x$  ,  $x \in X$  . Since  $X$  is locally compact , there exists an open set  $G$  in  $X$  such that  $x \in X$  and  $\bar{G}$  is compact. Let  $G^*$  is open in  $\frac{X}{R}$ . Let  $\{H_\alpha^*\}$  be an open cover of  $\bar{G}^*$  in  $\frac{X}{R}$ . Let  $H_\alpha$  denote the inverse image of  $H_\alpha^*$  in  $X$ . Then  $\{H_\alpha\}$  is an open cover of  $\bar{G}$  . Since  $\bar{G}$  is compact ,  $\bar{G} \subseteq H_{\alpha_1} \cup \dots \cup H_{\alpha_n}$ . Thus  $\bar{G}^*$  is compact . So  $\frac{X}{R}$  is locally compact .

### Lemma-6

If  $X$  and  $Y$  are locally compact disjoint topological spaces , then  $X \oplus Y$  is locally compact, and conversely.

#### Proof

Let  $X$  and  $Y$  be locally compact disjoint topological spaces. Let  $z \in X \oplus Y$ . Then  $z \in X$  or  $Y$  . Suppose  $z \in X$ . Since  $X$  is locally compact, there is an open subset  $G$  of  $X$  such that  $\bar{G}_X$  is compact in  $X$ . There  $\bar{G}_X$

is the closure of  $G$  in  $x$ . Since  $X \cap Y = \emptyset$ ,  $\bar{G}_X = \bar{G}_{X+Y} = \bar{G}$ , say. Let  $\{W_\alpha\}$  be an open cover of  $\bar{G}$ .  $X \oplus Y$ . Let  $\cup_\alpha W_\alpha \cap X$ . Then  $\{\cup_\alpha\}$  is an open cover of  $\bar{G}$ .  $\bar{G}$  being compact in  $X$  there exists  $V_{\alpha_1} \cup \dots \cup V_{\alpha_n}$ . So  $\bar{G} \subset W_{\alpha_1} \cup \dots \cup W_{\alpha_n}$ . Hence  $\bar{G} = \bar{G}_{X \oplus Y}$  is compact in  $X \oplus Y$ . Thus  $X \oplus Y$  is locally compact. Since  $X$  and  $Y$  are closed subsets of  $X \oplus Y$  both  $X$  and  $Y$  are locally compact.

#### Theorem 4

Let  $X$  and  $Y$  be disjoint topological spaces  $F, F'$ , then  $X \#_F Y$  is locally compact if and only if  $X-F, Y-F'$  and  $b(F)$  are locally compact.

#### Proof

First  $X$  and  $Y$  be locally compact. We identify  $X \#_F Y$  with  $(X-F) \cup b(F) \cup (Y-F) = (X-Y) \cup b(F) \cup (Y-X) = (X-Y) \oplus b(F) \oplus (Y-Y)$ . Since  $(X-Y), (Y-X)$  and  $b(F)$  are closed subsets of  $X$  and  $Y$ , Lemma 4 shows that they are locally compact. Hence, by Lemma 6,  $X \#_F Y$  is compact.

#### Theorem 5

Let  $X$  and  $Y$  be disjoint spaces with homeomorphic closed subsets  $F$  and  $F'$  respectively. If  $X$  and  $Y$  are Hausdorff then  $X \#_F Y$  is Hausdorff. If  $X \#_F Y$  is Hausdorff, then  $X-\text{int}(F)$  and  $Y-\text{int}(F')$  are Hausdorff.

#### Proof

Let  $X$  and  $Y$  be Hausdorff. Using the homeomorphism of  $F$  onto  $F'$  we may identify  $X \#_F Y$  with  $(X-\text{int}(F)) \cup (Y-\text{int}(F'))' = (X-F) \cup (Y-F) \cup b(F) = (X-F) \oplus (Y-F) \oplus b(F)$ . Let  $z_1, z_2 \in X \#_F Y$ . If  $z_1, z_2 \in X$ ,

then there exists, disjoint open sets  $G_1, G_2$  in  $X$  such that  $z_1 \in G_1$ ,  $z_2 \in G_2$ . Let  $G_1 = G_1 \cap (X\text{-int}(F))$ ,  $G_2^* = G_2 \cap (Y\text{-int}(F))$ .

Then  $G_1^*, G_2^*$  are disjoint open sets in  $X\text{-int}(F)$  and  $Y\text{-int}(F)$  and  $z_1 \in G_1^*, z_2 \in G_2^*$ . Let  $\bar{U}_1, \bar{U}_2$  be the images of  $G_1^*, G_2^*$  in  $X\#_F Y$  and contain  $z_1, z_2$  respectively. Hence is Hausdorff.

Now, Let  $X\#_F Y$  be Hausdorff. We write  $X\#_F Y = (X-F) \cup (Y-F) \cup b(F)$ . Let  $z_1, z_2 \in X\text{-int}(F)$ . Regarding  $z_1, z_2$  as elements of  $X\#_F Y$ , we can find disjoint open sets  $G_1, G_2$  in  $X\#_F Y$  which contain  $z_1$  and  $z_2$  respectively.  $G_1, G_2$ , the inverse images of  $G_1$  and  $G_2$  are open in  $X\text{-int}(F)$  and  $Y\text{-int}(F')$  respectively. Let  $\bar{G}_1, \bar{G}_2$  are disjoint open set in  $X\text{-int}(F)$ , and also  $z_1 \in \bar{G}_1, z_2 \in \bar{G}_2$ . Hence  $X\text{-int}(F)$  is Hausdorff. Similarly  $Y\text{-int}(F')$  is Hausdorff.

### Comment

The theorem is still valid if 'Hausdorff' is replaced by either 'normal' or 'regular'. The proofs are similar.

We recall that a metric space  $X$  is called a **Peano space** if  $X$  is compact, connected and locally connected []. Every Peano space is path-connected, and  $X$  is a Peano space if  $X$  is a Hausdorff and is a continuous image of the closed unit interval  $I=[0, 1]$  (see Simmons [15].

### Theorem 6

Let  $X$  and  $Y$  be two metric spaces and let  $F$  and  $F'$  two closed subsets of  $X$  and  $Y$  such that  $F$  and  $F'$  are isometric. Then  $X\#_F Y$  is a Peano space if and only if  $X\text{-int}(F)$  and  $Y\text{-int}(F')$  are Peano spaces.

First suppose  $X\text{-int}(F)$  and  $Y\text{-int}(F')$  be Peano spaces. Then,  $X\text{-int}(F)$  and  $Y\text{-int}(F')$  are connected, compact and locally . We regard  $X\text{-int}(F)$  and  $Y\text{-int}(F')$  are connected, when  $F$  and  $F'$  are identified via the homoeomorphism. So,  $(X\text{-int}(F)) + (Y\text{-int}(F'))$  connected, since  $(X\text{-int}(F)) \cap (Y\text{-int}(F')) = b(F) \neq \phi$ . As a quotient space of  $(X\text{-int}(F)) + (Y\text{-int}(F'))$ ,  $X\#_F Y$  is conceded. Now let  $\{V_\alpha\}$  and  $\{W_\beta\}$  be open covers of  $X\text{-int}(F)$  and  $Y\text{-int}(F')$  in  $X\#_F Y$ . Then  $\{V_\alpha\} \cup \{W_\beta\}$  is an open cover of  $X\#_F Y$ . Since  $X\text{-int}(F)$  and  $(Y\text{-int}(F'))$  are compact,  $\{V_\alpha\}$  and  $\{W_\beta\}$  have finite subcovers  $\{V_{\alpha_1}, \dots, V_{\alpha_m}\}$  and  $\{W_{\beta_1}, \dots, W_{\beta_n}\}$ . Then  $\{V_{\alpha_1}, \dots, V_{\alpha_m}, W_{\beta_1}, \dots, W_{\beta_n}\}$  is a finite subcover of  $\{V_\alpha\} \cup \{W_\beta\}$  for  $X\#_F Y$ , so that  $X\#_F Y$  is compact. By Theorem 2,  $X\#_F Y$  is locally connected. Hence  $X\#_F Y$  is a Peano space.

Conversely, let  $X\#_F Y$  be a Peano space. By Theorem 3, both  $X\text{-int}(F)$  and  $(Y\text{-int}(F'))$  are compact. Since  $X\#_F Y$  is identified with  $(X\text{-int}(F)) + (Y\text{-int}(F'))$ , Lemma 1 shows that both  $(X\text{-int}(F))$  and  $(Y\text{-int}(F'))$  (and hence  $(Y\text{-int}(F'))$ ) are connected. Lemma similarly proves that  $X\text{-int}(F)$  and  $Y\text{-int}(F')$  are locally connected. Hence both  $X\text{-int}(F)$  and  $Y\text{-int}(F')$  are Peano spaces.

The proof is thus complete.

## 5. External Sum

From two topological spaces  $X$  and  $Y$ , we shall now form a third space in a manner similar to the construction of a free product of groups, with amalgamation. This essentially consists in gluing homeomorphic closed subspace of the two spaces.



### Definition

Let  $X$  and  $Y$  be two disjoint topological spaces such that there are non-empty closed subspaces  $F$  and  $F'$  of  $X$  and  $Y$  respectively where  $F$  and  $F'$  are homeomorphic to each other. Let  $f : F \rightarrow F'$  be a homeomorphism. Define a relation  $R$  on  $X \cup Y$  as follows :

- (i) for each  $z \in (X - F) \cup (Y - F')$ ,  $zRz'$ , if and only if  $z = z'$ .
- (ii) for each  $x \in F$ ,  $xRx$ , and  $xRf(x)$ .
- (iii) for each  $y \in F'$ ,  $yRy$ , and  $yRf^{-1}(y)$ .

Consider  $X \cup Y$  as the sum  $X + Y$ . Then  $\frac{X+Y}{R}$  with the quotient topology will be called the **external sum** of  $X$  and  $Y$  and will be denoted by

$X \oplus_F Y$  or  $X \oplus_{F'} Y$ . Clearly, with the identification via the homeomorphism  $f$ ,  $X \oplus_F Y = (X-F) \oplus F \oplus (Y-F)$ .

If  $X$  and  $Y$  are disjoint subspaces of a topological space then we consider the usual or normal sum of  $X$  and  $Y$ , i.e.,  $X \cup Y$  with the subspace topology induced topology by  $Z$ , and then consider its quotient space modulo  $R$ . The resulting space will be called the **usual or normal external sum**.

We now study a few properties of the external sum.

### Theorem

$X \oplus_F Y$  is connected if and only if both  $X$  and  $Y$  are connected.

### Proof

First suppose  $X$  and  $Y$  are connected. Since  $X \oplus_F Y$  may be looked upon (with proper identification) as  $X \cup Y$  with  $X \cap Y = F \neq \emptyset$ , it follows that  $X \oplus_F Y$  is connected.

Conversely, suppose  $X \oplus_F Y$  is connected. If possible, let  $X$  or  $Y$ , say  $X$ , be disconnected. Then there are disjoint open sets  $G$  and  $H$  in  $X$  such that  $X = G \cup H$ . Then, with identification via the homeomorphism  $f: F \rightarrow F'$ ,  $X \oplus_F Y = X \cup Y = X \cup (Y - F) = G \cup H \cup (Y - F)$ . Here  $G, H$  are open sets in  $X$  and  $Y - F$  is an open set in  $Y$ . So by the definition of the topology of a sum and the quotient topology, each of  $G, H$  and  $Y - F$  is open in  $X \oplus_F Y$  and these are pair-wise disjoint. Hence  $X \oplus_F Y$  is disconnected. The contradiction proves that both  $X$  and  $Y$  are connected.

### Theorem

$X \oplus_F Y$  is locally connected if and only if both  $X$  and  $Y$  are locally connected.

### Proof

Let  $X \oplus_F Y$  be locally connected and let  $x \in G$  where  $G$  is open in  $X$ . Since under proper identification,  $X \oplus_F Y = X \cup Y = X + Y$ ,  $G$  is open in  $X \oplus_F Y$ . So, there exists a connected open set  $H$  in  $X \oplus_F Y$  such that  $x \in H \subseteq G$ . Since  $G \subseteq H$ ,  $H \subseteq X$ . Thus,  $H$  is a connected open subset of  $X$ . Hence  $X$  is locally connected. Similarly,  $Y$  is locally connected.

Now suppose both  $X$  and  $Y$  are locally connected. Let  $z \in X \oplus_F Y$ . Regarding  $X \oplus_F Y$  as  $X \cup Y$ , we see that  $z \in X$  or  $z \in Y$ , say  $z \in X$ . Let  $G$  be an open set in  $X \oplus_F Y$  such that  $z \in G$ . Let  $G' = G \cap X$ . Then,  $x \in G'$

and  $G'$  is open in  $X$ . Since  $X$  is locally connected, there is a connected open set  $H'$  in  $X$  such that  $x \in H' \subseteq G'$ . Now  $H'$  is also open in  $X \oplus_F Y = X \cup Y$ , and also,  $H'$  is connected as a subset of  $X \oplus_F Y$ , since a disconnection of  $H'$  in  $X \oplus_F Y$  automatically yields a disconnection of  $H'$  in  $X$ . Hence  $X \oplus_F Y$  is locally connected.

Our next result is

### Theorem

$X \oplus_F Y$  is compact if and only if both  $X$  and  $Y$  are compact.

### Proof

Let  $X \oplus_F Y$  be compact. Let  $\{G_\alpha\}$  be an open cover of  $X$ . If we regard  $X \oplus_F Y$  as  $X \cup Y$  with proper identification, then  $\{G_\alpha\} \cup \{Y\}$  is an open cover of  $X \oplus_F Y$ . Since  $X \oplus_F Y$  is compact for some  $n$ , there exist  $G_{\alpha_1}, \dots, G_{\alpha_n}$  such that  $G_{\alpha_1} \cup \dots \cup G_{\alpha_n} \cup Y = X \oplus_F Y$ . Hence  $X = G_{\alpha_1} \cup \dots \cup G_{\alpha_n} \cup X$ , so that  $X$  is compact. Similarly,  $Y$  is compact.

Conversely, let both  $X$  and  $Y$  be compact. Let  $\{G_\alpha\}$  be an open cover of  $X \oplus_F Y$ . We write  $X \oplus_F Y = X \cup Y$  and let  $H_\alpha = G_\alpha \cap X$  and  $L_\beta = G_\beta \cap Y$ . Then  $\{H_\alpha\}$  and  $\{L_\beta\}$  are open covers of  $X$  and  $Y$  respectively. Since  $X$  and  $Y$  are compact, there exist finite subcovers  $\{H_{\alpha_1}, \dots, H_{\alpha_m}\}$  and  $\{L_{\beta_1}, \dots, L_{\beta_n}\}$  of  $\{H_\alpha\}$  and  $\{L_\beta\}$  respectively. Then  $\{G_{\alpha_1} \cup \dots \cup G_{\alpha_m} \cup G_{\beta_1} \cup \dots \cup G_{\beta_n}\}$  is a finite subcover of  $\{G_\alpha\}$ . Hence  $X \oplus_F Y$  is compact.

### Theorem

Let  $X$  and  $Y$  be two compatible topological spaces. If  $Z$  is a topological space  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  are continuous maps such that  $f(t) = g(t)$ , for each  $t \in X \cap Y$ . Then  $g : X+Y \rightarrow Z$  given by  $h(x) = f(x)$  for each  $x \in X$  and  $h(y) = g(y)$  for each  $y \in Y$ , is continuous .

### Proof

Since  $X$  and  $Y$  are both open subsets of  $X + Y$ , the theorem follows from the pasting lemma.

### Theorem

Let  $X$  and  $Y$  be two disjoint topological spaces with  $F$  and  $F'$  two homeomorphic closed subspaces of  $X$  and  $Y$  respectively with  $\alpha : F \rightarrow F'$  a homeomorphism. Let  $Z$  be a topological space and  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  be continuous maps.

- (i) If  $f(t) = g\alpha(t)$ , for each  $t \in b(F)$ , then there is unique continuous map  $h : X \#_F Y \rightarrow Z$  induced by  $f$  and  $g$  .
- (ii) If  $f(t) = g\alpha(t)$ , for each  $t \in F$ , then there is a unique continuous map  $h : X \oplus_F Y \rightarrow Z$  induced by  $f$  and  $g$ .

### Proof

Both  $X \#_F Y$  and  $X \oplus_F Y$  are quotient spaces of  $X \oplus_F Y$ . Therefore (i) and (ii) follow from the previous theorem. The map  $h$  in that theorem

is easily seen to induce the maps  $h$  of (i) and (ii) because of the given conditions.

We recall the following result on continuous functions :

**Theorem (The Pasting Lemma) (Theorem 7.3 p 108, [61] )**

Let  $X = A \cup B$ , Where  $A$  and  $B$  are closed (or open) in  $X$ . Let  $f : A \rightarrow Y$  and  $g : B \rightarrow Y$  be continuous. If  $f(x) = g(x)$  for every  $x \in A \cap B$ , then  $f$  and  $g$  combine to give a continuous function  $h : X \rightarrow Y$ , defined by setting  $h(x) = f(x)$  if  $x \in A$  and  $h(x) = g(x)$ , if  $x \in B$ .

We used this to prove the following result on sums, connected sums and external sums.

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