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Some Topics in Fuzzy Topological Spaces

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Some Topics in Fuzzy Topological Spaces

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By

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Dedicated

To My Parents

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In fine, I am alone responsible for the errors and shortcomings in this study if there be any, I am sorry for that.

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CERTIFICATE

I have the pleasure in certifying that the M. Phill. thesis entitled "Some Topics in Fuzzy Topological Spaces" submitted by Md. Sahadat Hossain in fulfillment of the requirement for the degree of M. Phill. in Mathematics, University of Rajshahi, Rajshahi, Bangladesh has been completed under my supervision. I believe that the research work is an original one and it has not been submitted elsewhere for any degree.

I wish him a bright future and every success in life.

Supervisor

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STATEMENT OF ORIGINALITY

I declare that the contents in my M. Phill. thesis entitled " Some Topics in Fuzzy Topological spaces " is original and accurate to the best of my knowledge. I also declare that the materials contained in my research work have not been previously published or written by any person for any degree or diploma.

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CONTENTS

Introduction

The concept of fuzzy set was introduced by the American Mathematician L . A . Zadeh for the first time in 1965 . This provides a natural framework for generalizing many algebraic and topological concepts in various directions. Accordingly fuzzy groups , fuzzy ideals , fuzzy rings , fuzzy vector space , fuzzy measure , fuzzy topology , fuzzy topological groups, fuzzy topological vector space and many other branches have been developed all over the world during the last four decades . It is still developing in many directions. While reviewing the literature in fuzzy topology , we have seen the gap in separation axiom of fuzzy topological space , the counter part of which in ordinary topological space drew attention of several eminent mathematicians like Azad , K. K. ; Chang , C. L. ; Ali , D. M. ; Wuyts , P. ; Srivastava , A. K. and Lowen , R. etc. We aim to develop of theories of Fuzzy T_0 , Fuzzy T_1 , Fuzzy T_2 , Fuzzy R_0 , Fuzzy R_1 , Fuzzy Regular and Fuzzy Normal spaces analogous to its counter part in ordinary topology . The material of the thesis has been divided into six chapters , a brief scenario of which we present as follows.

Chapter one incorporates some of the basic definitions and results of fuzzy set, fuzzy topology and mappings. These results are ready references for the work in subsequent chapters . Results are stated without proof and can be seen in the papers referred to .

Our work starts from the second chapter. In the second chapter , we have introduced and studied some R_0 and R_1 properties in fuzzy topological spaces and we have established relations among them . Also we have studied some other properties of these concepts.

In the third chapter , We have introduced and studied some T_0 and T_1 properties in fuzzy topological spaces and we have established relations among them . Also we have studied some other properties of these concepts.

In the fourth chapter, we have introduced and studied some T_2 properties in fuzzy topological spaces and we have established relations among them . Also we have studied some other properties of these concepts.

In the fifth chapter, we have introduced and studied some Regular properties in fuzzy topological spaces and we have established relations among them. Also we have studied some other properties of these concepts.

In the sixth chapter, we have introduced and studied some Normal properties in fuzzy topological spaces and we have established relations among them. Also we have studied some other properties of these concepts.

CHAPTER:-1

Prerequisites

1.1. Introduction: - This chapter incorporates concepts and results of the Fuzzy sets, Fuzzy topological spaces and Fuzzy product topological spaces which are to be used as ready references for understanding the subsequent chapters. Most of the results are quoted from various research papers. Through the sequel, we make use the following notations.

Λ	:	Index set.
$I = [0, 1]$	:	Closed unit interval.
$I_1 = [0, 1)$	:	Right open unit interval.
$I_0 = (0, 1]$	:	Left open unit interval.
$u, v, w, ..$	:	Fuzzy sets.
(X, t)	:	Fuzzy topological space.
(X, T)	:	General topological space.
$\Pi_{i \in \Lambda} X_i$	:	Usual product of X_i.
$(X, t_1 \times t_2)$	:	Product of fuzzy topologies t_1 and t_2 on the set X.

$I_\alpha(t) = \{ u^{-1}(\alpha, t] : u \in t \}, \alpha \in I_1$ General topology on X.

1.2. This thesis deals with the study of fuzzy topological spaces. To present our work in a systematic way in this thesis, we consider in this chapter, various concepts and results on fuzzy sets and fuzzy topological spaces scattered in various research papers. For this we start with .

1.2.1. Definition:- Let X be a nonempty set and A be a subset of X. The function $1_A : X \longrightarrow [0, 1]$ defined by

$$1_A(x) = 1 \quad \text{if } x \in A$$
$$= 0 \quad \text{if } x \notin A$$

is called the characteristic function of A.

1.2.2. Definition [94]:- A function u from X into the unit interval I is called a fuzzy set in X. For every $x \in X$, $u(x) \in I$ is called the grade of membership (g.m.f) of x in u . Some authors say that u is a fuzzy subset of X instead of saying that u is a fuzzy set in X.

1.2.3. Definition [56]:- A fuzzy subset is empty if and only if grade of membership is identically zero in X. It is denoted by 0.

1.2.4. Definition [56] :- A fuzzy subset is whole if and only if its grade of member ship is identically one in X. It is denoted by 1.

1.2.5. Definition [94] :- Let X be a set and u and v be two fuzzy subsets of X. Then u is said to be subset of v if $u(x) \leq v(x)$ for every $x \in X$. It is denoted by $u \subseteq v$.

1.2.6. Definition [94] :- Let X be a set and u and v be two fuzzy subsets of X. Then u is said to be equal to v if and only if $u(x) = v(x)$ for every $x \in X$. It is denoted by $u = v$.

1.2.7. Definition [94] :- Let X be a set and u and v be two fuzzy subsets of X. Then u is said to be the complement of v if $v(x) = 1 - u(x)$, for every $x \in X$. It is denoted by u^c. Obviously $(u^c)^c = u$.

1.2.8. Definition [15] :- Let X be a set and u and v be two fuzzy subsets of X . Then the union w of u and v is a fuzzy subset of X , written as $w = u \cup v$ which is defined by

$$w(x) = (u \cup v)(x) = \max \{ u(x) , v(x) \} \text{ for every } x \in X.$$

In general, if Λ be an index set and $A = \{ u_i : i \in \Lambda \}$ be a family of fuzzy sets of X then the union $\cup u_i$ is defined by

$$(\cup u_i)(x) = \sup \{ u_i(x) : i \in \Lambda \}, \ x \in X.$$

1.2.9. Definition [15] :- Let X be a set and u and v be two fuzzy subsets of X. Then the intersection m of u and v is a fuzzy subset of X, written as $m(x) = (u \cap v)(x) = \min \{u(x), v(x)\}, \forall x \in X$, and $(\cap u_i)(x) = \inf \{u_i(x) : i \in \Lambda\}, x \in X$, where $\{ u_i, i \in \Lambda \}$.

1.2.10. Definition :- Let X be a set u and v be two fuzzy subsets of X. Then the difference of u and v is defined by $u - v = u \cap v^c$.

Laws of the algebra of fuzzy sets :

As in ordinary set theory, idempotent laws, associative law, commutative law, distributive laws, identity law, demorgan's laws hold in the case of fuzzy sets also. But the complement laws are not necessarily true. For example, if $X = \{ a, b, c \}$ and u is a fuzzy subset of X where is defined by

$u = \{ (a, .2), (b, .7), (c, 1) \}$,

then $u^c = \{ (a, .8), (b, .3), (c, 0) \}$

so $u \cup u^c = \{ (a, .8), (b, .7), (c, 1) \} \neq 1$,

$u \cap u^c = \{ (a, .2), (b, .3), (c, 0) \} \neq 0$.

Also in ordinary set theory $U \cap V = \phi$ iff $U \subset V^c$. But in fuzzy subsets reverse is not necessary true. For example if

$v = \{ (a, .6), (b, .2), (c, 0) \}$ then $u \subset v^c$,

$u \cap v = \{ (a, .2), (b, .2), (c, 0) \} \neq 0$.

1.3. Mapping and fuzzy subsets induced by mappings,

1.3.1. Definition [15] :- Let f be a mapping from a set X into a set Y and u be a fuzzy subset of X. Then f and u induced a fuzzy subset v of Y defined by

$$v(y) = \sup \{ u(x) \} \ \text{if} \ f^{-1}[\{y\}] \neq \phi, \ x \in X$$
$$= 0 \ \text{otherwise}.$$

1.3.2. Definition [15] :- Let f be a mapping from a set X into Y and v be a fuzzy subset of Y. Then the inverse of v written as $f^{-1}(v)$ is a fuzzy subset of X and is defined by

$$f^{-1}(v)(x) = v(f(x)), \text{ for } x \in X.$$

We now mention some properties of fuzzy subsets induced by mappings.

Let f be a mapping from X into Y, u be a fuzzy subset of X and v be a fuzzy subset of Y. Then the following properties are true **[15]**.

(a) $f^{-1}(v^c) = (f^{-1}(v))^c$ for any fuzzy subset v of Y.

(b) $f(u^c) = (f(u))^c$ for any fuzzy subset u of X.

(c) $v_1 \subset v_2 \Rightarrow f^{-1}(v_1) \subset f^{-1}(v_2)$, where v_1 and v_2 are two fuzzy subsets of Y.

(d) $u_1 \subset u_2 \Rightarrow f(u_1) \subset f(u_2)$, where u_1 and u_2 are two fuzzy subsets of X.

(e) $v \supset f(f^{-1}(v))$, for any fuzzy subset v of Y.

(f) $u \subset f^{-1}(f(u))$, for any fuzzy subset u of X.

(g) Let f be a function from X into Y and g be a function from Y into Z. Then $(g_\circ f)^{-1}(w) = f^{-1}(g^{-1}(w))$, for any fuzzy subset w in Z, where $(g_\circ f)$ is the composition of g and f.

1.3.3. Definition [56] :- A fuzzy point in X is a special type of fuzzy set in X with membership function $p(x) = r$, $p(y) = 0$, $\forall\, y \neq x$, where $0 < r < 1$. This fuzzy point is said to have support x and value r and this point is denoted by x_r or ${}_r1_x$.

1.3.4. Definition [56] :- A fuzzy point p is said to belong a fuzzy set u in X ($p \in u$) if and only if $p(x) < u(x)$ and $p(y) \leq u(y)$ $\forall\; y \neq x$. Evidently, every fuzzy set u can be expressed as the union of all the fuzzy points which belong to u.

1.3.5. Definition [56] :- Two fuzzy sets u and v in X are said to be intersected if and only if there exist a point $x \in X$ such that $(u \cap v)(x) \neq 0$. In this case we say that u and v intersect at x.

1.3.6. Definition [89] :- Let $I = [0, 1]$, X be a non empty set, and I^X be the collection of all mappings from X into I, ie the class of all fuzzy sets in X. A fuzzy topology on X is defined as a family t of members of I^X, satisfying the following conditions.

(i) $1, 0 \in t$,

(ii) If $u_i \in t$ for each $i \in \Lambda$, then $\cup_{i \in \Lambda} u_i \in t$.

(iii) If $u_1, u_2 \in t$ then $u_1 \cap u_2 \in t$.

The pair (X, t) is called a fuzzy topological space (fts, in short) and members of t are called t- open (or simply open) fuzzy sets. A fuzzy set v is called a t- closed (or closed) fuzzy set if $1 - v \in t$.

1.3.7. Example :- Let $X = \{a, b, c, d\}$, $t = \{0, 1, u, v\}$,

where $1 = \{(a, 1), (b, 1), (c, 1), (d, 1)\}$

$0 = \{(a, 0), (b, 0), (c, 0), (d, 0)\}$

$u = \{ (a, .2), (b, .5), (c, .7), (d, .9) \}$

$v = \{ (a, .3), (b, .5), (c, .8), (d, .95) \}$

Then (X , t) is a fuzzy topological space .

1.3.8. Definition [50] :- (According to lowen) A fuzzy topology on non empty set X is a collection t of fuzzy subsets of X such that

(**i**) all constant fuzzy subsets of X belong to t .

(**ii**) t is closed under formation of fuzzy union of arbitrary collection of members of t .

(**iii**) t is closed under formation of intersection of finite collection of members of t.

1.3.9. Definition [56] :- Let u be a fuzzy set in (X , t) . The interior of u is defined as the union of all t – open sets contained in u . It is denoted by u^{o}. Evidently u^{o} is the largest open fuzzy set contained in u and $(u^{o})^{o} = u^{o}$.

1.3.10. Definition [56] :- The intersection of all the t – closed set containing u is called the closure of u denoted by $\bar{u}$. Obviously $\bar{u}$ is the smallest closed set containing u and $\bar{\bar{u}} = \bar{u}$.

1.3.11. Definition [84] :- A fuzzy set n in a fts (X , t) is called a neighborhood of a point $x \in X$, if and only if there exit $u \in t$ such that $u \subseteq n$ and $u(x) = n(x) > 0$.

1.3.12. Example : - Let us consider the example 1.3.7 ,

and $n = \{ (a, .5), (b, .6), (c, .8), (d, .9) \}$. Now n is a neighborhood of $d \in X$. Since $u \in t$ such that $u \subseteq n$ and $u(d) = n(d) > 0$, n is a neighborhood of d . Similarly

$n_1 = \{ (a, .7), (b, .6), (c, .75), (d, .9) \}$ is a neighborhood of d . we denoted the family of all neighborhoods of x by N_x .

1.3.13. Definition [56] :-A fuzzy set u in a fts (X, t) is called a neighborhood of a fuzzy point x_r if and only if there exist a fuzzy set $u_1 \in t$ such that $x_r \in u_1 \subseteq u$. A neighborhood u is called an open neighborhood if u is open . The family consisting of all the neighborhoods of x_r is called the system of x_r .

1.4. Continuous map , open and closed map.

1.4.1. Definition [57] :- The function $f : (X, t) \longrightarrow (Y, s)$ is called fuzzy continuous if and only if for every $v \in s$, $f^{-1}(v) \in t$, the function f is called fuzzy homeomerphic if and only if f is bijective and both f and f^{-1} are fuzzy continuous.

1.4.2. Definition [53] :- The function $f : (X, t) \longrightarrow (Y, s)$ is called fuzzy open if and only if for each open fuzzy set u in (X, t) , $f(u)$ is open fuzzy set in (Y, s) .

1.4.3. Definition [57] :- The function $f : (X, t) \longrightarrow (Y, s)$ is called fuzzy closed if and only if for each closed fuzzy set u in (X, t), $f(u)$ is closed fuzzy set in (Y, s).

1.4.4. Proposition ([57] Theorem 1.1) :- Let $f : (X, t) \longrightarrow (Y, s)$ be a fuzzy continuous function , then the following properties hold :

(i) For every s – closed v , $f^{-1}(v)$ is t – closed.

(ii) For each fuzzy point p in X and each neighborhood u of f(u), then there exist a neighborhood v of p such that $f(v) = u$.

(iii) For any fuzzy set u in X, $f(\overline{u}) \subset (\overline{f(u)})$.

(iv) For any fuzzy set v in Y, $(\overline{f^{-1}(v)}) \subset f^{-1}(\overline{v})$.

1.4.5. Proposition ([53] Theorem 3.1) :- Let $f : (X, t) \longrightarrow (Y, s)$ be a fuzzy open function , then the following properties hold:

(i) $f(u^{\circ}) \subseteq (f(u))^{\circ}$, for each fuzzy set u in X .

(ii) $(f^{-1}(v))^{\circ} \subseteq f^{-1}(v^{\circ})$, for each fuzzy set v in Y.

1.4.6. Proposition ([53] Theorem 1.5) :- Let $f : (X, t) \longrightarrow (Y, s)$ be a function. Then f is closed if and only if $\overline{f(u)} \subseteq f(\overline{u})$ for each fuzzy set u in X.

1.5. Subspace topology, Base and Subbase.

1.5.1. Definition [56] :- Let (X, t) be a fuzzy topological space and A be an ordinary subset of X. The class $t_A = \{ u / A \ : u \in t \}$ determines a fuzzy topology on A . This topology is called the subspace fuzzy topology on A.

1.5.2. Definition [89] :- Let (X, t) be a fuzzy topological space . A subfamily B of t is a base for t if and only if each member of t can be express as the union of some members of B.

1.5.3. Definition[89] :- Let (X, t) be a fuzzy topological space . A subfamily S of t is a sub-base for t if and only if the family of finite intersection of members of S forms a base for t .

1.6. Product topology .

1.6.1. Definition [9] :- If u_1 and u_2 are two fuzzy subsets of X and Y respectively then the Cartesian product $u_1 \times u_2$ of two fuzzy subsets u_1 and u_2 is a fuzzy subsets of $X \times Y$ defined by $(u_1 \times u_2)(x, y) = \min(u_1(x), u_2(y))$, for each pair $(x, y) \in X \times Y$.

1.6.2. Definition [47] :- Let $\{ X_i , i \in \Lambda \}$, be any class of sets and let X denoted the Cartesian product of these sets, ie $X = \Pi_{i \in \Lambda} X_i$. Note that X consists of all points $p = < a_i , i \in \Lambda >$, where $a_i \in X_i$. Recall that, for each $j_o \in \Lambda$, we define the projection π_{jo} from the product set X to the coordinate space X_{jo}. ie $\pi_{jo} : X \longrightarrow X_{jo}$

by $\pi_{jo} (< a_i : i \in \Lambda >) = a_{jo}$,

These projections are used to defined the product topology.

1.6.3. Definition [89] :- If (X_1 , t_1) and (X_2 , t_2) be two fuzzy topological space and $X = X_1 \times X_2$ be the usual product and t be the coarsest fuzzy topology on X, then each projection $\pi_i : X \longrightarrow X_i$, $i = 1 , 2$. is fuzzy continuous. The pair (X , t) is called the product space of the fuzzy topological spaces (X_1 , t_1) and (X_2 , t_2) .

1.6.4. Proposition ([9] Theorem 3.6) :- If u is a fuzzy subset of a fuzzy topological space (X , t_1) and v is a fuzzy subsets of a fuzzy topological space (Y , t_2), then $\overline{u \times v} \subseteq \overline{u} \times \overline{v}$.

1.6.5. Definition [88] :- Let $\{ X_\alpha\}_{\alpha \in \Lambda}$ be a family of nonempty sets . Let $X = \Pi_{\alpha \in \Lambda} X_\alpha$ be the usual product of X_α 's and let π_α be the projection from X into X_α. Further assume that each X_α is an fts with fuzzy topology t_α . Now the fuzzy topology generated by $\{ \pi_\alpha^{-1}(b_\alpha) : b_\alpha \in t_\alpha , \alpha \in \Lambda \}$ as a sub basis , is called the product fuzzy topology on X . Clearly if w is a basis element in the product , then there exist $\alpha_1, \alpha_2 , \ldots\ldots \alpha_n \in \Lambda$ such that

$$w(x) = \min \{ b_\alpha (x_\alpha) : \alpha = 1 , 2 , 3 , \ldots\ldots n \}, \text{ where } x = (x_\alpha)_{\alpha \in \Lambda} \in X.$$

1.7 Good extension :

1.7.1. Definition [71] :- Let f be a real valued function on a topological space . If $\{ x : f(x) > \alpha \}$ is open for every real α , then f is called lower semi continuous function .

1.7.2. Definition [50] :- Let X be a nonempty set and T be a topology on X . Let $t = \omega(T)$ be the set of all lower semi continuous (lsc) functions from(X , T) to I (with usual topology). Thus $\omega(T) = \{ u \in I^X : u^{-1} (\alpha , 1] \in T \}$ for each $\alpha \in I_1$. It can be shown that $\omega(T)$ is a fuzzy topology on X.

Let P be the property of a topological space (X , T) and FP be its fuzzy topological analogue. Then FP is called a 'good extension' of P " iff the statement (X , T) has P iff(X , $\omega(T)$) has FP " holds good for every topological space (X , T).

Chapter : -2

R_0 and R_1 Fuzzy Topological Spaces

2.1 Introduction :-

In this chapter, we introduce and study some R_0 and R_1 properties in fuzzy topological spaces and obtain their several features .

2.1.1. Definition : -

Let (X, t) be fuzzy topological space and $\alpha \in I_1$.

(a) (X, t) is an $\alpha - R_0(i)$ space $\Leftrightarrow$ $\forall x, y \in X$, $x \neq y$, whenever $\exists$ $u \in t$ with $u(x) = 1$ and $u(y) \leq \alpha$, then $\exists$ $v \in t$ with $v(x) \leq \alpha$ and $v(y) = 1$.

(b) (X, t) is an $\alpha - R_0(ii)$ space $\Leftrightarrow \forall x, y \in X$, $x \neq y$, whenever $\exists$ $u \in t$ with $u(x) = 0$ and $u(y) > \alpha$, then $\exists$ $v \in t$ with $v(x) > \alpha$ and $v(y) = 0$.

(c) (X, t) is an α - R_0 (iii) space $\Leftrightarrow$ $\forall x, y \in X$, $x \neq y$, whenever $\exists$ $u \in t$ with $0 \leq u(x) \leq \alpha < u(y) \leq 1$, then $\exists$ $v \in t$ with $0 \leq v(y) \leq \alpha < v(x) \leq 1$.

(d) (X, t) is an R_0(iv) space $\Leftrightarrow$ $\forall x, y \in X, x \neq y$, whenever $\exists$ $u \in t$ with $u(x) < u(y)$, then , $\exists$ $v \in t$ with $v(x) > v(y)$.

2.1.2. Lemma:- Show that the properties $\alpha - R_0(i)$, α- R_0(ii) , $\alpha - R_0$(iii) and R_0 (iv) are all independent .

Proof: - For this , we give some examples .

2.1.3. Example :- Let $X = \{x, y\}$ and $u, v \in I^X$, where u, v are defined by $u(x) = 1$, $u(y) = 0$ and $v(x) = 0.5$, $v(y) = 1$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.6$, it is found that (X, t) is $\alpha - R_0(i)$, but (X, t) is not $\alpha - R_0(ii)$.

2.1.4. Example :- Let $X = \{x, y\}$ and $u, v \in I^X$, where u, v are defined by $u(x) = 0$, $u(y) = 1$ and $v(x) = 0.6$, $v(y) = 0$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.5$, we see that (X, t) is $\alpha - R_0(ii)$, but (X, t) is not $\alpha - R_0(i)$.

2.1.5. Example :- Let $X = \{x, y\}$ and $u, v \in I^X$, where u, v are defined by $u(x) = 1$, $u(y) = 0$ and $v(x) = 0.2$, $v(y) = 0.7$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.5$, it is easy to see that (X, t) is $\alpha - R_0(iii)$, but (X, t) is not $\alpha - R_0(i)$ and (X, t) is not $\alpha - R_0(ii)$.

2.1.6. Example :- Let $X = \{x, y\}$ and $u, v \in I^X$, where u, v are defined by $u(x) = 1$, $u(y) = 0$ and $v(x) = 0.2$, $v(y) = 0.4$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.5$, it is clear that (X, t) is $\alpha - R_0(iv)$, but (X, t) is not $\alpha - R_0(i)$, (X, t) is not $\alpha - R_0(ii)$ and (X, t) is not $\alpha - R_0(iii)$.

2.1.7. Example :- Let $X = \{x, y, z\}$ and $u, v, w \in I^X$, where u, v, w are defined by $u(x) = 1$, $u(y) = 1$, $u(z) = 0$ and $v(x) = 0$, $v(y) = 0$, $v(z) = 1$ and $w(x) = 0.9$, $w(y) = 0.5$, $w(z) = 0$. Consider the fuzzy topology t on X generated by $\{0, u, v, w, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.6$, it can be shown that (X, t) is $\alpha - R_0(i)$ and (X, t) is $\alpha - R_0(ii)$. It is observe that (X, t) is not $\alpha - R_0(iii)$, and (X, t) is not $R_0(iv)$, Since $w(x) > \alpha \geq w(y)$ but there does not exit $q \in t$ such that $q(x) \leq \alpha < q(y)$.

2.1.8. Example :-Let $X = \{ x , y , z \}$ and $u , v \in I^X$, where u , v are defined by $u(x) = 0.8$, $u(y) = 0.4$, $u(z) = 0.3$, and $v(x) = 0.3$, $v(y) = 0.8$, $v(z) = 0.2$. Consider the fuzzy topology t on X generated by $\{ 0 , u , v , 1 \} \cup \{ \text{Constants} \}$. For $\alpha = 0.5$ it can be shown that (X , t) is $\alpha - R_0(iii)$, but (X , t) is not $R_0(iv)$, since $u(y) > u(z)$ but we have no $q \in t$ such that $q(y) < q(z)$.

This completes the proof.

2.1.9. Theorem :- Prove that (X , t) is $0\text{-} R_0(ii) \Leftrightarrow (X , t)$ is $0\text{-} R_0(iii)$.

Proof:- Let (X , t) be $0 - R_0$ (ii) space . We shall prove that (X , t) is $0 - R_0$ (iii). Let $x , y \in X$ with $x \neq y$ and $u \in t$ such that $0 \leq u(x) \leq 0 < u(y) \leq 1$ ie $u(x) = 0$, $u(y) > 0$. Since (X , t) is $0 - R_0$ (ii) , then $\exists\, v \in t$ such that $v(x) > 0$, $v(y) = 0$. Then it is clear that $0 \leq v(y) \leq 0 < v(x) \leq 1$. Hence (X , t) is $0 - R_0$ (iii) .

Conversely, suppose that (X , t) is $0 - R_0$ (iii). We shall prove that (X , t) is $0 - R_0$ (ii). Let $x , y \in X$ with $x \neq y$ and $u \in t$ such that $u(x) = 0$ and $u(y) > 0$. It can be written as $0 \leq u(x) \leq 0 < u(y) \leq 1$. Since (X , t) is $0 - R_0$ (iii) , $\exists\, v \in t$ such that $0 \leq v(y) \leq 0 < v(x) \leq 1$ ie $v(y) = 0$ and $v(x) > 0$. Hence (X , t) is $0 - R_0$ (ii) .

This completes the proof.

2.1.10. Theorem:- Let (X , t) be a fuzzy topological space and $I_\alpha(t) = \{ u^{-1}(\alpha , 1] \; : \; u \in t\}$, then

(a) (X , t) is $\alpha - R_0$ (iii) $\Leftrightarrow$ $(X , I_\alpha(t))$ is R_0 .

(b) (X , t) is $\alpha - R_0$ (i) $<\not=>$ $(X , I_\alpha(t))$ is R_0 .

(c) (X , t) is α - R_0 (ii) $<\not=>$ $(X , I_\alpha(t))$ is R_0 .

(d) (X , t) is R_0(iv) $<\not=>$ $(X , I_\alpha(t))$ is R_0 .

Proof :- Let (X, t) be $\alpha - R_0$ (iii). We shall prove that $(X, I_\alpha(t))$ is R_0. Let $x, y \in X$, $x \neq y$ and $M \in I_\alpha(t)$ with $x \in M, y \notin M$ or $x \notin M$, $y \in M$. Suppose $x \in M$, $y \notin M$. We can write $M = u^{-1}(\alpha, 1]$, where $u \in t$. Then we have $u(x) > \alpha$, $u(y) \leq \alpha$, ie $0 \leq u(y) \leq \alpha < u(x) \leq 1$. Since (X, t) is $\alpha - R_0$(iii), for $\alpha \in I_1$, $\exists$ $v \in t$ such that $0 \leq v(x) \leq \alpha < v(y) \leq 1$, ie $v(x) \leq \alpha$, $v(y) > \alpha$. It follows that $x \notin v^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$ and also $v^{-1}(\alpha, 1] \in I_\alpha(t)$. Hence it is clear that $(X, I_\alpha(t))$ is R_0.

Conversely, suppose that $(X, I_\alpha(t))$ is R_0. We shall prove that (X, t) is $\alpha - R_0$ (iii). Let $x, y \in X$, $x \neq y$ and $u \in t$ with $0 \leq u(x) \leq \alpha < u(y) \leq 1$, ie $u(x) \leq \alpha$, $u(y) > \alpha$, it follows that $x \notin u^{-1}(\alpha, 1]$, $y \in u^{-1}(\alpha, 1]$, and $u^{-1}(\alpha, 1] \in I_\alpha(t)$. Since $(X, I_\alpha(t))$ is R_0, then $\exists$ $M \in I_\alpha(t)$ such that $x \in M, y \notin M$. We can write $M = v^{-1}(\alpha, 1]$ where $v \in t$, it follows that $v(x) > \alpha$, $v(y) \leq \alpha$, ie $0 \leq v(y) \leq \alpha < v(x) \leq 1$. Hence it is clear that (X, t) is $\alpha - R_0$ (i), ie (a) is proved.

Now, we give some examples.

2.1.11. Example :- Let $X = \{x, y, z\}$ and $u, v \in I^X$ where u, v are defined by $u(x) = 1$, $u(y) = 0$, $u(z) = 0.8$ and $v(x) = 0$, $v(y) = 1$, $v(z) = 0.7$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.6$, it is clear that (X, t) is $\alpha - R_0$ (i). Now $I_\alpha(t) = \{X, \Phi, \{x, z\}, \{y, z\}, \{z\}\}$. It is clear that $I_\alpha(t)$ is a topology on X and $(X, I_\alpha(t))$ is not R_0 space, since $y, z \in X$, $y \neq z$ and $\{x, z\} \in I_\alpha(t)$, with $z \in \{x, z\}$, $y \notin \{x, z\}$, but there is no $U \in I_\alpha(t)$ with $z \notin U$, $y \in U$.

2.1.12. Example :- Let $X = \{x, y, z\}$ and $u, v \in I^X$, where u, v are defined by $u(x) = 0.3$, $u(y) = 0$, $u(z) = 0.8$, and $v(x) = 0.8$, $v(y) = 1$, $v(z) = 0$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.5$, it is clear that (X, t) is $\alpha - R_0$ (ii) and (X, t) is also R_0 (iv). Now $I_\alpha(t) = \{X, \Phi, \{z\}, \{y\}, \{y, z\}\}$. It is clear that $I_\alpha(t)$ is a topology on X and $(X, I_\alpha(t))$ is not R_0 space, since $x, y \in X, x \neq y$ and $\{y\} \in I_\alpha(t)$ with $x \notin \{y\}$, $y \in \{y\}$, but there is no $U \in I_\alpha(t)$ with $x \in U$, $y \notin U$.

2.1.13. Example :- Let $X = \{x, y\}$ and $u, v, w \in I^X$, where u, v, w are defined by $u(x) = 1$, $u(y) = 0$, $v(x) = 0.4$, $v(y) = 0.9$, $w(x) = 0.7$, $w(y) = 0.3$. Consider the fuzzy topology t on X generated by $\{0, u, v, w, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.6$, it is clear that (X, t) is not $\alpha - R_0$ (i) and (X, t) is not $\alpha - R_0$ (ii). Now $I_\alpha(t) = \{X, \Phi, \{x\}, \{y\}\}$. Then we see that $I_\alpha(t)$ is a topology on X and $(X, I_\alpha(t))$ is R_0.

2.1.14. Example : - Let $X = \{x, y\}$ and $u, v \in I^X$, where u, v are defined by $u(x) = 0.4$, $u(y) = 0.5$, $v(x) = 0.3$, and $v(y) = 0.4$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constant}\}$. For $\alpha = 0.6$, we ~~that~~ see that (X, t) is not $\alpha - R_0$ (iv). Now $I_\alpha(t) = \{X, \Phi\}$. Then $I_\alpha(t)$ is a topology on X and $(X, I_\alpha(t))$ is R_0.

This completes the proof.

2.1.15. Theorem :-

Let (X, T) be a topological space. Then (X, T) is R_0, iff $(X, \omega(T))$ is $\alpha - R_0$ (p), where $p =$ i, ii, iii, iv .

Proof :- Let $(X, \omega(T))$ be $\alpha - R_0$(i). We shall prove that (X, T) is R_0. Let $x, y \in X$ with $x \neq y$ and $U \in T$ with $x \in U$, $y \notin U$. But by the definition of lsc, $1_U \in \omega(T)$ and $1_U(x) = 1$, $1_U(y) = 0$. Now we have $1_U \in \omega(T)$ with $1_U(x) = 1$, $1_U(y) \leq \alpha$. Since $(X, \omega(T))$ is $\alpha - R_0$ (i), for $\alpha \in I_1$, $\exists\ v \in \omega(T)$ such that $v(x) \leq \alpha$, $v(y) = 1$. Then $x \notin v^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$ as $v(x) \leq \alpha$, $v(y) = 1$, and also $\exists\ v^{-1}(\alpha, 1] \in T$. Hence it is clear that (X, T) is R_0 – space .

Conversely, suppose that (X, T) be a R_0–space. We shall prove that $(X, \omega(T))$ is $\alpha - R_0$ (i). Let $x, y \in X$ with $x \neq y$ and there exist $u \in \omega(T)$ such that $u(x) = 1$, $u(y) \leq \alpha$. Then $x \in u^{-1}(\alpha, 1]$, $y \notin u^{-1}(\alpha, 1]$ as $u(x) = 1$, $u(y) \leq \alpha$, and it is clear that $u^{-1}(\alpha, 1] \in T$. Since (X, T) is R_0, then $\exists\ V \in T$ such that $x \notin V, y \in V$, but $1_V \in \omega(T)$ and $1_V(x) = 0$, $1_V(y) = 1$, ie $\exists\ 1_V \in \omega(T)$ such that $1_V(x) \leq \alpha$, $1_V(y) = 1$. Hence it is clear that $(X, \omega(T))$ is $\alpha - R_0$ (i).

Hence (X, T) is $R_0 \Leftrightarrow (X, \omega(T))$ is $\alpha - R_0$(i).

In the same way we can prove that

(X, T) is $R_0 \Leftrightarrow (X, \omega(T))$ is $\alpha - R_0$(ii).

(X, T) is $R_0 \Leftrightarrow (X, \omega(T))$ is $\alpha - R_0$(iii).

(X, T) is $R_0 \Leftrightarrow (X, \omega(T))$ is R_0(iv).

Thus it is seen that $\alpha - R_0$ (p) is a good extension of its topological counter part (p = i, ii, iii, iv).

This completes the proof.

2.1.16. Theorem :- Let (X, t) be a fuzzy topological space and $A \subseteq X$,

$t_A = \{ u/A : u \in t \}$, then

(a) (X, t) is an $\alpha - R_0$ (i) $\Rightarrow (A, t_A)$ is an $\alpha - R_0$ (i).

(b) (X, t) is an $\alpha - R_0$ (ii) $\Rightarrow (A, t_A)$ is an $\alpha - R_0$ (ii).

(c) (X, t) is an $\alpha - R_0$ (iii) $\Rightarrow (A, t_A)$ is an $\alpha - R_0$ (iii)

(d) (X, t) is an R_0(iv) $\Rightarrow (A, t_A)$ is an R_0 (iv).

Proof :- First suppose that (X, t) is $\alpha - R_0$ (i). We shall prove that (A, t_A) is $\alpha - R_0$ (i). Let $x, y \in A$, with $x \neq y$ and $u \in t_A$ such that $u(x) = 1$, $u(y) \leq \alpha$, then also $x, y \in X$, $x \neq y$. But we can write $u = w/A$ where $w \in t$ and hence $w(x) = 1$, $w(y) \leq \alpha$. Since

(X, t) is $\alpha - R_0$ (i), then $\exists\ m \in t$ such that $m(x) \leq \alpha$, $m(y) = 1$. But from the definition $m/A \in t_A$, for every $m \in t$ and $m/A(x) \leq \alpha$, $m/A(y) = 1$. Thus (A, t_A) is $\alpha - R_0$ (i).

similarly (b), (c) and (d) can be proved.

This completes the proof.

2.1.17. Theorem :- Given (X_i, t_i), $i \in \Lambda$ be fuzzy topological spaces and $X = \prod_{i \in \Lambda} X_i$ and t be a product fuzzy topology on X. Then;

(a) $\forall\, i \in \Lambda$, (X_i, t_i) is $\alpha - R_0$(i) $\Leftrightarrow$ (X, t) is $\alpha - R_0$(i).

(b) $\forall\, i \in \Lambda$, (X_i, t_i) is $\alpha - R_0$(ii) $\Leftrightarrow$ (X, t) is $\alpha - R_0$(ii).

(c) $\forall\, i \in \Lambda$, (X_i, t_i) is $\alpha - R_0$(iii) $\Leftrightarrow$ (X, t) is $\alpha - R_0$(iii).

(d) $\forall\, i \in \Lambda$, (X_i, t_i) is R_0(iv) $\Leftrightarrow$ (X, t) is R_0(iv).

Proof :- Let (X_i, t_i), $i \in \Lambda$ be $\alpha - R_0$ (i). We shall prove that (X, t) is $\alpha - R_0$ (i). Let $x, y \in X$, with $x \neq y$ and $u \in t$ such that $u(x) = 1$, $u(y) \leq \alpha$. But we have $u(x) = \min\{u_i(x_i) : i \in \Lambda\}$ and $u(y) = \min\{u_i(y_i) : i \in \Lambda\}$ and hence we can find an $u_i \in t_i$ and $x_i \neq y_i$ such that $u_i(x_i) = 1$ and $u_i(y_i) \leq \alpha$. Since (X_i, t_i), $i \in \Lambda$ is α- R_0(i), $\alpha \in I_1$, then $\exists\ v_i \in t_i$, such that $v_i(x_i) \leq \alpha$, $v_i(y_i) = 1$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$ and hence $v_i(\pi_i(x)) \leq \alpha$, $v_i(\pi_i(y)) = 1$. It follows that $\exists\ u_i o \pi_i \in t$ such that $(v_i o\, \pi_i)(x) \leq \alpha$, $(v_i o\, \pi_i)(y) = 1$. Hence it is clear that (X, t) is $\alpha - R_0$ (i).

Conversely, suppose that (X, t) is $\alpha - R_0$ (i). We shall prove that. (x_i, t_i), $i \in \Lambda$, is $\alpha - R_0$ (i). Let for some $i \in \Lambda$, a_i be a fixed element in X_i, suppose that $A_i = \{x \in X = \prod_{i \in \Lambda} X_i \ /\ x_j = a_j$ for some $i \neq j\}$. So that A_i is the subset of X, and this implies that (A_i, t_{A_i}) is also the subspace of (X, t). Since (X, t) is $\alpha - R_0$(i), then (A_i, t_{A_i}) is also $\alpha - R_0$ (i) and A_i is a homeomorphic image of X_i. Hence it is clear that (X_i, t_i) is $\alpha - R_0$ (i) .ie (a) is proved.

Similarly (b), (c) and (d) can be proved.

2.2.1. Definition : -

Let (X, t) be a fuzzy topological space and $\alpha \in I_1$.

(a) (X, t) is said to be $\alpha - R_1(i) \Leftrightarrow \forall\, x, y \in X$ with $x \neq y$, whenever $\exists\, w \in t$ with $w(x) \neq w(y)$, then $\exists\, u, v \in t$ such that $u(x) = v(y) = 1$ and $u \cap v \leq \alpha$.

(b) (X, t) is said to be $\alpha - R_1(ii) \Leftrightarrow \forall\, x, y \in X$ with $x \neq y$, whenever $\exists\, w \in t$ with $w(x) \neq w(y)$, then $\exists\, u, v \in t$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$.

(c) (X, t) is said to be $\alpha - R_1(iii) \Leftrightarrow \forall\, x, y \in X$ with $x \neq y$, whenever $\exists\, w \in t$ with $w(x) \neq w(y)$, then $\exists\, u, v \in t$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$.

2.2.2. Lemma :- The following implications are true :

(X, t) is $\alpha - R_1(i)$ $\Rightarrow$ (X, t) is $\alpha - R_1(iii)$.

(X, t) is $\alpha - R_1(ii)$ $\Rightarrow$ (X, t) is $\alpha - R_1(iii)$.

Proof : - First, suppose that (X, t) is $\alpha - R_1(i)$. We shall prove that (X, t) is $\alpha - R_1(iii)$. Let $x, y \in X$ with $x \neq y$ and $w \in t$ such that $w(x) \neq w(y)$. Since (X, t) is $\alpha - R_1(i)$, for $\alpha \in I_1$, $\exists\, u, v \in t$ such that $u(x) = v(y) = 1$ and $u \cap v \leq \alpha$. Now it is clear that when $w \in t$ with $w(x) \neq w(y)$, $\exists\, u, v \in t$ such that $u(x) > \alpha$, $u(y) > \alpha$ and $u \cap v \leq \alpha$. Hence it is clear that (X, t) is $\alpha - R_1(iii)$.

Next, suppose that (X, t) is $\alpha - R_1(ii)$. We shall prove that (X, t) is $\alpha - R_1(iii)$. Let $x, y \in X$ with $x \neq y$ and $w \in t$ such that $w(x) \neq w(y)$. Since (X, t) is $\alpha - R_1(ii)$, for $\alpha \in I_1$, $\exists\, u, v \in t$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$, ie $u \cap v \leq \alpha$. Hence it is clear that (X, t) is $\alpha - R_1(iii)$.

Now, we gives some examples to show the non implications among α - R_1(i), α - R_1(ii) and α - R_1(iii).

2.2.3. Example :- Let $X = \{ x , y \}$ and $u , v \in I^X$, where u , v are defined by $u(x) = 1$, $u(y) = 0.4$, $v(x) = 0.4$, $v(y) = 1$. Consider the fuzzy topology t on X generated by $\{ 0 , u , v , 1 \} \cup \{$ Constants $\}$. For $\alpha > 0.4$, we see that (X , t) is α - R_1(i), but (X , t) is not α - R_1(ii).

2.2.4. Example : - Let $X = \{ x , y \}$ and $u , v \in I^X$, where u , v are defined by $u(x) = 0.9$ $v(y) = 0.9$ and $v(x) = 0$, $u(y) = 0$. Consider the fuzzy topology t on X generated by $\{ 0 , u , v , 1 \} \cup \{$ Constants $\}$. For every $\alpha < 0.9$. It is seen that (X , t) is α -R_1(ii), but (X , t) is not α - R_1(i).

2.2.5. Example: - Let $X = \{ x , y \}$, and $u , v \in I^X$, where u , v are defined by $u(x) = 0.8$ $v(y) = 0.8$ and $u(y) = 0.2$ $v(x) = 0.1$. Consider the fuzzy topology t on X generated by $\{ 0 , u , v , 1 \} \cup \{$ Constants $\}$. For every $0.2 < \alpha < 0.8$, we have that (X , t) is α - R_1(iii), but (X , t) is not α - R_1(i) and (X , t) is not α - R_1(ii).

This completes the proof.

2.2.6. Theorem :- If $0 \leq \alpha \leq \beta < 1$ then

(a) (X , t) is $\alpha - R_1$(i). $\Rightarrow$ (X , t) is β - R_1 (i).

(b) (X , t) is β - R_1 (ii). $\Rightarrow$ (X , t) is $\alpha - R_1$(ii).

(c) (X , t) is 0 - R_1 (ii). $\Leftrightarrow$ (X , t) is $0 - R_1$(iii).

Proof :- First , suppose that (X , t) is a α - R_1 (i) space . We shall prove that (X , t) is a β- R_1 (i) space . Let $x , y \in X$ with $x \neq y$ and $w \in t$ such that $w(x) \neq w(y)$. Since (X , t) is α -R_1 (i), for some $\alpha \in I_1$, $\exists\, u , v \in t$, such that $u(x) = 1 = v(y)$ and $u \cap v \leq \alpha$.

Since $0 \leq \alpha \leq \beta < 1$. Now we have $u \cap v \leq \beta$. Hence it is clear that (X, t) is a β - R_1 (i) space.

3.2.7. Example : - Let $X = \{x, y\}$, and $u, v \in I^X$, where u, v are defined by $u(x) = 1$, $v(y) = 1$ and $u(y) = 0.2$ $v(x) = 0.2$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.2$ and $\beta = 0.5$. It is clear that (X, t) is β - R_1(i), but (X, t) is not α - R_1(i).

Next, suppose that (X, t) is β - R_1(ii). We shall prove that (X, t) is α - R_1(ii). Let $x, y \in X$ with $x \neq y$ and $w \in t$ such that $w(x) \neq w(y)$. Since (X, t) is β - R_1(ii), for $\beta \in I_1$, $\exists\, u, v \in t$ such that $u(x) > \beta$, $v(y) > \beta$ and $u \cap v = 0$. This implies that $u(x) > \alpha$, $v(y) > \alpha$ as $0 \leq \alpha \leq \beta < 1$. Hence it is clear that (X, t) is α - R_1(ii).

3.2.8. Example : - Let $X = \{x, y\}$ and $u, v \in I^X$, where u, v are defined by $u(x) = 0.6$, $v(y) = 0.6$. $u(y) = 0$, $v(x) = 0$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.5$ and $\beta = 0.8$. It is clear that (X, t) is α - R_1(ii), but (X, t) is not β - R_1(ii).

Further, suppose that (X, t) is $0 - R_1$(ii). We shall prove that (X, t) is $0 - R_1$(iii). Let $x, y \in X$ with $x \neq y$ and $w \in t$ such that $w(x) \neq w(y)$. Since (X, t) is $0 - R_1$(ii), $\exists\, u, v \in t$ such that $u(x) > 0$, $v(y) > 0$ and $u \cap v = 0$, ie $u \cap v \leq 0$. Hence it is clear that (X, t) is $0 - R_1$(iii).

Conversely, suppose that (X, t) is 0- R_1(iii). We shall prove that (X, t) is $0 - R_1$(ii). Let $x, y \in X$ with $x \neq y$ and $w \in t$ such that $w(x) \neq w(y)$. Since (X, t) is $0 - R_1$(iii), $\exists\, u, v \in t$ such that $u(x) > 0$, $v(y) > 0$, $u \cap v \leq 0$, $u \cap v = 0$. Hence (X, t) is $0 - R_1$(ii). This completes the proof.

2.2.9. Theorem : - Let (X, t) be a fuzzy topological space, and $I_\alpha(t) = \{u^{-1}(\alpha, 1] : u \in t\}$ then

(a) (X, t) is an α - R_1 (i) $\Rightarrow$ $(X, I_\alpha(t))$ is a R_1.

(b) (X, t) is an α - R_1 (ii) $\Rightarrow$ $(X, I_\alpha(t))$ is a R_1.

(c) (X, t) is an α - R_1 (iii) $\Rightarrow$ $(X, I_\alpha(t))$ is a R_1.

Proof :- Let (X, t) be α - R_1(i) space. We shall prove that $(X, I_\alpha(t))$ is R_1- space. Let $x, y \in X$, with $x \neq y$ and $M \in I_\alpha(t)$ such that $x \in M$, $y \notin M$. We can write $M = w^{-1}(\alpha, 1]$, where $w \in t$. Then we see that $w(x) > \alpha$, $w(y) \leq \alpha$, therefore $w(x) \neq w(y)$. Since (X, t) is α - R_1, for $\alpha \in I_1$, $\exists$ $u, v \in t$, such that $u(x) = 1 = v(y)$ and $u \cap v \leq \alpha$. It follows that $\exists$ $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t)$ and $x \in u^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$ and $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$ as $u \cap v \leq \alpha$. Thus it is clear that $(X, I_\alpha(t))$ is R_1.

Next, suppose that (X, t) is α - R_1(ii). We shall prove that $(X, I_\alpha(t))$ is R_1- space. Let $x, y \in X$ with $x \neq y$ and $M \in I_\alpha(t)$ such that $x \in M$, $y \notin M$. So we can write $M = w^{-1}(\alpha, 1]$ where $w \in t$. Then we see that $w(x) > \alpha$, $w(y) \leq \alpha$, and hence $w(x) \neq w(y)$. Since (X, t) is α - R_1(ii), for $\alpha \in I_1$, $\exists\, u, v \in t$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$. It follows that $\exists$ $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t)$ and $x \in u^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$ as $u(x) > \alpha$, $v(y) > \alpha$ and $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$ as $u \cap v = 0$. Now it is clear that $(X, I_\alpha(t))$ is R_1- Space.

Finally, Suppose that (X, t) is α - R_1 (iii) space. We shall prove that $(X, I_\alpha(t))$ is a R_1 space. Let $x, y \in X$, with $x \neq y$ and $M \in I_\alpha(t)$ such that $x \in M$, $y \notin M$ or $x \notin M$, $y \in M$. Suppose that $x \in M$, $y \notin M$. We can write $M = w^{-1}(\alpha, 1]$, where $w \in t$. Now we have $w(x) > \alpha$, $w(y) \leq \alpha$ ie $w(x) \neq w(y)$. Since (X, t) is α -R_1(iii), for $\alpha \in I_1$, then $\exists$ u ,v $\in$ t such that u(x) > α , v(y) > α and $u \cap v \leq \alpha$. It follows that $\exists\, u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t)$ with $x \in u^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$ as u(x) > α , v(y) > α and $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$, since $u \cap v \leq \alpha$. Now it is clear that (X , t) is R_1.

Now we give some examples.

2.2.10. Example : - Let X = { x , y } and u , v , w $\in I^X$, where u , v and w are defined by u(x) = 1 , u(y) = 0.1 , v(x) = 0.4 , v(y) = 0.9 , w(x) = 0.7 , w(y) = 0.3. Consider the fuzzy topology t on X generated by { o , u , v , w , 1 } $\cup$ { Constants }. For α = 0.6 , it is clear that (X , t) is not α - R_1(i) and (X , t) is not α - R_1(ii) . Now $I_\alpha(t)$ = { X , ϕ , {x} , {y} }. Then clear that $I_\alpha(t)$ is a topology on X and (X , $I_\alpha(t)$) is R_1 –Space.

2.2.11. Example:- Let X = { x , y } and u , v $\in I^X$, where u , v are defined by u(x) = 0.4 , u(y) = 0.5 and v(x) = 0.3 , v(y) = 0.4 . Consider the fuzzy topology t on X generated by { 0 , u , v , 1 }$\cup${ Constants }. For α = 0.6, we see that (X , t) is not α -R_1(iii). Now $I_\alpha(t)$ = { X , ϕ }. Then $I_\alpha(t)$ is a topology on X and (X , $I_\alpha(t)$) is R_1- Space.
This completes the proof.

2.2.12. Theorem :- Let (X ,T) be a topological space . Consider the following statements:

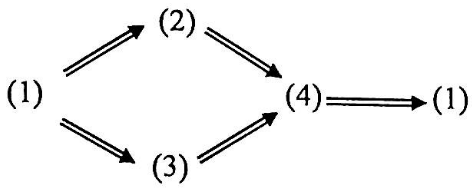
(1) (X , T) be a R_1 space .

(2) (X , ω(T)) be an α - R_1(i) space .

(3) (X , ω(T)) be an α - R_1(ii) space .

(4) (X , ω(T)) be an α - R_1(iii) space .

Then the following implications are true.



Proof :- First suppose that (X , T) be a R_1 -space . We shall prove that (X , ω(T)) be a α -R_1 (i) space. Let x , y ∈ X with x ≠ y and m ∈ ω(T) such that m(x) ≠ m(y) ie either m(x) < m(y) or m(x) > m(y) . Suppose that m(x)< r < m(y) . Then it is clear that m^{-1}(r , 1] ∈ T as m ∈ ω(T) and x ∉ m^{-1}(r , 1] , y ∈ m^{-1}(r , 1] . Since (X , T) is R_1 space then ∃ U, V ∈ T such that x ∈ U , y ∈ V and U ∩ V = ϕ. Since 1_U, 1_V are lower semi continuous function from (X , T) into I , then 1_U , 1_V ∈ ω(T) , and it is clear that $1_U(x) = 1$, $1_V(y) = 1$ and $1_U \cap 1_V = 0$ ie (X , ω(T)) is α - R_1 (i) space . Also it is clear that (X , ω(T)) is α - R_1 (ii) space .

Further, it is easy to show that (2) ⇒ (4) and (3) ⇒ (4) .

We, therefore prove that (4) ⇒ (1) .

Suppose that (X , ω(T)) is α - R_1 (iii). We shall prove that (X , T) is R_1 space . Let x , y ∈ X with x ≠ y and M ∈ T such that x ∈ M , y ∉ M or x∉ M , y ∈ M . Suppose x ∈ M , y ∉ M . But 1_M is lower semi continuous function from (X , T) into I , so 1_M ∈ ω(T) and 1_M (x) = 1 , 1_M(y) = 0 ie $1_M(x) \neq 1_M(y)$. Since (X , ω(T)) is α -R_1(iii) , for α ∈ I_1 then ∃ u , v ∈ ω(T) such that u(x) > α , v(y) > α and u ∩ v ≤ α . Now we is observed that u^{-1} (α , 1] , v^{-1} (α , 1] ∈ T such that , x ∈ u^{-1} (α , 1], y ∈ v^{-1} (α , 1] and u^{-1} (α , 1] ∩ v^{-1} (α , 1] = ϕ . Thus (X , T) is R_1 space .

This completes the proof.

Thus it is seen that α - $R_1(p)$ is a good extension of its topological counter part . (p = i , ii , iii,)

2.2.13. Theorem :- Let (X , t) be a fuzzy topological space, $A \subseteq X$ and $t_A = \{ u/A \ : u \in t \}$, then

(a) (X , t) is α - R_1 (i) $\Rightarrow$ (A , t_A) is α - R_1 (i) .

(b) (X , t) is α - R_1 (ii) $\Rightarrow$ (A , t_A) is α - R_1 (ii) .

(c) (X , t) is α - R_1 (iii) $\Rightarrow$ (A , t_A) is α - R_1 (iii) .

Proof :- Suppose (X , t) is α - R_1(iii) . We shall prove that (A , t_A) is α -R_1(iii). Let x , y $\in$ A with $x \neq y$ then x , y $\in$ X and $x \neq y$. Consider $m \in t_A$ with $m(x) \neq m(y)$. Then m can be written as w/A , where $w \in t$ and hence $w(x) \neq w(y)$. Since (X , t) is α - R_1 (iii) , for $\alpha \in I_1$, then $\exists$ u , v $\in$ t such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$. But we have $u/A \in t_A$, for every $u \in t$. Now we observed that $u/A(x) > \alpha$, $v/A(y) > \alpha$ and $u/A \cap v/A \leq \alpha$, since $u \cap v \leq \alpha$. Hence it is clear that (A , t_A) is α -R_1(iii) .

Similarly (a) and (b) can be proved .

2.2.14. Theorem :- Given (X_i, t_i), $i \in \Lambda$ be fuzzy topological spaces and $X = \Pi_{i \in \Lambda} X_i$ and t be the product topology on X , then

(a) $\forall i \in \Lambda$, (X_i , t_i) is α - R_1(i) $\Leftrightarrow$ (X , t) is α -R_1 (i) .

(b) $\forall i \in \Lambda$, (X_i , t_i) is α - R_1(ii) $\Leftrightarrow$ (X , t) is α -R_1 (ii) .

(c) $\forall i \in \Lambda$, (X_i , t_i) is α - R_1(iii) $\Leftrightarrow$ (X , t) is α -R_1 (iii) .

Proof :- Suppose that $\forall$ $i \in \Lambda$, (X_I , t_i) is α - R_1 (iii). We shall prove that (X , t) is α -R_1 (iii) . Let x , y $\in$ X with $x \neq y$ and $w \in t$ with $w(x) \neq w(y)$. But we have $w(x) = \min \{ w_i(x_i) \ : i \in \Lambda \}$, $w(y) = \min \{ w_i(y_i) \ : i \in \Lambda \}$. Hence we can find at least one

$w_i \in t_i$ and x_i, $y_i \in X_i$, with $x_i \neq y_i$ and $w_i(x_i) \neq w_i(y_i)$. Since (X_i, t_i), $i \in \Lambda$ is α- R_1 (iii), for $\alpha \in I_1$, then $\exists\ u_i, v_i \in t_i$ such that $u_i(x_i) > \alpha$, $v_i(y_i) > \alpha$ and $u_i \cap v_i \leq \alpha$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$ and hence $u_i(\pi_i(x)) > \alpha$, $v_i(\pi_i(y)) > \alpha$. It follows that $\exists\ u_i o \pi_i$, $v_i o \pi_i \in t$ such that $(u_i o \pi_i)(x) > \alpha$, $(v_i o \pi_i)(y) > \alpha$ and $(u_i o \pi_i) \cap (v_i o \pi_i) \leq \alpha$. Hence it is clear that (X, t) is α -R_1 (iii).

Conversely, suppose that (X, t) is α - R_1 (iii). We shall prove that (X_i, t_i), $i \in \Lambda$, is α -R_1(iii). Let for some $i \in \Lambda$, a_i be a fixed element in X_i, suppose that $A_i = \{ x \in X = \Pi_{i \in \Lambda} X_i \ / \ x_j = a_j$ for some $i \neq j \}$. So that A_i is a subset of X, and hence (A_i, t_{A_i}) is also a subspace of (X, t). Since (X, t) is α - R_1 (iii), then (A_i, t_{A_i}) is also α - R_1 (iii). Now we have A_i is homeomorphic image of X_i. Hence it is clear that (X_i, t_i) is α - R_1 (iii) ie (c) is proved.

Similarly (a) and (b) can be proved.

This completes the proof.

Chapter 3

T_0 and T_1 Fuzzy Topological spaces

3.1 Introduction :-

In this chapter, we introduce and study some T_0 and T_1 properties in fuzzy topological spaces and obtain their several features.

3.1.1. Definition : -

Let (X, t) be a fuzzy topological space and $\alpha \in I_1$.

(a) (X, t) is an α - $T_0(i)$ space $\Leftrightarrow \forall\, x, y \in X$, with $x \neq y$, $\exists\; u \in t$ such that $u(x) = 1$, $u(y) \leq \alpha$ or $\exists\; v \in t$ such that $v(x) \leq \alpha$, $v(y) = 1$.

(b) (X, t) is an α - $T_0(ii)$ space $\Leftrightarrow \forall\, x, y \in X$, with $x \neq y$, $\exists\; u \in t$ such that $u(x) = 0$, $u(y) > \alpha$ or $\exists\; v \in t$ such that $v(x) > \alpha$, $v(y) = 0$.

(c) (X, t) is an α - $T_0(iii)$ space $\Leftrightarrow \forall\, x, y \in X$, with $x \neq y$, $\exists\; u \in t$ such that $0 \leq u(x) \leq \alpha < u(y) \leq 1$ or $\exists\; v \in t$ such that $0 \leq v(y) \leq \alpha < v(x) \leq 1$.

(d) (X, t) is a $T_0(iv)$ space $\Leftrightarrow \forall\, x, y \in X$, with $x \neq y$, $\exists\; u \in t$ such that $u(x) \neq u(y)$.

3.1.2. Lemma: - The following implications are true :

(X, t) is α - $T_0(i)$ $\searrow$

$\qquad (X, t)$ is α - $T_0(iii)$ $\Rightarrow$ (X, t) is $T_0(iv)$.

(X, t) is α - $T_0(ii)$ $\nearrow$

Proof : - Let (X, t) be a fuzzy topological space and (X, t) is α - T_0 (i). We shall prove that (X, t) is α - T_0 (iii). Let $x, y \in X$ with $x \neq y$. Since (X, t) is α - T_0 (i), fore $\alpha \in I_1$, $\exists\, u \in t$ such that $u(x) = 1$, $u(y) \leq \alpha$. It follows that $0 \leq u(y) \leq \alpha < u(x) \leq 1$. Hence it is clear that (X, t) is α - T_0(iii).

Next, suppose that (X, t) is α - T_0 (ii). We shall prove that (X, t) is α - T_0(iii). Let $x, y \in X$ with $x \neq y$. Since (X, t) is α - T_0 (ii), for $\alpha \in I_1$, $\exists\, u \in t$ such that $u(x) = 0$, $u(y) > \alpha$. This implies that $0 \leq u(x) \leq \alpha < u(y) \leq 1$. Hence it is clear that (X, t) is α - T_0(iii).

Finally, suppose that (X, t) is α - T_0 (iii). We shall prove that (X, t) is T_0 (iv). Let $x, y \in X$ with $x \neq y$. Since (X, t) is α - T_0 (iii), for $\alpha \in I_1$, $\exists\, u \in t$ such that $0 \leq u(x) \leq \alpha < u(y) \leq 1$. Now we observe that $u(x) \neq u(y)$. Hence (X, t) is T_0 (iv).

Now, we give some examples to show the nonimplications among α - T_0 (i), α - T_0 (ii), α - T_0 (iii) and T_0 (iv).

3.1.3. Example: - Let $X = \{x, y\}$ and $u \in I^X$ be given by $u(x) = 0.6$, $u(y) = 0.8$. Consider the fuzzy topology t on X generated by $\{0, u, 1\} \cup \{$ Constants $\}$. For $\alpha = 0.7$ it is clear that (X, t) is α - T_0(iii) but (X, t) is not α - T_0(i). It is also clear that (X, t) is not α -T_0(ii).

3.1.4. Example : - Let $X = \{x, y\}$ and $u \in I^X$, where u is defined by $u(x) = 1$, $u(y) = 0.4$. Consider the fuzzy topology t on X generated by $\{0, u, 1\} \cup \{$ Constants $\}$. For $\alpha = 0.6$, we see that (X, t) is α -T_0(i) but (X, t) is not α -T_0(ii).

3.1.5. Example : - Let $X = \{x, y\}$ and $u \in I^X$, where u is defined by $u(x) = 0$, $u(y) = 0.7$. Consider the fuzzy topology t on X generated by $\{0, u, 1\} \cup \{$Constants$\}$. For $\alpha = 0.4$, it is clear that (X, t) is α - T_0(ii) but (X, t) is not α -T_0(i).

3.1.6. Example : - Let $X = \{ x , y \}$ and $u \in I^X$, where u is defined by $u(x) = 0.4$, $u(y) = 0.8$. Consider the fuzzy topology t on X generated by $\{ 0 , u , 1 \} \cup \{ \text{Constants} \}$. For $\alpha = 0.9$, we see that (X , t) is T_0(iv) but (X , t) is not α - T_0(iii).

This completes the proof.

3.1.7. Lemma :- If $0 \leq \alpha \leq \beta < 1$, then

(a) (X , t) is α - T_0 (i) $\Rightarrow$ (X , t) is β -T_0 (i).

(b) (X , t) is β - T_0 (ii) $\Rightarrow$ (X , t) is α -T_0(ii).

(c) (X , t) is 0 – T_0(ii) $\Leftrightarrow$ (X , t) is 0 – T_0(iii).

Proof :- Suppose that (X , t) be a fuzzy topological space and (X , t) is α - T_0(i) . We shall prove that (X , t) is β- T_0(i). Let $x , y \in X$ with $x \neq y$. Since (X , t) is α - T_0(i) , for $\alpha \in I_1$, $\exists$ $u \in t$ such that $u(x) = 1$. $u(y) \leq \alpha$. This implies that $u(x) = 1$, $u(y) \leq \beta$, since $0 \leq \alpha \leq \beta < 1$. Hence it is clear that (X , t) is β - T_0 (i).

3.1.8. Example: - Let $X = \{ x , y \}$ and $u \in I^X$, where u is defined by $u(x) = 1$, $u(y) = 0.7$. Consider the fuzzy topology t on X generated by $\{ 0 , u , 1 \} \cup \{ \text{Constants} \}$. For $\alpha = 0.6$ and $\beta = 0.8$, we see that (X , t) is β -T_0 (i) but (X , t) is not α -T_0 (i).

Next, suppose that (X , t) is β -T_0(ii). We shall prove that (X , t) is α - T_0 (ii) . Let $x , y \in X$ with $x \neq y$. Since (X , t) is β - T_0(ii), for $\beta \in I_1$, $\exists$ $u \in t$ such that $u(x) = 0$, $u(y) > \beta$. This implies that $u(x) = 0$, $u(y) > \alpha$, since $0 \leq \alpha \leq \beta < 1$. Hence it is clear that (X , t) is α - T_0 (ii).

3.1.9. Example:- Let $X = \{ x , y \}$ and $u \in I^X$, where u is defined by $u(x) = 0$, $u(y) = 0.5$. Consider the fuzzy topology t on X generated by $\{ 0 , u , 1 \} \cup \{ \text{Constants} \}$. For $\alpha = 0.4$ and $\beta = 0.7$, we see that (X , t) is α - T_0 (ii) , but (X , t) is not β - T_0(ii).

Finally, suppose that (X , t) is 0- T_0 (ii). We shall prove that (X , t) is 0 – T_0 (iii). Let x , y ∈ X with x ≠ y . Since (X , t) is 0 – T_0(ii), ∃ u∈ t such that u(x) = 0, u(y) > 0. This implies that $0 \le u(x) \le 0 < u(y) \le 1$. Hence (X , t) is 0 – T_0 (iii).

Conversely, suppose that (X , t) is 0-T_0(iii). We shall prove that (X , t) is 0 – T_0(ii). Let x , y ∈ X with x ≠ y . Since (X , t) is 0 –T_0(iii), ∃ u∈ t such that $0 \le u(x) \le 0 < u(y) \le 1$ ie u(x) = 0 , u(y) > 0 . Hence (X , t) ix 0 - T_0(ii).

This completes the proof.

3.1.10. Theorem :- Let (X , T) be a topological space. Consider the following statements:

(1) (X , T) be a T_0 – space.

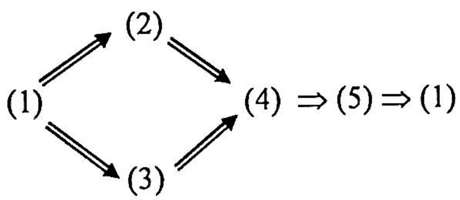
(2) (X , ω(T)) be an α- T_0(i) space .

(3) (X , ω(T)) be an α- T_0(ii) space .

(4) (X , ω(T)) be an α- T_0(iii) space .

(5) (X , ω(T)) be a T_0(iv) space .

Then the implications given below are true :



Proof :- Suppose (X , T) is a T_0 – topological space . We shall prove that (X , ω(T)) is α - T_0(i) fuzzy topological space. Let x , y ∈ X with x ≠ y. Since (X , T) is T_0 , ∃ U ∈ T such that x∈ U , y ∉ U. But from the definition of lsc , we have $1_U \in \omega(T)$ and $1_U(x) = 1$, $1_U(y) = 0$. Hence we see that (X , ω(T)) is α -T_0(i) space . Also we see that (X , ω(T)) is α - T_0 (ii) space.

Further, it is easy to show that (2) $\Rightarrow$ (4), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5).

We, therefore prove that (4) $\Rightarrow$ (1).

Suppose $(X, \omega(T))$ be a $T_0(iv)$ space. We shall prove that (X, T) is T_0 – space. Let $x, y \in X$ with $x \neq y$. Since $(X, \omega(T))$ is T_0 (iv), $\exists$ $u \in \omega(T)$ such that $u(x) \neq u(y)$ ie either $u(x) < u(y)$ or $u(x) > u(y)$. Suppose $u(x) < u(y)$. Then we can find a $r \in I_1$, such that $u(x) < r < u(y)$. We observe that $x \notin u^{-1}(r, 1]$, $y \in u^{-1}(r, 1]$, and by the definition of lsc, $u^{-1}(r, 1] \in T$. Hence (X, T) is T_0 – space.

Thus it is seen that $\alpha - T_0$ (p) is a good extension of its topological counter part (p = i, ii, iii, iv).

This completes the proof.

3.1.11. Theorem : - Let (X, t) be a fuzzy topological space, and

$I_\alpha(t) = \{ u^{-1}(\alpha, 1] \quad : u \in t \}$, then

(a) (X, t) is an α -$T_0(i) \Rightarrow (X, I_\alpha(t))$ is T_0.

(b) (X, t) is an α -$T_0(ii) \Rightarrow (X, I_\alpha(t))$ is T_0.

(c) (X, t) is an α -$T_0(iii) \Leftrightarrow (X, I_\alpha(t))$ is T_0.

Proof :- (a) Let (X, t) be a fuzzy topological space and (X, t) be α - $T_0(i)$. We shall prove that $(X, I_\alpha(t))$ is T_0 – space. Let $x, y \in X$ with $x \neq y$. Since (X, t) is α -$T_0(i)$, fore $\alpha \in I_1$, $\exists$ $u \in t$ such that $u(x) = 1$, $u(y) \leq \alpha$. Since $u^{-1}(\alpha, 1] \in I_\alpha(t)$, $y \notin u^{-1}(\alpha, 1]$ and $x \in u^{-1}(\alpha, 1]$, We have that $(X, I_\alpha(t))$ is T_0 – space.

(b) Again, suppose that (X, t) is α - $T_0(ii)$. We shall prove that $(X, I_\alpha(t))$ is T_0 – space. Let $x, y \in X$ with $x \neq y$. Since (X, t) is α - $T_0(ii)$, for $\alpha \in I_1$, $\exists$ $u \in t$ such that $u(x) = 0$,

$u(y) > \alpha$. Since $u^{-1}(\alpha, 1] \in I_\alpha(t)$, $x \notin u^{-1}(\alpha, 1]$ and $y \in u^{-1}(\alpha, 1]$, one can see that $(X, I_\alpha(t))$ is T_0 – space .

(c) Finally, suppose that (X, t) is α - T_0(iii). We shall prove that $(X, I_\alpha(t))$ is T_0 – space. Let $x, y \in X$ with $x \neq y$. Since (X, t) is α - T_0(iii) , for $\alpha \in I_1$, $\exists$ $u \in t$ such that $0 \le u(x) \le \alpha < u(y) \le 1$. Since $u^{-1}(\alpha, 1] \in I_\alpha(t)$, $x \notin u^{-1}(\alpha, 1]$ and $y \in u^{-1}(\alpha, 1]$, so it is clear that $(X, I_\alpha(t))$ is T_0 – space .

Conversely, suppose $(X, I_\alpha(t))$ be T_0 – space. We shall prove that (X, t) is a α - T_0(iii) space. Let $x, y \in X$ with $x \neq y$. Since $(X, I_\alpha(t))$ is T_0 – space , $\exists$ $u^{-1}(\alpha, 1] \in I_\alpha(t)$ such that $x \in u^{-1}(\alpha, 1]$ and $y \notin u^{-1}(\alpha, 1]$, where $u \in t$. Thus we have that $u(x) > \alpha$, $u(y) \le \alpha$, ie $0 \le u(y) \le \alpha < u(x) \le 1$, one can see that (X, t) is α - T_0(iii) space .

Now we give an example:-

3.1.12. Example : - Let $X = \{x, y\}$ and $u \in I^X$, where u is defined by $u(x) = 0.8$, $u(y) = 0.2$. Consider the fuzzy topology t on X generated by $\{0, u, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.6$ it is clear that (X, t) is not α - T_0(i) and (X, t) is not α - t_0(ii) . Now $I_\alpha(t) = \{X, \phi, \{x\}\}$. Then we see that $I_\alpha(t)$ is a topology on X and $(X, I_\alpha(t))$ is a T_0 space .

This completes the proof.

3.1.13. Theorem :- Let (X, t) be a fuzzy topological space , $A \subseteq X$, $t_A = \{u/A : u \in t\}$, then,

(a) (X, t) is α - T_0 (i) $\Rightarrow$ (A, t_A) is α - T_0 (i).

(b) (X, t) is α - T_0 (ii) $\Rightarrow$ (A, t_A) is α - T_0 (ii).

(c) (X, t) is α - T_0 (iii) $\Rightarrow$ (A, t_A) is α - T_0 (iii).

(d) (X, t) is T_0 (iv) $\Rightarrow$ (A, t_A) is T_0 (iv).

Proof :- Suppose that (X, t) be a fuzzy topological space and (X, t) is α - $T_0(i)$. We shall prove that (A, t_A) is α -$T_0(i)$. Let $x, y \in A$ with $x \neq y$, so that $x, y \in X$, as $A \subseteq X$. Since (X, t) is α - $T_0(i)$, for $\alpha \in I_1$, $\exists$ $u \in t$ such that $u(x) = 1$, $u(y) \leq \alpha$. For $A \subseteq X$, we have $u / A \in t_A$ and $u / A(x) = 1$ $u /A(y) \leq \alpha$ as $x, y \in A$. Hence it is clear that (A, t_A) is α -$T_0(i)$.

Similarly (b), (c) and (d) can be proved.

3.1.14. Theorem : - Given (X_i, t_i), $i \in \Lambda$ be fuzzy topological spaces and $X = \Pi_{i \in \Lambda} X_i$ and t be the product topology on X, then

(a) $\forall$ $i \in \Lambda$, (X_i, t_i) is α - $T_0(i) \Leftrightarrow (X, t)$ is α - T_0 (i).

(b) $\forall$ $i \in \Lambda$, (X_i, t_i) is α - $T_0(ii) \Leftrightarrow (X, t)$ is α - T_0 (ii).

(c) $\forall$ $i \in \Lambda$, (X_i, t_i) is α - $T_0(iii) \Leftrightarrow (X, t)$ is α - T_0 (iii).

(d) $\forall$ $i \in \Lambda$, (X_i, t_i) is $T_0(iv) \Leftrightarrow (X, t)$ is T_0 (iv).

Proof :- Suppose that $\forall$ $i \in \Lambda$, (X_i, t_i) is α - $T_0(i)$. We shall prove that (X, t) is α - $T_0(i)$. Let $x, y \in X$ with $x \neq y$, then $x_i \neq y_i$, for some $i \in \Lambda$. Since (X_i, t_i) is α - t_0 (i), for $\alpha \in I_1$, $\exists$ $u_i \in t_i$, $i \in \Lambda$ such that $u_i(x_i) = 1$ and $u_i(y_i) \leq \alpha$. But we have $\pi_i(x) = x_i$, and $\pi_i(y) = y_i$. Then $u_i(\pi_i(x)) = 1$ and $u_i(\pi_i(y)) \leq \alpha$, ie $(u_i \circ \pi_i)(x) = 1$ and $(u_i \circ \pi_i)(y) \leq \alpha$. It follows that $\exists$ $(u_i \circ \pi_i) \in t$ such that $(u_i \circ \pi_i)(x) = 1$, $(u_i \circ \pi_i)(y) \leq \alpha$. Hence it is clear that (X, t) is α - $T_0(i)$.

Conversely, suppose that (X, t) is α - $T_0(i)$ space. We shall prove that (X_i, t_i), $i \in \Lambda$ is α - $t_0(i)$. Let a_i be a fixed element in X_i, suppose that $A_i = \{ x \in X = \Pi_{i \in \Lambda} X_i : x_j = a_j$ for some $i \neq j \}$. So that A_i is a subset of X, and hence (A_i, t_{A_i}) is also a subspace of (X, t). Since (X, t) is α -$T_0(i)$ then (A_i, t_{A_i}) is also α-T_0 (i). Now we have A_i is homeomorphic image of X_i. Hence it is clear that (X_i, t_i), $i \in \Lambda$ is α -T_0 (i).

Similarly (b) , (c) and (d) can be proved .

3.1.15. Theorem:- Let (X, t) and (Y, s) be two fuzzy topological spaces and $f: X \longrightarrow Y$ be a one-one , onto and open map then,

(a) (X, t) is α -T_0(i) $\Rightarrow$ (Y, s) is α- T_0(i) .

(b) (X, t) is α -T_0(ii) $\Rightarrow$ (Y, s) is α- T_0(ii) .

(c) (X, t) is α -T_0(iii) $\Rightarrow$ (Y, s) is α- T_0(iii) .

(d) (X, t) is T_0(iv) $\Rightarrow$ (Y, s) is T_0(iv) .

Proof :- Suppose (X, t) be α -T_0(i) . We shall prove that (Y, s) is α -T_0(i) . Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since f is onto then , $\exists\ x_1, x_2 \in X$ with $f(x_1) = y_1$, $f(x_2) = y_2$ and $x_1 \neq x_2$ as f is one-one. Again since (X, t) is α-T_0(i) , for $\alpha \in I_1$, then $\exists\ u \in t$ such that $u(x) = 1$, $u(y) \leq \alpha$.

Now $f(u)(y_1) = \{ \operatorname{Sup} u(x_1) : f(x_1) = y_1 \}$

$= 1$.

$f(u)(y_2) = \{ \operatorname{Sup} u(x_2) : f(x_2) = y_2 \}$

$\leq \alpha$.

Since f is open then $f(u) \in s$ as $u \in t$. We observe that $\exists\ f(u) \in s$ such that $f(u)(y_1) = 1$, $f(u)(y_2) \leq \alpha$. Hence it is clear that (Y, s) is α - T_0(i) .

Similarly (b) , (c) and (d) can be proved .

3.1.16. Theorem :- Let (X, t) and (Y, s) be two fuzzy topological spaces and $f: X \longrightarrow Y$ be continuous and one-one map, then

(a) (Y, s) is α- T_0(i) $\Rightarrow$ (X, t) is α - T_0(i) .

(b) (Y, s) is α- T_0(ii) $\Rightarrow$ (X, t) is α - T_0(ii) .

(c) (Y, s) is α- T_0(iii) $\Rightarrow$ (X, t) is α - T_0(iii) .

(d) (Y , s) is T_0(iv) $\Rightarrow$ (X , t) is T_0(iv) .

Proof:- Suppose (Y , s) be α - T_0(i) . We shall prove that (X , t) is α -T_0(i) . Let $x_1, x_2 \in X$ with $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ in Y, as f is one-one. Since (Y , s) is α-T_0(i), for $\alpha \in I_1$, then $\exists\, u \in s$ such that $u(f(x_1)) = 1$, $u (f(x_2)) \leq \alpha$. This implies that $f^{-1}(u) (x_1) = 1$, $f^{-1}(u) (x_2) \leq \alpha$, since $u \in s$ and f is continuous then $f^{-1}(u) \in t$. Now it is clear that $\exists\, f^{-1}(u) \in t$ such that $f^{-1}(u) (x_1) = 1$, $f^{-1}(u) (x_2) \leq \alpha$. Hence (X , t) is α -T_0(i).

Similarly (b) , (c) and (d) can be proved.

3.2.1. Definition : -

Let (X , t) be a fuzzy topological space and $\alpha \in I_1$.

(a) (X , t) is an α - T_1(i) space $\Leftrightarrow$ $\forall\, x , y \in X$ with $x \neq y$, $\exists\, u , v \in t$ such that $u(x) = 1$, $u(y) \leq \alpha$ and $v(x) \leq \alpha$, $v(y) = 1$.

(b) (X , t) is an α -T_1(ii) space $\Leftrightarrow$ $\forall\, x , y \in X$ with $x \neq y$, $\exists\, u , v \in t$ such that $u(x) = 0$, $u(y) > \alpha$ and $v(x) > \alpha$, $v(y) = 0$.

(c) (X , t) is an α -T_1(iii) space $\Leftrightarrow$ $\forall\, x , y \in X$ with $x \neq y$, $\exists\, u , v \in t$ such that $0 \leq u(y) \leq \alpha < u(x) \leq 1$ and $0 \leq v(x) \leq \alpha < v(y) \leq 1$.

(d) (X , t) is a T_1 (iv) space $\Leftrightarrow$ $\forall\, x , y \in X$ with $x \neq y$, $\exists\, u , v \in t$ such that $u(x) < u(y)$ and $v(x) > v(y)$.

3.2.2. Lemma :- The following implications are true:

(X, t) is α-T_1(i). ↘

(X, t) is α-T_1(iii) $\Rightarrow (X, t)$ is T_1(iv).

(X, t) is α-T_1(ii) ↗

Proof:- Let (X, t) be α-T_1(i) fuzzy topological space. We shall prove that (X, t) is α-T_1 (iii). Let $x, y \in X$ with $x \neq y$. Since (X, t) is α-T_1(i), for $\alpha \in I_1$, $\exists\, u, v \in t$ such that $u(x) = 1$, $u(y) \leq \alpha$ and $v(x) \leq \alpha$, $v(y) = 1$. We see that $0 \leq u(y) \leq \alpha < u(x) \leq 1$ and $0 \leq v(x) \leq \alpha < v(y) \leq 1$. Hence it is clear that (X, t) is α-T_1(iii).

Next, suppose that (X, t) is α - T_1(ii). We shall prove that (X, t) is α-T_1(iii). Let $x, y \in X$ with $x \neq y$. Since (X, t) is α - T_1(ii), for $\alpha \in I_1$, $\exists\, u, v \in t$ such that $u(x) = 0$, $u(y) > \alpha$ and $v(x) > \alpha$, $v(y) = 0$. Now we see that $0 \leq u(x) \leq \alpha < u(y) \leq 1$ and $0 \leq v(y) \leq \alpha < v(x) \leq 1$. Hence (X, t) is α-T_1(iii).

Finally, suppose that (X, t) is α - T_1(iii). We shall prove that (X, t) is T_1(iv). Let $x, y \in X$ with $x \neq y$. Since (X, t) is α - T_1(iii), for $\alpha \in I_1$, $\exists\, u, v \in t$ such that $0 \leq u(x) \leq \alpha < u(y) \leq 1$ and $0 \leq v(y) \leq \alpha < v(x) \leq 1$. This implies that $u(x) < u(y)$ and $v(x) > v(y)$. Hence (x, t) is T_1(iv).

Now we give some examples to show the non implication among α - T_1(i), α - T_1(ii), α - T_1(iii) and T_1(iv).

3.2.3. Example:- Let $X = \{x, y\}$ and $u, v \in I^X$ where u, v are defined by $u(x) = 0.6$, $u(y) = 0.8$ and $v(x) = 0.8$, $v(y) = 0.6$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $= 0.7$, it is clear that (X, t) is α - T_1(iii). But (x, t) is not α - T_1(i) and (X, t) is not α-T_1(ii).

3.2.4. Example: - Let $X = \{ x , y \}$ and $u , v \in I^X$ where u , v are defined by $u(x) = 1$, $u(y) = 0.4$, and $v(x) = 0.4$, $v(y) = 1$. Consider the fuzzy topology t on X generated by $\{ 0, u , v , 1 \} \cup \{ \text{Constants} \}$. For $\alpha = 0.6$, we see that (X , t) is α -T_1(i), but (X , t) is not α -T_1(ii).

3.2.5. Example : - Let $X = \{ x , y \}$ and $u , v \in I^X$, where u , v are defined by $u(x) = 0$, $u(y) = 0.7$ and $v(x) =0.7$, $v(y) = 0$. Consider the fuzzy topology t on X generated by $\{ 0 , u , v , 1\} \cup \{ \text{Constants} \}$. For $\alpha = 0.6$, we see that (X , t) is α - T_1(ii), but (X , t) is not α - T_1(i).

3.2.6. Example : - Let $X = \{ x , y \}$ and $u , v \in I^X$, where u , v are defined by $u(x) = 0.4$, $u(y) = 0.6$ and $v(x) = 0.3$, $v(y) = 0.3$. Consider the fuzzy topology t on X generated by $\{ 0 , u , v , 1 \} \cup \{ \text{Constants} \}$. For $\alpha = 0.8$, we can see that (X , t) is T_1(iv) but (X , t) is not α - T_1(iii).

This completes the proof.

3.2.7. Theorem : - If $0 \le \alpha \le \beta < 1$, then

(a) (X , t) is α -T_1(i) $\Rightarrow (X , t)$ is β -T_1(i).

(b) (X , t) is β - T_1(ii) $\Rightarrow (X , t)$ is α -T_1(ii).

(c) (X , t) is 0 –T_1(ii) $\Leftrightarrow (X , t)$ is 0 – T_1(iii).

Proof : - (a) Let (X , t) be α - T_1(i) space. We shall prove that (X , t) is β - T_1(i). Let $x , y \in X$ with $x \neq y$. Since (X , t) is α - T_1(i), for $\alpha \in I_1$, $\exists\, u , v \in t$ such that $u(x) = 1$, $u(y) \le \alpha$, and $v(x) \le \alpha$, $v(y) = 1$. This implies that $u(x) = 1$, $u(y) \le \beta$ and $v(x) \le \beta$, $v(y) = 1$ as $0 \le \alpha \le \beta < 1$. Hence (X , t) is β - T_1(i).

3.2.8. Example: - Let $X = \{ x , y \}$ and $u , v \in I^X$ where u , v are defined by $u(x) = 1$, $u(y) = 0.7$ and $v(x) = 0.7$, $v(y) = 1$. Consider the fuzzy topology t on X generated by $\{ 0 , u , v , 1 \} \cup \{ \text{Constants} \}$. For $\alpha = 0.5$ and $\beta = 0.8$, it is clear that (X , t) is β -T_1(i) but (X , t) is not α - T_1(i).

(b) Next, suppose that (X , t) is β -T_1(ii). We shall prove that (X , t) is α - T_1(ii). Let $x , y \in X$ with $x \neq y$. Since (X , t) is β -T_1(ii), for $\beta \in I_1$, $\exists$ $u , v \in t$ such that $u(x) = 0$, $u(y) > \beta$ and $v(x) > \beta$, $v(y) = 0$. So, we see that $u(x) = 0$, $u(y) > \alpha$ and $v(x) > \alpha$, $v(y) = 0$ as $0 \leq \alpha \leq \beta < 1$. Hence (X , t) is α - T_1(ii).

3.2.9. Example : - Let $X = \{ x , y \}$ and $u , v \in I^X$, where u , v are defined by $u(x) = 0$, $u(y) = 0.5$, and $v(x) = 0.5$, $v(y) = 0$. Consider the fuzzy topology t on X generated by $\{ 0 , u , v , 1 \} \cup \{ \text{Constants} \}$. For $\alpha = 0.3$ and $\beta = 0.7$. It is clear that (X , t) is α - T_1(ii) but (X , t) is not β - T_1(ii).

(c) Finally, suppose (X , t) be $0 - T_1$(ii). We shall prove that (X , t) is 0-T_1(iii). Let $x , y \in X$ with $x \neq y$. Since (X , t) is $0 - T_1$(ii), $\exists$ $u , v \in t$ such that $u(x) = 0$, $u(y) > 0$ and $v(x) > 0$, $v(y) = 0$. So, we see that $0 \leq u(x) \leq 0 < u(y) \leq 1$ and $0 \leq v(y) \leq 0 < v(x) \leq 1$. Hence (X , t) is $0 - T_1$(iii).

Conversely, suppose that (X , t) is $0 - T_1$(iii). We shall prove that (X , t) is $0 - T_1$(ii). Let $x , y \in X$ with $x \neq y$. Since (X , t) is $0 - t_1$(iii), $\exists$ $u , v \in t$ such that $0 \leq u(x) \leq 0 < u(y) \leq 1$ and $0 \leq v(y) \leq 0 < v(x) \leq 1$. Thus we see that $u(x) = 0$, $u(y) > 0$ and $v(y) = 0$, $v(x) > 0$. Hence (X , t) is $0 - T_1$(ii).

This completes the proof.

3.2.10. Theorem: - Let (X , T) be a topological space. Consider the following statements.

(1) (X , T) be a T_1 – space .

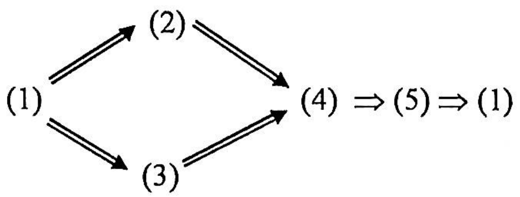
(2) (X , ω(T)) be an α - T_1(i) space .

(3) (X , ω(T)) be an α - T_1(ii) space .

(4) (X , ω(T)) be an α -T_1(iii) space .

(5) (X , ω(T)) be an T_1(iv) space .

Then the implications are true:



Proof :- Let (X , T) be a T_1 space. We shall prove that (X , ω(T)) is α - T_1(i). Let x , y ∈ X with x ≠ y. Since (X , t) is T_1 – space , ∃ U , V ∈ T such that x ∈ U , y ∉ U and x ∉ V , y ∈ V. But from the definition of lsc, 1_U , 1_V ∈ ω(T) and $1_U(x) = 1$, $1_U(y) = 0$ and $1_V(x) = 0$, $1_V(y) = 1$. Hence (X , ω(T)) is α -T_1(i) space , for any α ∈ I_1. Also it is clear that (X , ω(T)) is α - T_1(ii) .

Further, it is easy to show that (2) ⇒ (4) , (3) ⇒ (4) and (4) ⇒ (5) .

We, therefore prove that (5) ⇒ (1).

Suppose, (X , ω(T)) be T_1(iv) space . We shall prove that (X , T) is T_1 – space . Let x , y ∈ X with x ≠ y . Since (X , ω(T)) is T_1(iv) , ∃ u , v ∈ ω(T) such that $u(x) < u(y)$ and $v(x) > v(y)$. Let r , s ∈ I be such that $u(x) < r < u(y)$ and $v(x) > s > v(y)$. Then we have

$u^{-1}(r, 1], v^{-1}(s, 1] \in T$ and $x \notin u^{-1}(r, 1]$, $y \in u^{-1}(r, 1]$ and $x \in v^{-1}(s, 1]$, $y \notin v^{-1}(s, 1]$. Hence (X, T) is T_1-space.

This completes the proof.

Thus it is seen that α - $T_1(p)$ is a good extension of its topological counter part. $(p = i, ii, iii, iv)$

3.2.11. Theorem : - Let (X, t) be a fuzzy topological space, and

$I_\alpha(t) = \{ u^{-1}(\alpha, 1] \quad : u \in t \}$, then

(1) (X, t) is α -$T_1(i) \Rightarrow (X, I_\alpha(t))$ is T_1.

(2) (X, t) is α -$T_1(ii) \Rightarrow (X, I_\alpha(t))$ is T_1.

(3) (X, t) is α -$T_1(iii) \Leftrightarrow (X, I_\alpha(t))$ is T_1.

Proof : - Let (X, t) be α - $T_1(i)$. We shall prove that $(X, I_\alpha(t))$ is T_1. Let $x, y \in X$ with $x \neq y$. Since (X, t) is α - $T_1(i)$, for $\alpha \in I_1$, $\exists\, u, v \in t$ such that $u(x) = 1$, $u(y) \leq \alpha$ and $v(x) \leq \alpha$, $v(y) = 1$. Since $u^{-1}(\alpha, 1], v^{-1}(\alpha, 1] \in I_\alpha(t)$, and it is clear that $x \in u^{-1}(\alpha, 1]$, $y \notin u^{-1}(\alpha, 1]$ and $x \notin v^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$. Hence $(X, I_\alpha(t))$ is T_1- space.

Again, suppose that (X, t) is α - $T_1(ii)$ space. We shall prove that $(X, I_\alpha(t))$ is T_1- space. Let $x, y \in X$ with $x \neq y$. Since (X, t) is α -$T_1(ii)$, for $\alpha \in I_1$, $\exists\, u, v \in t$ such that $u(x) = 0$, $u(y) > \alpha$ and $v(x) > \alpha$, $v(y) = 0$. Since $u^{-1}(\alpha, 1], v^{-1}(\alpha, 1] \in I_\alpha(t)$ and it is clear that $x \notin u^{-1}(\alpha, 1]$, $y \in u^{-1}(\alpha, 1]$ and $x \in v^{-1}(\alpha, 1]$, $y \notin v^{-1}(\alpha, 1]$. Hence $(X, I_\alpha(t))$ is T_1 - Space.

Further, suppose that (X, t) is α -$T_1(iii)$. We shall prove that $(X, I_\alpha(t))$ is T_1. Let $x, y \in X$ with $x \neq y$. Since (X, t) is α -$T_1(iii)$, for $\alpha \in I_1$, $\exists\, u, v \in t$ such that

$0 \leq u(x) \leq \alpha < u(y) \leq 1$ and $0 \leq v(y) \leq \alpha < v(x) \leq 1$. Since $u^{-1}(\alpha, 1], v^{-1}(\alpha, 1] \in I_\alpha(t)$ and it is clear that $x \notin u^{-1}(\alpha, 1], y \in u^{-1}(\alpha, 1]$ and $x \in v^{-1}(\alpha, 1], y \notin v^{-1}(\alpha, 1]$. Hence $(X, I_\alpha(t))$ is T_1-Space.

Conversely, suppose that $(X, I_\alpha(t))$ is T_1- space. We shall prove that (X, t) is α-T_1(iii). Let $x, y \in X$ with $x \neq y$. Since $(X, I_\alpha(t))$ is T_1- space so $\exists\ M, N \in I_\alpha(t)$ such that $x \in M, y \notin M$ and $x \notin N, y \in N$, where $M = u^{-1}(\alpha, 1], N = v^{-1}(\alpha, 1]$, where $u, v \in t$. So it is clear that $u(x) > \alpha$, $u(y) \leq \alpha$ and $v(x) \leq \alpha$, $v(y) > \alpha$. This implies that $0 \leq u(y) \leq \alpha < u(x) \leq 1$ and $0 \leq v(x) \leq \alpha < v(y) \leq 1$. Hence (X, t) is α-T_1(iii).

Now, we give an example.

3.2.12. Example :- Let $X = \{x, y\}$ and $u, v, w \in I^X$, where u, v, w are defined by $u(x) = 1$, $u(y) = 0$, $v(x) = 0.42$, $v(y) = 0.95$, $w(x) = 0.15$, $w(y) = 0.32$. Consider the fuzzy topology t on X generated by $\{0, u, v, w, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.61$, it is cleat that (X, t) is not α-T_1(i) and (X, t) is not α-T_1(ii). Now $I_\alpha(t) = \{X, \phi, \{x\}, \{y\}\}$. Then we see that $I_\alpha(t)$ is a topology on X and $(X, I_\alpha(t))$ is T_1 space.

This completes the proof.

3.2.13. Theorem : - Let (X, t) be a fuzzy topological space, $A \subseteq X$,

$t_A = \{u/A \ : u \in t\}$, then

(a) (X, t) is α-T_1(i) $\Rightarrow$ (A, t_A) is α-T_1(i).

(b) (X, t) is α-T_1(ii) $\Rightarrow$ (A, t_A) is α-T_1(ii).

(c) (X, t) is α-T_1(iii) $\Rightarrow$ (A, t_A) is α-T_1(iii).

(d) (X, t) is T_1(iv) $\Rightarrow$ (A, t_A) is T_1(iv).

Proof : - Suppose (X, t) be an fts and (X, t) is α - $T_1(i)$. We shall prove that (A, t_A) is α - $T_1(i)$. Let $x, y \in A$ with $x \neq y$, then $x, y \in X, x \neq y$, as $A \subseteq X$, Since (X, t) is α -$T_1(i)$, for $\alpha \in I_1$, $\exists$ u , v $\in$ t such that $u(x) = 1$, $u(y) \leq \alpha$ and $v(x) \leq \alpha$, $v(y) = 1$. For $A \subseteq X$, we find that u/A , $v/A \in t_A$ and $u/A(x) = 1$, $u/A(y) \leq \alpha$ and $v/A(x) \leq \alpha$, $v/A(y) = 1$ as $x, y \in A$. Hence it is clear that (A, t_A) is α - $T_1(i)$.

Similarly (b), (c) and (d) can be proved.

This completes the proof.

3.2.14. Theorem : - Given (X_i, t_i) , $i \in \Lambda$ be fuzzy topological space and $X = \Pi_{i \in \Lambda} X_i$. Let t be the product fuzzy topology on X , then

(a) $\forall i \in \Lambda$, (X_i, t_i) is α - $T_1(i)$ $\Leftrightarrow$ (X, t) is α -$T_1(i)$.

(b) $\forall i \in \Lambda$, (X_i, t_i) is α - $T_1(ii)$ $\Leftrightarrow$ (X, t) is α -$T_1(ii)$.

(c) $\forall i \in \Lambda$, (X_i, t_i) is α - $T_1(iii)$ $\Leftrightarrow$ (X, t) is α -$T_1(iii)$.

(d) $\forall i \in \Lambda$, (X_i, t_i) is $T_1(iv)$ $\Leftrightarrow$ (X, t) is $T_1(iv)$.

Proof : - Suppose that $\forall i \in \Lambda$, (X_i, t_i) is α -$T_1(i)$. We shall prove that (X, t) is α $T_1(i)$. Let $x, y \in X$ with $x \neq y$, then $x_i \neq y_i$, for some $i \in \Lambda$. Since (X_i, t_i) is α - $T_1(i)$, for $\alpha \in I_1$, $\exists$ $u_i, v_i \in t_i$, $i \in \Lambda$, such that $u_i(x_i) = 1$, $u_i(y_i) \leq \alpha$ and $v_i(x_i) \leq \alpha$, $v_i(y_i) = 1$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$. Then $u_i(\pi_i(x)) = 1$, $u_i(\pi_i(y)) \leq \alpha$ and $v_i(\pi_i(x)) \leq \alpha$, $v_i(\pi_i(y)) = 1$. It follows that $\exists$ $u_i o \pi_i$, $v_i o \pi_i$ $\in$ t such that $(u_i o \pi_i)(x) = 1$, $(u_i o \pi_i)(y) \leq \alpha$ and $(v_i o \pi_i)(x) \leq \alpha$, $(v_i o \pi_i)(y) = 1$. Hence it is clear that (X, t) is α - $T_1(i)$.

Conversely, suppose that (X, t) is α - $T_1(i)$. We shall prove that (X_i, t_i) , $i \in \Lambda$ is α -$T_1(i)$. Let fore some $i \in \Lambda$, a_i be a fixed element in X_i, suppose that $A_i = \{ x \in X = \Pi_{i \in \Lambda} X_i : x_j = a_j \text{ for some } i \neq j \}$. So that A_i is a subset of X , and hence (A_i, t_{Ai}) is also a subspace of (X , t). Since (X , t) is α -$T_1(i)$, then (A_i , t_{Ai}) is also α -$T_1(i)$. Now we have A_i is a homeomorphic image of X_i . Hence (X_i , t_i) , $i \in \Lambda$, is α -$T_1(i)$.

Similarly (b) , (c) and (d) can be proved .

3.2.15. Theorem : -

(a)

$$\begin{array}{ccccc} (X,t) \text{ is } \alpha - T_1(i) & \Rightarrow & (X,t) \text{ is } \alpha - T_1(iii) & \Rightarrow & (X,t) \text{ is } T_1(iv). \\ \Downarrow & & \Downarrow & & \Downarrow \\ (X,t) \text{ is } \alpha - T_0(i) & \Rightarrow & (X,t) \text{ is } \alpha. - T_0(iii) & \Rightarrow & (X,t) \text{ is } T_0(iv). \end{array}$$

(b)

$$\begin{array}{ccccc} (X,t) \text{ is } \alpha - T_1(ii) & \Rightarrow & (X,t) \text{ is } \alpha - T_1(iii) & \Rightarrow & (X,t) \text{ is } T_1(iv) \\ \Downarrow & & \Downarrow & & \Downarrow \\ (X,t) \text{ is } \alpha - T_0(ii) & \Rightarrow & (X,t) \text{ is } \alpha - T_0(iii) & \Rightarrow & (X,t) \text{ is } T_0(iv). \end{array}$$

Proof : - The proofs of (a) and (b) are easy .

3.2.16. Theorem:- Let (X , t) and (Y , s) be two fuzzy topological spaces and $f : X \longrightarrow Y$ be a one-one , onto and open map then,

(a) (X , t) is α -$T_1(i)$ $\Rightarrow$ (Y , s) is α -$T_1(i)$.

(b) (X , t) is α -$T_1(ii)$ $\Rightarrow$ (Y , s) is α -$T_1(ii)$.

(c) (X , t) is α -$T_1(iii)$ $\Rightarrow$ (Y , s) is α -$T_1(iii)$.

(d) (X , t) is $T_1(iv)$ $\Rightarrow$ (Y , s) is $T_1(iv)$.

Proof :- Suppose (X , t) be α- $T_1(i)$. We shall prove that (Y , s) is α -$T_1(i)$. Let $y_1 , y_2 \in Y$ with $y_1 \neq y_2$. Since f is onto then $\exists\, x_1, x_2 \in X$ with $f(x_1) = y_1$, $f(x_2) = y_2$ and $x_1 \neq x_2$ as f is one-one . Again since (X , t) is α -T_1(i) , for $\alpha \in I_1$, $\exists$ u , v $\in$ t such that $u(x_1) = 1$, $u(x_2) \leq \alpha$ and $v(x_1) \leq \alpha$, $v(x_2) = 1$.

Now $f(u)(y_1) = \{ \text{Sup } u(x_1) \quad ; f(x_1) = y_1 \}$

$= 1.$

$f(u)(y_2) = \{ \text{Sup } u(x_2) \quad ; f(x_2) = y_2 \}$

$\leq \alpha$

and $f(v)(y_1) = \{ \text{Sup } v(x_1) \quad ; f(x_1) = y_1 \}$

$\leq \alpha$

$f(v)(y_2) = \{ \text{Sup } v(x_2) \quad ; f(x_2) = y_2 \}$

$= 1$

Since f is open then f(u), f(v) $\in$ s . Now it is clear that $\exists$ f(u) , f(v) $\in$ s such that $f(u)(y_1) = 1$, $f(u)(y_2) \leq \alpha$ and $f(v)(y_1) \leq \alpha$, $f(v)(y_2) = 1$. Hence (Y , s) is α- T_1(i) .

Similarly (b) , (c) and (d) can be proved .

3.2.17. Theorem:- Let (X , t) and (Y , s) be two fuzzy topological spaces and $f : X \longrightarrow Y$ be a continuous and one-one map then ,

(a) (Y , s) is α - T_1(i) $\Rightarrow$ (X , t) is α - T_1(i) .

(b) (Y , s) is α - T_1(ii) $\Rightarrow$ (X , t) is α - T_1(ii) .

(c) (Y , s) is α - T_1(iii) $\Rightarrow$ (X , t) is α - T_1(iii) .

(d) (Y , s) is T_1(iv) $\Rightarrow$ (X , t) is T_1(iv) .

Proof :- Suppose (Y , s) be α - T_1(i) . We shall prove that (X , t) is α -T_1(i) . Let $x_1, x_2 \in X$ with $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ in Y as f is one-one . Since (Y , s) is α -T_1(i) , for $\alpha \in I_1$, $\exists$ u , v $\in$ t such that $u(f(x_1)) = 1$, $u(f(x_2)) \leq \alpha$ and $v(f(x_1)) \leq \alpha$, $v(f(x_2)) = 1$. This implies that $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) \leq \alpha$ and $f^{-1}(v)(x_1) \leq \alpha$, $f^{-1}(v)(x_2) = 1$, since f is continuous and $u , v \in s$ then $f^{-1}(u) , f^{-1}(v) \in t$. Now it is clear that $\exists$ $f^{-1}(u) , f^{-1}(v) \in t$ such that $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) \leq \alpha$ and $f^{-1}(v)(x_1) \leq \alpha$, $f^{-1}(v)(x_2) = 1$. Hence (X , t) is α - $T_1(i)$.

Similarly (b) ,(c) and (d) can be proved.

Chapter : -4

T_2 Topological Space

4. Introduction:-

In this chapter, we introduce and study some T_2 properties in fuzzy topological spaces and obtain their several features.

4.1. Definition:-

Let (X , t) be a fuzzy topological space and $\alpha \in I_1$.

(a) (X , t) is α -T_2(i) space $\Leftrightarrow \forall\, x , y \in X$ with $x \neq y$, $\exists\, u , v \in t$ such that $u(x) = 1 = v(y)$ and $u \cap v \leq \alpha$.

(b) (X , t) is α - T_2(ii) space $\Leftrightarrow \forall\, x , y \in X$ with $x \neq y$, $\exists\, u , v \in t$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$.

(c) (X , t) is α - T_2(iii) space $\Leftrightarrow \forall\, x , y \in X$ with $x \neq y$, $\exists\, u , v \in t$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$.

4.2. Lemma :- The following implication are true .

(X , t) is α - T_2(i) $\Rightarrow$ (X , t) is α - T_2(iii).

(X , t) is α - T_2(ii) $\Rightarrow$ (X , t) is α - T_2(iii).

Proof:- Let (X , t) be a fuzzy topological space and (X , t) is α - T_2(i) . We shall prove that (X , t) is α - T_2(iii) . Let $x , y \in X$ with $x \neq y$. Since (X , t) is α - T_2(i) , for $\alpha \in I_1$,

$\exists$ u , v $\in$ t such that u(x) = 1 = v(y) and $u \cap v \leq \alpha$. So it is clear that u(x) > α , v(y) > α and $u \cap v \leq \alpha$. Hence it is clear that (X , t) is α - T_2(iii).

Next, suppose that (X , t) is α - T_2(ii). We shall prove that (X , t) is α - T_2(iii). Let x , y $\in$ X with x $\neq$ y . Since (X , t) is α - T_2(ii) , for $\alpha \in I_1$, $\exists$ u , v $\in$t such that u(x) > α , v(y) > α and $u \cap v = 0$. Now it is clear that u(x) > α, v(y) > α and $u \cap v \leq \alpha$. Hence (X , t) is α -T_2(iii).

Now, we give some examples to show the non implication among α - T_2 (i) , α - T_2 (ii) and α - T_2 (iii) .

4.3. Example :- Let X = { x , y } , and u , v $\in I^X$, where u and v are defined by u(x) = 0.7, u(y) = 0, v(x) = 0 and v(y) = 0.7. Consider the fuzzy topology t on X generated by { 0 , u , v , 1 } $\cup$ { Constants }. For α = 0.5 it is clear that (X , t) is α - T_2(ii) but (X , t) is not α- T_2(i).

4.4. Example :- Let X ={ x , y } and u , v $\in I^X$, where u, v are defined by u(x) = 1 = v(y) , and u(y) = 0.4 = v(x) . Consider the fuzzy topology t on X generated by { 0, u , v , 1 } $\cup$ {Constants }. For α = 0.6, we see that (X , t) is α - T_2(i) but (X , t) is not α- T_2(ii).

4.5. Examples :- Let X = { x , y }, and u , v $\in I^X$, where u , v are defined by u(x) = 0.9 , u(y) = 0.3 , v(x) = 0.2 and v(y) = 0.9 . Consider the fuzzy topology t on X generated by { 0, u , v , 1 }$\cup${ Constants }. For α = 0.5 , we get (X , t) is α - T_2(iii), but (X , t) is not α - T_2(i) and (X , t) is not α - T_2(ii).

4.6. Theorem :- If $0 \leq \alpha \leq \beta < 1$, then

(a) (X , t) is α - T_2(i) $\Rightarrow$ (X , t) is β - T_2(i) .

(b) (X , t) is β - T_2(ii) $\Rightarrow$ (X , t) is α - T_2(ii).

(c) (X , t) is 0 – T_2(ii) $\Leftrightarrow$ (X , t) is 0 – T_2(iii).

Proof :- Suppose (X , t) is α - T_2(i) . We shall prove that (X , t) is β - T_2(i). Let $x , y \in X$ with $x \neq y$. Since (X , t) is α - T_2(i) , for $\alpha \in I_1$, $\exists$ $u , v \in t$ such that $u(x) = 1 = v(y)$ and $u \cap v \leq \alpha$. This implies that $u(x) = 1 = v(y)$ and $u \cap v \leq \beta$ as $0 \leq \alpha \leq \beta < 1$. Hence (X , t) is β - T_2(i) .

4.7. Example :- Let $X = \{ x , y \}$ and $u , v \in I^X$, where u , v are defined by $u(x) = 1$, $u(y) = 0.6$, $v(x) = 0.7$, $v(y) = 1$. Consider the fuzzy topology t on X generated by $\{ 0 , u , v , 1 \} \cup \{ \text{Constants} \}$. For $\alpha = 0.3$ and $\beta = .08$, we get (X , t) is β - T_2(i) , but (X , t) is not α -T_2(i).

Next, suppose that (X , t) is β -T_2(ii). We shall prove that (X , t) is α - T_2(ii). Let $x , y \in X$, with $x \neq y$. Since (X , t) is β - T_2(ii) , for $\beta \in I_1$, $\exists$ $u , v \in t$ such that $u(x) > \beta$, $v(y) > \beta$ and $u \cap v = 0$. This implies that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$. as $0 \leq \alpha \leq \beta <1$. Hence it is clear that (X , t) is α - T_2(ii).

4.8. Example :- Let $X = \{ x , y \}$ and $u , v \in I^X$, where u , v are defined by $u(x) = 0.6$, $u(y) = 0$, $v(x) = 0$ and $v(y) = 0.7$. Consider the fuzzy topology t on X generated by $\{ 0 , u , v , 1 \} \cup \{ \text{Constants} \}$. For $\alpha = 0.5$ and $\beta = 0.9$, we get (X , t) is α - T_2(ii) , but (X , t) is not β - T_2(ii).

Finally, suppose that (X , t) is 0 – T_2(ii) . We shall prove that (X , t) is 0 – T_2(iii). Let $x , y \in X$, with $x \neq y$. Since (X , t) is 0 – T_2(ii) , $\exists$ $u , v \in t$ such that $u(x) > 0$, $v(y) >0$

and $u \cap v = 0$. This implies that $u(x) > 0$, $v(y) > 0$ and $u \cap v \leq 0$. Hence it is clear that (X, t) is $0 - T_2(iii)$.

Conversely, suppose that (X, t) is $0 - T_2(iii)$. We shall prove that (X, t) is $0 - T_2(ii)$. Let $x, y \in X$, with $x \neq y$. Since (X, t) is $0 - T_2(iii)$, $\exists\, u, v \in t$ such that $u(x) > 0$, $v(y) > 0$ and $u \cap v \leq 0$. This implies that $u(x) > 0$, $v(y) > 0$ and $u \cap v = 0$. Hence it is clear that (X, t) is $0 - T_2(ii)$.

This completes the proof.

4.9. Theorem :- Let (X, T) be a topological space. Consider the following statements:

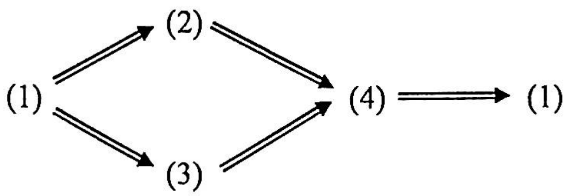
(1) (X, T) is a T_2 – space.

(2) $(X, \omega(T))$ is α - $T_2(i)$ space.

(3) $(X, \omega(T))$ is α - $T_2(ii)$ space.

(5) $(X, \omega(T))$ is α - $T_2(iii)$ space.

Then the following implications are true.



Proof : - Let (X, T) be a T_2 – space. We shall prove that $(X, \omega(T))$ is α - $T_2(i)$. Let $x, y \in X$, with $x \neq y$. Since (X, T) is T_2 – space, $\exists\, U, V \in T$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$. From the definition of lower semi continuous function 1_U, $1_V \in \omega(T)$ and $1_U(x) = 1$, $1_V(y) = 1$ and $1_U \cap 1_V = 0$. If $1_U \cap 1_V \neq 0$, then $\exists\, z \in X$ such that $(1_U \cap 1_V)(z) \neq 0 \Rightarrow 1_U(z) \neq 0$, $1_V(z) \neq 0$, $\Rightarrow z \in U$, $z \in V \Rightarrow z \in U \cap V \Rightarrow U \cap V \neq \phi$, a contradiction. So that $1_U \cap 1_V = 0$, and consequently $(X, \omega(T))$ is α -$T_2(i)$. Also, we see that $(X, \omega(T))$ is α - $T_2(ii)$.

Further, it is easy to show that (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4).

We, therefore prove that (4) $\Rightarrow$ (1).

Suppose that (X , ω(T)) is α - T_2(iii) space. We shall prove that (X , T) is T_2- space. Let x , y $\in$ X with x $\neq$ y . Since (X , ω(T)) is α - T_2(iii) , for some $\alpha \in I_1$, then $\exists$ u , v $\in \omega$(T) such that u(x) > α , v(y) > α and $u \cap v \leq \alpha$. Now we have that $u^{-1}(\alpha, 1], v^{-1}(\alpha, 1] \in T$, $\alpha \in I_1$ and $x \in u^{-1}(\alpha, 1], y \in v^{-1}(\alpha, 1]$. Moreover $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$. For if $z \in u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1]$, then $z \in u^{-1}(\alpha, 1]$ and $z \in v^{-1}(\alpha, 1] \Rightarrow u(z) > \alpha$ and $v(z) > \alpha \Rightarrow (u \cap v)(z) > \alpha$, a contradiction as $(u \cap v)(z) \leq \alpha$. Hence (X , T) is T_2 – space. This completes the proof.

Thus it is seen that α - T_2(p) is a good extension of its topological counter part (p = i , ii , iii,)

4.10. Theorem :- Let (X , t) be a fuzzy topological space, and

$I_\alpha(t) = \{ u^{-1}(\alpha, 1] : u \in t \}$.Then

(a) (X , t) is α - T_2(i) $\Rightarrow$ (X, I_α(t)) is T_2.

(b) (X , t) is α - T_2(ii) $\Rightarrow$ (X, I_α(t)) is T_2.

(c) (X , t) is α - T_2(iii) $\Leftrightarrow$ (X, I_α(t)) is T_2.

Proof : - Consider (X , t) be a fts and (X , t) is α - T_2(i) . We shall prove that (X , I_α(t)) is T_2 . Let x , y $\in$ X with x $\neq$ y . Since (X , t) is α - T_2 (i) , for $\alpha \in I_1$, $\exists$ u , v $\in$ t such that u(x) = 1, v(y) =1 and $u \cap v \leq \alpha$. But for every $\alpha \in I_1$, $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t)$ and also $x \in u^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$ and $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$, as $u \cap v \leq \alpha$. Hence it is clear that (X , I_α(t)) is T_2 – space .

Next, suppose that (X , t) is α - $T_2(ii)$. We shall prove that $(X , I_\alpha(t))$ is T_2-space. Let $x , y \in X$, with $x \neq y$. Since (X , t) is α - $T_2(ii)$, for $\alpha \in I_1, \exists\, u , v \in t$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$. But for every $\alpha \in I_1$, $u^{-1}(\alpha , 1]$, $v^{-1}(\alpha , 1] \in I_\alpha(t)$. So we have $x \in u^{-1}(\alpha , 1]$, $y \in v^{-1}(\alpha , 1]$ and $u^{-1}(\alpha , 1] \cap v^{-1}(\alpha , 1] = \phi$, as $u \cap v = 0$. Hence it is clear that $(X , I_\alpha(t))$ is T_2 – space .

Further, suppose that (X , t) is α - $T_2(iii)$. We shall prove that $(X , I_\alpha(t))$ is T_2-space. Let $x , y \in X$ with $x \neq y$. Since (X , t) is α - $T_2(iii)$, for $\alpha \in I_1, \exists\, u , v \in t$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$. But for every $\alpha \in I_1$, $u^{-1}(\alpha , 1]$, $v^{-1}(\alpha , 1] \in I_\alpha(t)$. So we have $x \in u^{-1}(\alpha , 1]$, $y \in v^{-1}(\alpha , 1]$ and $u^{-1}(\alpha , 1] \cap v^{-1}(\alpha , 1] = \phi$, as $u \cap v \leq \alpha$. Hence it is clear that $(X , I_\alpha(t))$ is T_2 – space.

Conversely, suppose that $(X , I_\alpha(t))$ is T_2 – space. We shall prove that (X , t) is α- $T_2(iii)$. Let $x , y \in X$ with $x \neq y$. Since $(X , I_\alpha(t))$ is T_2-space, $\exists\, U , V \in I_\alpha(t)$, such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Again since $U , V \in I_\alpha(t)$, so we get $u , v \in t$ such that $U = u^{-1}(\alpha , 1]$, $V = v^{-1}(\alpha , 1]$. This implies that $u(x) > \alpha$, $v(y) > \alpha$ and $u^{-1}(\alpha , 1] \cap v^{-1}(\alpha , 1] = \phi \Rightarrow (u \cap v)^{-1}(\alpha , 1] = \phi$ ie $u \cap v \leq \alpha$. So, we see that (X , t) is α -$T_2(iii)$.

Now we give an example.

4.11. Example :- Let $X = \{ x , y \}$, and $u , v \in I^X$, where u , v are defined by $u(x) = 0.8$, $u(y) = 0.2$, $v(x) = 0.3$ and $v(y) = 0.7$. Consider the fuzzy topology t on X generated by $\{ 0 , u , v , 1 \} \cup \{ \text{Constants} \}$. For $\alpha = 0.5$, we see that (X , t) is not α -$T_2(i)$ and (X , t) is not α - $T_2(ii)$. Now $I_\alpha(t) = \{ X , \phi , \{x\}, \{y\} \}$. Then we see that $I_\alpha(t)$ is a topology on X and $(X , I_\alpha(t))$ is T_2- space.

This completes the proof.

4.12. Theorem :- Let (X , t) be a fuzzy topological space and $A \subseteq X$,

$t_A = \{ u/A : u \in t \}$, then

(a) (X , t) is an α - $T_2(i)$ $\Rightarrow$ (A , t_A) is an α - $T_2(i)$.

(b) (X , t) is an α - $T_2(ii)$ $\Rightarrow$ (A , t_A) is an α - $T_2(ii)$.

(c) (X , t) is an α - $T_2(iii)$ $\Rightarrow$ (A , t_A) is an α - $T_2(iii)$.

Proof :- Suppose (X , t) be an α - $T_2(iii)$. We shall prove that (A , t_A) is an α - $T_2(iii)$. Let $x , y \in A$, with $x \neq y$, then $x , y \in X$, with $x \neq y$. Since (X , t) is α - $T_2(iii)$, for $\alpha \in I_1$, $\exists\, u , v \in t$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$. But we have $u/A , v/A \in t_A$ for every $u , v \in t$. This implies that $u/A(x) > \alpha$, $v/A(y) > \alpha$ and $u/A \cap v/A \leq \alpha$. Hence it is clear that $(A . t_A)$ is α - $T_2(iii)$.

Similarly, other cases can be proved.

4.13. Theorem :- Given $\{ (X_i , t_i) , i \in \Lambda \}$ be fuzzy topological spaces and $X = \Pi_{i \in \Lambda} X_i$ and t be the product fuzzy topology on X. Then

(a) $\forall\, i \in \Lambda$, (X_i , t_i) is an α - $T_2(i)$ $\Leftrightarrow$ (X , t) is an α -$T_2(i)$.

(b) $\forall\, i \in \Lambda$, (X_i , t_i) is an α - $T_2(ii)$ $\Leftrightarrow$ (X , t) is an α -$T_2(ii)$.

(c) $\forall\, i \in \Lambda$, (X_i , t_i) is an α - $T_2(iii)$ $\Leftrightarrow$ (X , t) is an α -$T_2(iii)$.

Proof :- Suppose $\forall\, i \in \Lambda$,(X_i , t_i) be an α -$T_2(iii)$. We shall prove that (X , t) is α-$T_2(iii)$. Let x , y be two distinct points in $X = \Pi_{i \in \Lambda} X_i$, then there exist an $x_i \neq y_i$ in X_i. Since (X_i , t_i) is an α -$T_2(iii)$, for $\alpha \in I_1$, $\exists\, u_i , v_i \in t_i$ such that $u_i(x_i) > \alpha$, $v_i(y_i) > \alpha$ and $u_i \cap v_i \leq \alpha$. But we have $\pi_i(x) = x_i$, $\pi_i(y) = y_i$, then $u_i(\pi_i(x)) > \alpha$, $v_i(\pi_i(y)) > \alpha$ and $(u_i \cap v_i) \circ \pi_i \leq \alpha$. Hence $(u_i \circ \pi_i) (x) > \alpha$, $(v_i \circ \pi_i) (y) > \alpha$ and $(u_i \circ \pi_i) \cap (v_i \circ \pi_i) \leq \alpha$. Put $u = u_i \circ \pi_i$,

$v = v_i \circ \pi_i$, then $u, v \in t$ with $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \le \alpha$. Hence it is clear that (X, t) is α - T_2(iii).

Conversely, suppose that (X, t) is α - T_2(iii). We shall prove that (X_i, t_i) is α -T_2(iii), for $i \in \Lambda$. For some $i \in \Lambda$, let a_i be a fixed element in X_i. Suppose that $A_i = \{ x \in X = \Pi_{i \in \Lambda} X_i : x_j = a_j \text{ for some } i \neq j \}$. Then A_i is a subsets of X and therefore (A_i, t_{A_i}) is a subspace of (X, t). Since (X, t) is α - T_2(iii) space. Then, we have also (A_i, t_{A_i}) is also α - T_2(iii) space. Further more, A_i is homeomorphic image of X_i. Hence it is clear that (X_i, t_i) is α -T_2(iii) space.

The proofs for (a) and (b) are similar.

4.14. Theorem:-

(a) (X, t) is α -T_2(i) $\Rightarrow$ (X, t) is α - T_2(iii)

$\Downarrow$ $\Downarrow$

(X, t) is α - T_1(i) $\Rightarrow$ (X, t) is α - T_1(iii) $\Rightarrow$ (X, t) is T_1(iv).

$\Downarrow$ $\Downarrow$ $\Downarrow$

(X, t) is α - T_0(i) $\Rightarrow$ (X, t) is α - T_0(iii) $\Rightarrow$ (X, t) is T_0(iv)

(b) (X, t) is α - T_2 (ii) $\Rightarrow$ (X, t) is α -T_2(iii).

$\Downarrow$ $\Downarrow$

(X, t) is α - T_1(ii) $\Rightarrow$ (X, t) is α - T_1(iii) $\Rightarrow$ (X, t) is T_1(iv).

$\Downarrow$ $\Downarrow$ $\Downarrow$

(X, t) is α - T_0(ii) $\Rightarrow$ (X, t) is α - T_0(iii) $\Rightarrow$ (X, t) is T_0(iv).

Proof : - The proofs of (a) and (b) are easy.

4.15. Theorem:- Let (X , t) and (Y , s) be two fuzzy topological spaces and $f : X \longrightarrow Y$ be a one-one , onto and open map , then

(a) (X , t) is α -T_2(i) $\Rightarrow$ (Y , s) is α -T_2(i) .

(b) (X , t) is α -T_2(ii) $\Rightarrow$ (Y , s) is α -T_2(ii) .

(c) (X , t) is α -T_2(iii) $\Rightarrow$ (Y , s) is α -T_2(iii) .

Proof:- Suppose (X , t) is α -T_2(i) . We shall prove that (Y , s) is α -T_2(i) . Let $y_1 , y_2 \in Y$ with $y_1 \neq y_2$. Since f is onto then , $\exists\ x_1 , x_2 \in X$ with $f(x_1) = y_1$, $f(x_2) = y_2$ and $x_1 \neq x_2$ as f is one-one . Again since (X , t) is α -T_2(i) , for $\alpha \in I_1$, $\exists$ u , v $\in$ t such that $u(x_1) = 1 = v(x_2)$ and $u \cap v \leq \alpha$.

Now $f(u)\,(y_1) = \{ \text{Sup}\ u(x_1) \quad : f(x_1) = y_1 \}$

$= 1$

$f(v)\,(y_2) = \{ \text{Sup}\ v(x_2) \quad : f(x_2) = y_2 \}$

and $f(u \cap v)\,(y_1) = \{ \text{Sup}\ (u \cap v)\,(x_1) \quad : f(x_1) = y_1 \}$

$f(u \cap v)\,(y_2) = \{ \text{Sup}\ (u \cap v)\,(x_2) \quad : f(x_2) = y_2 \}$

Hence $f(u \cap v) \leq \alpha \Rightarrow f(u) \cap f(v) \leq \alpha$.

Since f is open then f(u) , f(v) $\in$ s. Now it is clear that $\exists$ f(u) , f(v) $\in$ s such that $f(u)\,(y_1) = 1$, $f(v)\,(y_2) = 1$ and $f(u) \cap f(v) \leq \alpha$. Hence (Y , s) is α- T_2(i) .

Similarly (b) and (c) can be proved.

4.16. Theorem :- Let (X , t) and (Y , s) be two fuzzy topological spaces and $f : X \longrightarrow Y$ be continuous and one-one map then ,

(a) (Y , s) is α -T_2(i) $\Rightarrow$ (X , t) is α - T_2(i) .

(b) (Y , s) is α -T_2(ii) $\Rightarrow$ (X , t) is α - T_2(ii) .

(c) (Y , s) is α -T_2(iii) $\Rightarrow$ (X , t) is α - T_2(iii) .

Proof:- Suppose (Y , s) is α - $T_2(i)$. We shall prove that (X , t) is α -$T_2(i)$. Let x_1 , $x_2 \in X$ with $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ in Y , as f is one-one . Since (Y , s) is α - $T_2(i)$, for $\alpha \in I_1$, then $\exists$ u, v$\in$ s such that $u(f(x_1)) = 1 = v(f(x_2))$and $u \cap v \leq \alpha$. This implies that $f^{-1}(u)(x_1) = 1$, $f^{-1}(v)(x_2) = 1$ and $f^{-1}(u \cap v) \leq \alpha$ ie $f^{-1}(u) \cap f^{-1}(v) \leq \alpha$, since u , v $\in$ s and f is continuous then $f^{-1}(u)$, $f^{-1}(v) \in t$. Now it is clear that $\exists$ $f^{-1}(u)$, $f^{-1}(v) \in t$ such that $f^{-1}(u)(x_1) = 1$, $f^{-1}(v)(x_2) = 1$ and $f^{-1}(u) \cap f^{-1}(v) \leq \alpha$. Hence (X , t) is α -$T_2(i)$.

Similarly (b) and (c) can be proved.

Chapter : -5

Regular Fuzzy Topological Space

5. Introduction:-

In this chapter, we introduce and study some Regular property in fuzzy topological spaces and obtain their several features.

5.1. Definition:- Let (X , t) be a fuzzy topological space and $\alpha \in I_1$.

(a) (X , t) is an α - FR(i) space $\Leftrightarrow \forall\, w \in t^c, \forall\, x \in X$, with $w(x) < 1$, $\exists\, u, v \in t$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

(b) (X , t) is an α -FR(ii) space $\Leftrightarrow \forall\, w \in t^c, \forall\, x \in X$, with $w(x) < 1$, $\exists\, u, v \in t$ such that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

(c) (X , t) is an α - FR(iii) space $\Leftrightarrow \forall\, w \in t^c, \forall\, x \in X$, with $w(x) = 0$, $\exists\, u, v \in t$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

(d) (X , t) is an α -FR(iv) space $\Leftrightarrow \forall\, w \in t^c, \forall\, x \in X$, with $w(x) = 0$, $\exists\, u, v \in t$ such that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$, $u \cap v \leq \alpha$.

5.2. Theorem:- The following implications are true :

$$\begin{array}{ccccc} & & (X, t) \text{ is } \alpha\text{ - FR(ii)} & & \\ & \nearrow & & \searrow & \\ (X, t) \text{ is } \alpha\text{ - FR(i)} & & & & (X, t) \text{ is } \alpha\text{ -FR(iv).} \\ & \searrow & & \nearrow & \\ & & (X, t) \text{ is } \alpha\text{ - FR(iii)} & & \end{array}$$

Proof:- First , suppose that (X , t) is α - FR(i). We shall prove that (X , t) is α - FR(ii) . Let $w \in t^c$, $x \in X$, with $w(x) < 1$. Since (X , t) is α -FR(i) , for $\alpha \in I_1$, $\exists$ u , v $\in$ t such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Now, we see that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Hence it is clear that (X , t) is α -FR(ii) .

Next, suppose that (X , t) is α -FR(i) . We shall prove that (X , t) is α - FR(iii) . Let $w \in t^c$, $x \in X$, with $w(x) = 0$. Then we have $w(x) < 1$. Since (X , t) is α - FR(i) , for $\alpha \in I_1$ $\exists$ u , v $\in$ t such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Now it is clear that (X , t) is α -FR(iii).

Again, suppose that (X , t) is α -FR(ii) . We shall prove that(X , t) is α -FR(iv). Let $w \in t^c$, $x \in X$, with $w(x) = 0$, Then clearly $w(x) < 1$. Since (X , t) is α -FR(ii) , for $\alpha \in I_1$, $\exists$ u , v $\in$ t such that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Hence it is clear that (X , t) is α -FR(iv).

Further, suppose that (X , t) is α - FR(iii) . We shall prove that (X , t) is α -FR(iv). Let $w \in t^c$ and $x \in X$, with $w(x) = 0$. Since (X , t) is α- Fr(iii) , for $\alpha \in I_1$, $\exists$ u , v $\in$ t such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Hence one can see that (X , t) is α -FR(iv).

Now , we give some examples to show the non implication among α- FR(i) , α - FR(ii) , α- FR(iii) and α - FR(iv) .

5.3. Example :- Let $X = \{ x , y \}$ and $u , v \in I^X$, where u , v are defined by $u(x) = 0.9$, $u(y) = 0$, $v(x) = 0.5$ and $v(y) = 1$. Consider the fuzzy topology t on X generated by

$\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $w = 1 - u$ and $\alpha = 0.7$, we see that (X, t) is α - FR(ii) but (X, t) is not α-FR(i) .

5.4. Example :- Let $X = \{x, y\}$ and $u, v \in I^X$, where u, v are defined by $u(x) = 0.2$, $u(y) = 0.3$, $v(x) = 0.3$, $v(y) = 0.2$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $w = 1 - u$ and $\alpha = 0.5$, we see that (X, t) is α - FR(iii) and (X, t) is α -FR(iv) , but (X, t) is not α - FR(ii). As there do not exist any $u, v \in t$ such that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

5.5. Example :- Let $X = \{x, y\}$ and $u, v, w \in I^X$, where u, v and w are defined by $u(x) = 0.9$, $u(y) = 0$, $v(x) = 0.5$, $v(y) = 1$, $w(x) = 1$, $w(y) = 0$. Consider the fuzzy topology t on X generated by $\{0, u, v, w, 1\} \cup \{\text{Constants}\}$. For $\alpha = 0.6$ and $p = 1 - w$, it is seen that (X, t) is α - FR(iv) but (X, t) is not α - FR(iii).

This completes the proof.

5.6. Theorem :- Let $0 \leq \alpha \leq \beta < 1$, then

(a) (X, t) is α -FR(i) $\Rightarrow$ (X, t) is β - FR(i) .

(b) (X, t) is α - FR(iii) $\Rightarrow$ (X, t) is β - FR(iii).

Proof :- First , suppose that (X, t) is α - FR(i). We shall prove that (X, t) is β - FR(i) . Let $w \in t^c$ and $x \in X$ with $w(x) < 1$. Since (X, t) is α - FR(i) , for $\alpha \in I_1$, $\exists\, u, v \in t$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Since $\alpha \leq \beta$, then $u \cap v \leq \beta$. So it is observed that (X, t) is β - FR(i).

Next, suppose that (X, t) is α - FR(iii) . We shall prove that (X, t) is β - FR(iii) . Let $w \in t^c$, $x \in X$ with $w(x) = 0$. Since (X, t) is α - FR(iii) , for $\alpha \in I_1$, $\exists\, u, v \in t$ such that $u(x) = 1$, $u(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Since $0 \leq \alpha \leq \beta < 1$, then $u \cap v \leq \beta$. Now it can be written as $\forall\, w \in t^c$, $\forall\, x \in X$, with $w(x) = 0$, $\exists\, u, v \in t$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \le \beta$. Hence it is clear that (X, t) is β - FR(iii).

Now, we give an example.

5.7. Example :- Let $X = \{x, y\}$ and $u, v \in I^X$, where u, v are defined by $u(x) = 1$, $u(y) = 0$, $v(x) = 0.7$, $v(y) = 1$. Consider the fuzzy topology t on X generated by $\{0, u, v, 1\} \cup \{\text{Constants}\}$. For $w = 1 - u$, $\alpha = 0.75$, $\beta = 0.6$. We see that (X, t) is β - FR(i) and (X, t) is β - FR(iii), but (X, t) is not α - FR(i) and (X, t) is not α -FR(iii).

5.8. Theorem :- Let (X, t) be a fuzzy topological space and

$$I_\alpha(t) = \{\, u^{-1}(\alpha, 1] \;:\; u \in t \,\}, \text{ then}$$

$$(X, t) \text{ is } 0 - \text{FR(i)} \Rightarrow (X, I_0(t)) \text{ is Regular.}$$

Proof :- Consider (X, t) be a 0 –FR(i). We shall prove that $(X, I_0(t))$ is Regular. Let V be a closed set in $I_0(t)$ and $x \in X$ such that $x \notin V$. Then $V^c \in I_0(t)$ and $x \in V^c$. So by the definition of $I_0(t)$, there exists an $u \in t$ such that $V^c = u^{-1}(0, 1]$ ie $u(x) > 0$. Since $u \in t$, then u^c is closed fuzzy set in t and $u^c(x) < 1$. Since (X, t) is 0 –FR(i), then $\exists\, v, w \in t$ such that $v(x) = 1$, $w \ge 1_{(u^c)^{-1}\{1\}}$, $v \cap w = 0$.

(a) Since $v, w \in t$ then $v^{-1}(0, 1], w^{-1}(0, 1] \in I_0(t)$ and $x \in v^{-1}(0, 1]$

(b) Since $w \ge 1_{(u^c)^{-1}\{1\}}$ then $w^{-1}(0, 1] \supseteq \left(1_{(u^c)^{-1}\{1\}}\right)^{-1}(0, 1]$.

(c) And $v \cap w = 0$, mean $(v \cap w)^{-1}(0, 1] = v^{-1}(0, 1] \cap w^{-1}(0, 1] = \phi$.

Now, we have

$$\left(1_{(u^c)^{-1}\{1\}}\right)^{-1}(0,1] = \{x : 1_{(u^c)^{-1}\{1\}}(x) \in (0,1]\}$$

$$= \{x : 1_{(u^c)^{-1}\{1\}}(x) = 1\}$$

$$= \{x : x \in (u^c)^{-1}\{1\}\}$$

$$= \{x : u^c(x) = 1\}$$

$$= \{x : u(x) = 0\}$$

$$= \{x : x \notin V^c\}$$

$$= \{x : x \in V\}$$

$$= V$$

Put $W = v^{-1}(0,1]$ and $W^* = w^{-1}(0,1]$, then $x \in W$, $W^* \supseteq V$ and $W \cap W^* = \phi$.

Hence it is clear that (X , $I_0(t)$) is Regular.

5.9. Theorem :- Let (X , t) be a fuzzy topological space $A \subseteq X$, and $t_A = \{ u/A : u \in t \}$,

then

$$1_{((u/A)^c)^{-1}\{1\}}(x) = (1_{(u^c)^{-1}\{1\}}/A)(x)$$

Proof :- Let w be a closed fuzzy set in t_A ie $w \in t_A^c$, then $u/A = w^c$, where $u \in t$.

Now we have

$$1_{((u/A)^c)^{-1}\{1\}}(x) = \begin{cases} 0 & \text{if} \quad x \notin ((u/A)^c)^{-1}\{1\} \\ 1 & \text{if} \quad x \in ((u/A)^c)^{-1}\{1\} \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad x \notin \{y : (u/A)^c(y) = 1\} \\ 1 & \text{if} \quad x \in \{y : (u/A)^c(y) = 1\} \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad (u/A)^c(x) < 1 \\ 1 & \text{if} \quad (u/A)^c(x) = 1 \end{cases}$$

$$= \begin{matrix} 0 & \text{if} \quad w(x) < 1 \\ 1 & \text{if} \quad w(x) = 1 \end{matrix}$$

Again $1_{(u^c)^{-1}\{1\}}(x) = \begin{cases} 0 & if \quad x \notin (u^c)^{-1}\{1\} \\ 1 & if \quad x \in (u^c)^{-1}\{1\} \end{cases}$

$$= \begin{cases} 0 & if \quad x \notin \{y : u^c(y) = 1\} \\ 1 & if \quad x \in \{y : u^c(y) = 1\} \end{cases}$$

$$= \begin{cases} 0 & if \quad u^c(x) < 1 \\ 1 & if \quad u^c(x) = 1 \end{cases}$$

Now $(1_{(u^c)^{-1}\{1\}}/A)(x) = \begin{cases} 0 & if \quad (u^c/A)(x) < 1 \\ 1 & if \quad (u^c/A)(x) = 1 \end{cases}$

$$= \begin{cases} 0 & if \quad (u/A)^c(x) < 1 \\ 1 & if \quad (u/A)^c(x) = 1 \end{cases}$$

$$= \begin{cases} 0 & if \quad w(x) < 1 \\ 1 & if \quad w(x) = 1 \end{cases}$$

Hence it is clear that $1_{((u/A)^c)^{-1}\{1\}}(x) = (1_{(u^c)^{-1}\{1\}}/A)(x)$.

5.10. Theorem :- Let (X, t) be a fuzzy topological space and $A \subseteq X$ and

$t_A = \{ u/A : u \in t \}$, then

(a) (X, t) is α-FR(i) $\Rightarrow$ (A, t_A) is α-FR(i).

(b) (X, t) is α-FR(ii) $\Rightarrow$ (A, t_A) is α-FR(ii).

(c) (X, t) is α-FR(iii) $\Rightarrow$ (A, t_A) is α-FR(iii).

(d) (X, t) is α-FR(iv) $\Rightarrow$ (A, t_A) is α-FR(iv).

Proof :- Let (X, t) be α-FR(i). We shall prove that (A, t_A) is α-FR(i). Let w be a closed fuzzy set in t_A, and $x^* \in A$ such that $w(x^*) < 1$. This implies that $w^c \in t_A$ and $w^c(x^*) > 0$. So there exists an $u \in t$ such that $u/A = w^c$ and clearly u^c is closed in t and $u^c(x^*) = (u/A)^c(x^*) = w(x^*) < 1$, ie $u^c(x^*) < 1$. Since (X, t) is α-FR(i), for $\alpha \in I_1$, $\exists\, v, v^* \in t$

such that $v(x^*) = 1$, $v^* \geq 1_{(u^c)^{-1}\{1\}}$ and $v \cap v^* \leq \alpha$. Since $v , v^* \in t$, then $v/A , v^*/A \in t_A$ and $v/A(x^*) = 1$, $v^*/A \geq (1_{(u^c)^{-1}\{1\}}/A)$ and $v/A \cap v^*/A = (v \cap v^*)/A \leq \alpha$.

But $1_{(u^c)^{-1}\{1\}}/A = 1_{((u/_A)^c)^{-1}\{1\}} = 1_{w^{-1}\{1\}}$, then $v^*/A \geq 1_{w^{-1}\{1\}}$. Hence it is clear that (A , t_A) is α - FR(i).

The proofs of (b) , (c) and (d) are similar.

5.11. Theorem :- Let (X , T) be a topological space. Consider the following statements .

(1) (X , T) is a Regular space.

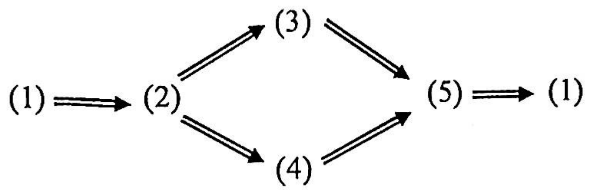
(2) (X , ω(T)) is α -FR(i) .

(3) (X , ω(T)) is α -FR(ii) .

(4) (X , ω(T)) is α -FR(iii) .

(5) (X , ω(T)) is α -FR(iv) .

Then



Proof :- First , suppose that (X , T) be regular space . We shall prove that (X , ω(T)) is α -FR(i) . Let w be a closed fuzzy set in ω(T) and $x \in X$ such that $w(x) < 1$, then $w^c \in \omega(T)$ and $w^c(x) > 0$. Now we have $(w^c)^{-1}(0 ,1] \in T$, $x \in (w^c)^{-1} (0 , 1]$. Also it is clear that $[(w^c)^{-1}(0, 1]]^c = w^{-1}\{1\}$ be a closed in T and $x \notin w^{-1}\{1\}$. Since (X , T) is Regular , then $\exists$ $V , V^* \in T$ such that $x \in V , V^* \supseteq w^{-1}\{1\}$ and $V \cap V^* = \phi$. But by the definition of lower semi continuous function $1_V , 1_{V^*} \in \omega(T)$ and $1_V(x) = 1$, $1_{V^*} \supseteq 1_{w^{-1}\{1\}}$, $1_V \cap 1_{V^*} = 1_{V \cap V^*} = 0$. Put $u = 1_V$ and $v = 1_{V^*}$, then , it is clear that $u(x) = 1$, $v \supseteq 1_{w^{-1}\{1\}}$ and $u \cap v \leq \alpha$.

Hence (X , ω(T)) is α - FR(i).

It can easily be shown that (2) $\Rightarrow$ (3) , (3) $\Rightarrow$ (5) , (2) $\Rightarrow$ (4) , (4) $\Rightarrow$ (5) .

We therefore prove that (5) $\Rightarrow$ (1).

Let $(X , \omega(T))$ be α - FR(iv) . We shall prove that (X , T) is Regular space . Let $x \in X$, V be a closed set in T , such that $x \notin V$. This implies that $V^c \in T$ and $x \in V^c$. But from the definition of $\omega(T)$, $1_{V^c} \in \omega(T)$, and $(1_{V^c})^c = 1_V$ closed in $\omega(T)$ and $1_V(x) = 0$. Since $(X , \omega(T))$ is α - FR(iv) , for $\alpha \in I_1$, $\exists$ u , v $\in \omega(T)$ such that $u(x) > \alpha$, $v \geq 1_{(1_V)^{-1}\{1\}}$ $= 1_V$ and $u \cap v \leq \alpha$. Since u , v $\in \omega(T)$, then $u^{-1}(\alpha , 1]$, $v^{-1}(\alpha , 1] \in T$ and $x \in u^{-1}(\alpha , 1]$. Since $v \geq 1_V$, then $v^{-1}(\alpha , 1] \supseteq (1_V)^{-1} (\alpha , 1] = V$, and $u \cap v \leq \alpha$ implies $(u \cap v)^{-1}(\alpha , 1] = u^{-1}(\alpha , 1] \cap v^{-1} (\alpha , 1] = \phi$. Now from the above it is clear that (X , T) is Regular space.

Thus it is seen that α - FR(p) is a good extension of its topological counter part . (p = i , ii , iii, iv).

5.12. Theorem: - Let (X , t) and (Y , s) be two fuzzy topological spaces and $f : X \longrightarrow Y$ be continuous , one-one , onto and open map then ,

(a) (X , t) is α - FR(i) $\Rightarrow$ (Y , s) is α - FR(i) .

(b) (X , t) is α - FR(ii) $\Rightarrow$ (Y , s) is α - FR(ii) .

(c) (X , t) is α - FR(iii) $\Rightarrow$ (Y , s) is α - FR(iii) .

(d) (X , t) is α - FR(iv) $\Rightarrow$ (Y , s) is α - FR(iv) .

Proof :- Suppose (X , t) be α - FR(i) . We shall prove that (Y , s) is α - FR(i) . Let $w \in s^c$ and $p \in Y$ such that $w(p) < 1$, $f^{-1}(w) \in t^c$ as f is continuous and $x \in X$ such that $f(x) = p$ as f is one-one and onto . Hence $f^{-1}(w) (x) = w (f(x)) = w(p) < 1$. Since (X , t) is α - FR(i) , for

$\alpha \in I_1$, then $\exists$ u , v $\in$ t such that $u(x) = 1$, $v(y) = 1$, $y \in \{ f^{-1}(w) \}^{-1}\{1\}$ and $u \cap v \leq \alpha$.

This implies that

$$f(u)(p) = \{ \text{Sup } u(x) \ : f(x) = p \} = 1 .$$

and $f(v) f(y) = \{ \text{Sup } v(y) \} = 1$ as $f(f^{-1}(w)) \subseteq w \Rightarrow f(y) \in w^{-1}\{1\}$

and $f(u \cap v) \leq \alpha$ as $u \cap v \leq \alpha \Rightarrow f(u) \cap f(v) \leq \alpha$.

Now it is clear that $\exists$ f(u) , f(v) $\in$ s such that $f(u)(x) = 1$, $f(v)(f(y)) = 1$, $f(y) \in w^{-1}\{1\}$ and $f(u) \cap f(v) \leq \alpha$. Hence (Y , s) is α - FR(i).

Similarly (b) , (c) and (d) can be proved .

5.13. Theorem:- Let (X , t) and (Y , s) be two fuzzy topological spaces and $f : X \longrightarrow Y$ be a continuous , one-one , onto and closed map then ,

(a) (Y , s) is α - FR(i) $\Rightarrow$ (X , t) is α - FR(i) .

(b) (Y , s) is α - FR(ii) $\Rightarrow$ (X , t) is α - FR(ii) .

(c) (Y , s) is α - FR(iii) $\Rightarrow$ (X , t) is α - FR(iii) .

(d) (Y , s) is α - FR(iv) $\Rightarrow$ (X , t) is α - FR(iv) .

Proof:- Suppose (Y , s) be α - FR(i) . We shall prove that (X , t) is α - FR(i) . $w \in t^c$ and $x \in X$ with $w(x) < 1$, then $f(w) \in s^c$ as f is closed and we find $p \in Y$ such that $f(x) = p$ as f is one-one. Now we have $f(w)(p) = \{ \text{Sup } w(x) \ : f(x) = p \} < 1$. Since (Y , s) is α - FR(i) , for $\alpha \in I_1$, then $\exists$ u , v $\in$ s such that $u(f(x)) = 1$, $v(y) = 1$, $y \in (f(w))^{-1}\{1\}$ and $u \cap v \leq \alpha$. This implies that $f^{-1}(u), f^{-1}(v) \in t$ as f is continuous and u , v $\in$ s. Now $f^{-1}(u)(x) = u(f(x)) = u(p) = 1$ and $f^{-1}(v)(q) = v(f(q)) = v(y) = 1$ as $f(q) = y$, $y \in (f(w))^{-1}\{1\}$ ie $f(p) \in (f(w))^{-1}\{1\}$ $\Rightarrow q \in w^{-1}\{1\}$ and $f^{-1}(u) \cap f^{-1}(v) = f^{-1}(u \cap v) \leq \alpha$ as $u \cap v \leq \alpha$. Now we observe that

$\exists\, f^{-1}(u)$, $f^{-1}(v) \in t$ such that $f^{-1}(u)(x) = 1$, $f^{-1}(v)(q) = 1$, $q \in w^{-1}\{1\}$ and $f^{-1}(u) \cap f^{-1}(v) \leq \alpha$.

Hence (X , t) is α -FR(i) .

Similarly (b) , (c) and (d) can be proved.

Chapter : - 6

Normal Fuzzy Topological Space

6. Introduction:-

In this chapter, we introduce and study Normal property in fuzzy topological spaces and obtain their several features.

6.1. Definition :- Let (X , t) be a fuzzy topological space and $\alpha \in I_1$.

(a) (X , t) is α - FN(i) space $\Leftrightarrow \forall\, w , w^* \in t^c$, with $w \cap w^* \leq \alpha$, $\exists\, u , u^* \in t$ such that $u(x) = 1$, $\forall\, x \in w^{-1}\{1\}$, $u^*(y) = 1$, $\forall\, y \in w^{*\,-1}\{1\}$ and $u \cap u^* = 0$.

(b) (X , t) is α - FN(ii) space $\Leftrightarrow \forall\, w , w^* \in t^c$ with $w \cap w^* = 0$, $\exists\, u , u^* \in t$ such that $u(x) = 1$, $\forall\, x \in w^{-1}\{1\}$, $u^*(y) = 1$, $\forall\, y \in w^{*\,-1}\{1\}$ and $u \cap u^* \leq \alpha$.

(c) (X , t) is a α - FN(iii) space $\Leftrightarrow \forall\, w , w^* \in t^c$ with $w \cap w^* \leq \alpha$, $\exists\, u , u^*, \in t$ such that $u(x) = 1$, $\forall\, x \in w^{-1}\{1\}$, $u^*(y) = 1$, $\forall\, y \in w^{*-1}\{1\}$ and $u \cap u^* \leq \alpha$.

6.2. Theorem :- The following implications are true:

(X , t) is 0 - FN(i) $\Leftrightarrow (X , t)$ is 0 - FN(ii) $\Leftrightarrow (X , t)$ is 0 -FN(iii) .

Proof:- Suppose (X , t) be α -FN(i). We shall prove that (X , t) is 0 -FN(ii). Let $w , w^* \in t^c$ with $w \cap w^* = 0$. This implies that $w \cap w^* \leq 0$. Since (X , t) is 0 - FN(i), for $\alpha \in I_1$, $\exists\, u , u^* \in t$ such that $u(x) = 1$, $\forall\, x \in w^{-1}\{1\}$, $u^*(y) = 1$, $\forall\, y \in w^{*\,-1}\{1\}$ and $u \cap u^* = 0$. It can be written as $u \cap u^* \leq 0$. Hence it is clear that (X , t) is α - FN(ii).

Conversely, suppose that (X , t) is 0 - FN(ii) . We shall prove that (X , t) is0- FN(i) . Let $w , w^* \in t^c$ with $w \cap w^* \le 0$, ie $w \cap w^* = 0$. Since (X , t) is 0 –FN(ii) , $\exists\, u , u^* \in t$ such that $u(x) = 1 , \forall\, x \in w^{-1}\{1\}$, $u^*(y) = 1 , \forall\, y \in w^{*-1}_{\{1\}}$, and $u \cap u^* \le 0$, ie $u \cap u^* = 0$. Hence it is clear that (X , t) is 0 – FN(i) .

Next, suppose that (X , t) is 0 – FN(ii) . We shall prove that (X , t) is 0 – FN(iii). Let $w , w^* \in t^c$ with $w \cap w^* \le 0$, ie $w \cap w^* = 0$. Since (X , t) is 0 – FN(ii) , $\exists\, u , u^* \in t$ such that $u(x) = 1, \forall\, x \in w^{-1}\{1\}$, $u^*(y) = 1, \forall\, y \in w^{*-1}\{1\}$and $u \cap u^* \le 0$. Hence it is clear that (X , t) is α-FN(iii).

Conversely, suppose that (X , t) is 0 – FN(iii) . We shall prove that (X , t) is 0 – FN(ii). Let $w , w^* \in t^c$ with $w \cap w^* = 0$, ie $w \cap w^* \le 0$. Since (X , t) is 0 – FN(iii) , $\exists\, u , u^* \in t$ such that $u(x) = 1, \forall\, x \in w^{-1}\{1\}$, $u^*(y) = 1, \forall\, y \in w^{*-1}\{1\}$ and $u \cap u^* \le 0$. Hence it is clear that (X , t) is α-FN(ii).

6.3. Theorem :- If $0 \le \alpha \le \beta < 1$, then

(a) (X , t) is β - FN(i) $\Rightarrow (X , t)$ is α - FN(i) .

(b) (X , t) is α - FN(ii) $\Rightarrow (X , t)$ is β - FN(ii).

Proof :- First , suppose that (X , t) is β - FN(i) . We shall prove that (X , t) is α - FN(i) . Let $w , w^* \in t^c$ with $w \cap w^* \le \alpha$. Since $0 \le \alpha \le \beta < 1$, then $w \cap w^* \le \beta$. Since (X , t) is β - FN(i), for $\beta \in I_1$, $\exists\, u , u^* \in t$ such that $u(x) = 1, \forall\, x \in w^{-1}\{1\}$, $u^*(y) = 1, \forall\, y \in w^{*-1}\{1\}$ and $u \cap u^* = 0$. Hence it is clear that (X , t) is a α - FN(i).

Next, suppose that (X , t) is α - FN(ii) . We shall prove that (X , t) is β - FN(ii) . Let $w , w^* \in t^c$ with $w \cap w^* = 0$. Since (X , t) is α -FN(ii) , $\exists\, u , u^* \in t$ such that $u(x) = 1,\ x \in w^{-1}\{1\}$, $u^*(y) = 1 , y \in w^{*-1}\{1\}$ and $u \cap u^* \le \alpha$. Since $0 \le \alpha \le \beta <1$, then $u \cap u^* \le \beta$. Hence it is clear that (X , t) is β - FN(ii) .

6.4. Theorem :- Let (X , t) be a fuzzy topological space , and $I_\alpha(t) = \{ u^{-1}(\alpha , 1] \ : u \in t \}$, then

(X , t) is $0 -$ FN(iii) space $\Rightarrow (X , I_0(t))$ is Normal space .

Proof :- Suppose that (X , t) be a α - FN(iii) space . We shall prove that $(X , I_0(t))$ is Normal space . Let V , V^* be closed set in $I_0(t)$, and $V \cap V^* = \phi$. Then $V^c , V^{*c} \in I_0(t)$ and $(V \cap V^*)^c = V^c \cup V^{*c} = X$. Since $V^c , V^{*c} \in I_0(t)$, then , $\exists\ u , u^* \in t$ such that $V^c = u^{-1}(0 , 1]$ and $V^{*c} = u^{*-1}(0 , 1]$ and $u^{-1}(0 , 1] \cup u^{*-1}(0 , 1] = V^c \cup V^{*c} = X$. Hence $(u \cup u^*)^{-1}(0 , 1] = X$. Now we find $u^c , u^{*c} \in t^c$ such that $((u \cup u^*)^{-1} (0 , 1])^c = \phi$. This implies that $(u \cup u^*)^c = u^c \cap u^{*c} = 0$. Since (X , t) is 0 - FN(iii) , $\exists\ v , v^* \in t$ such that $v \geq 1_{(u^c)^{-1}\{1\}}$, $v^* \geq 1_{((u^*)^c)^{-1}\{1\}}$, $v \cap v^* = 0$. But from the definition of $I_0(t)$, $v^{-1}(0 , 1]$, $v^{*-1}(0 , 1] \in I_0(t)$, and we get $v^{-1}(0 , 1] \supseteq 1_{(u^c)^{-1}\{1\}} (0 , 1]$, $v^{*-1}(0 , 1] \supseteq 1_{((u^*)^c)^{-1}\{1\}} (0 , 1]$, $(v \cap v^*)^{-1}(0 , 1] = \phi$. Put $W = v^{-1}(0 , 1]$, $W^* = v^{*-1}(0 , 1]$ in $I_0(t)$. Then finally we find, $W \supseteq V$, $W^* \supseteq V^*$ and $W \cap W^* = \phi$. Hence it is clear that $(X , I_0(t))$ is Normal space.

6.5. Theorem :- Let (X , T) be a topological space consider the following statements:

(a) (X , T) be a Normal space

(b) $(X , \omega(T))$ be α - FN(i) space .

(c) $(X , \omega(T))$ be α - FN(ii) space .

(d) $(X , \omega(T))$ be α - FN(iii) space .

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a) .

Proof :- First suppose that (X , T) be a Normal space . We shall prove that $(X , w(T))$ be α - FN(i) space . Let w , w^* be closed in $w(T)$ and $w \cap w^* \leq \alpha$. Then it is clear that $w^c, w^{*c} \in w(T)$ and $(w \cap w^*)^c \geq 1 - \alpha > 0$. But from the definition of $w(T)$, $(w^c)^{-1}(0 , 1]$,

$(w^{*c})^{-1}(0,1] \in T$. Now we see that $((w \cap w^*)^c)^{-1}(0,1] = X$, and we also see that $((w^c)^{-1}(0,1])^c = w^{-1}\{1\}$ and $((w^{*c})^{-1}(0,1])^c = (w^*)^{-1}\{1\}$ be closed in T. Now $(((w \cap w^*)^c)^{-1}(0,1])^c = (w \cap w^*)^{-1}\{1\} = w^{-1}\{1\} \cap w^{*-1}\{1\} = \phi$. Since (X, t) is fuzzy Normal, then, $\exists\, V, V^* \in T$ such that $V \supseteq w^{-1}\{1\}$, $V^* \supseteq w^{*-1}\{1\}$ and $V \cap V^* = \phi$. But from the definition of $w(T)$, $1_V, 1_{V^*} \in w(T)$, $1_V \geq 1_{w^{-1}\{1\}}$, $1_{V^*} \geq 1_{(w^*)^{-1}\{1\}}$ and $1_{V \cap V^*} = 0$.

Hence $(X, w(T))$ is α - FN(i) space.

It can easily be shown that (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

We, therefore prove that (d) $\Rightarrow$ (a).

Suppose, that $(X, w(T))$ is α - FN(iii). We shall prove that (X, T) is Normal space. Let $V, V^* \in T^c$ and $V \cap V^{*c} = \phi$. Then it is clear that $1_V, 1_{V^*}$ be closed in $w(T)$ and $1_{V \cap V^*} = 0$. Since $(X, w(T))$ is α - FN(iii), then $\exists\, u, u^* \in w(T)$ such that $u \geq 1_V$, $u^* \geq 1_{V^*}$ and $u \cap u^* \leq \alpha$. But from the definition of $w(T)$, $u^{-1}(\alpha, 1]$, $u^{*-1}(\alpha, 1] \in T$ an $u^{-1}(\alpha, 1] \supseteq (1_V)^{-1}(\alpha, 1] = V$, $u^{*-1}(\alpha, 1] \supseteq (1_{V^*})^{-1}(\alpha, 1] = V^*$ and $u^{-1}(\alpha, 1] \cap u^{*-1}(\alpha, 1] = (u \cap u^*)^{-1}(\alpha, 1] = \phi$. Hence it is clear that (X, T) is Normal space.

Thus it is seen that α - FN(p) is a good extension of its topological counter part . $(p = i, ii, iii)$

6.6. Theorem :- Let (X, t) and (Y, s) be two fuzzy topological spaces and $f: X \longrightarrow Y$ be a continuous, one-one, onto and open map then,

(a) (X, t) is α -FN(i) $\Rightarrow$ (Y, s) is α - FN(i).

(b) (X, t) is α -FN(ii) $\Rightarrow$ (Y, s) is α - FN(ii).

(c) (X, t) is α -FN(iii) $\Rightarrow$ (Y, s) is α - FN(iii).

Proof :- Suppose (X, t) be α - FN(i). We shall prove that (Y, s) is α -FN(i). Let $w, w^* \in s^c$ with $w \cap w^* \le \alpha$ then $f^{-1}(w)$, $f^{-1}(w^*) \in t^c$ as f is continuous. Now $f^{-1}(w \cap w^*) \le \alpha$ as $w \cap w^* \le \alpha \Rightarrow f^{-1}(w) \cap f^{-1}(w^*) \le \alpha$. Since (X, t) is α - FN(i), for $\alpha \in I_1$, then $\exists\, u, u^* \in t$ such that $u(x) = 1$, $x \in (f^{-1}(w))^{-1}\{1\}$, $u^*(y) = 1$, $y \in (f^{-1}(w^*))^{-1}\{1\}$ and $u \cap u^* = 0$. This implies that $f(u)$, $f(v) \in s$ as f is open.

Now $f(u)(p) = \{ \text{Sup}\, u(x) \quad ; f(x) = p \}$, $f^{-1}(p) \in (f^{-1}(w))^{-1}\{1\}$

ie $f(u)(p) = 1$, $p \in w^{-1}\{1\}$

and $f(u^*)(q) = \{ \text{Sup}\, u^*(y) \quad ; f(y) = q \}$, $f^{-1}(q) \in (f^{-1}(w))^{-1}\{1\}$

ie $f(u^*)(q) = 1$, $q \in w^{*-1}\{1\}$ and $f(u) \cap f(u^*) = f(u \cap u^*) = 0$ as $u \cap u^* = 0$.

Now it is clear that $\exists\, f(u)$, $f(u^*) \in s$ such that $f(u)(p) = 1$, $p \in w^{-1}\{1\}$, $f(u^*)(q) = 1$, $q \in w^{*-1}\{1\}$ and $f(u) \cap f(u^*) = 0$. Hence (Y, s) is α - FN(i).

Similarly (b) and (c) can be proved.

6.7. Theorem :- Let (X, t) and (Y, s) be two fuzzy topological spaces and $f : X \longrightarrow Y$ be continuous, one-one, onto and closed map then,

(a) (Y, s) is α - FN(i) $\Rightarrow$ (X, t) is α - FN(i).

(b) (Y, s) is α - FN(ii) $\Rightarrow$ (X, t) is α - FN(ii).

(c) (Y, s) is α - FN(iii) $\Rightarrow$ (X, t) is α - FN(iii).

Proof:- Suppose (Y, s) be α - FN(i). We shall prove that (X, t) is α - FN(i). Let $w, w^* \in t^c$ with $w \cap w^* \le \alpha$, then $f(w)$, $f(w^*) \in s^c$ as f is closed and $f(w) \cap f(w^*) = f(w \cap w^*) \le \alpha$, as $w \cap w^* \le \alpha$. Since (Y, s) is α -FN(i), for $\alpha \in I_1$, then $\exists\, u, u^* \in s$ such that $u(x) = 1$, $x \in (f(w))^{-1}\{1\}$, $u^*(y) = 1$, $y \in (f(w^*))^{-1}\{1\}$ and $u \cap u^* = 0$. This implies that $f^{-1}(u)$, $f^{-1}(u^*) \in t$ as f is continuous and $u, u^* \in s$.

Now $f^{-1}(u)(p) = u(f(p)) = u(x) = 1$ as $f(p) = x \in (f(w))^{-1}\{1\}$

ie $f^{-1}(u)(p) = 1$, $p \in w^{-1}\{1\}$

$f^{-1}(u^*)(q) = u^*(f(q)) = u^*(y) = 1$ as $f(q) = y \in (f(w^*))^{-1}\{1\}$

ie $f^{-1}(u^*)(q) = 1$, $q \in w^{*-1}\{1\}$

and $f^{-1}(u) \cap f^{-1}(u^*) = f^{-1}(u \cap u^*) = 0$.

Now it is clear that $\exists\, f^{-1}(u), f^{-1}(u^*) \in t$ such that $f^{-1}(u)(p) = 1$, $p \in w^{-1}\{1\}$, $f^{-1}(u^*)(q) = 1$, $q \in w^{*-1}\{1\}$ and $f^{-1}(u) \cap f^{-1}(u^*) = 0$. Hence (X, t) is α - FN(i).

Similarly (b) and (c) can be proved.

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