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By

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DEDICATED

TO

MY PARENTS

Certificate

Certified that the thesis entitled "On Some Aspects of Topology" submitted by Swapan Kumar Das in fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics, University of Rajshahi, Rajshahi, has been completed under our supervision. We believe that this research work is an original one and it has not been submitted elsewhere for any degree.

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Statement of Originality

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University, and to the best of my knowledge and belief, does not contain any material previously published or written by another person except where due reference is made in the text.

Swapan Kumar Das

Abstract

A topological space is a non-empty set X together with a collection $\mathcal{T}$ of subsets of X satisfying the conditions:

(i) X, $\Phi \in \mathcal{T}$,

(ii) the union of any class of sets in $\mathcal{T}$ belongs to $\mathcal{T}$,

(iii) the intersection of a finite number of sets belongs to $\mathcal{T}$,

$\mathcal{T}$ is called a topology on X.

The thesis is a study of several variants of topology obtained by generalizing some its aspects, viz, the conditions (ii) and (iii). The variants which have been considered here are the following:

(1) a U-structure, (a topology in which the condition (iii) is omitted),

(2) an I-structure, (a topology in which the condition (ii) is omitted),

(3) a CU-structure, (a U-structure in which 'any class' in (ii) is replaced by 'a countable class'),

(4) a CUI-structure, (a topology in which 'any class' in (ii) is replaced by 'a countable class'),

(5) an FU-structure, (a U-structure in which ‘any class’ in (ii) is replaced by ‘a finite class’),

(6) an FUI-structure, (a topology in which ‘any class’ in (ii) is replaced by ‘a finite class’).

X together with the above structures (1) - (6) have been called a U- space, an I-space, a CU-space, a CUI-space, an FU-space and a FUI-space respectively.

Among these, U-spaces and I-spaces have been defined and studied earlier by others and have been called **supratopological spaces** and **infratopological spaces** respectively. Our studies of these spaces in this thesis have considerably larger breadth and depth.

The thesis has been divided into seven chapters. The first six chapters give detailed study of general properties, different kinds of compactness and compactification, several kinds of connectedness, various separation properties, projectives in some categories of U-spaces. The U-space version of most of the well-known and the important theorems for topological spaces have been proved to be valid. Very many suitable examples and counter examples have been constructed. In the last chapter the other kinds of the above-mentioned spaces have been dealt with. A few properties have been established and a few examples have been provided. Most of the properties proved for U-spaces do hold for these spaces as well. But these have not been stated and proved to avoid monotony or repetitions.

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CHAPTER – 1

U-spaces and U-continuous Functions

Introduction

The concept of a U-space in a less general form had been considered earlier by some authors as a supra- topological space in [4], [9], [27], [28], [38]. In this chapter we have introduced the notions: U-spaces and three types of continuous functions for these spaces. We have obtained some characterizations and proved some properties of U-spaces and continuous functions. While some of the properties of U-spaces studied here have been studied by the above-mentioned authors for supra-topological spaces, we have probed deeper and proved newer properties for the more general set-up, namely, U-spaces. We have also defined compact U-spaces, Hausdorff U-spaces and studied their properties.

Semi-open sets, pre-open sets, α-open sets, β-open sets, δ-open sets, locally open sets and locally closed sets play an important part in the researches of generalizations of continuity in topological spaces. The collections of some these sets form U-structures while the others do not. We have verified these facts.

Preliminaries

Definition 1.1 A topology on a set X is a collection $\mathscr{T}$ of subsets of X having the following properties:

(i) Φ and X are in $\mathscr{T}$

(ii) Any union of members of $\mathscr{T}$ is in $\mathscr{T}$.

(iii) Any intersection of finite members of $\mathscr{T}$ is in $\mathscr{T}$.

The ordered pair (X, $\mathscr{T}$) is called a **topological space.** Shortly we can write X. The members of $\mathscr{T}$ are called open sets and the complement of an open set is called a closed set.

Example 1.1 Let X = {a, b, c}, $\mathscr{T}_1$ = {{a},{a, b}, X, Φ}, $\mathscr{T}_2$ = {X, Φ, {a, b}, {b, c}, {b}}. Then $\mathscr{T}_1$ and $\mathscr{T}_2$ are topologies on X.

Example 1.2 Let X = $\mathbb{R}$, the set of all real numbers, and $\mathscr{T}$ = { $\mathbb{R}$, Φ, all unions of intervals}. Then $\mathscr{T}$ is a topology on $\mathbb{R}$, called the **usual topology** on $\mathbb{R}$. $\mathbb{R}$, together with the usual topology, will be called **the real line.**

Definition 1.2 For a topological space X and a subset A of X, the closure of A and the interior of A denoted by **ClA** and **IntA** respectively are defined by **ClA** = the intersection of all closed supersets of A, **IntA** = the union of all open subsets of A.

Definition 1.3 [28] A subset A is said to be **pre-open** if A $\subseteq$ Int (Cl(A)).

Every open set is pre-open but the converse is not true.

Example 1.3 Let X be the real line $\mathbb{R}$ and A = $\mathbb{Q}$, the set of all rational numbers. Then $\mathbb{Q}$ is not open in X, but $\mathbb{Q} \subseteq$ Int (Cl($\mathbb{Q}$)) = $\mathbb{R}$. So that $\mathbb{Q}$ is pre-open.

The family of all pre-open sets in X is denoted by PO(X).

Definition 1.4 [19] A subset A is said to be **semi-open** set if A$\subseteq$ Cl(Int(A)).

Clearly, every open set is semi-open. However, the converse is not true.

Example 1.4 X = The real line $\mathbb{R}$ and A = (0, 1] or [0, 1) or [0, 1]. Then A is not open. Now Cl (Int(A)) = [0, 1], and so, A is semi-open.

The family of all semi-open sets in X is denoted by SO(X).

Definition 1.5 [34] $A \subseteq X$, A is an **α-open** set if $A \subseteq Int(Cl(Int(A)))$.

Every open set is α-open set but the converse is not true.

Example 1.5 Let X = {a, b, c, d} and $\mathscr{T}$ = {X, Φ, {a, b}, {a, b, c}} and let A = {a, b, d}. Then A is not open but Int (Cl (Int(A))) = X and so $A \subseteq$ Int (Cl (Int(A))), i.e. A is an α-open set.

Example 1.6 Let X = The real line $\mathbb{R}$, A = (0, 1] ∪ (1, 2) is not open but Int (Cl (Int(A))) = (0, 2) ⊇ A. Therefore A is an α-open set.

The family of all α-open set in X is denoted by α(X).

Definition 1.6 [34] A subset A is said to be **β-open set** if $A \subseteq$ Cl Int (Cl(A))).

Every open set is β-open set but the converse is not true.

Examples 1.7, 1.8 and 1.9 prove this statement.

Example 1.7 A is open ⇒ A = Int(A) ⇒ $A \subseteq Int(Cl(A)) \subseteq Cl (Int(Cl(A)))$. ⇒ A is β-open.

Example 1.8 If X = The real line $\mathbb{R}$ and A = (0, 1], then A is not open.

However, since Cl (Int(Cl(A))) = [0, 1] & (0, 1] ⊆ [0, 1], A is β-open.

Example 1.9 Let X = {a, b, c, d}, $\mathscr{T}$ = {X, Φ, {a, b, c}} and let A= {a, d}. Then A is not open, but Cl (Int(Cl(A))) = X $\supseteq$ A and A is a β-open set.

The family of all β-open set in X will be denoted by β(X).

Definition 1.7 [40] A $\subseteq$ X is a **δ-open set** if Int(Cl(A)) $\subseteq$ Cl(Int(A)).

Every open set is **δ**-open but the converse is not true.

Example 1.10 Let X = The real line $\mathbb{R}$ and let A = (0, 1]. Then A is not open. However, Cl(Int(A)) = [0, 1] and Int(Cl(A)) = (0, 1). Therefore Int(Cl(A)) $\subseteq$ Cl(Int(A)). Hence A is δ-open.

The family of all δ-open set in X is denoted by δ(X).

Definition 1.8 [23] Let X be a topological space with topology $\mathscr{T}$ and A be a subset of X. A is said to be **locally open** if A = G ∪ F, for an open subset G and a closed subset F of X.

Every open set is locally-open set but the converse is not true.

Example 1.11 Let X = The real line $\mathbb{R}$ and let A = (0, 2]. Then A is not open, but A = (0, 1) ∪ [1, 2], and so, A is locally open. Also, If A = (0, 1], then A is not open, but A = (0, 1) ∪ [½, 1], then A is locally open.

The family of all locally-open set in X will be denoted by LO(X).

Definition 1.9 [17] A subset A of a topological space X is said to be **b-open** (resp. ***b- open, b**- open, **b- open**) set if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$ (resp. $A \subseteq Cl(Int(A)) \cap Int(Cl(A))$, $A \subseteq Int(Cl(Int(A))) \cup Cl(Int(Cl(A)))$, $A \subseteq Int(Cl(Int(A))) \cap Cl(Int(Cl(A)))$).

Definition 1.10 [6] $A \subseteq X$, A is said to be **locally closed** if $A = G \cap F$, for some open subset G and closed subset F of X.

Every open set is locally-closed set but the converse is not true.

Example 1.12 Let X = The real line $\mathbb{R}$ and let A = (0, 2]. Then A is not open, but A = (0, 3) ∩ [-1, 2], and so, A is locally closed. Also, If A = (0, 1], A is not open in $\mathbb{R}$, but (0, 1] = (0, 2) ∩ [-1, 1], A is locally closed set.

The family of all locally-closed set in X will be denoted by LC(X).

Definition 1.11 [36] A subfamily M of the power set P(X) of a nonempty set X is called a **minimal structure (briefly M-structure) on X** if, $\Phi \in M$ and $X \in M$.

By (X, M), we denote a nonempty subset X with a minimal structure M on X and call it **M-space**. Each member of M is said to be **M-open** and complement of an M-open set is said to be **M-closed set**.

Example 1.13 Let X = {a, b, c, d}, $\mathscr{M}$ = {X, Φ, {a, b}, {b, c}}.Then (X, $\mathscr{M}$) is an M - space.

U-space

Definition 1.12 A U-**structure** on a nonempty set X is a collection $\mathcal{U}$ of subsets of X having the following properties:

(i) Φ and X are in $\mathcal{U}$,

(ii) Any union of members of $\mathcal{U}$ is in $\mathcal{U}$.

The ordered pair (X, $\mathcal{U}$) is called a **U-space**. A U-space which is not a topological space is called a **proper U-space**. The members of $\mathcal{U}$ are called **U-open set** and the complement of U-open set is called **U- closed set**.

A U- structure and a U-space have been called a supratopology and a supratopological space respectively by some authors (see [4], [9], [27], [38])

In general we have

Topological space ⇒ U-space ⇒ M-space

Topological space ⇍ U-space ⇍ M-space

Example 1.14 Let X = {a, b, c, d}, $\mathcal{U}$ = {X, Φ, {a, b}, {a, c}, {a, b, c}}. Here (X, $\mathcal{U}$) is a U-space but not a topological space.

Example 1.15 Let X= {a, b, c, d} and $\mathscr{M}$ = {X, Φ, {a, b}, {b, c}}. Then (X, $\mathscr{M}$) is an M- space but not U-space and also not a topological space.

Example 1.16 Let X be a totally ordered set with an order relation ≤ and $\mathcal{U}$ is the set of all unions of the form {x∈X: x < a} and {x∈X: x > b}. Then $\mathcal{U}$ is called **order U-structure on X.**

Example 1.17 Let ℝ denote the real numbers and let $\mathcal{U}$ consist of the empty set, all open rays and their unions, then (ℝ, $\mathcal{U}$) is a U-space. This U-space will be called the **usual U-space** ℝ. We note that $\mathcal{U}$ is not a topology on ℝ, since (2,3) = (- ∞, 3)∩(2, ∞)∉ $\mathcal{U}$.

Definition 1.13 If (X,$\mathcal{U}$) is a U-space and $\Phi \neq A \subseteq X$.

Let $\mathcal{U}_A$ = {A∩G | G∈ $\mathcal{U}$ } is a U-structure in A. For, $\bigcup_{\alpha}(A \cap G_{\alpha}) = A \cap (\bigcup_{\alpha} G_{\alpha})$ and $\bigcup_{\alpha} G_{\alpha} \in \mathcal{U}$. Then (A, $\mathcal{U}_A$) is a **U-space and is called a U-subspace of (X, $\mathcal{U}$).**

Also, we say that A **is a U-subspace of X.**

Example1.18 Let X =(0,1) and $\mathcal{U}$ the union of the sets{(0,b) :b∈ℝ,0 < b < 1} and {(a,1) : a∈ℝ, 0 < a < 1}. Then (X, $\mathcal{U}$) is a U-space but not a topological space , since $\left(\frac{1}{2},1\right)\cap\left(0,\frac{2}{3}\right)=\left(\frac{1}{2},\frac{2}{3}\right)\notin\mathcal{U}$. In fact this is the U-space obtained by considering (0, 1) as a U-subspace of $\mathbb{R}$ with the usual U-structure.

In the usual U-space $\mathbb{R}$, every singleton set {a} is closed in $\mathbb{R}$, since {a} = $(-\infty, a]\cap[a, \infty)$. **However, every finite set need not be closed.**

Definition 1.14 A sub collection $\mathcal{B}$ of $\mathcal{P}$(X) is called a **U-base** of a U-space X if any U-open set of X can be written as a union of members of $\mathcal{B}$. In this case we called the U-space X is generated by $\mathcal{B}$.

Example1.19 Let X = {a, b, c, d,e}, $\mathcal{U}$ = {X, Φ, {a},{a, b},{b, c},{a, b, c}}.

Then β = { X, Φ, {a},{a, b},{b, c}}.

Remark 1.1 Let X be a topological space. Let the classes of all b-open (resp. *b-open, b**-open, **b-open) sets in X be denoted by b(X) (resp. *b(X), b**(X), **b(X)). We shall now consider which of (X, PO(X)), (X, SO(X)), (X, β(X)), (X, LO(X)), (X, LC(X)), (X, α(X)), (X, δ(X)) and (X, b(X)), (X, *b(X)), (X, b**(X)), (X, **b(X)) are **M-spaces and which are U-spaces,** where the notations are usual**:**

(i): (X, α(X)) is a topological space, [34]. So, it is both an M-space and U-space.

(ii):- (X, PO(X)) is a U-space but not a topological-space, [12].

(iii):- (X, SO(X)) is a U-space but not a topological-space, ([23], Them.15(i), (ii)).

(iv):- (X, β(X)) is a U-space, but not a topological space, ([23], Them.18(i)).

(v):- (X, LO(X)), (X, LC(X)), (X, δ(X)) are not U-spaces but are M-spaces, ([23], Them. 16(i), 17(i), 19(i)).

(vi) (X, b(X)), (X, *b(X)), (X, b**(X)), (X, **b(X)) are U-**spaces, [17].**

Remark 1.2 Let (X, $\mathcal{U}$) be a U-space. Let $\mathcal{T}_{\mathcal{U}}$ denote the topology generated by $\mathcal{U}$ on X. This will be called the **topology induced by $\mathcal{U}$**. Also, for any sub-collection or super-collection $\mathcal{U}$ of $\mathcal{T}$ in $\mathcal{P}$(X) which is closed under union is a U-structure on X. (X, $\mathcal{U}$) is supratopology on X, associated with $\mathcal{T}$. A. S. Mashhour and others have considered and studied these supratopologies associated with a topology. We have dealt with U-spaces in general.

Definition 1.15 Let (X, $\mathcal{U}$) be a U-space. For a subset A of X, the **U-closure of A** ($_U$**Cl(A)**) and the **U-interior of A** ($_U$**Int(A)**) are defined as follows:

$$_U\mathbf{Cl(A)} = \cap\{F : A \subseteq F, F^c \in \mathcal{U}\},\ _U\mathbf{Int(A)} = \cup\{U : U \subseteq A, U \in \mathcal{U}\}.$$

Clearly, we have $_U$**Cl(A)** is U- closed and $_U$**Int(A)** is U- open.

Lemma 1.1 Let X be a U-space and A a subset of X. Then $x \in {}_U Cl(A)$ if and only if $G \cap A \neq \Phi$, for every U-open set G containing x.

Proof:

Necessity: Suppose that there exists a U-open set G containing x such that $G \cap A = \Phi$. Then $A \subseteq G^c$. Since G is U-open, G^c is U-closed. Therefore ${}_U Cl(A) \subseteq G^c$.

Hence $x \notin {}_U Cl(A)$.

Sufficiency: Suppose that $x \notin {}_U Cl(A)$. There exists a U-closed set F in X such that $A \subseteq F$ and $x \notin F$. Thus there exists a U-open set F^c in X which contains x and is such that $F^c \cap A = \Phi$.

Lemma 1.2 Let X be a U-space. For subsets A and B of X, the following hold:

(i) ${}_U Cl(A^c) = ({}_U Int(A))^c$, ${}_U Int(A)^c = ({}_U Cl(A))^c$

(ii) ${}_U Cl(\Phi) = \Phi$, ${}_U Cl(X) = X$, ${}_U Int(\Phi) = \Phi$ and ${}_U Int(X) = X$

(iii) If $A \subseteq B$ then ${}_U Cl(A) \subseteq {}_U Cl(B)$ and ${}_U Int(A) \subseteq {}_U Int(B)$

Proof:

(i) 1st Part:

Let $x \in {}_U Cl(A^c)$. This implies that for every neighborhood V of x, $V \cap A^c \neq \Phi \Rightarrow V \not\subset A$.

So x is not in ${}_U Int(A)$. This implies that $x \in ({}_U Int(A))^c$.

Therefore, ${}_{U}Cl(A^c) \subseteq ({}_{U}Int(A))^c$.

Again let $x \in ({}_{U}Int(A))^c \Rightarrow x \notin {}_{U}Int(A)$ So there does not exist any U-open set V containing x such that $V \subseteq A$. This implies that for every V containing x, $V \cap A^c \neq \Phi$. Hence $x \in {}_{U}Cl(A^c)$.

Therefore $({}_{U}Int(A))^c = {}_{U}Cl(A^c)$.

(i) 2nd Part:

$x \in {}_{U}Int(A^c)$. There exists U-open set V such that $x \in V \subseteq A^c \Rightarrow V \cap A = \Phi$ which means $x \notin {}_{U}Cl(A)$. Therefore $x \in ({}_{U}Cl(A))^c$.

Hence ${}_{U}Int(A^c) \subseteq ({}_{U}Cl(A))^c$

Again let $x \in ({}_{U}Cl(A))^c \Rightarrow x \notin {}_{U}Cl(A)$. Hence there exists U-open set V such that $V \cap A = \Phi \Rightarrow x \in V \subseteq A^c$. Hence $x \in {}_{U}Int(A^c$

Therefore, $({}_{U}Cl(A))^C = {}_{U}Int(A)^C$.

Proof (ii) is Obvious.

Proof (iii) is Obvious.

Continuous functions

As in the case of supratopological spaces [27], we define 3 types of continuity in the following.

Definition 1.16 Let **($X,\mathscr{U}$)** and **($Y,\mathscr{U}'$)** be two U-spaces. A function f: X $\rightarrow$Y is said to be **U-continuous** if for each U-open set H in Y, f^{-1}(H) is a U-open set in X.

Example 1.20 Let X={a, b, c, d }, $\mathscr{U}$ = {X, Φ, {a}, {a, b}, {a, c, d},{b, c, d}}

Y = {p, q, r}, $\mathscr{U}'$ = {Y, Φ, {p},{p, q},{p, r},{q, r}}. Let f: X $\rightarrow$Y be defined by f(a) = p, f(b) = q, f(c) = r, f(d) = r. Then f is U-continuous.

Definition 1.17 Let (X, $\mathscr{U}$) be a U-space and (Y, $\mathscr{T}$) a topological space. A function f: X $\rightarrow$Y is said to be $\bar{U}$ **-continuous** if for each open set H in Y, f^{-1}(H) is U-open set in X.

Example 1.21 Let X= {a, b, c}, $\mathscr{U}$ = {X, Φ, {a},{b, c},{a, c}}. Y **=** {p, q, r}, $\mathscr{T}$ = {Y, Φ, {p}, {p, q}, {p, r}}. Then (X, $\mathscr{U}$) is a U-space and (Y, $\mathscr{T}$) is a topological space. The function f: X $\rightarrow$Y is defined by f(a) = r, f(b) = q, f(c) = q. Then f is $\bar{U}$ -continuous.

Definition 1.18 Let **(X, $\mathscr{T}$)** be a topological space and **(Y, $\mathscr{U}$)** be a U-space. A function f: X $\rightarrow$Y is said to be U^* **-continuous** if for each U-open set H in Y, f^{-1}(H) is open set in X.

Example 1.22 Let X = {a, b, c, d}, $\mathscr{T}$ = {X,Φ,{a},{b},{c},{a, b},{b, c}, {a, c},{c, d},{a, b, c}, {a, c, d},{b, c, d}}. Then (X, $\mathscr{T}$) is a topological space.

Y = {p, q, r }**,** $\mathscr{U}$ = {Y,Φ,{p},{p, q},{p, r}, {q, r}}.

(Y, $\mathscr{U}$) is a U-space but not a topological space.

The function f: X $\rightarrow$Y is defined by f(a) = p, f(b) = q, f(c) = r, f(d) = r.

Then f is U^* -continuous.

Theorem 1.1 [27, p-503]. Let **(**X, $\mathscr{U}$**)** and **(**Y, $\mathscr{U}'$) be two U-spaces. For a function f: X $\rightarrow$Y the following properties are equivalent:

1) f is U-continuous ;

2) $f^{-1}(H) = {}_U\mathrm{Int}(f^{-1}(H))$ for every H $\in$ $\mathscr{U}'$;

3) $f({}_U\mathrm{Cl}(A)) \subseteq {}_U\mathrm{Cl}(f(A))$ for every subset A of X;

4) ${}_U\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}({}_U\mathrm{Cl}(B))$ for every subset B of Y ;

5) $f^{-1}({}_U\mathrm{Int}(B)) \subseteq {}_U\mathrm{Int}(f^{-1}(B))$ for every subset B of Y;

6) ${}_U\mathrm{Cl}(f^{-1}(K)) = f^{-1}(K)$ for every subset K of Y such that $K^c \in \mathscr{U}'$.

Proof:

(1) $\Rightarrow$ (2)

Let H $\in \mathcal{U}'$ and x $\in f^{-1}(H)$. Then f(x) $\in$ H. There exists G $\in \mathcal{U}$ containing x such that f(G) $\subseteq$ H. Thus x $\in$ G $\subseteq f^{-1}(H)$. This implies that x $\in {}_U\text{Int}(f^{-1}(H))$.

This shows that $f^{-1}(H) \subseteq {}_U\text{Int}(f^{-1}(H))$. By Lemma 1.2, we have ${}_U\text{Int}(f^{-1}(H)) \subseteq f^{-1}(H)$. Therefore, $f^{-1}(H) = {}_U\text{Int}(f^{-1}(H))$.

(2) $\Rightarrow$ (3)

Suppose that A is any subset of X and x $\in {}_U\text{Cl}(A))$ and H $\in \mathcal{U}'$ containing f(x). Then x $\in f^{-1}(H) = {}_U\text{Int}(f^{-1}(H))$.There exists G $\in \mathcal{U}$ such that x $\in$ G $\subseteq f^{-1}(H)$. Since x $\in {}_U\text{Cl}(A))$, by Lemma-1.1 $G \cap A \neq \Phi$ and $\Phi \neq f(G \cap A) \subseteq f(G) \cap f(A) \subseteq V \cap f(A)$. Since H $\in \mathcal{U}'$ containing f(x), f(x) $\in {}_U\text{Cl}(f(A))$ and hence $f({}_U\text{Cl}(A)) \subseteq {}_U\text{Cl}(f(A))$.

(3)$\Rightarrow$ (4)

Let B be any subset of Y.

Then we have $f({}_U\text{Cl}(f^{-1}(B))) \subseteq {}_U\text{Cl}(f(f^{-1}(B))) \subseteq {}_U\text{Cl}(B)$.

Therefore, we obtain ${}_U\text{Cl}(f^{-1}(B)) \subseteq f^{-1}({}_U\text{Cl}(B))$.

(4) $\Rightarrow$ (5)

Let B be any subset of Y .Then we have

$({}_U\text{Int}(f^{-1}(B))^c = {}_U\text{Cl}(f^{-1}(B^c)) \subseteq f^{-1}({}_U\text{Cl}(B^c)) = f^{-1}({}_U\text{Int}(f(B)))^c = (f^{-1}({}_U\text{Int}(B)))^c$.

Therefore, we obtain $f^{-1}({}_U\mathrm{Int}(B)) \subseteq {}_U\mathrm{Int}(f^{-1}(B))$.

(5) $\Rightarrow$ **(6)**

Let K be any subset of Y such that $K^c \in \mathcal{U}'$. by (5), we have $(f^{-1}(K))^c = f^{-1}({}_U\mathrm{Int}(K^c)) \subseteq {}_U\mathrm{Int}(f^{-1}(K^c)) = {}_U\mathrm{Int}(f^{-1}(K))^c = ({}_U\mathrm{Cl}(f^{-1}(K)))^c$.

Therefore, we have ${}_U\mathrm{Cl}(f^{-1}(K)) \subseteq f^{-1}(K) \subseteq {}_U\mathrm{Cl}(f^{-1}(K))$.

Thus, we obtain ${}_U\mathrm{Cl}(f^{-1}(K)) = f^{-1}(K)$.

(6) $\Rightarrow$ (1)

Let $x \in X$ and $H \in \mathcal{U}'$ containing f(x). By (6),

We have $(f^{-1}(H))^c = f^{-1}(H^c) = {}_U\mathrm{Cl}(f^{-1}(H^c)) = {}_U\mathrm{Cl}(f^{-1}(H))^c = ({}_U\mathrm{Int}(f^{-1}(H)))^c$. Hence we have $x \in f^{-1}(H) = {}_U\mathrm{Int}(f^{-1}(H))$. Therefore, there exists $G \in \mathcal{U}$ such that $x \in G \subseteq f^{-1}(H)$.

Thus $x \in G \in \mathcal{U}$ and $f(G) \subseteq H$. This shows that f is U-continuous.

Compact and Hausdorff U-spaces

Definition 1.19 Let (X, $\mathcal{U}$) be a U-space. A **U open cover** of a subset K of X is a collection {G_α} of U-open sets such that $K \subseteq \bigcup_\alpha G_\alpha$.

Definition 1.20 A U-space X is said to be **compact** if for every U-open cover of X has a finite sub-cover.

A subset K of a U-space X is said to be **compact** if every U-open cover of K has finite sub-cover.

Example 1.23 Let X = $\mathbb{N}$, $\mathcal{U}$ = {**2** $\mathbb{N}$**, 4** $\mathbb{N}$**, 8**$\mathbb{N}$**, 16**$\mathbb{N}$**, ……. 2**^{n} $\mathbb{N}$**, ...,**$\mathbb{N}$, Φ}. Then X is a compact U- space.

Let Φ ≠ A ⊆ X and $\mathfrak{G}$ be a U-open cover of A. Let n_0 be the smallest +ve integer such that $\mathbf{2}^{n_0}\mathbb{N} \in \mathfrak{G}$**.** Then $A \subseteq \mathbf{2}^{n_0}\mathbb{N}$**.** So {$\mathbf{2}^{n_0}\mathbb{N}$} is a finite sub-cover of $\mathfrak{G}$**.** Therefore every subset of X is compact.

Example 1.24 Let X = $\mathbb{N}$ and $\mathcal{U}$ = {m $\mathbb{N}$**:** m∈ $\mathbb{N}$ **}**∪**{**Φ}. Then X is a compact U- space.

Heine-Borel Theorem is an important result for compactness in Topology. This states that a subspace A of the real line $\mathbb{R}$ is compact if and only if A is closed and bounded.

However, the corresponding theorem does not hold for the usual U-space $\mathbb{R}$. For, $\mathbb{N}$ is a compact subspace of the usual U-space $\mathbb{R}$ but it is neither U-closed nor bounded.

As for topological spaces, we have

Theorem 1.2 Let **(X, $\mathcal{U}$**) and **(Y, $\mathcal{U}$** ') be two U-spaces. If f: X $\to$Y is a U-continuous function and B is a compact subspace of U-space X, then f(B) is compact.

Proof: Let $\{H_i : i \in I\}$ be any U-open cover of f(B). For each $x \in B$, there exists $i(x) \in I$ such that $f(x) \in H_{i(x)}$. Since f is U-continuous, there exists U-open set G(x) containing x such that $f(G(x)) \subseteq H_{i(x)}$. The family $\{G(x): x \in B\}$ is a U-open cover of B. Since B is compact, there exists a finite number of points, say $x_1, x_2, x_3, \ldots\ldots, x_n$ in B such that $B \subseteq \cup\{G(x_k) : x_k \in B, 1 \le k \le n\}$. Therefore, we have $f(B) \subseteq \cup\{f(G(x_k)) : x_k \in B, 1 \le k \le n\} \subseteq \cup\{H_{i(x_k)} : x_k \in B, 1 \le k \le n\}$.Thus f(B) is compact.

We can similarly prove that the following two results:

Theorem 1.3 Let (X, $\mathcal{U}$) be a U-space and (Y,$\mathcal{T}$) a topological space. If f: X $\to$Y is a $\overline{U}$ -continuous function and B is a compact subspace of U-space X, then f(B) is compact.

Theorem 1.4 Let (X, $\mathscr{T}$**)** be a topological space and (Y, $\mathscr{U}$) be a U-space. If f: X $\to$Y is a U^*-continuous function and B is a compact subspace of U-space X, then f(B) is compact.

Theorem 1.5 Every closed subspace of a compact U-space is compact.

Proof: Let X be a compact U-space and F be U-closed subspace of X. Let $\{V_i\}$ be U-open cover of F. Therefore F$\subseteq \cup V_i$ and $V_i = G_i \cap F$, where G_i is a U-open set of X. Therefore $F^c \cup \{G_i\}$ is a U-open cover of X. Since X is a compact U-space, there exists $i_1, i_2, i_3, \ldots\ldots\ldots, i_n$ such that

X = $F^c \cup G_{i_1} \cup G_{i_2} \cup \ldots\ldots\ldots \cup G_{i_n}$ $\therefore F \subseteq V_{i_1} \cup V_{i_2} \cup \ldots\ldots. \cup V_{i_n}$. Therefore F is compact.

Definition 1.21 A U-space X is **Hausdorff U-space** if for each x, y $\in$X, with x$\neq$y, there exists disjoint U-open sets G and H in X such that x$\in$G, y$\in$H.

Example 1.25 Let X = {a, b, c, d}, $\mathscr{U}$ = {{a},{d},{b, c},{b, d}, {a, d}, {a, c},{a, b, c},{b, c, d},{a, c, d}, {a, b, d}, X, Φ}.

Then (**X**, $\mathscr{U}$) is a Hausdorff U-space.

Example 1.26 The usual U-space $\mathbb{R}$ is Hausdorff , for any x, y $\in \mathbb{R}$, with x$\neq$y (say x < y), there exist two disjoint U- open sets (- $\infty, \frac{x+y}{2}$) and ($\frac{x+y}{2}, \infty$) containing x and y respectively.

Example 1.27 (Example of a U-space which is not Hausdorff)

Let X be an infinite set and $\mathscr{U} = \{X, \Phi, \{G \subseteq X \mid G^c$ is a singleton set$\}\}$ **.** Then (**X**, $\mathscr{U}$) is a proper U-space which is not Hausdorff.

Theorem 1.6 **E**very subspace of a Hausdorff U-space is Hausdorff.

Proof: It is trivial.

In topology we have

Theorem 1.7 **E**very compact subspace of a Hausdorff space is closed.

However, we note that the following.

Remark 1.3 **A** compact subset of a Hausdorff U-space need not be closed.

Its truth is proved by the following example:

Example 1.28 Let A = $\{1,2,3\} \subseteq \mathbb{R}$**,** then clearly A is a compact U-space, but it is not closed. Because every U-closed set in $\mathbb{R}$ is of the form $[b, \infty)$, or $(-\infty, a]$ or their intersection.

Definition 1.22 Let $(X, \mathcal{U})$ and $(Y, \mathcal{U}')$ be two U-spaces. A function $f: X \to Y$ is said to have a **strongly U-closed graph** (resp. **U-closed graph**) if for each $(x, y) \in (X \times Y) - G(f)$, there exists $V \in \mathcal{U}$ containing x and $W \in \mathcal{U}'$ containing y such that $[V \times {}_U Cl(W)] \cap G(f) = \Phi$ (resp. $[V \times W] \cap G(f)) = \Phi$).

Lemma 1.3 Let $(X, \mathcal{U})$ and $(Y, \mathcal{U}')$ be two U-spaces. A function $f: X \to Y$ has a strongly U-closed graph (resp. U-closed graph) if and only if for each $(x,y) \in (X \times Y) - G(f)$, there exist $V \in \mathcal{U}$ containing x and $W \in \mathcal{U}'$ containing y such that $f(V) \cap {}_U Cl(W) = \Phi$ (resp. $f(V) \cap W = \Phi$).

Proof: It is clear from the above definition. Since $f(x) \notin {}_U Cl(W)$ for any $x \in V$. Therefore, $f(V) \cap {}_U Cl(W) = \Phi$.

Theorem 1.8 Let $(X, \mathcal{U})$ and $(Y, \mathcal{U}')$ be two U-spaces. If a function $f: X \to Y$ is a U-continuous function and $(Y, \mathcal{U}')$ is a Hausdorff U-space, then G(f) is strongly U-closed.

Proof: Suppose that $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is a Hausdorff U-space, there exist disjoint sets V and W in $\mathcal{U}'$ containing y and f(x) respectively. By Lemma-1.1 we have ${}_U Cl(V) \cap W = \Phi$. Since f is U-continuous, there exists $U \in \mathcal{U}$ containing x such that $f(U) \subseteq W$.

This implies that $f(U) \cap {}_U Cl(V) = \Phi$ and by Lemma-1.3 G(f) is strongly U-Closed.

Theorem 1.9 Let (X, $\mathcal{U}$) and (Y, $\mathcal{U}'$) be two U-spaces. A function f: X $\rightarrow$Y is a surjective function with a strongly U-closed graph, then (Y, $\mathcal{U}'$) is a Hausdorff U-space.

Proof : Let y_1 and y_2 be any distinct points of Y. Then there exists $x_1 \in X$ such that $f(x_1) = y_1$.Then we have $(x_1, y_2) \in (X \times Y) - G(f)$. Since G(f) is strongly U-closed, there exists $V \in \mathcal{U}$ containing x, and $W \in \mathcal{U}'$ containing y_2 such that $f(V) \cap {}_U Cl(W) = \Phi$. Therefore, we have $y_1 = f(x_1) \in f(V) \subseteq ({}_U Cl(W))^c$. By Lemma-1.3 there exists $K \in \mathcal{U}'$ such that $y_1 \in K$ and $K \cap W = \Phi$. Moreover, we have $y_2 \in W$. This shows that (Y, $\mathcal{U}'$) is a Hausdorff U-space.

CHAPTER – 2

Separation and compactness in U-spaces

Introduction

In this chapter we have generalized to U-spaces the concepts of T_0-space, T_1-space, T_2-space, completely Hausdorff space, regular space, completely regular space, $T_{3\frac{1}{2}}$-space, normal space, T_4-space, completely normal space, locally compact space, compactification, and some results on topological spaces occurring in Munkres [33] and Majumdar & Akhter [24]. We have defined product of U-spaces, and given an example of a U-space which is regular but not Hausdorff and of a Hausdorff U-space which is not regular. We have generalized Tychonoff's theorem to U-spaces.

Separation in U-spaces

Definition 2.1 A U-space X is **T_0 -U-space** if for each x, y$\in$ X, with x $\neq$ y, there exist two distinct U-open sets G and H in X such that x$\in$G, y$\in$H.

Example 2.1 Let X = {a, b, c, d}, $\mathcal{U}$ = {{a}, {d}, {b, c}, {b, d}, {a, d}, {a, c}, {a, b, c}, {a, b, d}, {b, c, d}, {a, c, d}, X, Φ }. Then (X, $\mathcal{U}$) is a

T_0- U-space.

But (X, $\mathcal{U}$) is not a topological space.

Definition 2.2 A U- space X is **T_1 -U-space** if for each x, y$\in$ X, with x $\neq$ y, there exist two U-open sets G and H in X such that x$\in$G, y$\notin$ G and x$\notin$H, y$\in$H.

Example 2.2 Let X be an infinite set. Let $\mathcal{U}$ consist of the sets $\{a\}^c$, for each a$\in$X, and their unions. Clearly, X, $\Phi \in \mathcal{U}$. Then (X, $\mathcal{U}$) is a T_1-U-space. However, (X, $\mathcal{U}$) is not a topological space. Since $\{a\}^c \cap \{b\}^c = \{a, b\}^c \notin \mathcal{U}$.

Example 2.3 Let X = {a, b, c}, $\mathcal{U}$ = {{a, b}, {a, b, c}, {a, c}, Φ}. Then (X, $\mathcal{U}$) is a T_0 -U-space but not T_1-U-space.

Here T_1-U-space $\Rightarrow$ T_0 -U- space, but T_0 -U- space $\not\Rightarrow$ T_1-U-space.

Theorem 2.1 [24](Theorem-1.3, p. 100)

A U- space X is T_1-U-space iff every subset of X which consisting of exactly one point of X is U-closed.

Proof: Let X be a T_1-U-space and $x \in X$. We shall show that $X - \{x\}$ is U-open. Let $y \in X - \{x\}$. Since X is a T_1-U-space, for each $y \in X$, $y \neq x$, there exist U-open set G_y such that $y \in G_y$ but $x \notin G_y$. So, $G_y \subseteq X - \{x\}$. Therefore $X - \{x\}$ is U-open.

Conversely let every subset containing one point of X be U-closed and let $x, y \in X$ and $x \neq y$. Since $\{x\}$ and $\{y\}$ are U-closed, $G = X - \{y\}$, $H = X - \{x\}$ are U-open and $x \in G$, $y \notin G$ and $x \notin H$, $y \in H$. Therefore X is a T_1-U-space.

Definition 2.3 A Hausdorff U-space is called a **T_2-U-space**.

Example 2.4 Let X = {a, b, c}, $\mathcal{U}$ = {X, Φ, {a}, {b},{b, c},{a, c}, {a, b}}.

Then (X, $\mathcal{U}$) is U-space not a topological space. Here a and b are separated by {a} and {b, c}, b and c are separated by {b} and {c, a}, c and a are separated by {a} and {b, c}. Here (X, $\mathcal{U}$) is a T_2-U-space.

(X, $\mathcal{U}$) in Ex.-2.2 is a T_1-U-space but it is not a T_2-U-space.

Hence every T_2-U-space is a T_1-U-space, but not conversely.

Definition 2.4 Let (X, $\mathcal{U}_X$) and (Y, $\mathcal{U}_Y$) be U- spaces. (X×Y, $\mathcal{U}$), where $\mathcal{U}$ is a collection of subsets of X×Y, is called **the product of X with Y** if $\mathcal{U}$ is the U-structure on X×Y generated by $\left(\bigcup_{x\in X}\{\pi_x^{-1}G_x\}\right)\cup\left(\bigcup_{y}\{\pi_y^{-1}G_y\}\right)$, π_x: X×Y→X, π_y: X×Y→Y are the projection maps.

Hence if (X×Y, $\mathcal{U}$) is the product of (X, $\mathcal{U}_X$) with (Y, $\mathcal{U}_Y$), then $\mathcal{U}$ is the smallest U-structure on X×Y such that the projection maps π_x: X×Y→X and π_y: X×Y→Y are U-continuous.

In general, let {X_α, $\mathcal{U}_\alpha$}be any non-empty family of U-spaces. Then, ($\prod_\alpha X_\alpha$, $\mathcal{U}$), where $\mathcal{U}$ is a collection of subsets of $\prod_\alpha X_\alpha$, is called **the product of {X_α, $\mathcal{U}_\alpha$}** if $\mathcal{U}$ is the U-structure on $\prod_\alpha X_\alpha$ generated by $\bigcup_\alpha\{\pi_\alpha^{-1}(U_\alpha)|U_\alpha\in\mathcal{U}_\alpha\}$, where $\pi_\alpha:\prod_\alpha X_\alpha\to X_\alpha$ is the projection map.

It follows therefore

Theorem 2.2 (X×Y, $\mathcal{U}$) is the product of (X, $\mathcal{U}_1$) with (Y, $\mathcal{U}_2$) if and only if $\mathcal{U}$ is the U-structure generated by {G_1×Y: $G_1\in\mathcal{U}_1$}∪{X×G_2: $G_2\in\mathcal{U}_2$}.

Our next theorems are generalizations of (Theorem- 2.2- 2.4, p. 102-103) in [24]

Theorem 2.3 The product of any nonempty class of Hausdorff U-space is Hausdorff.

Proof: Let $\{X_i\}$ be the product of a nonempty class of Hausdorff U-spaces X_i and $X = \prod X_i$. Suppose x, y$\in$ X, x $\neq$ y. If x = $\{x_i\}$ and y = $\{y_i\}$ are two distinct points in X, then we must have $x_{i_0} \neq y_{i_0}$ for at least one index i_0. Since X_{i_0} is a Hausdorff U-space, there exist two disjoint U-open sets U and V of X_{i_0} such that $x_{i_0} \in$ U and $y_{i_0} \in$V. Let G = $\prod_i G_i$ and H =$\prod_i H_i$, where U = G_{i_0} and V = H_{i_0} and for $i \neq i_0$, $G_i \cup H_i = X_i$. Thus G and H are two disjoint U-open sets of X and x $\in$G and y$\in$H.

Therefore X is Hausdorff.

Definition 2.5 Let (X, $\mathcal{U}$) be a U-space and R an equivalence relation on X. For each U$\in$ $\mathcal{U}$, let $U' = \{\text{cls } x \mid x \in \mathcal{U}\}$. Let $\mathcal{U}' = \{U' \mid U \in \mathcal{U}\}$. Then $\mathcal{U}'$ is a U-structure on $\frac{X}{R}$. $(\frac{X}{R}, \mathcal{U}')$ will be called the **usual U-space** $\frac{X}{R}$**,** unless otherwise stated, $\frac{X}{R}$ will denote this U-space.

Theorem 2.4 Let X be a U-space and R is an equivalence relation of X. If R is a U-closed subset of the product space $X \times X$, then $\frac{X}{R}$ is Hausdorff.

Proof: Let p : $X \to \frac{X}{R}$ be a projection mapping i. e. p(x) = clsx. Let z, $z' \in \frac{X}{R}$.

So z = p(x), z′ = p(x′), where x, $x' \in X$. Since R is a U-closed subset of $X \times X$, there exist two U-open sets U and V such that $(x, x') \in U \times V \subseteq R'$. Since p is a U-open mapping, p(U), p(V) are U-open. Clearly, $z \in p(U)$, $z' \in p(V)$. Since $U \times V \subseteq R'$, $p(U) \cap p(V) = \Phi$. Hence $\frac{X}{R}$ is Hausdorff.

Theorem 2.5 Let X be a U- space and Y a Hausdorff U-space and let f : $X \to Y$ be a U-continuous mapping. Then $\frac{X}{R(f)}$ is Hausdorff.

[Here R(f) is an equivalence relation of X, given by $(x, x') \in R(f) \Rightarrow$ f(x) = f(x′)].

Proof: Let clsx and clsy be two distinct elements of $\frac{X}{R(f)}$. So f(x) and f(y) are two distinct elements of Y. Since Y is Hausdorff, there exist two disjoint U-open sets G and H of Y such that $f(x) \in G$ and $f(y) \in H$. Since f is U-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint U-open sets of X. Hence $x \in f^{-1}(G)$ and $y \in f^{-1}(H)$.

Again p : X $\rightarrow \frac{X}{R(f)}$ is a projection mapping, this implies that p(f $^{-1}$(G)) and p(f $^{-1}$(H)) are two disjoint U-open sets of $\frac{X}{R(f)}$ containing clsx and clsy respectively. Hence $\frac{X}{R(f)}$ is Hausdorff.

Definition 2.6 A U-space X is said to be **U-T$_2\frac{1}{2}$ space or, completely Hausdorff** if for each x, y$\in$ X, with x $\neq$ y, there exist U-open sets G and H such that x $\in$G and y$\in$H and $\overline{G} \cap \overline{H} = \Phi$.

Example 2.5. Let X = {a, b, c, d}, $\mathcal{U}$ = { X, Φ, {a, b}, {a, c}, {a, d}, { b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. Then X is a **proper completely Hausdorff U-space.**

Definition 2.7 A U-space X is called **regular** if for any U-closed set F of X and any point x$\in$ X, such that x$\notin$F there exist two disjoint U-open sets G and H such that x $\in$G and F$\subseteq$H.

For U- spaces 'Hausdorff' and 'regular' are independent concepts.

Example 2.6 (A proper U-space which is regular but not Hausdorff).

Let X = {a, b, c, d}, $\mathcal{U}$ = {X, Φ, {a},{d},{a, d},{a, b, c}, {b, c, d}}. Then (X, $\mathcal{U}$) is a proper U-space. Here the U-closed sets are X, Φ, {a},{d}, {b, c},{a, b, c},{b, c, d}. We easily see that X is a regular but it is not

Hausdorff, since b and c cannot be separated by disjoint U-open sets. Also (X, $\mathcal{U}$) is not a topological space.

Example 2.7 (A proper U-space which is Hausdorff but not regular).

Let X = $\mathbb{R}$ and $\mathcal{U}$ is a structure generated by $\mathcal{U}_1 \cup \mathcal{U}_2$, where $\mathcal{U}_1$ is the usual space on $\mathbb{R}$ and $\mathcal{U}_2 = \{\mathbb{Q}^c\}$, where $\mathbb{Q}$ is the set of all rational numbers.

Then (X, $\mathcal{U}$) is a Hausdorff U- space, since $\mathcal{U}_1 \subseteq \mathcal{U}$.

If F = $\mathbb{Q}$ and x is an irrational number, then F is U-closed, since $\mathbb{Q}^c \in \mathcal{U}_2$ and $x \notin F$. But x and F can not be separated by disjoint U-open sets.

Here (X, $\mathcal{U}$) is not regular.

Thus a Hausdorff U-space need not be regular.

We now generalize theorems of [24](P. 104- 106).

Theorem 2.6 Any U-space X is regular iff for each $x \in X$ and each U-open set G containing x, there exists a U-open set H of X such that $x \in H \subseteq \overline{H} \subseteq G$.

Proof: Let X be regular U-space and let F = G′ . Then F is U-closed and $x \notin F$.

Since X is regular, there exist U-open sets V_1, V_2 such that $x \in V_1$, $F \subseteq V_2$ and $V_1 \cap V_2 = \Phi$. This implies that $V_1 \subseteq V_2' \subseteq F'$. Therefore $\overline{U} \subseteq \overline{V_2'} = V_2' \subseteq G$. If we write V_1 = H. Then we get $x \in H \subseteq \overline{H} \subseteq G$. Now let for every $x \in X$ and for every U-open set G, there exist U-open sets H, such that $x \in H \subseteq \overline{H} \subseteq G$.

Let F is U-closed and $x \notin$ F. $\Rightarrow F'$ is U-open and $x \in F'$. According to the condition there exist U-open set H, such that $x \in H \subseteq \overline{H} \subseteq F'$. Let $(\overline{H})' = W$.

Then W is U-open, $F \subseteq$ W and $W \cap H = \Phi$.

Theorem 2.7 The product of collection of nonempty regular U-space is regular.

Proof: Let $\{X_i\}$ be collection of nonempty regular U- space and $X = \prod X_i$.

We shall show that X is regular. Let $x \in G$, G is a U-open set of X. Then $x = \{x_i\}$ and G is a U-open basic subset containing $\prod_i G_i$ where $x \in \prod G_i$.

Therefore G_i is a U-open set of X_i containing x_i. Since X_i is regular, there exist U-open set V_i , where $x \in V_i$, $\overline{V_i} \subseteq G_i$. Now let $V = \prod Vi$. Then $x \in$ V and $\overline{V} = \prod \overline{V_i} \subseteq \prod G_i \subseteq G$.

Hence X is regular.

Theorem 2.8 Every subspace of regular U-space is regular.

Proof: Let X be a regular U-space and $Y \subseteq$ X. Let $y \in Y$ and B is a U-closed set of Y, such that $y \notin B$. Since B is U-closed in Y, there exist a U-closed subset F of X such that $B = F \cap Y$. So, $y \notin F$. Since X is regular, there exist disjoint U-open sets G and H such that $y \in$ G and $F \subseteq H$. Let $V_1 = G \cap Y$, $V_2 = H \cap Y$.

Therefore V_1 and V_2 are disjoint U-open sets where $y \in V_1$ and $B \subseteq V_2$.

Hence Y is regular.

Theorem 2.9 Let X be regular T_1-U-space. R is an equivalence relation of X. If the projection mapping $p : X \to \frac{X}{R}$ is U-closed, then R is U-closed subset of $X \times X$.

Proof: We shall show that R' is U-open. Let $(x, y) \in R'$. It is sufficient to show that there exist two U-open sets W_1 and W_2 of X such that $x \in W_1$ and $y \in W_2$ and $W_1 \times W_2 \subseteq R'$.

This implies that $p(W_1) \cap p(W_2) = \Phi$. Since $(x, y) \in R'$, $p(x) \neq p(y)$, i. e. $x \notin p^{-1}(p(y))$, again since {y} is U-closed and p is U-closed mapping. So p(y) is U-closed. Since p is U-continuous, then $p^{-1}(p(y))$ is U-closed. Thus there exists disjoint U-open sets W_1 and V such that $x \in W_1$ and $p^{-1}(p(y)) \subseteq V$. Since p is a closed mapping, there exist U-open set G containing p(y) such that $p^{-1}(p(y)) \subseteq p^{-1}(G) \subseteq V$.

If we consider $p^{-1}(G) = W$, then $W_1 \times W_2$ is a U-open set of R'.

Theorem 2.10 Let X be regular T_1-U-space. R is an equivalence relation of X and $p : X \to \frac{X}{R}$ is U-closed and open mapping. Then $\frac{X}{R}$ is Hausdorff.

Proof: Since $p : X \to \frac{X}{R}$ is U-closed, R is a U-closed subset of $X \times X$.

Let $p(x), p(y) \in \frac{X}{R}$. So, $x, y \notin R$. Since R is a U-closed set of $X \times X$, there exist two U-open sets V and W of X such that $x \in V$ and $y \in W$ and $V \times W \subseteq R'$.

Therefore p(x) $\in$ p(V), p(y) $\in$ p(W). Since p is U- open, p(V) and p(W) are U-open set of $\frac{X}{R}$ and V$\times$W $\subseteq$R$'$ provides p(V) $\cap$p(W) = Φ.

Hence $\frac{X}{R}$ is Hausdorff.

Definition 2.8 A U-space X is said to be **completely regular** if and if for any U-closed subset F of X and x$\in$X which does not belongs to F, there exists a U-continuous function f: X $\to$ [0, 1] such that f(x) = 0 and f(F) = 1. Here [0, 1] is considered as a subspace of the usual U-space $\mathbb{R}$.

Example 2.8. Let X = [0, 1] and $\mathcal{U}$ = {X, Φ,{{[(a, 1)], [(0, b)] | 0$\leq$ a, b $\leq$ 1} and their unions}}. Then the U-open sets of X are X, Φ, and the sets of the form [(0,b)],[(a,1)] and [(0,b)]$\cup$[(a,1)], b < a.

Hence, the U-closed sets of X are of the form X, Φ, [(0, a)], [(b, 1)] and [(a, b)], a < b. Here [(a, b)] stands for any of (a, b), (a, b], [a, b) and [a, b].

Clearly, (X, $\mathcal{U}$) is a proper U-space.

Let F be a proper U-closed set, i.e., Φ$\neq$ F$\neq$ X. Let c$\in$X, c$\notin$F.

Then,(i) F = [(a, b)], for some 0$\leq$ a, b$\leq$ 1, a < b; or,

(ii) F = [(0, b)], or, (iii) F = [(a, 1)], 0$\leq$ a, b$\leq$ 1.

We now consider Y = [0, 1] as a subspace of the usual U-space $\mathbb{R}$. We first consider case (i) Define f: X$\to$Y by

(α) f(x) = 1, x$\in$(c, 1],

$= 0$, $x \in [0, c]$, if c is on the left of F;

(β) f(x) = 1, $x \in [c, 1)$,

$= 0$, $x \in (c, 1]$, if c is on the right of F.

Then in both the cases of (α) and (β), f is U-continuous and f(F) = 1, f(c) = 0.

Next, we consider the case (ii)

Define f: $X \to Y$ by

f(x) = 1, $x \in [c, 1]$,

$= 0$, $x \in (0, c)$;

Then f is U-continuous and f(F) = 1, f(c) = 0.

Finally, we consider the case (iii)

Define f: $X \to Y$ by f(x) = 1, $x \in [0, c]$,

$= 0$, $x \in (c, 1]$.

Here again f is U-continuous and f(F) = 1, f(c) = 0.

Hence (X, $\mathcal{U}$) is completely regular.

Comment 2.1

The above U-space X of Example 2.8 is also Hausdorff, normal and regular. We prove these below:

(i) Let x, $y \in X$, $x \neq y$. Then for the disjoint U-open sets $G_1 = [0, \frac{x+y}{2})$ and $G_2 = (\frac{x+y}{2}, 1]$, $x \in G_1$, $y \in G_2$. **Thus, X is Hausdorff**.

(ii) Let F_1 and F_2 be two disjoint U-closed sets in X. We shall show that there are disjoint U-open sets G_1, and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$. We see that F_1 is the form [0, a)], or [(b, 1], or [(a, b)].

If F_1= [0, a)], F_2 is the form (a, 1], or [(c, 1], or [(c, d)], for some c > a. In the first two cases, both F_1 and F_2 are U-open sets also, we take $G_1 = F_1$, $G_2 = F_2$.

If F_2 = [(c, d)], we take $G_1 = F_1$, $G_2 = (\frac{a+c}{2}, 1]$.

Here X is normal.

(ii) Similarly, we can prove that **X is regular**.

Definition 2.9 A regular U-space X is called **T_3-U-space** if for each singleton subset of X is U-closed.

Definition 2.10 A T_1-U- space X is said to be **$T_{3\frac{1}{2}}$-U-space** if X is completely regular.

Theorem 2.11 [24](Theorem- 3.8, p. 107)

Every completely regular U-space is regular.

Proof: Let X be a completely regular U-space. F is a U-closed set of X and $x \in X$ which does not belongs to F, there exists a U-continuous function f: X $\rightarrow$ [0, 1] such that f(x) = 0 and f(F) = 1.

Let a, b $\in$[0, 1] and a < b. Then [0, a] and [b, 1] are two disjoint U-open set of [0, 1]. $\therefore$ x $\in f^{-1}$[0, 1] and F $\subseteq f^{-1}$[b, 1].

Therefore X is regular.

One can prove that **a subspace of regular (a completely regular) U-space and a product of regular (a completely regular) U-spaces is regular (completely regular).**

Definition 2.11 A U-space X is said to be **normal** if for each pair disjoint U-closed sets F_1 and F_2 , there exist U-open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \Phi$.

Theorems in U-spaces corresponding to the standard theorems regarding regular, normal and completely regular topological spaces can be shown to be valid. In particular, Urysohn's Lemma and Tietze Extension Theorem have their analogues for U-spaces.

We shall give here examples to show that proper regular and normal U-spaces exist and are distinct.

Example 2.9 (A proper U-space which is normal and regular.)

Let X = {a, b, c, d}, $\mathcal{U}$ = {X, Φ, {a}, {d}, {a, d}, {a, b, c},{b, c, d}}.

(X, $\mathcal{U}$) is a proper U-space , since {a, b, c}$\cap${b, c, d} = {b, c}$\notin \mathcal{U}$.

Closed sets are X, Φ, {a}, {d}, {b, c}, {b, c, d}, {a, b, c}.

Here {b, c}$\subseteq${a, b, c} and {d}$\subseteq${d}. {b, c} and {d} are U-closed and disjoint and there exist disjoint U-open sets containing {b, c} and {d} respectively. Similarly, we can show that for any pair of disjoint closed sets, there exist disjoint U-open sets containing them respectively. Hence X is normal.

Here {b, c, d} is a closed set, a$\notin${b, c, d} and there exist disjoint U-open sets containing a and {b, c, d} respectively. The other cases being trivially satisfied, X is regular.

We note that **the U-space X in the above example is regular but not a T_3-U-space.**

Example- 2. 10 (A proper U-space which is normal and regular.)

Let X = {a, b, c}, $\mathcal{U}$ = {X, Φ, {a}, {c}, {a, b}, {b, c}, {a, c}}.

(X, $\mathcal{U}$) is a proper U-space , since {a, b}$\cap${b, c} = {b}$\notin$ $\mathcal{U}$.

Closed sets are X, Φ, {a}, {b}, {c}, {b, c}, {a, b}.

Here {a}$\subseteq${b, c} and {b, c}$\subseteq${b, c}. {a} and {b, c} are U-closed and disjoint and there exist disjoint U-open sets containing {a} and {b, c} respectively. Similarly, we can show that for any pair of disjoint closed sets, there exist disjoint U-open sets containing them respectively. Hence X is **normal.**

Here {b, c} is a closed set, a$\notin$ {b, c} and there exist disjoint U-open sets containing {b, c} and {a} respectively. **So, X is regular.**

Example - 2.11 (A proper U-space which is normal but not regular.)

Let X = {a, b, c, d}, $\mathcal{U}$ = {X, Φ, {a, b},{a, c}, {a, b, c}}.

(X, $\mathcal{U}$) is a proper U-space , since {a, b} ∩ {a, c}={a} ∉ $\mathcal{U}$.

Closed sets are X, Φ, {a}, {c, d}, {b, d}, {d}.

Here b ∉ {c, d}, a ∉ {c, d} but none of these can be separated by disjoint U-open sets. Hence (X, $\mathcal{U}$) is **not regular.**

However, (X, $\mathcal{U}$) is **normal,** since there are no pair of disjoint U-closed sets.

We give below another such example.

Example 2. 12 Let X = {a, b, c, d, e}, $\mathcal{U}$ = {X, Φ,{a, b},{c, d, e}, {b, d, e}, {a, b, d, e}}. (X, $\mathcal{U}$) is a proper U-space , since {a, b} ∩ {b, d, e}={b} ∉ $\mathcal{U}$.

Closed sets are X, Φ, {c}, {a, c}, {a, b}, {c, d, e}.

Here b ∉ {a, c}, d ∉ {a, c} but none of these can be separated by disjoint U-open sets. Hence (X, $\mathcal{U}$) is **not regular**.

Here {a, b} ∩ {c} = Φ,{a, b} ⊆ {a, b} and {c} ⊆ {c, d, e}; {a, b} ∩ {c, d, e} = Φ, {a, b} ⊆ {a, b} and {c, d, e} ⊆ {c, d, e}. So, (X, $\mathcal{U}$) is **normal.**

Example - 2.13 (A proper U-space which is not normal and not regular.)

Let X = {a, b, c, d, e}, $\mathcal{U}$ = {X, Φ, {d, e},{a, b, c, d}}.

(X, $\mathcal{U}$) is a proper U-space but not topological space.

Closed sets are X, Φ, {e}, {a, b, c}.

(X, $\mathcal{U}$) is **not normal,** since {a, b, c} & {e} are disjoint U-closed sets which can not be separated.

Here b $\notin$ {e}, d $\notin$ {a, b, c} but none of these can be separated by disjoint U-open sets. Hence (X, $\mathcal{U}$) is **not regular**.

Theorem 2.12 [24](Theorem- 3.10, p. 108)

A U-space X is normal if and if for each U-closed set F and each U-open set G containing F, there exists a U-open set H of X such that

$F \subseteq H \subseteq \overline{H} \subseteq G$.

Proof: Let X be a normal U-space and F a U-closed set of X and G a U-open set containing F. Then G′ is U-closed set in X disjoint from F.

Since X is normal, there exist disjoint U-open sets V and H such that $G' \subseteq V$ and $F \subseteq H$. Therefore $H \subseteq V'$. Again since V′ is U- closed, $\overline{H} \subseteq V'$.

Hence $F \subseteq H \subseteq \overline{H} \subseteq G$.

Again let F be a U-closed set of X and for each U-open set G containing F, there exists a U-open set H such that $F \subseteq H \subseteq \overline{H} \subseteq G$. let A and B be two disjoint U-closed set of X. So, $A \subseteq B'$ and B′ is a U-open set. Therefore there exists a U-open set V such that $A \subseteq V \subseteq \overline{V} \subseteq B'$. Let $W = (\overline{V})'$.

This implies that V and W are two disjoint U-open sets such that $A \subseteq V$ and $B \subseteq W$.

Hence X is normal.

Definition 2.12 A normal U-space with T_1 property is called **T_4-U-space**.

Every T_4 -U-space is T_3-U-space but T_3-U-space may be or not T_4-U-space.

A normal U-space may be or not T_2-U-space and T_2-U-space may be or not normal U-space.

Example – 2.14 (A U-space X is T_2-U-space but not regular and normal.)

Let $F_1 = \{q \in \mathbb{Q} : q < 3\}$ and $F_2 = \{q \in \mathbb{Q} : q \geq 3\}$. Then F_1 and F_2 are U-closed subsets of X and $F_1 \cap F_2 = \Phi$. F_1 and F_2 can not be separated as a disjoint U-open set.

Hence the U-space X is **not normal**.

We now generalize theorems of [24](p. 110- 120)

Theorem 2.13 Every second countable regular U-space is normal.

Proof: Let X be a second countable regular U-space and β be a countable base of X. Let A and B are two disjoint U-closed set of X. Since X is a regular U-space, for each $x \in A$ there exist U-open sets G and H of X such that $x \in H \subseteq \overline{H} \subseteq$ G and $G \cap B = \Phi$. For U-open set V, there exist a U-open sets with β basis containing x and contained in V. $\{G_n\}$ is a countable collection of U-open sets covering of A, and for each n, $\overline{G_n} \cap B = \Phi$.

Similarly, $\{H_n\}$ is a countable collection of U-open sets covering of B, and for each n, $\overline{H_n} \cap A = \Phi$. Let $G = \cup G_n$ and $H = \cup H_n$. Then $A \subseteq G$, $B \subseteq H$ but may not be $G \cap H = \Phi$ (they need not be disjoint).

Therefore for each n, suppose $G'_n = G_n - \bigcup_{i=1}^{n} \overline{H_i}$ and $H'_n = H_n - \bigcup_{i=1}^{n} \overline{G_i}$.

From the definition of G_n, H_n and G'_n, H'_n. It is clear that the collection $\{G'_n\}$ and $\{H'_n\}$ are U-open cover of A and B respectively, where G'_n and H'_n are disjoint to each other. Let $G' = \cup G'_n$ and $H' = \cup H'_n$, then $A \subseteq G'$, $B \subseteq H'$ and $G' \cap H' = \Phi$. Because if $x \in G' \cap H'$ then for any i, j; $x \in G_i' \cap H_i'$. Here $i \leq j$ or, $j \leq i$. Let $i \leq j$. Since $H_j' = H_j - \bigcup_{k=1}^{j} \overline{G_k}$, $G_i' \cap H_j' = \Phi$. This is contradiction. A similar contradiction arise if $j \leq i$. Therefore X is a normal U-space.

Theorem 2.14 (The generalized form of Urysohn lemma)

Let X be a normal U-space and A, B be disjoint U-closed subsets of X. Then there exists a U-continuous function $f : X \rightarrow [0, 1]$ such that $f(A) = 0$, $f(B) = 1$.

Proof: Since A and B are disjoint U-closed subsets of X, $A \subseteq B'$, B' is U-open. Since X is normal. According to the Theorem 2.12, there exist a U-open set $U_{\frac{1}{2}}$ such that $A \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq B'$.

Again A and $\overline{U_{\frac{1}{2}}}$ are U-closed sets and $U_{\frac{1}{2}}$ and B' are U-open sets respectively containing A and $\overline{U_{\frac{1}{2}}}$. So there exist two U-open sets $U_{\frac{1}{4}}$ and $U_{\frac{3}{4}}$ such that $A \subseteq U_{\frac{1}{4}} \subseteq \overline{U_{\frac{1}{4}}} \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq U_{\frac{3}{4}} \subseteq \overline{U_{\frac{3}{4}}} \subseteq B'$.

If we continue this process, for each rational number of the form

$t = \frac{m}{2^n}$; {m = 1, 2, 3, --------(2^n – 1), and n = 1, 2, 3, ---------}. We have an U-open set of the form U_t such that $t_1 < t_2$. $\Rightarrow$ A $\subseteq U_{t_1} \subseteq \overline{U_{t_1}} \subseteq U_{t_2} \subseteq \overline{U_{t_2}} \subseteq B'$.

We now define a function f : X$\rightarrow$[0, 1] ; f(x) = 0, if for each t, x$\in$ U_t

= sup {t : x $\neq$ U_t }

It is clear that f(A) = 0 and f(B) = 1.

Now we show that f is U-continuous. All intervals of the form [0, a) and (a, 1], where 0 < a < 1, constitute a U-open sub base for [0, 1]. It is easy to see that f(x) < a iff x$\in$ U_t for t < a; i.e. f^{-1}([0, a)) = {x : f(x) < a} = $\bigcup_{t<a} U_t$, which is an U-open set. Again {x : f(x) $\le$ a} = $\bigcup_{t<a} \overline{U_t} = \overline{\bigcup_{t<a} U_t}$

Therefore f^{-1}((0, a]) = {x : f(x) > a} = $\left(\overline{\bigcup_{t>a} U_t}\right)'$, which is an U-open set.

Hence f is U-continuous.

Theorem 2.15 If A and B are two disjoint U-closed sets of a U-space X and if there is a U-continuous function f : X$\rightarrow$[0, 1] such that f(A) = 0, f(B) =1, then X is normal.

Proof: Let A and B are two disjoint closed sets of a U-space X. Then there is a U-continuous function f : X$\rightarrow$[0, 1] such that f(A) = 0, f(B) =1. Let a, b$\in$[0, 1] and a < b. Then [0, a) and (b, 1] are two disjoint U-open sets. Since f is U-continuous, f^{-1}([0, a)) and f^{-1}((b, 1]) are two disjoint U-open sets of X and A $\subseteq$ f^{-1}([0, a)), B$\subseteq$ f^{-1}((b, 1]).

Therefore X is normal.

Theorem 2.16 (The generalized form of the Tietze extension theorem.)

Let X be a normal U-space and F be a U-closed subspace of X and $f : F \to [a, b]$ is a U-continuous real function. Then f has a U-continuous extension $\overline{f} : X \to [a, b]$, i.e. there is a U-continuous function $\overline{f} : X \to [a, b]$ such that $\overline{f} \mid F = f$.

Proof: If a = b then f is constant function and in this reason $\overline{f}$ is also constant function with the same value. So let a < b. We may clearly assume that [a, b] is the smallest U-closed interval which contains the range of f. Furthermore, the device used in the proof of theorem enables us to assume that a = - 1 and b = 1. i. e. $f : F \to [-1, 1]$ and [- 1, 1] is the smallest closed interval which contains the range of f.

Let $f_0 = f$ and $A_0 = \{x : f_0(x) \leq -\frac{1}{3}\}$ and $B_0 = \{x : f_0(x) \geq \frac{1}{3}\}$. Then A_0 and B_0 are disjoint nonempty closed subsets of F. Since F is a U-closed subset of X, then A_0 and B_0 are U-closed subsets of X. According to the Theorem 2.14, there exists a U-continuous function

$g_0 : X \to [-\frac{1}{3}, \frac{1}{3}]$ such that $g_0(A_0) = -\frac{1}{3}$ and $g_0(B_0) = \frac{1}{3}$.

Let $f_1 = f_0 - g_0$ (here $g_0 \mid F = g_0$).Then $f_1 : F \to [-1, 1]$ is U-continuous function and $|f_1(x)| \leq \frac{2}{3}$. If $A_1 = \{x \mid f_1(x) \leq (-\frac{1}{3})(\frac{2}{3})\}$ and $B_1 = \{x \mid f_1(x) \geq (\frac{1}{3})(\frac{2}{3})\}$.

Then A_1 and B_1 are two non-empties disjoint U-closed subsets of F.

Then in the same way as above there exists a U-continuous function $g_1: X \to [-\frac{2}{9}, \frac{2}{9}]$ such that $g_1(A_1) = (-\frac{1}{3})(\frac{2}{3}) = -\frac{2}{9}$ and $g_1(B_1) = (\frac{1}{3})(\frac{2}{3}) = \frac{2}{9}$.

Let $f_2 = f_1 - g_1 = f_0 - (g_0 + g_1)$. Then $|f_2(x)| \leq \left(\frac{2}{3}\right)^2$.

With the help of this process we get $\{f_n\}$ sequence function on F and $\{g_n\}$ sequence function on X, where $|f_n(x)| \leq \left(\frac{2}{3}\right)^n$, $|g_n(x)| \leq \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n$ and

$f_n = f_0 - (g_0 + g_1 + g_2 + \text{-------} + g_{n-1})$.

For this subtraction consider $g_i \mid F$. Assume that $s_n = g_0 + g_1 + g_2 + \text{-------} + g_{n-1}$.

Since $|g_n(x)| \leq \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n$ and $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n = 1$, $\{s_n\}$ converges uniformly.

Therefore, limit of $\{s_n\}$ is $\overline{f}$ and $\overline{f}$: $X \to R$ is a U-continuous function and $|s(x)| \leq 1$.

Again since for each x, $|f_n(x)| \leq \left(\frac{2}{3}\right)^n$, then $f_n(x) \to 0$. So $s_n \to f_0$ on F. i. e. the value of $\overline{f}$ and f are equal on F.

Hence $\overline{f}$ is a U-continuous extension of f.

Theorem 2.17 Let X be U-space and Y be Hausdorff U-space and let A be a subspace of X. If f : A $\rightarrow$Y is a U-continuous mapping, then f has no more than one U-continuous extension $\overline{A} \rightarrow$Y.

Proof: If possible, let g, h: $\overline{A} \rightarrow$Y be two U-continuous extension of f.

Then A has a limit point x such that g(x) $\neq$h(x). Since Y is a Hausdorff U-space, there exist two disjoint U-open sets G and H of Y such that g(x)$\in$G and h(x)$\in$H. Since g, h are continuous, so g^{-1}(G) and h^{-1} (H) are U-open sets of $\overline{A}$ and x $\in$ $g^{-1}(G) \cap h^{-1}(H)$.

Now x is a limit point of A, $(g^{-1}(G) \cap h^{-1}(H)) \cap A \neq \Phi$.

Let $a \in (g^{-1}(G) \cap h^{-1}(H)) \cap A$, then g(a) $\in$G and h(a) $\in$H. Since g | A = h | A, g(a) = h(a) $\in G \cap H$ which is contradicts.

Theorem 2.18 Let f : X $\rightarrow$Y be a U-continuous mapping, where X is a U-space and Y is a Hausdorff U-space. Prove that graph of f i.e.

{(x, f(x)), x$\in$X} is a U-closed subspace of product space $X \times Y$.

Proof: Let A = {(x, f(x)), x$\in$X}. We shall show that A is U-closed. i.e. A′ is U-open. Suppose (x, y) $\in$A′, then y$\neq$ f(x). Since Y is a Hausdorff U-space, there exist two disjoint U-open sets G and H such that y$\in$G and f(x) $\in$H.

So (x, y) $\in f^{-1}(H) \times G$.

It is enough to show that $f^{-1}(H) \times G \subseteq A'$ for showing A′ is U-open.

Let $f^{-1}(H) \times G \not\in A'$, then $(x_0, y_0) \in f^{-1}(H) \times G$. But $(x_0, y_0) \not\in A'$.

i. e. $(x_0, y_0) \in A$.

Then y_0 = f(x_0) and $x_0 \in f^{-1}(G) \Rightarrow x_0 \in f^{-1}(H) \cap f^{-1}(G)$. i. e. f($x_0$) $\in H \cap G$ which is contradicts.

Therefore $f^{-1}(H) \times G \subseteq A'$.

Theorem 2.19 If X is Hausdorff U-space and $p \in X$, then the intersection of all U-closed sets of X containing p equal to {p} and the intersection of all U-open sets of X containing p is equal to {p}.

Proof: Let X be a Hausdorff U-space. Since {p} is a U-closed set of Hausdorff U-space. Therefore, intersection of all U-closed sets containing p is {p}.

Again let the intersection of all U-open sets containing p is A. Obviously, $p \in A$.

If $A \neq \{p\}$, then $q \in A$ where $p \neq q$. Since X is Hausdorff, there exist disjoint U-open set G, H such that $p \in G$ and $q \in H$. But $q \in A \Rightarrow q \in G$ which is contradicts. Hence A = {p}.

Theorem 2.20 Let f : $X \to Y$, g : $Y \to X$ be U-continuous and g.f = 1_x. If Y is a Hausdorff U-space, then X is a Hausdorff U-space and f(X) is U-closed of Y.

Proof: Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Since g.f = 1_x, f is 1-1. This implies that

$f(x_1) \neq f(x_2)$. Since Y is a Hausdorff U-space, there exist disjoint U-open set G and H of Y such that $f(x_1) \in G$ and $f(x_2) \in H$. Since f is U-continuous, $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint U-open sets of X and $x_1 \in f^{-1}(G)$ and $x_2 \in f^{-1}(H)$.

Hence X is Hausdorff.

Theorem 2.21 Every infinite Hausdorff U-space has countable infinite discrete U-subspaces.

Proof: Let X be an infinite Hausdorff U-space. Let x_1 and x_2 be distinct two points of X. Then there exist two disjoint U-open sets G_1 and G_2 of X such that $x_1 \in G_1$ and $x_2 \in G_2$.

Let $x_3 \in X$ which is separate from x_1 and x_2. Then there exist U-open sets H_1, H_2, H_3 and H_4 such that $x_1 \in H_1$, $x_2 \in H_2$, $x_3 \in H_3$ and $x_3 \in H_4$ and $H_1 \cap H_3 = \Phi$. Let $H_2 \cap H_4 = \Phi$. Suppose $H_3 \cap H_4 = U_3$, $H_1 = U_1$ and $H_2 = U_2$. Then U_1, U_2 and U_3 are disjoint U-open sets. Since X is an infinite, by using induction principle, we have for every $n \geq 1$, $x_1, x_2, x_3, \text{-------}, x_n \in X$ and $U_1, U_2, U_3, \text{-------}, U_n$ are U-open sets such that for each $x_i \in U_i$ and for $i \neq j$, $x_i \neq x_j$ and $U_i \cap U_j = \Phi$, (i, j = 1, 2, 3, ------, n).

Let $Y = \{ x_1, x_2, x_3, \text{-------} \}$. Then Y is a countable infinite U-subspace whose U-open sets are $\{x_i\} = Y \cap U_i$.

Definition 2.13. Let X be a U-space and let $\{x_n\}$ be a sequence in X. An element $x \in X$ is called a **limit of $\{x_n\}$** if, for each U-open set G of X with $x \in G$, then there exists a positive integer n_0 such that for each positive integer $n > n_0$, $x_n \in G$.

Theorem 2.22 The limit of every convergent sequence of Hausdorff U-space is unique.

Proof: Let X be a Hausdorff U-space and $\{x_n\}$ be a convergent sequence of X. Assume that $x_n \to x$, $x_n \to y$ and $x \neq y$. Since X is a Hausdorff U-space, there exist two disjoint U-open sets G and H of X such that. $x \in G$ and $y \in H$. Since x_1 and x_2 are limits of $\{x_n\}$, there exist two natural numbers n_1, n_2 such that $n > \max\{n_1, n_2\}$, then $x_n \in G$ and $x_n \in H$. Therefore $G \cap H \neq \Phi$ which is contradicts.

Definition 2.14 A U-space X is said to be **completely normal** if every subset of X is normal.

Theorem 2.23 A U-space X is completely normal iff for any two subsets A and B, $\overline{A} \cap B = \Phi$ and $A \cap \overline{B} = \Phi$, then X is separated by two disjoint U-open sets A and B.

Proof: Let X be completely normal. Assume that A and B are two subsets of U-space X such that $\overline{A} \cap B = \Phi$ and $A \cap \overline{B} = \Phi$. Let $Y = X - \overline{A} \cap \overline{B}$, then $A, B \subseteq Y$. Since $\overline{A} \cap Y$ and $\overline{B} \cap Y$ are two disjoint closed subsets of Y, there exist disjoint U-open sets G, H of Y such that $\overline{A} \cap G \subseteq G$ and $\overline{B} \cap Y \subseteq H$. Clearly, $A \subseteq G$, $B \subseteq H$. Since Y is a U-open set of X, G and H are also U-open sets of X.

Now let for two subsets A and B of X, $\overline{A} \cap B = \Phi$ and $A \cap \overline{B} = \Phi$, then X is separated by disjoint U-open sets A and B.

Let Y be a U-subspace of X and A, B are two disjoint U-closed subsets of Y. Since A and B are U-closed of Y, then in Y, $\overline{A} \cap B = \Phi$ and $A \cap \overline{B} = \Phi$. Since A and B are U-closed subsets of Y, if closure of A and B are $\overline{A}$ and $\overline{B}$ respectively, then $\overline{A} \cap B = \Phi$ and $A \cap \overline{B} = \Phi$.

According to the condition there exist disjoint U-open sets G and H such that A $\subseteq$G and B$\subseteq$H. Therefore in Y, A and B are separated by disjoint U-open sets Y$\cap$G and Y$\cap$H.

Theorem 2.24 Let Y be a Hausdorff U-space and for each point y of Y, the closure of every U-open set containing y is regular. Then Y is regular.

Proof: Let F be a U-closed subset of Y and y$\in$Y, and y$\notin$F. Since Y is a Hausdorff, for each f$\in$F there exist disjoint U-open sets U_f and V_f such that y$\in$ U_f and f$\in$$V_f$. Since every $\overline{V_f}$ such that f $\in$$W_f$ $\subseteq$$\overline{W_f}$ $\subseteq$$V_f$.

Therefore there exists a U-open set G_f of Y such that W_f = $G_f \cap \overline{V_f}$ = $G_f \cap V_f$.

So, W_f is a U-open set in Y.

Now let W = $\bigcup_{f\in F} W_f$. Then W containing F and $\overline{W'}$containing Y are U-open sets of Y. Clearly, W and $\overline{W'}$are disjoint. It is enough to show that $\overline{W'}$ containing Y for completing the proof. If y$\notin$$\overline{W'}$, then y$\in$$\overline{W}$. Therefore for every U-open set G of Y containing y, G$\cap$W$\neq\Phi$. i.e. for any f, G$\cap$ W_f $\neq \Phi$. i. e., y$\in$$\overline{W_f}$ $\subseteq$$V_f$ which is contradiction.

Theorem 2.25 If X is a normal and A is a U-closed subset of X. G is a U-open set of X containing A. Then there is a open F_σ set V such that A $\subseteq$ V $\subseteq$ G.

Proof: Since G is a U-open set containing U-closed set A, X is normal there exist U-open set $V_{\frac{1}{2}}$ such that A $\subseteq V_{\frac{1}{2}} \subseteq \overline{V_{\frac{1}{2}}} \subseteq$ G. Again since A $\subseteq V_{\frac{1}{2}}$ and $\overline{V_{\frac{1}{2}}} \subseteq$ G, there exist U-open sets $V_{\frac{1}{4}}$ and $V_{\frac{3}{4}}$,

where A $\subseteq V_{\frac{1}{4}} \subseteq \overline{V_{\frac{1}{4}}} \subseteq V_{\frac{1}{2}} \subseteq \overline{V_{\frac{1}{2}}} \subseteq V_{\frac{3}{4}} \subseteq \overline{V_{\frac{3}{4}}} \subseteq$ G.

Repeatedly we use this process and we get a sequence $\{\overline{V_t}\}$ of closed sets of X, where t = $\frac{m}{2^n}$, n = 1, 2, 3, --------; m = 1, 2, 3, ------,(2^n – 1) and if $t_1 < t_2$ then A $\subseteq V_{t_1} \subseteq \overline{V_{t_1}} \subseteq V_{t_2} \subseteq \overline{V_{t_2}} \subseteq$ G. Let V = $\bigcup_{m,n} \overline{V_{t_n}}$. Then V is a F_σ set V and A $\subseteq$ V $\subseteq$ G.

Now we shall show that V is U-open. Let x $\in$ V then for any t_n, x $\in \overline{V_{t_n}}$. So, x $\in V_{t_{n+1}}$. i. e. x is a interior point of V. Therefore V is a U-open set.

Compact U-spaces

Theorem 2.26. Let (X$\times$Y, $\mathcal{U}$) be the U-product of (X, $\mathcal{U}_1$) with (Y, $\mathcal{U}_2$). Then X$\times$Y is compact if X and Y are compact.

Proof: Let C = $\{G_\alpha\}_{\alpha \in A}$ be a U-cover of X$\times$Y. Then for each α,

$G_\alpha = \bigcup_{i\in I}(G_{1,\alpha_i} \times Y) \cup \bigcup_{j\in J}(X \times G_{2,\alpha_j})$ for some G_{1,α_i} 's in $\mathcal{U}_1$ and G_{2,α_j} 's in $\mathcal{U}_2$.

Therefore,

$X\times Y = \bigcup_{\alpha\in A}[\bigcup_{i\in I}(G_{1,\alpha_i} \times Y)]\cup[\bigcup_{j\in J}(X\times G_{2,\alpha_j})]=[\bigcup_{\alpha\in A}\bigcup_{i\in I}(G_{1,\alpha_i} \times Y)]\cup[\bigcup_{\alpha\in A}\bigcup_{j\in J}(X\times G_{2,\alpha_j})]$

Then $C_1 = \{G_{1,\alpha_i}\}_{\alpha\in A, i\in I}$ is a U_1-cover of X and $C_2 = \{G_{2,\alpha_j}\}_{\alpha\in A, j\in J}$ is a U_2-cover of Y. Since X and Y are compact, C_1 and C_2 have some finite sub covers, say $\{G_{1,\alpha_r,i_s}\}_{1\le r\le u,\ 1\le s\le v}$ and $\{G_{1,\alpha_{r'},i_s'}\}_{1\le r'\le u',\ 1\le s'\le v'}$ then $\{G_{1,\alpha_r} \times G_{2,\alpha_{r'}}\}_{1\le r\le u, 1\le r'\le u'}$ is a finite sub cover of C. $\therefore$ $X\times Y$ is compact.

Definition 2.15 A U-space X is said to be l**ocally compact** if for each $x\in X$ there exists a U-open set G containing x of X whose closure is compact.

Example 2.15 The U-space $\mathbb{R}$ is locally compact. Because, for a neighborhood of any real number x of the form $S_a(x) = (-\infty, x + a)$, a > o. $\overline{S_a}(x) = [-\infty, x + a]$ is compact. However, $\mathbb{R}$ is not a compact U-space, since the U-open cover $\{(-\infty, a) \mid a\in \mathbb{R}\}$ of $\mathbb{R}$ does not have a finite sub cover.

Every compact U-space is locally compact but locally compact U-space need not be compact.

Theorem 2.27 Every locally compact Hausdorff U-space is regular.

Proof: Let X be a locally compact Hausdorff U-space. Then X has one point compactification X_∞ and it is Hausdorff and compact U-space.

Since every compact Hausdorff U-space is regular, X_∞ is a regular U-space.

Since the U-subspace of regular U-space is regular.

Therefore X is regular U-space as X_∞ is U-subspace of X.

Definition 2.16 If Y is a compact Hausdorff U-space and X is a proper U-subspace of Y whose closure equals to Y, then Y is said to be a **compactification** of U-space X.

Two compactifications Y_1 and Y_2 of U-space X are said to be **equivalent** if there is a U-homeomorphism h: $Y_1 \to Y_2$ such that h(x) = x for every $x \in X$.

If Y - X equals to a single point, then Y is called the **one-point-compactification** of X.

Theorem 2.28 A U-space X has a one- point-compactification if and only if X is locally compact but not itself compact.

Proof: To see this, let X be a U-locally compact U-space but not itself compact, and let Y = {y}, where $y \notin X$. Let $Z = X \cup Y$. Declare a subset V to be U-open in Z if either V is U-open in X or V is the K^c the complement of a compact U-space K in X. Then Z becomes a compact U-space, and is the one-point-compactification of X. Z will be denoted by X_∞ (as in topology) and y denoted by ∞.

Example 2.16 ([33], p. 185)

The one-point-compactification of the usual U-space $\mathbb{R}$ is homeomorphic with the circle. The **one-point-compactification** of $\mathbb{R}^2$ is homeomorphic to the sphere S^1.

Let S^1 denote the unit circle $\{(x, y) \in \mathbb{R}^2: x^2 + y^2 = 1\}$ regarded as a U-subspace of the product $\mathbb{R} \times \mathbb{R}$ of the usual U-space $\mathbb{R}$ with itself. The imbedding h: $(0, 1) \to S^1$ given by $h(t) = (\cos 2\pi t) \times (\sin 2\pi t)$ induces a

compactification. This is equivalent to the one-point-compactification of the U-space X.

Theorem 2.29 [24](p. 93) If X is a Hausdorff U-space then X_∞ is also a Hausdorff U-space.

Proof: For proving this theorem it is enough to show that for any point x of X there exist two U-open sets G and H of X_∞ such that $x \in G$, $\infty \in H$ and $G \cap H = \Phi$.

Let $x \in X$, then there exists a U-open set G such that $x \in G$ and $\overline{G}$ is a compact of U-space X. Let $H = Y - \overline{G}$, then G and H are U-open sets of Y and $x \in G$, $\infty \in H$ and $G \cap H = \Phi$.

Definition 2.17 [24](p- 134) . Let A and B be two U-spaces and h: $A \to B$ is a U-continuous, open and one-to-one map. Then h(A) is a U-homeomorphic subspace of A contained in B. Here **A is called U-imbedded in B with U-imbedding h**.

If A and h(A) are identified with each other, then A is a U-subspace of B.

Definition 2.18 A compact Hausdorff U-space Y is equivalently called a c**ompactification(see p-60) of a U-space X** if there is a U-imbedding h: X→Y such that h(X) is U-dense in Y. i. e. if Y is an extension U-space of h(X).

Example 2.17 Let Y be the U-space [0,1] obtained by regarding (0,1) as a U-subspace of the usual U-space $\mathbb{R}$. Then Y is a compactification of (0,1) obtained by adding one point at each end.

Example 2.18 Let Y = [-1,1]× [-1,1] be a U-subspace of $\mathbb{R}^2$. Here $\mathbb{R}^2$ is the product $\mathbb{R} \times \mathbb{R}$ of the usual U-space $\mathbb{R}$. Let h: (0,1) →Y be a map defined by h(x) = x×sin(1/x). Then h: X → h(X) is a U-homeomorphism and $\overline{h(X)}$ is the topologist's sine curve. The U-imbedding h gives rise to a compactification of (0,1) quite different from the one-point-compactification and the above two-point-compactification of (0,1). It is obtained by adding one point at the right-hand end of h(X), and an entire line segment of points at the left- hand end. $\overline{h(X)}$ is compact and Hasusdorff U- space .

Therefore ($\overline{h(X)}$, h) is a compactification of the U-space X.

Remark 2.1 Let X = (0, 1) and let X be a U-subspace of the usual U-space $\mathbb{R}$. A bounded U-continuous function f: (0,1) → $\mathbb{R}$ is extendable to the one- point-compactification of U-space if and only if the limits

$\lim_{x \to 0+} f(x)$ and $\lim_{x \to 1-} f(x)$ exist and are equal.

We conclude the paper with generalization of a theorem in Munkres [33] (p- 237)

Theorem 2.30. Let X be a U-space. Let h: $X \to Z$ be a U-imbedding of X in the compact Hausdorff U-space Z. Then there exists a corresponding compactification Y of U-space X; which has the property that there is a U-imbedding H: $Y \to Z$ that equals h on X.

Proof: Given h, let X_o denote the U-subspace h(X) of Z, and Y_o denote its closure in Z. Then Y_o is a compact Hausdorff U-space and $\overline{X_o} = Y_o$; therefore, Y_o is an compactification of X_o.

We now construct a U-space Y containing X such that the pair (X, Y) is U-homeomorphic to the pair (X_o, Y_o). Let us choose a set A disjoint from X that is in bijective correspondence with the set $Y_o - X_o$ under some map k: $A \to Y_o - X_o$. Define $Y = X \cup A$, and define a bijective correspondence H: $Y \to Y_o$ by the rule H(x) = h(x) for $x \in X$, $H(\alpha) = k(\alpha)$ for $\alpha \in A$.

Make Y into a U-space by declaring V to be U-open in Y if and only if H(V) is U-open in Y_o. The map H is automatically a U-homeomorphism; and the U-space X is a U-subspace of Y because H equals the U-homeomorphism h when restricted to the U-subspace X of Y. By expanding the range of H, we obtain the required U-imbedding of Y into Z.

Chapter- 3

Connectedness in U-spaces

Introduction

In this chapter, we have introduced the concepts of connectedness in U-spaces. The concepts of a component, total disconnectedness, local connectedness, path-connectedness, local path-connectedness, connectedness im kleinen in the topological spaces ([24], [33]) have been generalized to the case of U-spaces.

We have constructed many examples and proved a number of theorems involving these concepts.

Definition 3.1 Let X be the usual U-space $\mathbb{R}$. A U-space X is said to be **connected** if X can not be written as a disjoint union of two nonempty U-open sets. i.e. if there do not exist nonempty U-open sets G and H such that $G \cap H = \Phi$ and $G \cup H = X$.

If X is not connected U-space then, it is called **disconnected U-space**.

Let A be a nonempty subset of X. Then A is said to be **connected** if A is connected as a U-subspace of X. Thus, A is connected if there do not exist U-open sets G and H in X such that

$A \cap G \neq \Phi, A \cap H \neq \Phi, (A \cap G) \cap (A \cap H) = \Phi$ and $(A \cap G) \cup (A \cap H) = A$. Or, $(A \subseteq G \cup H)$

The empty set Φ and singleton sets {p} are always connected U-space.

Example 3.1 We consider $\mathbb{N}$ as a U-subspace of the usual U-space $\mathbb{R}$. Let $n_0 \in \mathbb{N}$. Let G = $\{r \in \mathbb{N} : -\infty < r < n_0 + 1\}$ and H = $\{r \in \mathbb{N} : n_0 < r < \infty\}$. Then G and H are U- open subsets of $\mathbb{N}$ and $G \cup H = \mathbb{N}$. $\mathbb{N}$ is a disconnected U-space.

Similarly we can prove that $\mathbb{Z}$ is a disconnected U-space.

We prove here that $\mathbb{Q}$ is disconnected.

Example 3.2 Let A = $\mathbb{Q}$. Since $\sqrt{2}$ is irrational, G = $(-\infty, \sqrt{2})$ *and* $H = (\sqrt{2}, \infty)$ are U-open in the usual U-space $\mathbb{R}$. Now,

$\Phi \neq G \cap A = \{q \in \mathbb{Q}: q < \sqrt{2}\}$, $\Phi \neq H \cap A = \{q \in \mathbb{Q}: q > \sqrt{2}\}$.

So $(G \cap A) \cap (H \cap A) = \Phi$ and $(G \cap A) \cup (H \cap A) = \mathbb{Q}$. Therefore $\mathbb{Q}$ is a disconnected U-subspace of $\mathbb{R}$.

Example 3.3 $\mathbb{R}$, $(-\infty, a)$, (b, ∞) and (a, b), $(a, b]$, $[a, b)$ and every interval in $\mathbb{R}$ are connected subsets of usual U-space $\mathbb{R}$. In fact, these are the only connected U-subspace of $\mathbb{R}$.

The following theorems generalize the corresponding theorems about topological spaces [24](p. 70 - 78). **Here we only give the statements of the theorem**. The proofs are almost exactly similar to those for topological spaces. The proof of Theorem 3.10 (Theorem 1.9, [24]) has been given to show that the arguments really hold. Also we have proved the proofs of the theorems about the continuous images, since these are different here from those in topology.

Theorem 3.1 If (X, $\mathcal{U}$) is a U-space and A and B are connected U-subspace of X such that $A \cap B \neq \Phi$, then $A \cup B$ is connected.

Theorem 3.2 Let (X, $\mathcal{U}$) be a U-space and $\{A_i\}_{i \in I}$ a collection of connected U-subspace of X . If $\bigcap_{i \in I} A_i \neq \Phi$, then $\bigcup_{i \in I} A_i$ is connected.

Theorem 3.3 The U-space $\mathbb{R}$ and each interval of $\mathbb{R}$ is connected and these are the only connected U-subspace of $\mathbb{R}$.

Theorem 3.4 A U-continuous image of a connected U-space is connected.

Proof: Let X be a connected U-space and Y a U-space and f: $X \rightarrow Y$ is a U-continuous mapping. We shall show that f(X) is connected. If f(X) is not connected, let f(X) = (f(X) $\cap$ G) $\cup$ (f(X) $\cap$ H) be separation of f(X). G and H be nonempty U-open sets of Y and f is a U-continuous function.

Therefore $f^{-1}(G)$ and $f^{-1}(H)$ are U-open sets of X and

$X = (X \cap f^{-1}(G)) \cup (X \cap f^{-1}(H)) = f^{-1}(G) \cup f^{-1}(H)$, $f^{-1}(G) \neq \Phi$, $f^{-1}(H) \neq \Phi$ and $f^{-1}(G) \cap f^{-1}(H) = \Phi$.

Hence X is disconnected, contradicting the assumption.

Therefore f(X) is connected.

Theorem 3.5 A Ū-continuous image of a connected U-space is connected.

Proof: Let X be a U-space and Y a connected space and f: $X \rightarrow Y$ is a

Ū-continuous mapping. We shall show that f(X) is connected. If f(X) is not connected, let $f(X) = (f(X) \cap G) \cup (f(X) \cap H)$ be separation of f(X). G and H be nonempty open sets of Y and f is a Ū-continuous function.

Therefore $f^{-1}(G)$ and $f^{-1}(H)$ are U- open sets of X and

$X = (X \cap f^{-1}(G)) \cup (X \cap f^{-1}(H)) = f^{-1}(G) \cup f^{-1}(H)$, $f^{-1}(G) \neq \Phi$, $f^{-1}(H) \neq \Phi$ and $f^{-1}(G) \cap f^{-1}(H) = \Phi$.

Hence X is disconnected, contradicting the assumption.

Therefore f(X) is connected.

Theorem 3.6 A U*-continuous image of a connected U-space is connected.

Proof: Let X be a connected space and Y a U-space and f: $X \rightarrow Y$ is a

U*-continuous mapping. We shall show that f(X) is connected U-space. If f(X) is not connected, let $f(X) = (f(X) \cap G) \cup (f(X) \cap H)$ be separation of f(X). G and H be nonempty U-open sets of Y and f is a U*- continuous function.

Therefore $f^{-1}(G)$ and $f^{-1}(H)$ are open sets of X and

$X = (X \cap f^{-1}(G)) \cup (X \cap f^{-1}(H)) = f^{-1}(G) \cup f^{-1}(H)$, $f^{-1}(G) \neq \Phi$, $f^{-1}(H) \neq \Phi$ and $f^{-1}(G) \cap f^{-1}(H) = \Phi$.

Hence X is disconnected, contradicting the assumption.

Therefore f(X) is connected.

Theorem 3.7 Let X be a connected U-space. Then there exists no U-closed-open subsets of X except X and Φ.

Theorem 3.8 Let X be a U-space and A is a connected U-subspace of X .If B is a U-subspace of X such that $A \subseteq B \subseteq \overline{A}$, then B is connected;

in particular $\overline{A}$ is connected.

Theorem 3.9 A U-space X is disconnected if and only if there exists a $\overline{U}$-continuous mapping X onto the discrete two point space {0,1}.

Proof: Let X be a U-space and E is the discrete two point space {0, 1}. Suppose that X is disconnected. Then X has two disjoint U-open sets G and H such that $X = G \cup H$.

Let us define a map $f: X \to E$ such that $f(x) = 0, x \in G$

$$= 1, x \in H$$

Also G and H are U-open sets. This implies that f is $\overline{U}$-continuous.

Conversely, suppose that there exists a $\overline{U}$-continuous map f: X $\to$ E. Then $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are disjoint U-open sets of X and $X = f^{-1}(\{0\}) \cup f^{-1}(\{1\})$. So X is disconnected.

Theorem 3.10 A finite Cartesian product of connected U-spaces is connected.

Proof: Let X_1, X_2, X_3, -------, X_n be connected U-spaces and

$X = X_1 \times X_2 \times X_3 \times \text{-------} \times X_n$.

We shall use induction rule.

Let n = 2, then $X = X_1 \times X_2$. Let $(a, b) \in X_1 \times X_2$. Since $X_1 \times \{b\}$ and X_1 and for each $x_1 \in X_1$, $\{x_1\} \times X_2$ and X_2 are homeomorphic. So $X_1 \times \{b\}$ and $\{x_1\} \times X_2$ are U-connected.

Again since $(x_1, b) \in (X_1 \times \{b\}) \cap (\{x_1\} \times X_2)$.

This implies that $U_{x_1} = (X_1 \times \{b\}) \cup (\{x_1\} \times X_2)$ is connected.

Let $U = \bigcup_{x_1 \in X_1} U_{x_1}$. This union is connected because it is the union of a collection of connected U-spaces that have the point (a, b) in common. Since this union $U = X_1 \times X_2$, the space $X_1 \times X_2$ is connected.

Now let $X_1 \times X_2 \times X_3 \times \text{-------} \times X_{n-1}$ be connected for n > 2.

Since $X_1 \times X_2 \times X_3 \times \text{-------} \times X_{n-1} \times X_n$ and $(X_1 \times X_2 \times X_3 \times \text{-------} \times X_{n-1}) \times X_n$ are homeomorphic, $X_1 \times X_2 \times X_3 \times \text{-------} \times X_n$ is connected as in the case of $X_1 \times X_2$.

Theorem 3.11 Let $\{X_i\}_{i \in I}$ be collection of nonempty connected U-space and $X = \prod X_i$, then X is connected.

Theorem 3.12 (The generalized form of the Intermediate value theorem).

Let f: $X \to Y$ be a $\overline{U}$-continuous map, where X is a connected U-space and Y is an ordered set with the order U-structure. If a and b are two points of X and if r is a point of Y lying between f(a) and f (b), then there exists a point c of X such that f(c) = r.

Proof: Let $A = f(X) \cap \{y \in Y: y < r\}$ and $B = f(X) \cap \{y \in Y: r < y\}$. So,

$A \cap B = \Phi$ and $A \neq \Phi$, $B \neq \Phi$ because f(a) $\in A$ and f(b) $\in B$. Since A and B are U-open, if there were no point c of X such that f(c) = r, then $f(X) = A \cup B$ and f(X) is disconnected, contradicting the fact that the image of a connected U-space under a $\overline{U}$-continuous map is connected.

Definition 3.2 Let X be a U-space. A subset M of X is said to be **U-component or connected component** if (i) M is connected, (ii) if A is a connected subset of X such that $M \subseteq A \subseteq X$, then A = M or A = X, i.e. M is a maximal subset of a U-space X.

Example - 3.4 Let $X = [3, 5) \cup (6, 9)$ be a subspace of the usual U-space $\mathbb{R}$. Here X is a disconnected U-space and [3, 5) and (6, 9) are two components of U-space.

Example- 3.5 $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are subspaces of usual U-space $\mathbb{R}$. Singleton subsets are the only components of the above U-subspaces.

Example 3.6 Let X = {a, b, c, d, e}, $\mathcal{U}$ = {X, Φ,{e},{a, b}, {c, d}, {a, b, c}, {a, b, e}, {c, d, e},{a, b, c, d}, {a, b, c, e}}. Then X is a disconnected U-space and {a, b}, {c, d}, {e}, are the components of X.

Theorem 3.13 Let X be a U-space.

(i) Every connected U-closed-open subset of X is a component of X.

(ii) Every component of X is U-closed.

(iii) Every element of X is contained in a unique component of X.

(iv) Every connected subset of U-space X is contained in a unique component of U-space X.

Definition 3.3 Let X be a U-space. A U-space X is called **totally disconnected** U-space if for every pair of distinct points x and y (x $\neq$ y), there exists a non-empty disjoint U-open set A, B such that X = A $\cup$ B with x $\in$ A and y $\in$ B.

Example 3.7 The U-subspaces $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Q}'$ (the set of irrational numbers) of the U-space $\mathbb{R}$ are totally disconnected U-spaces.

We shall prove the truth of the statement here.

(i) Let m, n $\in \mathbb{N}$ with m < n.

Then, {1, 2, 3,........., m} $\cup$ {m + 1,m + 2, m + 3,.........} is a disconnection of $\mathbb{N}$. Here, {1, 2, 3, ...m} = $\mathbb{N} \cap$ (- ∞, m + $\frac{1}{2}$) and

$\{m + 1, m + 2, m + 3, ...\} = \mathbb{N} \cap (m + \frac{1}{2}, \infty)$ are U-open subsets of $\mathbb{N}$ which contain m and n respectively.

Thus $\mathbb{N}$ is totally disconnected.

(ii) The proof that $\mathbb{Z}$ is totally disconnected is similar.

(iii) Let a, $b \in \mathbb{Q}$ with a < b. Then there exists an irrational number x such that a < x < b. Then, $A \cup B$, where $A = \{y \in \mathbb{Q} : y < x\}$ and

$B = \{y \in \mathbb{Q} : y > x\}$ is a disconnection of $\mathbb{Q}$. Then $a \in A$, $b \in B$, and

$A = \mathbb{Q} \cap (-\infty, x)$, $B = \mathbb{Q} \cap (x, \infty)$. So that A and B are U-open in $\mathbb{Q}$. Hence $\mathbb{Q}$ is totally disconnected.

(iv) We can prove similarly that $\mathbb{Q}'$ is totally disconnected.

Example 3.8 Every discrete U-space consisting of more than one element is totally disconnected. This is obvious**.**

Theorem 3.14 The U-components of totally disconnected U-spaces consists of exactly one element.

Proof: Let X be a totally disconnected U-space. It is enough to prove that every U-subspace of X with two distinct elements is disconnected. Let x, $y \in X$ and $x \neq y$. Since X is totally disconnected, there exist $X = A \cup B$ such that $x \in A$ and $y \in B$. Thus $\{x, y\} = (A \cap \{x, y\}) \cup (B \cap \{x, y\})$.

Hence {x, y} is disconnected.

Definition 3.4 A U-space X is said to be l**ocally connected** if for every $x \in X$, and for every neighborhood G of x, there is a connected U-open set V of X, such that $x \in V \subseteq G$. X is a locally connected U-space if and only if X is locally connected U-space at each of its points.

Our Theorems 3.15-3.22 are generalizations of theorems in ([24], P-123-131)

Theorem 3.15 Every U-open subspace of a locally connected U-space is locally connected.

Proof: Let X be a locally connected U-space and G be a U-open subspace of X. Let H be a U-open set containing a point x of G. Since G is U-open, so H is a U-open set of X. Since X is locally connected, there exists a connected U-open set V in X which contains x and is contained in H. Also V is a U-open set of G.

Hence G is locally connected.

Theorem 3.16 The image of a locally connected U-space under a mapping which is both U-continuous and U-open is locally connected.

Proof: Let X be a locally connected U-space and Y be a U-space. Let $f : X \to Y$ be U-continuous, U-open and onto mapping. Let $y \in Y$ and G be a U-open set of Y containing y. For each $x \in X$, $y = f(x)$ and $f^{-1}(G)$ is U-open set of X containing x.

Since X is locally connected, there exists a connected U-open set V of $f^{-1}(G)$ containing x. i.e. $x \in V \subseteq f^{-1}(G)$. Since f is U-open and U-continuous.

f(V) is a connected U-open set of Y and f(x) = y = f(V). Since V $\subseteq$ f $^{-1}$(G), f(V) $\subseteq$ G.

Hence f(X) = Y is locally connected.

Theorem 3.17 The product space of two locally connected U-spaces is locally connected.

Proof: Let X and Y be locally connected U-space. We shall show that X $\times$ Y is locally connected.

Let (x, y) $\in$ X $\times$ Y and G be a U-open set of X $\times$ Y containing (x, y). Since projection mapping π_X : X $\times$ Y $\rightarrow$ X is U-open, π_X (G) is a U-open set containing x. Since X is locally connected, so there exists a connected U-open set V_1 of X containing x of π_X (G).

Again π_Y (G) is a U-open set and there exists a connected U-open set V_2 of Y containing y of a locally connected U-open set π_Y (G). Therefore $V_1 \times V_2$ is a connected U-open set of X $\times$ Y containing (x, y) and $V_1 \times V_2 \subseteq$ G.

Hence X $\times$ Y is locally connected.

Theorem 3.18 A U-space X is locally connected if and only if for each U- component of every U-open set of X is U-open.

Proof: Let X be a locally connected U-space and let G be a U-open set in X. According to the above Theorem 3.15 G is locally connected. Let C be a component of G and let a $\in$ C. Since G is a locally connected U-space, there exists connected U-open set V of G containing a. i.e. $V \subseteq G$. Since C is a component and $a \in C$, $V \subseteq C$. Therefore C is U-open in X.

Conversely, suppose that U-components of U-open sets in X are open. Suppose $x \in X$ and a neighborhood G of x. Let C be the U-component of G containing x. C is U-open in X by hypothesis. So C is a connected U-open set of G containing x.

Hence X is locally connected.

Definition 3.5 Let X be a U-space and let $f : [0,1] \rightarrow X$ be a U-continuous mapping.

If f(0) = x, f(1) = y, then f is called **path** from x to y.

Definition 3.6 Any U-space X is called **path connected U-space** if there is a path in X from x to y.

Definition 3.7 [24](p. 131)

A U- space X is said to be **locally path connected U-space** at x if for every open set G of x have a open subset V which is path connected U-space containing x.

If X is locally path connected at each of its points, then it is said to be **locally path connected U-space.**

Example 3.9 Each interval and each ray in the usual U-space $\mathbb{R}$ are connected, locally connected, path-connected and locally path-connected U-spaces.

Each of the subspaces [-1, 0) $\cup$ (0, 1] and [1, 2] $\cup$ [3, 4] of $\mathbb{R}$ is neither connected nor path-connected but each is both locally connected and locally path-connected.

Example 3.10 Let **C = ([**0,1] $\times\{0\})\cup(\{\frac{1}{n}:n\in\mathbb{Z}\}\times[0,1])\cup(\{0\}\times[0,1])$ be a U-subspace of $\mathbb{R}^2$ and let D = C - $\{0\}\times(0,1)$ be a U-space. Here C is the union of connected U-subset I_α, where I_α = [0,1] $\times\{0\}$, $\left\{\frac{1}{n}\right\}\times[0,1]$ or $\{0\}\times[0,1]$. Since each I_α is connected and $I_\alpha\cap\left(\bigcup_{\alpha\neq\beta}I_\beta\right)\neq\Phi$. Therefore C is a connected U- space and also D is a connected U-space and $\overline{D}$ = C. If p is any point on $\{0\}\times[0,1]$, then for any open sphere S_ϵ(p) with centered at p there exist a U-open set G $\subset$ S_ϵ(p) such that G is disconnected. Therefore, C is not locally connected U-space. If we consider p is (0,1), then similarly we can show that D is locally disconnected.

Theorem 3.19 Every path connected U-space is connected.

Proof: Suppose X be a path connected U-space and $x_o \in X$. Then for any $x \in X$ there is a path from x_o to x. That means there exists a U-continuous mapping f : I$\to$X such that f(0) = x_o , f(1) = x_o. Since I is a connected U-space, f(I) is connected U-subset of X. Therefore x_o and x are contained in same component of X. Since for any $x \in X$ true that X has only one component.

Therefore X is connected.

Theorem 3.20 The image of a U-continuous mapping of path connected U-space is path connected.

Proof: Let X be a path connected U- space and Y be a U-space. Let $\phi : X \to Y$ be a onto U-continuous mapping. We shall show that Y is a path connected U-space. Let y_1 and y_2 are two points of Y. Then there exists $x_1, x_2 \in X$ such that $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$.

Since X is a path connected U-space, there exists a U-continuous mapping $f : I \to X$ such that $f(0) = x_1$ and $f(I) = x_2$. Then $\phi(f(0)) = y_1$, $\phi(f(I)) = y_2$.

So $\phi f : I \to Y$ is a U-continuous mapping i.e. Y is path connected.

Theorem 3.21 The product space of any finite number of path connected U-spaces is path connected.

Proof: Let $X_1, X_2, X_3, ---------, X_n$ be path connected U-spaces and

$X = X_1 \times X_2 \times X_3 \times -------- \times X_n$. Suppose $x, y \in X$, then $x = (x_1, x_2, x_3, -------, x_n)$ and $y = (y_1, y_2, y_3, --------, y_n)$, $x_i, y_i \in X_i$. Since each X_i path connected, there exists a U-continuous mapping $f_i : [0, 1] \to X_i$ such that $f_i(0) = x_i$, $f_i(1) = y_i$.

If $f : [0, 1] \to X$ defined by $f(a) = (f_1(t), f_2(t), ------, f_n(t))$,
then $f(0) = (x_1, x_2, x_3, -------, x_n) = x$ and $f(1) = (y_1, y_2, y_3, --------, y_n) = y$.

We shall show that f is U-continuous. Let G be a U-open set of X, then π_i(G) = G_i, G_i is a U-open set of X_i, where π_i(G): X$\to X_i$ is a projection mapping. Since f_i is U-continuous, $f_i^{-1}(G_i)$ is a U- open set of [0, 1].

Now $f^{-1}(G) = f_1^{-1}(G_1) \cap f_2^{-1}(G_2) \cap$--------$\cap f_n^{-1}(G_n)$.

Therefore f^{-1}(G) is U-open. i.e. f is U-continuous.

Remark 3.1 The closure of a path connected subsets of U-space may not be path connected.

Example 3.11 Let S = $\{(x, \sin\frac{1}{x}) : 0 < x \le 1\}$ be a subset of the product U-space $\mathbb{R}\times\mathbb{R}$. Then it is a path connected U-space but the closure $\overline{S}$ is not a path connected U-space.

Here, we see that $\overline{S}$ is connected but not path connected.

Definition 3.8 Let X be a U-space and a and b be two separate point of X. A finite sequence of subsets A_1, A_2, A_3, ------, A_n of X is called **simple chain** from a to b if a only belongs to A_1 and b only belongs to A_n and $A_i \cap A_j \neq \Phi$ iff $|i-j| \le 1$.

Theorem 3.22. If a and b are two separate points of connected U-space X and {U_α} is a U-open cover of X, then there exists a simple chain of U_α from a to b.

Proof: Let $\{U_\alpha\}$ be a U-open cover of X and let Y be a collection of points y of X and there exists a simple chain of U_α from a to y. Then Y is U-open.

Because if $y \in Y$ and U_1, U_2, U_3, -----, U_n ($U_i \in \{U_\alpha\}$) from a to y is a simple chain, then U_1, U_2, U_3, ------, U_n or U_1, U_2, U_3, ------, U_{n-1} from a to y' for each $y' \in U_n$ is a simple chain. So, $y' \in Y$ and $U_n \subseteq Y$. Therefore Y is a U-open set.

Now we shall show that Y is closed. Suppose y be a limit point of Y. Then there is a point y' ($y' \neq y$) of Y in each U-open set U containing y.

Therefore the exists a simple chain $U_1, U_2, U_3, \ldots\ldots, U_n$ from a to y. Now we can consider y' is a point of Y in which n is the smallest.

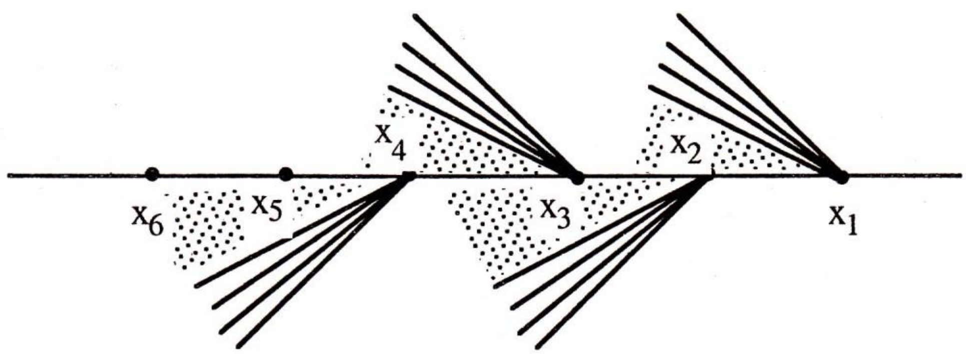
Since $U_n \cap U \neq \Phi$, $U_1, U_2, U_3, \ldots, U_n, U$ from a to y will be simple chain if for each i < n, $U_i \cap U = \Phi$. Because if $U_i \cap U \neq \Phi$ ($i_0 < n$) and let $y'' \in U_{i_0} \cap U$. Then $y'' \in Y$ and $U_1, U_2, U_3, \ldots, U_{i_0}$ from a to y'' is a simple chain. Since $i_0 < n$ which is contradictory to the smallest n. So $U_1, U_2, U_3, \ldots, U_n, U$ from a to y is a simple chain. i.e. $y \in Y$.

Hence theorem is proved.

Definition 3.9 Let X be a U-space and $x \in X$, X is said to be **connected im kleinen at x** if , for each U-open set V of X with $x \in V$, there exists $W \subseteq V$ which containing x and is such that, for each $y \in W$, there is a connected subset C of W with x, $y \in C$.

If a U-space X is locally connected at x, then it is connected im kleinen at x. However, the converse need not be true.

pThe example given by the following figure of the U-subspace of $X \times X$, where X is the usual U-space $\mathbb{R}$, is connected im kleinen at x_1, x_2, x_3, x_4, ... but it is not locally connected at these points.



CHAPTER - 4

Paracompact U-spaces

Introduction

The concept of paracompactness for topological spaces was defined by Dieudonne [10]. This concept has been proved to be very important and useful.

In this chapter the notion of paracompact U-spaces has been introduced and a number of sufficient conditions for paracompactness for such spaces have been established.

In connection with paracompactness of U-spaces, we have generalized the concepts of refinement, locally finite, countably locally finite, star and barycentric refinements in U-spaces, and proved the U-space-versions of a few theorems concerning paracompact topological spaces ([11], [24] and [33]). A few relevant examples have been provided.

Paracompactness in U-spaces

We start with a few necessary definitions in U-spaces which generalize the corresponding topological concepts.

Definition 4.1 Let $\mathcal{G}$ be a collection of subsets of the U-space X. A collection $\mathcal{B}$ of subsets of X is said to be a U-**refinement** of $\mathcal{G}$ (or is said to be refine $\mathcal{G}$) if for each element B of $\mathcal{B}$, there is an element $G \in \mathcal{G}$, such that $B \subseteq G$. If the elements of $\mathcal{B}$ are open sets, we call $\mathcal{B}$ a U-**open refinement** of $\mathcal{G}$; if they are closed sets, we call $\mathcal{B}$ a U-**closed refinement** of $\mathcal{G}$.

Definition 4.2 A collection $\mathcal{G}$ of subsets of a U-space X is l**ocally finite** if every point of X has a neighborhood that intersects only finitely many elements of $\mathcal{G}$.

Thus, for a U-space X and a collection $\{A_\alpha\}$ of subsets of X, $\{A_\alpha\}$ is **locally finite** if, for each $x \in X$, there exists a U-open set G containing x such that

$G \cap A_\alpha \neq \Phi$, for only a finite number of α's.

Locally finite collections are also called **neighborhood- finite.**

Example - 4.1 Let $X = \mathbb{N}$ and let $\mathcal{U}$ consist of X, Φ and all subsets of $\mathbb{N}$ of the form $G_n = \{n, n + 1, n + 2, n + 3\}$ and their unions. **Then (X, $\mathcal{U}$) is a proper U- space**, since $G_1 \cap G_2 = \{2, 3, 4\} \notin \mathcal{U}$. Let $\mathcal{C}$ denote the family of sets $C_k = \{n \in \mathbb{N} \mid n \geq k\}$, $k \in \mathbb{N}$. Let $x \in X$. Then $x = n_0$, for some $n_0 \in \mathbb{N}$. For the neighborhood, $G_{n_o} = \{n_0, n_0 + 1, n_0 + 2, n_0 + 3\}$ of x, $G_{n_o} \cap C_k \neq \Phi$, only for $k = 1, 2, 3, \cdots, n_0 + 3$.

Hence $\mathcal{G}$ is locally finite.

Definition 4.3 A collection $\mathcal{G}$ of subsets of a U-space X is said to be **countably locally finite** if $\mathcal{G}$ is a countable union of locally finite collections $\mathcal{G}_n$ i.e., $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$.

Example - 4.2 Let (X, $\mathcal{U}$) be the proper U-space of example 4.1. For each positive integer k and m, let $\mathcal{G}_{k,m} = \{n \in \mathbb{N} \mid n \geq \frac{k}{m}\}$, Let $\mathcal{G}_m = \{\mathcal{G}_{k,m}\}_{k \in N}$.

For each m, $\mathcal{G}_m$ is locally- finite.

Therefore $\mathcal{G} = \bigcup_m \mathcal{G}_m$ **is countably locally- finite.**

Definition 4.4 A U-space X is **paracompact** if X is Hausdorff and every U-open cover $\mathcal{G}$ of X has a locally finite U-open refinement of $\mathcal{G}$ that covers X.

Clearly, any compact Hausdorff U-space is paracompact.

We now give a non- trivial example of a paracompact U-space.

Example 4.3 Let $X = \mathbb{Z}$ and $\mathcal{U}$ = The collection of all A_n's and their unions, where for $n \in \mathbb{Z}$, $A_n = \{x \in X : n \leq x \leq n + 3\}$. Then, $\mathcal{U}$ is a U-structure but not a topology, since $A_1 \cap A_2 = \{x \in X : 2 \leq x \leq 4\}$ which does not belong to $\mathcal{U}$.

Also, $(X, \mathcal{U})$ is Hausdorff. For, if $m, n \in \mathbb{Z}$, $m \neq n$, then let $m < n$, $m \in A_{m-3}$, $n \in A_n$, and $A_{m-3} \cap A_n = \Phi$.

We shall now show that every U-open cover of X has a locally finite refinement. Let $\mathcal{G}$ be a U-open cover of X. For each $x \in X$, $x \in A_{n,x} \subseteq G_x$, for some $A_{n,x} \in A_n$, where G_x is a member of $\mathcal{G}$. (Such $A_{n,x}$ and G_x exist. G_x exists because $\mathcal{G}$ is a U-open cover of X. And, by definition, G_x is a union of a class of A_n's at least one of which must contain x. Call this A_n $A_{n,x}$).

Let $\mathcal{A} = \{A_{n,x} : x \in X\}$. Then $\mathcal{A}$ is a **refinement of** $\mathcal{G}$ which covers X. Let $x_0 \in X$ and let $G = A_{n, x_0}$. Then G is a U-open set containing x_0 and G intersects only seven members of $\mathcal{A}$,

viz, A_{n-3, x_0} , A_{n-2, x_0}, A_{n-1, x_0}, A_{n, x_0}, A_{n+1, x_0}, A_{n+2, x_0}, A_{n+3, x_0} .

Thus, $\mathcal{A}$ **is locally finite refinement** of $\mathcal{G}$ which covers X.

Hence X is a **paracompact U-space** which is not a **topological space**.

It is clear that an infinite number of such proper paracompact U-spaces can be similarly constructed.

Our next example is a proper U-space which is not paracompact but in which every U-open cover has a locally finite refinement that covers X.

Example 4.4 Let $X = \mathbb{Z}$, fix $x_o \in \mathbb{Z}$. For each $x \in \mathbb{Z}$, let $A_x = \{x_o, x, x + 1, x + 2\}$. Let $\mathcal{U}$ be the collection of Φ, all A_x's, $x \in \mathbb{Z}$ and their unions. Then $(X, \mathcal{U})$ is U-space, but not a topological space. Since $A_{x+1} \cap A_{x+5} = \{x_o\} \notin \mathcal{U}$.

Let $\mathcal{G}$ be an **U-open cover of X**. Let $x \in X$. Then there is a $G_x \in \mathcal{G}$ such that $x \in G_x$ and so $x \in A_y$, for some $y \in X$. Let $\mathcal{D}$ be the collection of all sets B_y's such that for some y_o, $B_{y_o} = A_{y_o}$, and for each $y \neq y_o$, $B_y = A_y - \{x_o\}$.

Then $\mathcal{D}$ is a refinement of $\mathcal{G}$ and covers X. Now A_y is a U-open set containing x and it is clear that A_y intersects only a finite number of B_y's.

Thus $\mathcal{D}$ is a locally finite refinement of $\mathcal{G}$.

We now note that $(X, \mathcal{U})$ is not Hausdorff, since for each $x, y \in Z$, $A_x \cap A_y \neq \Phi$.

Hence X is not paracompact.

We recollect that **the usual U-space** $\mathbb{R}$ is $\mathbb{R}$ with **the U-structure** consisting of all subsets of $\mathbb{R}$ of the forms $(-\infty, a)$ and (b, ∞) and their unions.

Remark - 4.1 $\mathbb{R}$ with the usual topology is paracompact. But the usual U- space $\mathbb{R}$ is not paracompact. We prove its truth below:

For $\mathcal{G}$ = {(-∞, a)|a∈ ℝ } is an open cover of ℝ. If x∈ ℝ, and x∈G with G is U-open, then G is the form $\bigcup_{i,j}[(-\infty, a_i) \cup (b_j, \infty)]$, for some a_i's, b_j's, and x belongs to some $(-\infty, a_i)$ or, some (b_j, ∞).

If $\mathcal{D}$ is a refinement of $\mathcal{G}$ which covers ℝ, then $\mathcal{D}$ is a collection of sets of the form (-∞, c), where c < a, for each a with (- ∞, a) ∈ $\mathcal{G}$. Clearly, $\mathcal{D}$ is an infinite collection of U-open sets, and G meets infinitely many members of $\mathcal{D}$. So $\mathcal{D}$ is not **locally finite.**

Thus $\mathcal{G}$ has no locally finite refinement.

Let (X, $\mathcal{U}$) be a U-space and $\mathcal{T} = \mathcal{T}_{\mathcal{U}}$ be the topology generated by $\mathcal{U}$ on X. Then we have the following theorem**.**

Theorem 4.1 If (X, $\mathcal{U}$) is paracompact, then (X, $\mathcal{T}_{\mathcal{U}}$)= (X, $\mathcal{T}$) is paracompact.

Proof: Clearly (X, $\mathcal{T}$) is **Hausdorff** if (X, $\mathcal{U}$) is Hausdorff. Let $\mathcal{G}$ be an open cover of X in (X, $\mathcal{T}$). For each x∈X, there exists G_x in $\mathcal{G}$ such that $x \in G_x$. Then G_x contains a set H_x such that $x \in H_x$ and H_x is the intersection of a finite collection of sets $U_{1,x}, U_{2,x}, \ldots, U_{r,x}$ in $\mathcal{U}$. Choose any $U_{i,x}$ and call it U_x.

Let $\mathcal{D} = \{U_x : x \in X\}$. Then, $\mathcal{D}$ is a U-cover of X.

Since (X, $\mathcal{U}$) is paracompact, $\mathcal{D}$ has a locally finite refinement say $\mathcal{D}'$ which covers X. For each y in X, let $y \in V_y \in \mathcal{D}$. Let $H_y = G_y \cap V_y$.

Then $\mathcal{G}^{/}$ ={H_y: y∈X} is a open cover of X. $\mathcal{G}^{/}$ is a locally finite refinement of $\mathcal{G}$, since $\mathcal{D}^{/}$ is a **locally finite refinement** of $\mathcal{D}$. Thus (X, $\mathcal{T}_{\mathcal{U}}$) is **paracompact**.

Our next theorems are generalizations of theorems for topological spaces.
Theorem 4.2 ([24], Theorem 9.1, p. 160,161)

Every paracompact Hausdorff U- space X is normal.

Proof: Let X be a paracompact U-space. Firstly, we shall show that X is regular. Let x∈ X and B be a U-closed subset of X, where x∉ B. Since X is Hausdorff, for every b∈B there exist two disjoint U-open sets U_b, V_b such that x∈ U_b and b∈, V_b . So x$\notin \overline{V_b}$ .Then $\mathcal{G}$ = $\{V_b\}_{b\in B}\cup\{X - B\}$ is a U-open cover of X. Since X is paracompact, there exists a locally finite refinement $\mathcal{D}$ of $\mathcal{G}$ which is a U-open cover of X. Let $\mathcal{E}$ be the subcollection of $\mathcal{D}$ consisting of all those members of $\mathcal{D}$ which intersect B. Then $\mathcal{E}$ is a U-open cover of B. Since for every b∈B, x$\notin \overline{V_b}$, so for every E∈$\mathcal{E}$, x $\notin \overline{E}$.

Let W = $\bigcup_{E\in\xi} E$**,** then W is U-open set of B. We shall now show that

$\overline{W}=\bigcup_{E\in\xi}\overline{E}$. Obviously, $\bigcup_{E\in\xi}\overline{E}\subseteq\overline{W}$. If possible, suppose x∈$\overline{W}$. Then for every U-open set G containing x, G∩W≠Φ. since E is locally finite, G intersects only a finite number of members say E_1, E_2, E_3,------,E_r of E.

Let W_1= $E_1\cup E_2\cup E_3\cup$-------$\cup E_r$ and W_2 = $\bigcup_{E\in\xi,\ E\neq E_1,E_2,E_3,\ldots\ldots,E_r} E$.

So, $G \cap W_2 = \Phi$. This implies that $x \notin \overline{W}_2$. Since $\overline{W} = \overline{W}_1 \cup \overline{W}_2$, $x \in \overline{W}_1 = \overline{E}_1 \cup \overline{E}_2 \cup \overline{E}_3 \cup$-------$\cup \overline{E}_r$. So, $\overline{W} \subseteq \bigcup_{E \in \xi} \overline{E}$

Thus, $\overline{W} \subseteq \bigcup_{E \in \xi} \overline{E}$. But this is a contradiction, since $x \notin \overline{E}$ for each E. So, $x \in \overline{W}$.

Hence $\overline{W'}$ is a U- open set containing x. Therefore X is regular.

Now let A and B be two U-closed subsets. Since X is regular, for every $a \in A$ and for B there exist disjoint U-open set U_a and V_a such that $a \in U_a$ and $B \subseteq V_a$. One merely repeats the same argument, there exists a U-open set $W = \bigcup_{E \in \xi} E$ containing A, where (i) $\mathfrak{E}$ is a locally finite U-open cover of A and (ii) Every $\overline{E} \cap B = \Phi$. Since $\mathfrak{E}$ is locally finite, $\overline{W} = \bigcup_{E \in \xi} \overline{E}$, and $B \subseteq \overline{W}$.Hence X is normal.

Theorem 4.3 ([24], Theorem 9.2)

Every U-closed subspace of a paracompact U-space is paracompact.

Proof: Let X be a paracompact U-space and Y a U-closed subspace of X. Obviously, Y is Hausdroff. Let $\mathfrak{C}' = \{C'_\alpha\}$ be a U-open cover set of Y. Then for each $C'_\alpha = C_\alpha \cap Y$, where C_α is a U-open set of X. Now suppose $\mathfrak{C} = \{C_\alpha\}$.

Then $\mathfrak{D} = \{C_\alpha\} \cup \{Y^c\}$ is a U-open cover of X. Since X is paracompact, there exists a locally finite U-open cover $\mathfrak{E}$ of X which is a refinement of $\mathfrak{D}$. $\mathfrak{G}$ is the subcollection of those members of $\mathfrak{E}$ which are not subsets of Y^c and $\mathcal{G}$ is refinement of $\mathcal{C}$. For this reason $\mathcal{G}$ is a U-open cover of Y and a refinement of $\mathcal{C}'$. Since $\mathcal{E}$ is locally finite, $\mathcal{G}$ is locally finite.

Hence Y is **paracompact**.

Remark- 4.2. A U-subspace of a paracompact U-space need not be paracompact.

Since this statement is true about topological spaces (see [24], p. 108,161), it is also true about U-spaces.

For proof, we need to define a special U-structure on $\mathbb{R}$ which is called the lower limit U-structure. This U-space is denoted by $\mathbb{R}_l$.

Definition 4.5 Let $\mathcal{C}$ be the collection of subsets of the form [a, b) = {n $|a \le x < b$ }, where a < b, the U-structure generated by $\mathcal{C}$ is called **the lower limit U-structure on $\mathbb{R}$.**

Theorem 4.4 Product of two paracompact U-spaces need not be paracompact.

Proof: As for topological spaces one can be shown that the U-space $\mathbb{R}_l$ is paracompact, but $\mathbb{R}_l \times \mathbb{R}_l$ is not normal, and hence, not paracompact.

Our next theorems generalize a few more results for topological spaces. The proofs however are almost similar to those for topological spaces. So, we omit proofs of some of these theorems in some cases.

Theorem 4.5 ([24], Lemma 9.3) Let X be a regular U-space and let $\mathcal{C}$ be a U-open cover of X. Consider the following conditions on $\mathcal{C}$: $\mathcal{C}$ has a refinement which is

(i) a U-open cover of X and countably locally finite,

(ii) a cover of X and locally finite,

(iii) a U-closed cover of X and locally finite,

(iv) a U-open cover of X and locally finite.

Among the above four conditions on $\mathcal{C}$, the following implications hold; (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii).

Proof: It is trivial that (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii)

Let $\mathcal{G}$ be a U-open cover of X. Let $\mathcal{B}$ be an U-open refinement of $\mathcal{G}$ that covers X and is countably locally finite i.e. $\mathcal{B} = \cup \mathcal{B}_n$, where $\mathcal{B}$ is a locally finite. Let $V_i = \bigcup_{U \in B} G$ and for each $n \in \mathbb{N}$ and each $G \in \mathcal{B}$, define

$S_n(G) = G - \bigcup_{i<n} V_i$.

Let $\mathcal{C}_n = \{S_n(G) \mid G \in \mathcal{B}_n\}$. Then $\mathcal{C}_n$ is refinement of $\mathcal{B}_n$, because $S_n(G) \subseteq G$, for each $G \in \mathcal{B}_n$ Let $\mathcal{C} = \cup \mathcal{C}_n$. We shall show that $\mathcal{C}$ is a locally finite collection refinement of $\mathcal{G}$, covers X. Suppose $x \in X$. We shall show that for any $S_n(G)$, $x \in S_n(G)$ a neighborhood of x that intersects only a finite elements of $\mathcal{C}$. Since $\mathcal{B}$ covers X, there is a smallest positive integer number n_0 such that $x \in G \in \mathcal{B}_{n_0}$. Since x does not belong to any member of $\mathcal{B}_i$ for $i < n_0$, $x \in S_{n_0}(G) \in \mathcal{C}$. Since each collection $\mathcal{B}_n$ is locally finite, we can choose for each n = 1, 2, 3, ...n_0 a neighborhood W_n of x that intersects only finitely many members of $\mathcal{B}_n$ .Now if W_n intersects the member $S_n(V)$ of $\mathcal{C}$, W_n must intersect the member V of $\mathcal{B}_n$, since $S_n(V) \subset V$.

Therefore, W_n intersects only finitely many members of $\mathcal{C}$. Furthermore, because $G \in \mathcal{B}_n$, G does not intersect any element of $\mathcal{C}_n$, for $n > n_0$. As a result, the neighborhood $W_1 \cap W_2 \cap W_3 \cap \text{----------} \cap W_{n_0} \cap G$ of x intersects only finitely many elements of $\mathcal{C}$.

(iii) $\Rightarrow$ (iv)

Let $\mathcal{G}$ be a U-open cover of X. Using (iii) Choose $\mathcal{B}$ a refinement of $\mathcal{G}$ that is locally finite and a U-closed cover of X. Now we consider for every $B \in \mathcal{B}$ a U-open set $D(B) \supseteq B$ that the collection $\{D(B) | B \in \mathcal{B}\}$is also locally finite refinement of $\mathcal{G}$. Since B is locally finite. For every $x \in X$, there exist a neighborhood N_x of x that intersect finite members of $\mathcal{B}$. Then $\{N_x \mid x \in X\}$ is a U-open cover of X.

According to (iii) There is a refinement $\mathcal{C}$ of $\{N_x \mid x \in X\}$ which is a U-closed cover of X. Clearly, for every $C \in \mathcal{C}$ intersects finite members $B \in \mathcal{B}$. For each $B \in \mathcal{B}$, let $\mathcal{C}(B) = \{C : C \in \mathcal{C} \text{ and } C \subseteq X - B\}$.

Again let, E(B) = X - $\bigcup_{C\in\xi(B)} C$. Then $\bigcup_{C\in\xi(B)} C$ is closed by a lemma 8.1 of [24] which has the following statement:

"Let $\{A_\alpha\}$ be locally finite collection of subsets of X. Then

(a) Any subcollection of $\{A_\alpha\}$ is locally finite.

(b) $\{\overline{A_\alpha}\}$ is locally finite.

(c) $\overline{\bigcup_\alpha A_\alpha} = \bigcup_\alpha \overline{A_\alpha}$."

So E(B) is an U-open set. According to the definition E(B)$\supseteq$B. The collection {E(B)}is a U-open cover of X. For each B$\in\mathcal{B}$, F(B)$\in\mathcal{G}$, where F(B)$\supseteq$B.

Let $\mathcal{D}$ = {E(B) $\cap$F(B)|B$\in$ $\mathcal{B}$ }. Then the collection $\mathcal{D}$ is refinement of $\mathcal{G}$ and U-open cover of X. Since B$\subseteq$ E(B) $\cap$F(B) and B is a U-open cover of X. Suppose x$\in$X. Now we shall show that $\mathcal{D}$ is locally finite. Since $\mathcal{C}$ is locally finite, there exists a neighborhood W of x that intersects only a finite number of members of C, (say) C_1, C_2, C_3, ------, C_n. Since $\mathcal{C}$ is U-cover of X, so, W$\subseteq$ $C_1\cup C_2\cup C_3\cup$------$\cup C_n$.

Now if any member C of $\mathcal{C}$ intersects the set E(B) $\cap$F(B), then

C $\not\subset$X - B. Therefore C intersects B. Since C intersects a finite number of members B, so C will intersect at most this number of members of $\mathcal{D}$.

Therefore W will also intersect finite number of members of $\mathcal{D}$.

Now if we write E(B) $\cap$F(B), the collection $\mathcal{D}$ = {D(B)|B$\in$ $\mathcal{B}$} is refinement of $\mathcal{G}$ and is a locally finite open cover of X.

Comment 4.1 The properties (i) - (iv) of the above Theorem 4.5 can also be stated as :

(a) Each U-open covering of X has a U-open refinement that can be decomposed into an at most countable collection of locally finite families of U-open sets.

(b) Each U-open covering of X has a locally finite refinement, consisting of sets not necessarily either U-open or U-closed.

(c) Each U-open covering of X has a U-closed locally finite refinement.

(d) X is paracompact.

Dugunji [11] uses these properties in ([11], Theorem 2.3)

Theorem 4.6 ([24], Theorem 9.5) If a locally compact Hausdorff U-space X is a countable union of compact U-spaces then X is paracompact.

Proof: Let X be a locally compact Hausdorff U-space and X = $\bigcup_n C_n$, where C_n is compact. Let for each n, $C_n \subseteq C_{n+1}$ (We can assume this, for otherwise we can consider C'_n instead of C_n where $C'_n = \bigcup_{i=1}^{n} C_i$). At first we shall show that X = $\cup W_n$, where W_n is U-open, $\overline{W_n}$ is compact and $\overline{W_n} \subseteq$Wn+1. Let $x \in C_1$ and let G_x be a neighborhood of x, where $\overline{G_x}$ is compact. Then $\{G_x\}_{x\in C_1}$ is a U-open cover of C_1**.** Since C_1 is compact**,** there is a finite U-open subcover **{**$G_{x_1}, G_{x_2}, ------, G_{x_n}$**}** of C_1. Let $W_1 = \bigcup_{i=1}^{n} G_{x_i}$.

Therefore $\overline{W_1}$ is compact, this implies that $C_2 \cup \overline{W_1}$ is compact. Suppose W_2 is a U-open set of $C_2 \cup \overline{W_1}$ obtained in the same way as the U-open set W_1 of C_1. So $\overline{W_2}$ is compact, $C_2 \subseteq W_2$ and $\overline{W_1} \subseteq W_2$. Let, for each m< n, the U-open set W_m be defined in a similar member such that $C_m \subseteq W_m$, $\overline{W_m}$ is compact and $\overline{W_m} \subseteq W_{m+1}$. Proceeding as before we get for each positive integer $n \geq 2$ a U-open set W_n of $C_n \cup \overline{W_{n-1}}$ **,** where $\overline{W_n}$ is compact and $\overline{W_{n-1}} \subseteq W_n$.

Let $\mathscr{W}$ = {G_α} be a U-open cover of X and $K_n = \overline{W_n} - W_{n-1}$. Then K_n is compact. Now for every $x \in K_n$, there is a neighborhood V_x of x such that for any α, $V_x \subseteq G_\alpha$.Assume that $V_x \subseteq W_{n+1}$, since $\overline{W_n} \subseteq W_{n+1}$ and $V_x \cap W_{n-2} = \Phi$, since $\overline{W_{n-2}} \subseteq W_{n-1}$.Since K_n is compact, so there is a finite cover

$\mathscr{D}_n$ = {V_{x_1}, V_{x_2} -----V_{x_n}} of K_n. We denote by $\mathscr{V}$ the union of the finite covers $\mathscr{D}_n$ of K_n for all n. Then $\mathscr{V}$ is a U- open cover of X and since $V_x \in \mathscr{V}$ is contained in a $G_\alpha \in \mathscr{W}$. $\mathscr{V}$ is refinement of $\mathscr{W}$. Suppose $x \in X$. Then there exists a least natural number n such that $x \in \overline{W_n}$. Since $x \notin W_{n-1}$, so, $x \in K_n$. As a result there is a neighborhood $V \in \mathscr{V}$ which intersect only finite member of those members of $\mathscr{V}$ which covers K_{n-2}, K_{n-1},K_n and K_{n+1}.

Theorem 4.7 ([24], Theorem 9.6) A locally compact Hausdorff U-space with a countable basis is paracompact.

To prove the next theorem we need a lemma.

Lemma 4.1 ([11], Lemma 3.3) If X, Y are U-spaces with X normal, and p: X$\rightarrow$ Y is a U-continuous U-closed surjection, then Y is too normal.

Proof: Let A and B be two disjoint U-closed sets in Y. Since p is U-continuous, $p^{-1}(A)$ and $p^{-1}(B)$ are disjoint U-closed sets in X. X being normal, there are disjoint U-open sets G and H in X such that $p^{-1}(A) \subseteq G$, $p^{-1}(B) \subseteq H$. Since p is U-closed, p(G) and p(H) are disjoint U-open sets in Y whith A$\subseteq$p(G), B$\subseteq$p(H). Thus Y is normal.

Theorem 4.8 ([11], Theorem 2.6) Every U-continuous closed image of a paracompact U-space is paracompact.

Proof: Let X and Y be U-spaces with X paracompact, and let p: X$\rightarrow$Y be U-continuous U-closed surjection mapping. Let $\{G_\alpha \mid \alpha \in \mathscr{A}\}$ be any U-open covering of Y. Since X is normal and p is U-continuous, U-closed and surjection, Y is normal. By Theorem 4.5 and comment 4.1it suffices to show that$\{G_\alpha \mid \alpha \in \mathscr{A}\}$ has an U-open refinement which can be decomposed into at most countably many locally finite families. We assume $\mathscr{A}$ is well-ordered and begin by constructing a U-open covering $\{V_{\alpha,n} \mid (\alpha,n) \in \mathscr{A} \times Z^+\}$ of X such that:

(i). For each n, $\{\overline{V}_{\alpha,n} \mid \alpha \in \mathscr{A}\}$ is a U-covering of X and a precise locally finite refinement of $\{p^{-1}(G_\alpha) \mid \alpha \in \mathscr{A}\}$.

(ii). If $\beta > \alpha$ then $p(\overline{V}_{\beta,\, n+1}) \cap p(\overline{V}_{\alpha,n}) = \Phi$.

Proceeding by induction, we take a precise U-open locally finite refinement of $\{p^{-1}(G_\alpha)\}$ and shrink it by normality of X to get $\{\overline{V}_{\alpha,1}\}$. Assuming $\{V_{\alpha,i}\}$ to be defined for all $i \le n$, let $W_{\alpha,n+1} = p^{-1}(G_\alpha) - p^{-1}p(\bigcup_{\lambda<\alpha}\overline{V}_{\lambda,n})$. Each $W_{\alpha,n+1}$ is U-open, since by local finiteness $\bigcup_{\lambda<\alpha}\overline{V}_{\lambda,n}$ is U-closed and p is a U-closed map.

Furthermore, $\{W_{\alpha,n+1} \mid \alpha \in \mathscr{A}\}$ is a U-covering of X: given $x \in X$, let α_o be the first index for which $x \in p^{-1}(G_\alpha)$; then $x \in W_{\alpha_o,n+1}$, since $p^{-1}p(\overline{V}_{\lambda,n}) \subset p^{-1}(G_\lambda)$ for each λ. Taking a precise, U-open locally finite refinement of $\{W_{\alpha,n+1} \mid \alpha \in \mathscr{A}\}$, shrink it to get $\{\overline{V}_{\alpha,n+1}\}$. Clearly, condition (i) holds, and since $\overline{V}_{\beta,n+1}$ is not in the inverse image of any $p(\overline{V}_{\alpha,n})$ for $\alpha < \beta$, condition (ii) is also satisfied.

For each n and α, let $H_{\alpha,n} = Y - p(\bigcup_{\beta\neq\alpha}V_{\beta,n})$ which is an U-open set. We have

(a) $H_{\alpha,n} \subset p(\overline{V}_{\alpha,n}) \subset G_\alpha$ for each n and α. Indeed,

$$p^{-1}(H_{\alpha,n}) = X - p^{-1}p(\bigcup_{\beta\neq\alpha}V_{\alpha,n}) \subset X - p^{-1}p(X - \overline{V}_{\alpha,n}) \subset \overline{V}_{\alpha,n} \subset p^{-1}(G_\alpha).$$

(b) $H_{\alpha,n} \cap H_{\beta,n} = \Phi$ for each n whenever $\alpha \neq \beta$.

In fact, $y \in H_{\alpha,n} \Rightarrow y \in p(\overline{V}_{\alpha,n})$ and is in no other $p(\overline{V}_{\beta,n})$.

c) $\{H_{\alpha,n} \mid (\alpha,n) \in \mathscr{A} \times Z^+\}$ is an U-open covering of Y. Let $y \in Y$ be given; for each fixed n there is, because of (i), a first α_n with $y \in p(\overline{V}_{\alpha_n,n})$;

choosing now $\alpha_k = \min\{\alpha_n \mid n \in Z^+\}$, we have y$\in$p($\overline{V}_{\alpha_k, k}$). If $\beta < \alpha_k$, then the definition of α_k shows y$\notin$p($\overline{V}_{\beta, k+1}$); if $\beta > \alpha_k$, then by (ii), we find that y$\notin$ p($\overline{V}_{\beta, k+1}$); therefore we conclude that y$\in$ $H_{\alpha_k, k+1}$.

To complete the proof, we need only modify the $H_{\alpha,n}$ slightly to assure locally finiteness for each n. Choose a precise U-open locally finite refinement of { p^{-1} ($H_{\alpha,n}$)$\mid(\alpha,n) \in \mathscr{A} \times Z^+$}, and shrink it to get an U-open locally finite covering {$K_{\alpha,n}$} satisfying p($\overline{K}_{\alpha,n}$)$\subset H_{\alpha,n}$. For each n, let S_n = {y $\mid$some nbd of y intersects at most one $H_{\alpha,n}$ }; S_n is U-open and contains the U-closed $\bigcup_\alpha p(\overline{K}_{\alpha,n})$ = p ($\bigcup_\alpha \overline{K}_{\alpha,n}$),so by normality of Y we find an U-open G_n with $\bigcup_\alpha p(\overline{K}_{\alpha,n}) \subset G_n \subset \overline{G_n} \subset S_n$ .The U-open covering {$G_n \cap H_{\alpha,n} \mid (\alpha,n) \in \mathscr{A} \times Z^+$}, with the decomposition { $G_n \cap H_{\alpha,n} \mid \alpha \in \mathscr{A}$} for n = 1, 2, 3, ------- satisfies the conditions of Theorem 4.5 and Comment 4.1 for the given {G_α }.

Definition 4.6 Let $\mathscr{G}$ = {$G_\alpha \mid \alpha \in \mathscr{A}$} be a covering of U-space X. For any B$\subset$X the set $\cup$ { $G_\alpha \mid B \cap G_\alpha \neq \Phi$ } is called the U-**star of B** with respect to $\mathscr{G}$, and is denoted by St (B, $\mathscr{G}$).

Definition 4.7 A **U-**covering $\mathscr{B}$ is called a U-**barycentric refinement of a U-covering** $\mathscr{G}$ whenever the covering {St (x, $\mathscr{B}$) $\mid x \in X$ } refines $\mathscr{G}$.

Theorem 4.9 ([11], Theorem 3.2) Let X be normal U-space, and $\mathcal{G} = \{G_\alpha \mid \alpha \in \mathcal{A}\}$ a locally finite U-open covering. Then $\mathcal{G}$ has an U-open barycentric refinement.

Proof: Shrink $\mathcal{G}$ to an U-open covering $\mathcal{B} = \{V_\alpha \mid \alpha \in \mathcal{A}\}$ such that $\overline{V}_\alpha \subset G_\alpha$ for each α; clearly, $\mathcal{B}$ is also locally finite. For each $x \in X$, define $W(x) = \bigcap \{G_\alpha \mid x \in \overline{V}_\alpha\} \cap \bigcap \{\complement \overline{V}_\beta \mid x \in \overline{V}_\beta\}$.

We show that $\mathcal{B}^* = \{W(x) \mid x \in X\}$ is the required U-open covering. Note that each W(x) is U-open: the locally finiteness of $\mathcal{B}$ assures that the first term is a finite intersection and that the last term, $\complement \cup \overline{V}_\beta$ is a U-open set. Next, $\mathcal{B}^*$ is a U-covering, since $x \in W(x)$ for each $x \in X$. Finally, fix any $x_o \in X$ and choose a $\overline{V}_\alpha$ containing x_o. Now, for each x such that $x_o \in W(x)$, we must have $x \in \overline{V}_\alpha$ also, otherwise $W(y) \subset \complement \overline{V}_\alpha$; and because $x \in \overline{V}_\alpha$, we conclude that $W(x) \subset G_\alpha$. Thus, $St(x_o, \mathcal{B}^*) \subset G_\alpha$, and the proof is complete.

Definition 4.8 A **U-** covering $\mathcal{B} = \{V_\beta \mid \beta \in \mathscr{B}\}$ is called a U-star refinement of the U-covering $\mathcal{G}$ whenever the U-covering $\{St(V_\beta, \mathcal{B} \mid \beta \in \mathscr{B}\}$ refines $\mathcal{G}$.

Theorem 4.10 ([11], Theorem 3.4) A U-barycentric refinement $\mathcal{B}^*$ of a U-barycentric refinement $\mathcal{B}$ of $\mathcal{G}$ is a U-star refinement of $\mathcal{G}$.

Theorem 4.11 ([11], Theorem 3.5) A T_1-U-space X is paracompact if and only if each U-open covering has an U-open barycentric refinement.

Proof: Only the sufficiency requires proof. We first show that any U-open covering $\mathcal{G} = \{G_\alpha \mid \alpha \in \mathcal{A}\}$ has a refinement as in Theorem- 4.5 and Comment 4.1.

Let $\mathcal{G}^*$ be an U-open star refinement of $\mathcal{G}$ and let $\{\mathcal{G}_n \mid n \geq 0\}$ be a sequence of U-open coverings, where each $\mathcal{G}_{n+1}$ U-star refines $\mathcal{G}_n$ and $\mathcal{G}_0$ U-Star refines $\mathcal{G}^*$. Define a sequence of U-covering inductively by $\mathcal{B}_1 = \mathcal{G}_1$, $\mathcal{B}_2 = \{St(V, \mathcal{G}_2) \mid V \in \mathcal{B}_1\}$......, $\mathcal{B}_n = \{St(V, \mathcal{G}_n) \mid V \in \mathcal{B}_{n-1}\}$, ...

Each $\mathcal{B}_n$ is an U-open refinement of $\mathcal{G}_o$; in fact, each covering $\{St(V, \mathcal{G}_n) \mid V \in \mathcal{B}_n\}$ refines $\mathcal{G}_o$: this is true for n = 1 and, proceeding by induction, if it is true for n = k – 1, its truth for n = k follows by noting that whenever $V = St(V_o, \mathcal{G}_k)$ for some $V_o \in \mathcal{B}_{k-1}$, then $St(V_o, \mathcal{G}_k) = St[St(V_o, \mathcal{G}_k), \mathcal{G}_k] \subset St(V_o, \mathcal{G}_{k-1})$ because $\mathcal{G}_k$ is a U-star refinement of $\mathcal{G}_{k-1}$.

ow well-order X and for each $(n, x) \in Z^+ \times X$ define

$E_n(x) = St(x, \mathcal{B}_n) - \cup \{St(z, \mathcal{B}_{n+1}) \mid z \text{ precedes } x\}$.

Then $\mathcal{D} = \{E_n(x) \mid (n, x) \in Z^+ \times X\}$ is a U-covering: given $p \in X$, the set

$A = \{z \mid p \in \bigcup_{i=1}^{\infty} St(z, \mathcal{B}_i)\}$ is not empty, since $p \in A$; if x is the first member of A, then $p \in St(x, \mathcal{B}_n)$ for some $n \in Z^+$ and $p \in St(z, \mathcal{B}_{n+1})$ for all z preceding x, so $p \in E_n(x)$. Moreover, since $\mathcal{B}_n$ refines $\mathcal{G}_0$, we find that $\mathcal{D}$ refines $\mathcal{G}^*$.

Each $G \in \mathcal{G}_{n+1}$ can meet at most one $E_n(x)$: for, if $G \cap E_n(x) \neq \Phi$, then there is a $V \in \mathcal{B}_n$ with $x \in V$ and $V \cap G \neq \Phi$, so $x \in V \cup G \subset V_o \in \mathcal{B}_{n+1}$ and $G \subset St(x, \mathcal{B}_{n+1})$. Thus, if $E_n(x)$ is the first set G meets, it cannot meet any $E_n(p)$ for p following x. Now let $W_n(x) = St(E_n(x), \mathcal{G}_{n+2})$.

Then $\mathcal{B}^* = \{W_n(x) \mid (n, x) \in Z^+ \times X\}$ is clearly an U-open covering of X. Furthermore, $\mathcal{B}^*$ refines $\mathcal{G}$ because $\mathcal{D}$ refines $\mathcal{G}^*$.

Finally, for each fixed $n \in Z^+$, the family $\{W_n(x) \mid x \in X\}$ is locally finite: indeed, each $G \in \mathcal{G}_{n+2}$ can meet at most one $W_n(x)$, because $G \cap W_n(x) \neq \Phi$, if, and only if, $E_n(x) \cap St(G, \mathcal{G}_{n+2}) \neq \Phi$ and $St(G, \mathcal{G}_{n+2})$ is contained in some $G_o \in \mathcal{G}_{n-1}$ which we know can meet at most one $E_n(x)$.

The theorem will follow from Theorem 4.5 and Comment 4.1, once we show that X is regular U-space. To this end, let $B \subset X$ be U-closed and $x \notin B$. Since in a T_1-U-space each point is a U-closed set, $\mathcal{G} = \{X - x, \complement B\}$ is an U-open covering. Let $\mathcal{B}$ be an U-open star refinement. Then $St(x, \mathcal{B})$ and $St(B, \mathcal{B})$ are the required disjoint neighborhoods of x and B: for if there were a V containing x and a V' meeting B such that $V \cap V' \neq \Phi$, then $St(V, \mathcal{B})$ would contain x and points of B, which is impossible. The theorem is proved.

Definition 4.9 Let $\mathcal{G} = \{G_\alpha \mid \alpha \in \mathcal{A}\}$ be an U-open covering of X. A sequence $\{\mathcal{G}_n \mid n \in Z^+\}$ of U-open coverings is called U-**locally starring for** $\mathcal{G}$ if for each $x \in X$ there exists an nbd V(x) and $n \in Z^+$ such that $St(V, \mathcal{G}_n) \subset$ some G_α.

Theorem 4.12 ([11], Theorem 3.7) A T_1-U-space is paracompact if and only if each U-open covering $\mathcal{G}$ there exists a sequence $\{\mathcal{G}_n \mid n \in Z^+\}$ of U-open coverings that is U-locally starring for $\mathcal{G}$.

Proof: "Only if" is trivial. "If": We can assume that $\mathcal{G}_{n+1} \prec \mathcal{G}_n$ for each $n \in Z^+$. Let $\mathcal{B} = \{V \text{ open in } X \mid \exists\, n : [V \subset G \in \mathcal{G}_n] \wedge [\, St(V, \mathcal{G}_n) \subset \text{ some } G_\alpha]\}$. For each $V \in \mathcal{B}$, let n(V) be the smallest integer satisfying the condition. Because $\{\mathcal{G}_n \mid n \in Z^+\}$ is locally starring for $\mathcal{G}$, it follows that $\mathcal{B}$ is a U-open covering; we will show that $\mathcal{B}$ is in fact a U-barycentric refinement of $\mathcal{G}$.

Let $x \in X$ be fixed, let $n(x) = \min\{n(V) \mid (x \in X) \wedge (V \in \mathcal{B})\}$, and let $V_o \in \mathcal{B}$ be a set containing x such that $n(V_o) = n(x)$.

For any $V \in \mathcal{B}$ containing x, we have $n(V) \geq n(x)$, and consequently $St(x, \mathcal{B}) \subset \bigcup \{St(x, \mathcal{G}_i) \mid i \geq n(x)\}$. Since $\mathcal{G}_{i+1} \prec \mathcal{G}_i$ for each i, this shows $St(x, \mathcal{B}) \subset St(x, \mathcal{G}_{n(x)}) \subset St(V_o, \mathcal{G}_{n(V_o)}) \subset \text{ some } G_\alpha$.

Therefore by Theorem-4.11, X is paracompact.

CHAPTER - 5

Projectives in some categories of Hausdorff U-spaces

Introduction

We have started this chapter with some definitions in the theory of categories. We have generalized to U-spaces the concepts of projective topological spaces, Stone Čceh compactification, perfect maps, extremally disconnected spaces. We have also generalized to U-spaces some results on topological spaces occurring in [8], [10], [13], [14], [24] , [31], [33], and [37]. These concepts and results have been used later in the chapter.

We have next introduced the notion of projectiveness in some categories of Hausdorff U-spaces. A few important properties of such U-spaces have been studied. A number of interesting examples have been constructed to prove non- trivialness of such results.

For most of the cases the proofs of the above generalizations to U-spaces run parallel to those for topological spaces. But we have given the proofs in detail to show that these really do hold in the present cases.

We have constructed 2 examples of proper projective U-spaces which are locally compact but not compact and two examples of proper projective compact U-spaces.

In this chapter a U-space will mean a Hausdorff U-space, unless otherwise mentioned.

Projectives in some categories of Hausdorff U-space

Definition 5.1 A category consists of

(i) A class C of objects A, B, C,…..

(ii) For each pair of objects A, B a set hom(A, B) whose elements are called morphism, with the property that

(a) $\alpha \in$hom(A, B) and $\beta \in$hom(B, C) implies there exists $\gamma \in$hom (A, C) which is written $\gamma = \beta\,\alpha$;

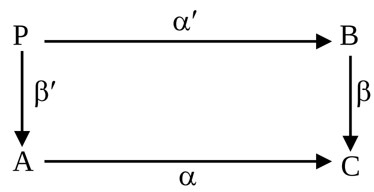
(b) For each A$\in$ C, there exists $1_A \in$hom(A, A) such that for each B$\in$ C and for each $\alpha \in$hom(A, B), $\alpha = \alpha\, 1_A$, and $\alpha = 1_B\, \alpha$,

(c) Let $\alpha \in$hom(A, B), $\beta \in$hom(B, C), $\gamma \in$hom(C, D) then $\gamma(\beta\,\alpha) = (\gamma\,\beta)\alpha$.

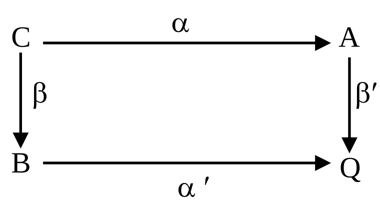
If $\beta\,\alpha = \gamma\,\alpha \Rightarrow \beta = \gamma$ then α is an epimorphism or, epic (onto);

If $\alpha\,\beta = \alpha\,\gamma \Rightarrow \beta = \gamma$, then α is a monomorphism or, monic (one – to – one).

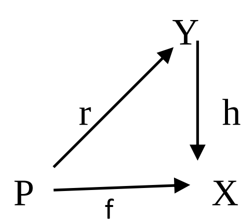
Definition 5.2 [8](p- 89) Let A and B be groups and C be a subgroup of A and B. Let two maps be α: A$\to$C and β: B$\to$C. We consider the different ways of completing them to a commutative square. We can regard these commutative squares as objects of another category, a morphism being a map between the new vertices so as to obtain a wedge with commutative faces. A final object in this category is called **pullback** of α, β.



Definition 5.3 [8](p- 90) Let A and B be two groups and C is a subgroup of A and B, then their **pushout** is the 'free product of A and B amalgamating C '. We note that the product of A and B is a special case of the pullback, obtained by taking C to be a final object, and the co product is a special case of the pushout obtained by taking C to be an initial object. Sometimes the pullback is called 'fibre product' and the **pushout** 'fibre sum'.



Definition 5.4 [14](p- 3) A U-space P is **projective**, if for any pair of U-spaces, X, Y and pair of U-continuous maps h: Y $\rightarrow$X and f: P$\rightarrow$X, with h onto, there exists a U-continuous map r: P$\rightarrow$ Y such that hr(p) = f(p) for every p$\in$P.



Definition 5.5 [37](p- 7) A U-continuous function f: X $\rightarrow$Y where X and Y are arbitrary Hausdorff U-spaces is called U-**perfect** if f is U-closed and the set $f^{-1}(y)$ is compact for each y in Y.

Definition 5.6 [13](p- 482) A U-space X is **extremally disconnected** if the closure of every U-open set is U-open.

Definition 5.7 A U-space Y is an **extension U-space** of another space X if X is dense in Y.

The generalization of the construction of the Stone- Čech compactification for a completely regular U-space

Let X be a completely regular T_1-U-space. Let $\{f_\alpha\}_{\alpha\in A}$ be the collection of all bounded U-continuous real-valued function on X, indexed by some index set A.

For each $\alpha \in$A, choose $I_\alpha = (-\infty, \sup f_\alpha(X)]$, $J_\alpha = [\inf f_\alpha(x), \infty)$ regarded as U-subspaces of the usual U-space $\mathbb{R}$. Then define h:X$\rightarrow \prod_{\alpha\in A} I_\alpha$ by the rule $h(x)= (f_\alpha(x))_{\alpha\in A}$. Since X is completely regular T_1-U-space, for two distinct points x_1, x_2, $\{x_2\}$ is U-closed and $x_1 \notin \{x_2\}$ so there exists f_α such that

$f_\alpha(x_1) \neq f_\alpha(x_2)$.

Hence $h(x_1) \neq h(x_2)$. Therefore h is one-one. Since $f_\alpha : X \rightarrow I_\alpha$ is U-continuous, it follows from the definition of $\prod_{\alpha\in A} I_\alpha$ that h is U-continuous.

We shall show that h is U-open. Let V_1 be a U-open set of X and $y_0 \in h(V_1)$. Let $x_0 \in V_1$ such that $h(x_0) = y_0$. Since X is completely regular, there exists f_α such that $f_\alpha(x_0) \in (-\infty, \sup f_\alpha(X))$ and $f_\alpha(X - V_1) = \sup f_\alpha(X)$. Let

$V_2 = \pi_\alpha^{-1}(-\infty$, sup $f_\alpha(X))$. Then V_2 is U-open in $\prod_{\alpha \in A} I_\alpha$, and $W = V_2 \cap h(X)$ is a U-open set of h(X).

We shall show that $y_0 \in W \subset h(V_1)$. Since $y_0 \in h(V_1) \subseteq h(X)$ and $\pi_\alpha h(x_0) = f_\alpha(x_0)$, $y_0 = h(x_0) = \pi_\alpha^{-1} f_\alpha(x_0) \subseteq V_2$. Therefore $y_0 \in W$.

Let $y \in W$. Then for any $x \in X$, y = h(x) and $\pi_\alpha(y) \in (-\infty$, sup $f_\alpha(X))$. Since $\pi_\alpha(y) = \pi_\alpha h(x) = f_\alpha(x)$ and $f_\alpha(X - V_1)$ = sup $f_\alpha(X)$, so $x \in V_1$, i.e., $y = h(x) \in h(V_1)$.

Therefore h: $X \to \prod_{\alpha \in A} I_\alpha$ is an U-imbedding. Hence ($\overline{h(X)}$, h) is a compactification of X. $\overline{h(X)}$ is usually written $\beta(X)$ and is called the **generalized form of Stone- Čech compactification** of X.

Definition 5.8 [37](p- 8) Let **P** be the category of all paracompact U-spaces and perfect U-maps and **T** be the category of all Tychonoff U-spaces and perfect U-maps. It is to be noted that both of these categories contain **C**, the category of compact U-spaces and U-continuous maps, as a full subcategory. **P** is also a full subcategory of **T**.

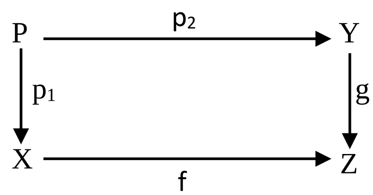
Theorem 5.1 [37](p- 8) The category **P** has pullbacks.

Proof: Let f: $X \to Z$ and g: $Y \to Z$ be two morphisms in the category **P** (that is X, Y, Z be paracompact U-spaces and f, g are perfect U-maps). We have to show the existence of a pullback diagram for f and g.

Let $P = \{(x, y) \in X \times Y : f(x) = g(y)\}$ and p_1 and p_2 be the projection on X and Y respectively. Suppose there exist $p'_1 : P' \to X$ and $p'_2 : P' \to Y$ such that $fp'_1 = gp'_2$.

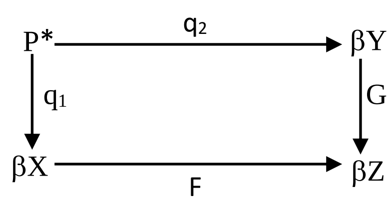
Define $h: P' \to X \times Y$ as follows:

$h(t) = (p'_1(t), p'_2(t))$, $t \in P'$. Since $fp'_1 = gp'_2$, $h(t) \in P$ that is, $h: P' \to P$ such that $p_1 h = p'_1$ and $p_2 h = p'_2$. It is easy to see that the map h is unique. Thus the diagram



is a pullback for f and g. We show that this diagram belongs to **P**, that is, that the maps p_1 and p_2 are U-perfect.

Consider the pullback diagram



for the maps F: $\beta X \to \beta Z$ and G: $\beta Y \to \beta Z$ where F and G are the extensions of the map f and g onto βX and βY respectively (βX, βY and βZ are the generalization of Stone- Čech compactifications and η_X, η_Y and η_Z are reflector maps of X, Y and Z respectively).

We have $F\eta_X = \eta_Z f$, $G\eta_Y = \eta_Z g$ and $P^* = \{(x^*, y^*) \in \beta X \times \beta Y : F(x^*) = G(y^*)\}$. q_1 and q_2 are projections of P* to βX and βY respectively.

Again, let p_1^*: $\beta P \to \beta X$, p_2^*: $\beta P \to \beta Y$ be the extensions of p_1 and p_2 onto βP. Hence $\eta_X p_1 = p_1^* \eta_P$, $\eta_Y p_2 = p_2^* \eta_P$. Since $fp_1 = gp_2$, $\eta_Z fp_1 = \eta_Z gp_2$. Note that $Fp_1^* \eta_P = F\eta_X p_1 = \eta_Z fp_1$ and $Gp_2^* \eta_P = G\eta_Y p_2 = \eta_Z gp_2$.

Therefore, $Fp_1^* \eta_P = Gp_2^* \eta_P$. Since $\eta_P(P)$ is U- dense in βP

we have $Fp_1^* = G p_2^*$ on βP.

From the definition of pullback there exists a (unique) mapping h: $\beta P \to P^*$ such that $p_1^* = q_1h$ and $p_2^* = q_2h$. Again, for the maps $\eta_X p_1$: $P \to \beta X$ and

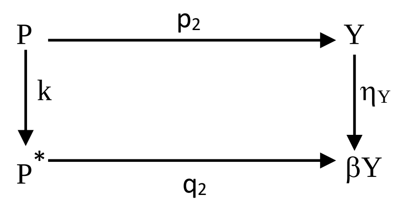
$\eta_Y p_2$:$P \to \beta Y$, we have $F\eta_X p_1 = G\eta_Y p_2$ (this equality is already noted earlier).

From the definition of pullback once again we get a map k: $P \to P^*$such that $\eta_X p_1 = q_1k$ and $\eta_Y p_2 = q_2k$. It is easy to see that the map k is as follows:

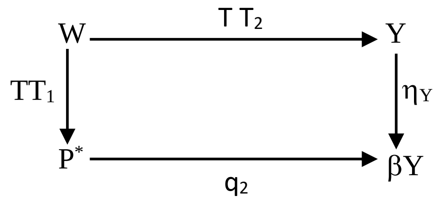
$k(x, y) = (\eta_X p_1(x, y), \eta_Y p_2(x, y)) = (\eta_X (x), \eta_Y (y))$, $(x, y) \in P$. k clearly turns out to be a U-homeomorphism into P^*. Moreover it is not difficult to notice that $k = h\eta_P$. Now k is a U-homeomorphism of P onto $k(P) \subset P^*$. From the property of generalization form of Stone- Čech compactification it follows that

(i) $h(\beta P - \eta_P(P)) \subset \overline{k(P)} - k(P) \subset P^*$.

Now $q_2k = \eta_Y p_2$, that is,



is a commutative diagram. So we consider the pullback diagram for $q_2 : P^* \to \beta Y$ and $\eta_Y : Y \to \beta Y$ say



Where W is given by $\{(s, y) \in P^* \times Y: q_2(s) = \eta_Y(y)\}$ and π_1, π_2 are the respective projections to P* and Y. Since $q_2(s) = q_2(x^*, y^*) = y^*$, $q_2(s) = \eta_Y(y)$ implies $y^* = \eta_Y(y)$.

Consequently, $W = \{((x^*, \eta_Y(y)), y) \in P^* \times Y: \eta_Y(y) = y^*\}$

$$= \{((x^*, \eta_Y(y)), y) \in (\beta X \times \beta Y) \times Y : F(x^*) = G(\eta_Y(y))\}.$$

If $F(x^*) = G(\eta_Y(y))$ then $F(x^*) = G(\eta_Y(y)) = \eta_Z g(y)$. Since f is a U-perfect map, $F(\beta X - \eta_X(x)) \subset \beta Z - \eta_Z(Z)$. As a consequence, $x^* \in \eta_X(x)$, that is, $x^* = \eta_X(x)$ for some $x \in X$. So we have

$W = \{((\eta_X(x), \eta_Y(y)), y) \in (\beta X \times \beta Y) \times Y : F(\eta_X(x)) = G(\eta_Y(y))\}$. Again $\eta_Z g(y) = G(\eta_Y(y)) = F(\eta_X(x)) = \eta_Z f(x)$ and this naturally implies $f(x) = g(y)$.

We then get,

(ii) $W = \{((\eta_X(x), \eta_Y(y)), y) \in (\beta X \times \beta Y) \times Y : f(x) = g(y)\}$

$$= \{k(x, y), y): (x, y) \in P \text{ and } p_2(x, y) = y\}.$$

Since $\eta_Y p_2 = q_2 k$ there exist a unique map $j: P \to W$ as follows:

$$j(x, y) = (k(x, y), p_2(x, y)), (x, y) \in P.$$

Easy to see from (ii) that j(P) = W. In fact j is a U-homeomorphism of P and W. Now W is, by construction, a U-closed subset of $P^* \times Y$ which is paracompact U-space (as P* is compact U-space and Y is paracompact U-space). As a result W is paracompact U-space. This makes P paracompact U-space and J is a U-isomorphism of P and W in the category **P**. We then obtain that the diagram

$$\begin{array}{ccc} P & \xrightarrow{p_2} & Y \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ P^* & \xrightarrow[q_2]{} & \beta Y \end{array}$$

is a pullback diagram. Note that η_Y is a one-one map, that is, η_Y is a U-monomorphism. From the definition of inverse image we see that $P = q_2^{-1}(Y)$ as a sub object of P*.In terms of sets this means that

$k(P) = q_2^{-1}(\eta_Y\ (Y))$. As a result

$q_2(P^*- k(P)) \subset \beta Y - \eta_Y\ (Y)$. We know from (i) that

$h(\beta P - \eta_P\ (P)) \subset \overline{k(P)} - k(P) \subset P^*- k(P)$, so that

$p_2^*(\beta P - \eta_P(P)) = q_2 h\ (\beta P - \eta_P\ (P)) =$

$$q_2[h(\beta P - \eta_P\ (P))] \subset q_2(P^*- k(P)) \subset \beta Y - \eta_Y\ (Y).$$

Hence, by the characterization of Henriksen and Isbell mentioned at the beginning, p_2 is a U-perfect map. Similarly, p_1 is a U-perfect map.

The proof of Theorem - 5.1 also yields the following theorem.

We generalize the theorems, Lemmas and Corollary of [13] (p- 482- 484)

Theorem 5.2 In any category of U-spaces and maps satisfying conditions

(a) all admissible maps are U-continuous,

(b) if A is an admissible space and {p, q} is a two element space, then $A\times\{p, q\}$ and the projection map of this U-space onto A are admissible,

(c) if A is an admissible space and B is a U-closed subspace of A, then B and the inclusion map of B into A are admissible, a projective U-space is extremally disconnected.

Proof: Let X be a projective U-space in such a category. Let G be any U-open subset of X; we must prove $\overline{G}$ is U-open.

In $X\times\{p, q\}$ consider the U-closed set

$Y = ((X - G) \times \{p\})\cup(\overline{G}\times\{q\})$, and its inclusion map i. Let π be the projection of $X\times\{p, q\}$ onto X. Our hypothesis on the category implies that $\pi \circ i$ is an admissible map of Y onto X and that the identity ϕ is an admissible map of X into X. Since X is projective U- space, there is an admissible map ψ of X into Y such that $\phi = \pi \circ i \circ \psi$. Because $\pi\circ i$ is one -to-one on $G\times\{q\}$ it is clear that $\psi(x) = \langle x,q\rangle$ for $x\in G$; from the continuity of ψ follows

$\psi(x) = \langle x,q\rangle$ for $x\in \overline{G}$ .Similarly, for $x\notin \overline{G}$, $\psi(x) = \langle x, p\rangle$.

Thus we have proved $\overline{G} =\psi^{-1}(\overline{G}\times\{q\})$. Since ψ is U-continuous and $\overline{G}\times\{q\}$ is U-open in Y, $\overline{G}$ is U-open in X as required.

Theorem 5.3 In an extremally disconnected U-space no sequence is convergent unless it is ultimately constant.

Proof: Suppose that the sequence $\{x_n\}$ converges to p in the extremally disconnected U-space X. Assume this sequence is not ultimately constant, we shall deduce a contradiction.

First we construct inductively a disjoint sequence $\{U_i\}$ of U-open sets in X such that each U_i contains a member $x_{n(i)}$ of the given sequence, where $\{n(i)\}$ is an increasing sequence of integers. Let n(1) be an index for which $x_{n(1)} \neq p$, and choose a U-open set U_1 such that $x_{n(1)} \in U_1$ but $p \notin \overline{U_1}$. Suppose we have chosen disjoint U-open sets $U_1, U_2, U_3, \ldots\ldots, U_k$ and increasing integers $n_1, n_2, n_3, \ldots\ldots, n_k$ such that $x_{n(i)} \in U_i$ and $p \notin \overline{U_i}$ for i = 1,2,3,……..,k. Then $V = X - (\overline{U_1} \cup \overline{U_2} \cup \overline{U_3} \cup \ldots\ldots\ldots\ldots \cup \overline{U_k})$ is an U-open neighborhood of p, so $x_q \in V$ for all sufficiently large q. By a suitable choice of $n_{(k+1)}$ we shall have $n_{(k+1)} > n_k$, $x_{n(k+1)} \in V$ but $x_{n(k+1)} \neq p$ since the original sequence is not ultimately constant. Choose an U-open set W such that

$x_{n(k+1)} \in W$ but $p \notin \overline{W}$, and let $U_{k+1} = W \cap V$. This completes the inductive construction.

Let $G = \cup U_{2q}$. Since X is extremally disconnected U-space, $\overline{G}$ is an U-open set, and $p \in \overline{G}$ being the limit of $\{x_{n(2q)}\}$. Thus $\overline{G}$ is a neighborhood of p, so $x_r \in \overline{G}$ for all large r; in particular, $x_{n(s)} \in \overline{G}$ for some odd integer s. Since U_s is a neighborhood of $x_{n(s)}$, $U_s \cap G$ is not empty, contrary to the definition of G and disjointness of the U's.

Definition 5.9 A U-space is said to have a **countable basis at x** if there is a countable collection B of neighborhoods of x such that each neighborhood of x contains at least one of the elements of B.

A U-space that has a countable basis at each of its points is said to **satisfy the first countability axiom**, or to be **first-countable**.

Corollary 5.1 In a category in which all Hausdorff U-spaces satisfy the first axiom of countability and properties

(a) all admissible maps are U-continuous,

(b) if A is an admissible space and {p, q} is a two-element space, then $A \times \{p, q\}$ and the projection map of this space onto A are admissible,

(c) if A is an admissible space and B is a U-closed subspace of A, then B and the inclusion map of B into A are admissible hold, then every projective Hausdorff U-space is discrete topological Hausdorff U-spaces.

Lemma 5.1 Let A and E be U-spaces. Suppose f is a U-continuous map of E onto A such that $f(E_o) \neq A$ for any proper closed subset E_o of E.

Then, for any U-open set $G \subset E$, $f(G) \subset \overline{A - f(E-G)}$.

Proof: There is nothing to prove if G is empty. Suppose otherwise, let a be any point of f(G), and let N be any U-open neighborhood of a.

The lemma will follow if we prove that $N \cap (A - f(E\text{-}G))$ is not empty. Because $G \cap f^{-1}(N)$ is a nonempty U-open subset of E, $f(E - (G \cap f^{-1}(N))) \neq A$.

Take $x \in A - f(E - (G \cap f^{-1}(N)))$; clearly $x \in A - f(E - G)$. Since f is onto, $x = f(y)$ where evidently $y \in (G \cap f^{-1}(N))$. Therefore $x = f(y) \in f(f^{-1}(N)) = N$,

so $x \in N \cap (A - f(E - G))$, and the latter set is not empty.

Lemma 5.2 In an extremally disconnected U-space, if U_1 and U_2 are disjoint U-open sets, then $\overline{U_1}$ and $\overline{U_2}$ are also disjoint.

Proof: First, $\overline{U_1}$ and U_2 are disjoint because U_2 is U-open; then $\overline{U_1}$ and $\overline{U_2}$ are disjoint because $\overline{U_1}$ is U-open.

Lemma 5.3 Let A be an extremally disconnected Hausdorff compact U-space, and let E be a compact U-space. Suppose f is a U-continuous map of E onto A such that $f(E_o) \neq A$ for any proper U-closed subset E_o of E.

Then f is a U-homeomorphism.

Proof: We need only show that f is one-to one. Suppose, on the contrary, that x_1 and x_2 are distinct points of E for which $f(x_1) = f(x_2)$. Let G_1 and G_2 be disjoint U-open neighborhoods of x_1 and x_2 respectively. Both the sets $E - G_1$ and $E - G_2$ are compact, so $A - f(E - G_1)$ and $A - f(E - G_2)$ are U-open.

The latter sets are disjoint because $E = (E - G_1) \cup (E - G_2)$. By the Lemma- 5.2, $\overline{A - f(E - G_1)}$ and $\overline{A - f(E - G_2)}$ are disjoint. On the other hand, it follows from Lemma- 5.1 that $f(x_1) = f(x_2)$ is a point common to these sets. This contradiction establishes Lemma- 5.3.

Lemma 5. 4 [13](p- 484) Let A and D be compact Hausdorff U-spaces, and let f map D continuously onto A. Then D contains a compact U-subspace E such that f(E) = A but $f(E_o) \neq A$ for any proper U-closed subset E_o of E.

Proof: This is a well known consequence of Zorn's Lemma.

Theorem- 5.4 In the category of compact U-spaces and U-continuous maps, the projective U-spaces are precisely the extremally disconnected U-spaces.

Proof: To prove that all projective U-spaces in the category are extremally disconnected U-space, we have only to verify the conditions of Theorem-5.2. We turn to the opposite inclusion.

Let A be an extremally disconnected compact U-space, let B and C be compact U-spaces, let f be a U-continuous map of B onto C, and let ϕ be a U-continuous map of A into C. We must prove that there exists a U-continuous map ψ of A into B such that $\phi = f\psi$.

In the space $A \times B$ consider D = {(a, b) $|\phi(a) = f(b)$}. This set is clearly closed and therefore compact U-space. Since f is onto, the projection π_1 of $A \times B$ onto A carries D onto A. By Lemma- 5.4 there is a U-closed subset E of D such that $\pi_1(E) = A$ but $\pi_1(E_o) \neq A$ for any proper U- closed subset E_o of E. Let ρ be the restriction of π_1 to E. Lemma-5.3 asserts that ρ is a U-homomorphism. Let $\psi = \pi_2 \rho^{-1}$, where π_2 is the projection of $A \times B$ into B; this is the required map. Suppose $a \in A$; since $\rho^{-1}(a) \in D$, $f(\pi_2(\rho^{-1}(a))) = \phi(\pi_1(\rho^{-1}(a))) = \phi(a)$.

Thus $\phi = f\pi_2 \rho^{-1} = f\psi$; this completes the proof.

Definition 5.10 A map is said to be **U-proper** if and only if it is U-continuous and the inverse image of every compact U- space is compact.

Example- 5.1 (Proper extremally disconected compact U- space).

Let X = {a, b, c, d}, $\mathcal{U}$ = {X, Φ, {a, b}, {a, c}, {a, b, c}, {d}, {a, b, d}, {a, c, d}}. Since {a, b} ∩ {a, c}={a} ∉ U. $\mathcal{U}$ is a U-structure.

Then (X, $\mathcal{U}$) is a proper U-space.

Here $\overline{\{a,b\}} = \overline{\{a,c\}} = \overline{\{a,b,c\}}$ = {a, b, c}, $\overline{\{d\}}$ = {d}, $\overline{\{a,b,d\}}$ = X, $\overline{\{a,c,d\}}$ = X .

Hence X is a proper extremally disconnected and compact U-space.

Example - 5.2 (a proper projective compact U-space)

Let X = {a, b, c, d} and $\mathcal{U}$ = {X,Φ, {a, b},{a, c}, {a, d},{b, c}, {b, d}, {c, d},{b, c, d}, {a, c, d}, {a, b, d}, {a, b, c}}. Then X is a proper U-space which is clearly, Hausdorff, compact and extremally disconnected U-space. Thus X is a proper projective compact U-space.

Example - 5.3 Let X = $\mathbb{N}$ be U-space, n_o is a fixed element of $\mathbb{N}$ and

let $\mathcal{U}$ ={{ $\mathbb{N}$,Φ}∪{{n∈ $\mathbb{N}$ | n ≤ n_o}, {n∈ $\mathbb{N}$ | n > n_o}, {n∈ $\mathbb{N}$ | n < n_o + 3}, {n∈ $\mathbb{N}$ | n ≥ n_o + 3}, | n_o ∈ $\mathbb{N}$ }}, and their unions.

Now { n∈ $\mathbb{N}$ | n < n_o + 3}∩{n∈ $\mathbb{N}$ | n > n_o } = { n_o + 1, n_o + 2}∉ $\mathcal{U}$.

Thus $\mathcal{U}$ is a U-structure but not a topology, and so, **(X,$\mathcal{U}$) is a proper U-space.**

(i) **X is clearly compact.**

(ii) **X is Hausdorff.** For, if $n_1, n_2 \in \mathbb{N}$ and $n_1 \neq n_2$, say $n_1 < n_2$, then $n_1 \in U_1 = \{1,2,3,.........,n_1\}$, $n_2 \in U_2 = \{n \in \mathbb{N} \mid n > n_1\}$ and $U_1 \cap U_2 = \Phi$.

(iii) **X is extremally disconnected** U- space**,** since, for each U-open set G of X, $\overline{G}$ = G is U-open.

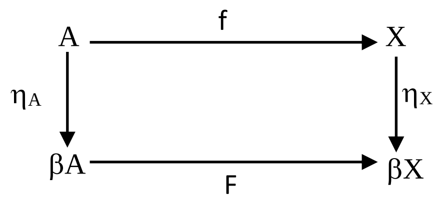
Hence by Theorem 5.5, X is a proper projective compact U-space.

Definition 5.11 If $A \subset X$, a **U-retraction** of X onto A is a U-continuous map r: X $\to$A such that $r|$ A is the identity map of A. If such a map r exists, we say that A is a U-**retract** of X.

We now generalize the theorems of ([37], p- 11-12)

Theorem 5.5 Let X be any extremally disconnected object from the category **P**. Any perfect U-mapping f: $A \to X$ of another object A onto X is a U-retraction.

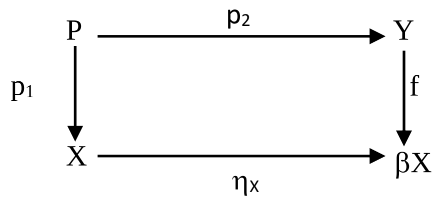
Proof: We have f: $A \to X$ onto. Then we can draw the following diagram



Where F is the unique U-continuous extension of f onto βA taking values in βX. Since f is a surjection, F is also onto. But βX is extremally disconnected U-space and F is an onto map. Since βX is projective U-space in the category **C**. F is a U-retraction, that is there exists a mapping g: $\beta X \rightarrow \beta A$ such that $Fg = 1_{\beta X}$ = the identity map on βX. Since f is a perfect U-map, $F(\beta A - \eta_A(A))$ $= \beta X - \eta_X(X)$. Therefore, $g(\eta_X(X)) \subset \eta_A(A)$. Put $h = \eta_A^{-1} g \eta_X : X \rightarrow A$. Now $fh(x) = f\,\eta_A^{-1} g \eta_X(x)$. But $F(g\eta_X(x)) = \eta_X(x)$ and $g(\eta_X(x)) \in \eta_A(A)$, that is, $g(\eta_X(x)) = \eta_A(a)$ for some $a \in A$. Therefore, $\eta_X(x) = F(\eta_A(a)) = \eta_X f(a)$. So, $a = \eta_A^{-1}(\eta_A(a)) = \eta_A^{-1} g \eta_X(x)$ and x = f(a) and hence, $f(a) = f\eta_A^{-1} g \eta_X(x) = x$. Consequently fh(x) = x for each $x \in X$, that is, $fh = 1_X$.Naturally f is a U-retraction.

Theorem 5.6 The category **P** has projectives that is any paracompact U-space is the perfect U-image of a projective U-space object. In fact, for every object X there is a projective U-space objects P and an onto U-perfect mapping $p_1: P \rightarrow X$ such that p_1 maps no proper U-closed subspace of P onto X. For any other such object P′ and $p'_1: P' \rightarrow X$ there is an U-isomorphism e: $P \rightarrow P'$ such that $p_1 = p_1'e$.

Proof: Let X be any object of **P**. Look at βX, the Stone - Čech compactification of X. There exists an extremally disconnected compact U-space Y and a U-continuous onto map f: $Y \rightarrow \beta X$ such that $f(S) \neq \beta X$ for any proper U-closed subspace S of Y. Consider the pull- back diagram



for the morphisms $\eta_X : X \to \beta X$ and f: $Y \to \beta X$,

where P = {(x, y) $\in X \times Y$: $\eta_X(x) = f(y)$} and p_1 and p_2 are projections to X and Y respectively. We do not claim that this is a pullback in **P**. Clearly, $\eta_X p_1 = fp_2$. Since η_X is a U-monomorphism, p_2 is U-monomorphism. Since f is onto, p_1 is onto. Again, P is a U-closed subset of $X \times Y$ and the latter is paracompact U-space P is, hence, paracompact U-space. p_1 is also U-closed so that p_1 becomes a perfect U-map. $fp_2 = \eta_X p_1 \Rightarrow fp_2(P) = \eta_X(X)$. Let W = p_2(P). Since f is a U-closed map, $f\overline{(p_2(P))} = f(\overline{W}) = \beta X$. Observe that $\overline{W}$ is a U-closed subset of Y and f($\overline{W}$) = βX. From the choice of Y it follows that $\overline{W}$ = Y, that is,W = p_2(P) is dense in Y. Y is extremally disconnected U-space rendering W extremally disconnected U-space. Now it is not very difficult to see that p_2 is a U-perfect map onto W. Since P is paracompact U-space and p_2 is a U-perfect map onto W, W is a paracompact U- space.

By Theorem-5.5, p_2 is a U-retraction. Since p_2 is a U-monomorphism and a U-retraction also, it is an U-isomorphism, that is p_2 is a U-homeomorphism of P and W. Thus P is an extremally disconnected paracompact U-space. So P is projective U-space due to “In the category **P**, the projective objects are precisely the extremally disconnected paracompact U-spaces”. Since p_1 is a U-perfect map of P onto X, X is a U-perfect image of a projection object. Let Q be a proper U-closed subset of P. Then p_2(Q) is a proper U-closed subset of p_2(P)= W. Write p_2(Q) = W(F) where F is a

U-closed subset of Y. Since $p_2(Q)$ is a proper U-closed subset of W, F is a proper U-closed subset of Y.

If $p_1(Q)= X$, then $\eta_X(X) = \eta_X p_1(Q)= fp_2(Q)=f(W(F)) \subset f(F)$. Since f is a U-closed map of X onto βX, f(F) is a U-closed and hence equals βX. This is a contradiction. Consequently P enjoys the property that no proper U-closed subspace of P is mapped onto X by p_1.

If possible let P' be a projective paracompact U-space with a U-perfect map p_1': $P' \to X$ such that $p_1'(P') = X$ and if Q is any proper U-closed subspace of P' then $p_1'(Q) \neq X$. Then there exist a morphism $e : P \to P'$ and a morphism e': $P' \to P$ such that $p_1 = p_1'e$ and $p_1' = p_1e'$. Then $p_1(P) = X = p_1'(P') \Rightarrow p_1'e(P) = X = p_1e'(P')$. Naturally, e and e' are onto; we shall show that $ee' = 1_p$, that is, e is a U-co-retraction. If $ee' \neq 1_p$, there exists a proper U-closed subset S of P such that $d^{-1}(S) \cup S = P$ where $d = e'e$.

Obviously, $d(d^{-1}(S)) \subset S$ whence $p_1d(d^{-1}(S)) \subset p_1(S)$. But $p_1d = p_1e'e = p_1'e = p_1$, hence $p_1(S) \supset p_1d(d^{-1}(S)) = p_1(d^{-1}(S))$; so that $p_1(S) = p_1(P) = X$, a contradiction as S is a proper U-closed subset of P. We thus conclude that e is a U-co-retraction. Already e is a U-retraction; hence e is a U-isomorphism, that is, e is a U-homeomorphism of P onto P'.

Theorem 5.7 [14](p- 7) Let P be a compact Hausdorff U-space. Then P is projective if and only if for every compact Hausdorff U-space W and U- continuous g: $W \to P$, onto, there exists a U-continuous s: $P \to W$ such that $g_o s(p) = p$.

Proof: Assume that P is projective U-space and let s be a lifting of the identity map on P.

Conversely, assume that P is projective U- space and let X and Y be U-spaces and h: $Y \to X$ and f: $P \to X$, U-continuous map with h onto. Then there exists a U-continuous map r: $P \to Y$ such that $h_\circ r(p) = f(p)$ for every $p \in P$.

Let $W = \{(p, y) \in P \times Y: f(p) = h(y)\}$ and define $g : W \to P$ by $g(p, y) = p$ and q: $W \to Y$ by $q(p, y) = y$. If s: $P \to W$ is as above then $r = q_\circ s$ is a lefting of f.

Theorem 5.8 [31](p- 70) If P is a U-retract of P′ and P′ is projective, then P is projective.

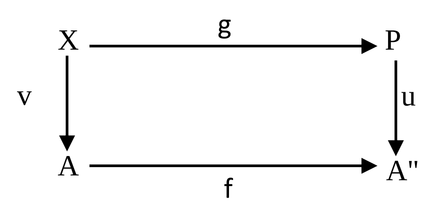
Proof: Let $P \to P' \to P = 1_P$. If $A \to A''$ is an U-epimorphism and $P \to A''$ is any morphism, then using projectivity of P′ we have $P \to A'' = P \to P' \to P \to A'' = P \to P' \to A \to A''$ for some morphism $P' \to A$. This establishes U-projectivity of P.

Theorem 5.9 [31] (p-70) If P is projective U-space in A, then every U-epimorphism $A \to P$ is a U-retraction. Conversely if P has the property that every U-epimorphism $A \to P$ is a U-retraction, and if A ether has projective or is abelian, then P is projective U- space.

Proof: If P is projective U-space, then given a U-epimorphism $A \to P$ there is a morphism $P \to A$ such that $P \to A \to P$ is 1_P. In other words $P \to A$ is a U-retraction.

Conversely, suppose that every U-epimorphism A $\to$P is a U-retraction.

If A has projective then we may take A projective and then it follows from Theorem 5.8. On the other hand, if A is abelian, then, given an U-epimorphosm f: A $\to$A" and a morphism u:P$\to$A", we can form the pullback diagram

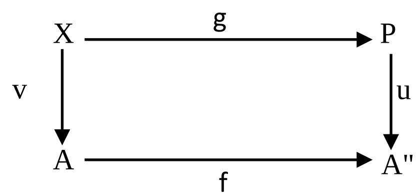


we know that g is an U-epimorphism. Then by assumption we can find h: P $\to$X such that gh = 1_P. Then we have fvh = ugh = u. This proves that P is projective U-space.

Theorem 5.10 [37](p- 12) In the category **P**, the projective U-space objects are precisely the extremally disconnected paracompact U-spaces.

Proof: If P is projective U-space, then given a U-epimorphism A $\to$P there is a morphism P$\to$A such that P $\to$ A $\to$P is 1_P. In other words P $\to$A is a U-retraction.

Conversely, suppose that every U-epimorphism A $\to$P is a U-retraction. If A has projective then we may take A projective U-space and then it follows from Theorem 5.8. On the other hand, if A is abelian, then, given an U-epimorphosm f: A $\to$A" and a morphism u:P$\to$A", we can form the pullback diagram



we know that g is an U-epimorphism. Then by assumption we can find h: P $\rightarrow$X such that gh = 1_P. Then we have fvh = ugh = u. This proves that P is projective U-space. Therefore the projective U-space objects of P are the objects for which perfect U-maps onto them are U-retraction.

Hence the theorem follows from theorems 5.5, 5.8 and 5.9.

Let X be any extremally disconnected U-space object from the category P. By theorem- 5.5 we can prove that any U-perfect mapping f: A$\rightarrow$X of another object A onto X is a U-retraction.

By theorem- 5.8 ‘If P is a U-retract of P′ and P′ is projective U-space, then P is projective U-space’ And theorem- 5.9 “If P is projective U-space in A, then every U-epimorphism A$\rightarrow$P is a U-retraction. Conversely if P has the property that every U-epimorphism A$\rightarrow$P is a U-retraction, and if A has projective U- space, then P is projective U- space.” P is projective U-space.

Hence the theorem is proved.

Examples of proper projective U-spaces which are locally compact but not compact.

Example- 5.4 Let X = $\mathbb{R}$, $\mathcal{U}$ ={X, Φ, $(-\infty,\frac{1}{2})$,[0,1), $[\frac{1}{2},1)$, [1,2),..,[n, n + 1), …., and their unions}.

(i) Then (X, $\mathcal{U}$) is a U-space but not a topological space.

Since $(-\infty,\frac{1}{2}) \cap [0, 1) = [0, \frac{1}{2}) \notin \mathcal{U}$.

(ii) **X is not compact**, since $\mathcal{C}$ = {$(-\infty,\frac{1}{2})$,[0, 1), [1, 2),…..,[n, n + 1),….,} is U-open cover of X but it has no finite sub cover.

(iii) **X is locally compact**. For let $x_o \in X$. If $x_o < \frac{1}{2}$, then $(-\infty,\frac{1}{2})$ is a neighborhood of x_o whose closure is $(-\infty,1)$, which is compact U-space, since every U-open cover of $(-\infty,\frac{1}{2})$ must contain either X or both $(-\infty,\frac{1}{2})$ and $[\frac{1}{2},1)$ and each such cover is clearly finite.

If $x \geq \frac{1}{2}$, $x \in [n, n + 1)$ for some $n \in \{0\} \cup \mathbb{N}$. Then $\overline{[n,n+1)} = [n, n + 1)$ which is obviously compact, since [n, n + 1) is U-closed.

(iv) All the U-open sets except $(-\infty,\frac{1}{2})$ and [0, 1) are both U-open and U-closed & so the U-closure of any union of these is U-open. Also, $\overline{\left(-\infty,\frac{1}{2}\right)} = (-\infty, 1)$, $\overline{[0,1)} = (-\infty, 1)$.

Hence the closure of every U-open set is U-open.

Thus X is extremally disconnected U-space, and so, X is projective U-space.

Example 5.5 Let $X = \mathbb{Z}$, $\mathcal{U}$ ={X, Φ,{n∈ $\mathbb{Z}$| - ∞< n≤1}, {0,1,2}, {3,4,5},{6,7,8} and their unions}. X is a proper U-space.

For {n∈ Z | - ∞< n≤1}∩{0,1,2} = {0, 1}∉ $\mathcal{U}$.

(i) **X is not compact.** For the U-open cover

{{n∈ Z | - ∞< n≤1},{0,1,2}, {3,4,5},{6,7,8},…………….} has no finite sub cover .

(ii) However, **X is locally compact**. To sec this, let $x_o \in X$. If $x_o \le 1$, the {n∈ $\mathbb{Z}$| - ∞< n≤1} is a U-open neighborhood of x_0 and its closure is {n∈ $\mathbb{Z}$ | - ∞ < n≤2} which is clearly compact. If x_0 >1, then for x_0 = 2, {0, 1, 2} is a U-open neighborhood of x_0 and its closure is

{n∈ $\mathbb{Z}$ | - ∞< n ≤ 2} which again is U-compact, and for x_0 = n >2,

x∈{3r, 3r + 1, 3r + 2} for some positive r, and this set is a U-open neighborhood of x_0. Also, it is its own closure.

Clearly it is compact.

Thus X is locally compact U-space.

(iii) The sets {3r, 3r + 1, 3r + 2} are both U-open and U-closed for each r ≥1, $\overline{\{n \in Z \mid -\infty < n \le 1\}}$ = {n∈ $\mathbb{Z}$ | - ∞< n ≤ 2} =

{n∈ $\mathbb{Z}$ | - ∞ < n ≤ 1}∪{0, 1, 2}

(iv) which is U-open. Also, $\overline{\{0,1,2\}} = \{n \in \mathbb{Z} \mid -\infty < n \leq 2\}$ is U-open, as before.

Hence X is extremally disconnected U-space.

Therefore X is projective U-space.

Cover of compact Hausdorff U-space

We now generalize definitions of [14] (p- 7 - 8)

Definition 5.12 Let X be a compact Hausdorff U-space. A pair (C, f) is called a **U-cover of X**, provided that C is a compact Housdorff U-space and f: C $\rightarrow$ X is a U-continuous map that is onto X.

Definition 5.13 Let X and C be compact Housdorff U-space and f: C $\rightarrow$X a U-continuous map that is onto X. A pair (C, f) is called a **U-essential cover of X** if it is a U-cover and whenever Y is a compact, Hausdorff U-space h: Y $\rightarrow$C is U-continuous and f(h(y)) = X, then necessarily h(Y) = C.

Definition 5.14 Let X and C be compact Housdorff U-space and f: C $\rightarrow$X a U-continuous map that is onto X. A pair (C, f) is called a **U-rigid cover of X** if it is a U-cover and the only U-continuous map h: C $\rightarrow$C satisfying f(h(c)) = f(c) for every $c \in C$ is the identity map.

Theorem 5.11 Let X be a compact Hausdorff U-space and let (C, f) be a U- essential cover of X. Then (C, f) is a U- rigid cover of X.

Proof: Let h: C $\rightarrow$C satisfy f(h(c)) = f(c) for every $c \in C$. Let C_1 = h(C) which is a compact U-subset of C that still maps onto X. The inclusion map of i: $C_1 \rightarrow C$ satisfies, $f(i(C_1))$ = X and hence must be onto C. Thus h(C) = C.

Next, we claim that if $G \subseteq C$ is any non- empty U-open set, then $G \cap h^{-1}(G)$ is non- empty. For assume to the contrary, and let F = C \ G. Then F is compact U-space and given any $c \in G$ there exist $y \in h^{-1}(G)$ with h(y) = c. Hence, $y \in F$ and f(c) = f(h(y)) = f(y). Thus f(F) = X, again contradicting the essentiality of C. Thus, for every U-open set G, we have that $G \cap h^{-1}(G)$ is non-empty.

Now fix any $c \in C$ and for every neighborhood G of c pick $x_G \in G \cap h^{-1}(G)$. We have that the net $\{x_G\}$ converges to c. Hence, by continuity, $\{h(x_G)\}$ converges to h(c). But since $h(x_G) \in G$ for every G, we also have that $\{h(x_G)\}$ converges to c. Thus, h(c) = c and since c was arbitrary, C is U-rigid cover of X.

Theorem 5.12 Let (C, f) be a U-cover of X with C a projective U-space.

Then (C, f) is a U-essential cover if and only if (C, f) is a U-rigid cover.

Proof: We already have that a U-essential cover is always a U-rigid cover. So assume that (C, f) is a U-rigid cover. Let h: Y $\rightarrow$ C with f(h(Y))= X. Since C is projective, then there exists a map s: C $\rightarrow$ Y with (f$\circ$h)$\circ$s = f. We have hos : C $\rightarrow$C and f(h$\circ$s(c))= f(c) and so by rigidity, h$\circ$s(c)= c for every c$\in$C. In particular, h must be onto and so C is U-essential cover.

Theorem 5.13 Let (Y, f) be a U-cover of X and let C $\subset$ Y be a minimal, compact U-subset of Y that maps onto X. Then (C, f) is a U-rigid, essential cover of X.

Proof: First, we prove U-essential. Given any compact Hausdorff U-space Z and h: Z $\rightarrow$C such that f(h(Z)) = X, we have that h(Z) $\subseteq$ C is compact U-space and hence h(Z) = C by minimality.

Since (C, f) is a U-essential cover of X, by the above results it is also a U-rigid cover.

CHAPTER- 6

Anti-Hausdorff U-spaces

Introduction

In this chapter the concept of anti-Hausdorff U-spaces has been introduced and a few important properties of such spaces have been studied. A number of interesting examples have been constructed to prove non- trivialness of such results.

We have generalized some results on anti-Hausdorff topological spaces in [25] to U-spaces.

Definition 6.1 A U-space X with $|X| \geq 2$ is said to be **anti-Hausdorff U-space**, if for every pair of distinct points x, y in X and pair of distinct U-open sets G and H such that $x \in G$, $x \in H$, $G \cap H \neq \Phi$, i.e., if no two distinct points can be separated by disjoint U-open sets.

Here, $|X|$ denoted the number of elements of X. A anti-Hausdorff U-space which is not a topological space will be called a non-trivial anti-Hausdorff U-space. Otherwise it is called trivial. It is easily seen that an anti-Hausdorff U-space X is non-trivial only $|X| \geq 3$.

Example 6.1 Let X = {a, b, c}, $\mathcal{U}_1$ = {X, Φ, {a, b},{a, c}} and $\mathcal{U}_2$ = {X, Φ,{b, c},{a, c}}.

Then (X, $\mathcal{U}_1$) and (X, $\mathcal{U}_2$) are non-trivial anti-Hausdorff U-spaces.

Example 6.2 Let X = {a, b, c, d} and $\mathcal{U}_1$ = {X, Φ, {a, b, c},{a, d}}, $\mathcal{U}_2$ = {X, Φ,{a, b},{a, c},{a, b, c}}. Then (X, $\mathcal{U}_1$) and (X, $\mathcal{U}_2$) are non-trivial anti-Hausdorff U-spaces.

Example 6.3 Let X = $\mathbb{N}$, $\mathcal{U}$ = {X, Φ, {1, 2, 3},{1, 4, 5}, {1, 2, 3, 4, 5}}.

Then (X, $\mathcal{U}$) is a non-trivial anti-Hausdorff U-space.

Example 6.4 Let X = $\mathbb{R}$, $\mathcal{U}$ = {X, Φ, $\mathbb{N}$, $\mathbb{Z}$, 2 $\mathbb{Z}$, $\mathbb{N} \cup 2\mathbb{Z}$ }.

Then (X, $\mathcal{U}$) is a non-trivial anti-Hausdorff U-space.

Theorem 6.1 A U-subspace of a non-trivial anti-Hausdorff U-space need not be anti-Hausdorff.

Proof: Let us consider the U-space (X, $\mathcal{U}$), where X = {a, b, c, d} and $\mathcal{U}$ = {X, Φ,{a, b},{a, c}, {a, b, c}}.

Then (X, $\mathcal{U}$) is a non-trivial anti-Hausdorff U-space, since there is no pair of disjoint non-empty U-open sets in X. Now let Y = {b, c}.

Then as a subspace of X, Y has the U-structure, $\mathcal{U}$ = {Y, Φ, {b},{c},{b, c}}.

Obviously, Y is not anti-Hausdorff U-space.

Theorem 6.2 If A and B are two non-trivial anti-Hausdorff U-subspaces of a U-space X, then the subspace $A \cap B$ need not be a non-trivial anti-Hausdorff U-space.

Proof: Let X= {a, b, c, d, e, f }, $\mathcal{U}$ = {X, Φ, {a, b, c},{ b, c, d},{a, b, c, d}, {b, c, d, e, f} }. Clearly (X, $\mathcal{U}$) is a non-trivial U-space. Let A = {a, c, d, f } and B = {a, b, d, f}. Then A and B are U-subspace of X with

$\mathcal{U}_A$ = {A, Φ, {a, c},{c, d}, {a, c, d}, {c, d, f}}, $\mathcal{U}_B$ = {B, Φ,{a, b},{b, d}, {a, b, d},{b, d, f}}.

Clearly both A and B are non-trivial anti-Hausdorff U-subspaces of X. **Now $A \cap B$ = {a, d, f }** and $\mathcal{U}_{A \cap B}$ ={$A \cap B$, Φ, {a}, {d},{a, d},{d, f}}. Then $A \cap B$ is a trivial U-space, which is not anti-Hausdorff.

Thus $A \cap B$ is not a non- trivial anti-Hausdorff U-space.

In the situation of Theorem-6.2, it is also possible that $A \cap B$ is a non-trivial anti-Hausdorff U-space as is shown by the following example.

Example 6.5 Let X= {a, b, c, d, e}, $\mathcal{U}$ = {X, Φ, {a, b},{a, b, c},{a, c, d, e}}. Clearly (X, $\mathcal{U}$) is a non-trivial U-space. Let A = {a, b, c, d} and

B = {a, b, c, e}. Then A and B are U-subspace of X with $\mathcal{U}_A$ = {A, Φ, {a, b}, {a, b, c},{a, c, d}}, $\mathcal{U}_B$ = {B, Φ, {a, b},{a ,b, c},{a, c, e}}.

Clearly both A and B are non-trivial anti-Hausdorff U-subspaces.

Now $A \cap B$ = {a, b, c} and $\mathcal{U}_{A \cap B}$ = {$A \cap B$, Φ,{a, b}, {a, c}} which is a non-trivial anti-Hausdorff U-space.

Remark 6.1 If A_1 and A_2 are two non-trivial subspaces of a non-trivial U-space X, then the subspace $A_1 \cap A_2$ may be non-trivial anti-Hausdorff U-space even if neither A_1 nor A_2 is so.

Proof: Let X = {a, b, c, d, f }, $\mathcal{U}$ = { X,Φ,{a}, {b, c},{c, d}, {a, b, c}, {a, c, d}, {f}, {b, c, f}, {c, d, f }, {a, f }, {a, b, c, f }, {a, c, d, f }, {b, c, d}, {a, b, c, d},{b, c, d, f}}.

Clearly U is U-structure on X.

Let A_1 = {a, b, c, d} and A_2 = {b, c, d, f }.

Then the U-structure $\mathcal{U}_{A_1}$ and $\mathcal{U}_{A_2}$ on A_1 and A_2 respectively are

$\mathcal{U}_{A_1}$= {A_1, Φ, {a}, {b, c}, {c, d}, {a, b, c}, {a, c, d}, {b, c, d}} and

$\mathcal{U}_{A_2}$ = {A_2, Φ, {f}, {b, c}, {c, d}, {b, c, f}, {b, c, d}, {c, d, f }}.

Clearly both A_1 and A_2 are non-trivial subspaces of a U-space X, neither of which is anti-Hausdorff.

Now $A_1 \cap A_2$ = {b, c, d} and $\mathcal{U}_{A_1 \cap A_2}$ ={$A_1 \cap A_2$, Φ, {b, c}, {c, d}}.

Thus $A_1 \cap A_2$ is a non- trivial anti- Hausdorff U-space.

Theorem 6.3 Let A_1 and A_2 be two anti-Hausdorff U-spaces with U-structures $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively. Then ($A_1 \cup A_2$, $\langle \mathcal{U}_1 \cup \mathcal{U}_2 \rangle$) need not be anti-Hausdorff U-space.Here $\langle \mathcal{U}_1 \cup \mathcal{U}_2 \rangle$ is the U-structure generated by $\mathcal{U}_1 \cup \mathcal{U}_2$ in $A_1 \cup A_2$.

Proof: Let A_1 = {a, c, d, e} $\mathcal{U}_1$ = {A_1, Φ, {a},{a, c}, {a, c, d},{a, d, e}}, and A_2 = {b, c, d, e} $\mathcal{U}_2$ = {A_2, Φ, {b},{b, c},{b, c, d},{b, d, e}}. Then (A_1, $\mathcal{U}_1$) and (A_2, $\mathcal{U}_2$) are non-trivial anti-Hausdorff U-spaces.

Then A = $A_1 \cup A_2$ = {a, b, c, d, e}. Let $\mathcal{U}$ be the U-structure on A generated by $\mathcal{U}_1 \cup \mathcal{U}_2$, i.e., $\mathcal{U}$ ={A, A_1, A_2, Φ,{a},{b},{a, c},{b, c}, {a, c, d}, {a, d, e},{b, c, d}, {b, d, e},{a, b},{a, b, c},{a, b, c, d},{a, b, d, e}}. So, in (X, $\mathcal{U}$), a $\in$ {a}, b $\in$ {b} with {a},{b} $\in \mathcal{U}$ and {a} $\cap$ {b} = Φ.

Hence (X, $\mathcal{U}$) is not an anti- Hausdorff U-space.

Theorem 6.4 Every U-continuous image of an anti- Hausdorff U-space is an anti-Hausdorff U-space.

Proof: Let X, Y be two U-spaces where X is anti-Hausdorff U-space. Let f be a U-continuous map of X onto Y. Let y_1 and y_2 be two distinct points of Y, and let H_1 and H_2 be two U-open sets in Y such that $y_1 \in H_1$, $y_2 \in H_2$. Since f is onto there exist x_1, x_2 in X such that $f(x_1) = y_1$, $f(x_2) = y_2$. Let $G_1 = f^{-1}(H_1)$, $G_2 = f^{-1}(H_2)$. Since f is U-continuous, both G_1 and G_2 are U-open sets. Since X is anti-Hausdorff U-space, $G_1 \cap G_2 \neq \Phi$. Let $x \in G_1 \cap G_2$, then $f(x) \in H_1 \cap H_2$. Thus $H_1 \cap H_2 \neq \Phi$. So, Y is anti-Hausdorff U-space.

Definition 6.2 Let (X, $\mathcal{U}$) be U-space and R an equivalence relation on X. The equivalence class for each $x \in X$ is denoted by $\bar{x}$. We define U-structure $\overline{\mathcal{U}}$ on the collection of equivalence classes $\frac{X}{R}$ of X with respect to R as follows. Any subset $\bar{V}$ of $\frac{X}{R}$ will be a member of $\overline{\mathcal{U}}$ iff $\{x \in X | \bar{x} \in \bar{V}\} \in \mathcal{U}$, i.e., the collection of equivalence classes of every U-open set V of X is U-open in $\frac{X}{\mathrm{R}}$ and these are the only member of $\frac{X}{\mathrm{R}}$.

This U-structure $\overline{\mathcal{U}}$ is called the identification U-structure or the quotient U-structure on X and $(\frac{X}{\mathrm{R}}, \overline{\mathcal{U}})$ is called the identification U-space or **the quotient U-space of X with respect to R.**

Example 6.6. The torus (surface of doughnut) can be constructed by taking a rectangle and identifying its edges together appropriately.

Corollary 6.2. If X is an anti-Hausdorff U-space and R is an equivalence relation on X, then the quotient U-space $\frac{X}{R}$ is anti-Hausdorff U-space.

Proof: It follows from the definition of quotient U-space that the map f : X$\rightarrow \frac{X}{R}$ given by f(x) = cls x is continuous and onto. The proof is then obvious.

Definition 6.3 A U-space X is said to be **U-irreducible** if every pair of non-empty U-open sets in X intersect.

Thus a U-space X is U-irreducible if, for every pair of non-empty U-open sets V, W in X, $V \cap W \neq \Phi$.

Theorem 6.5 Let X be a U-space. For the statements:

(i) X is anti-Hausdorff. U-space,

(ii) X is U-irreducible,

(iii) Every non- empty U-open set in X is connected U-space,

(iv) Every non- empty U-open set in X is dense in X,

following implications hold**:** (i) $\Leftrightarrow$(ii), (iii) $\Rightarrow$ (ii) and (ii) $\Leftrightarrow$ (iv).

Proof: We first prove (i) $\Leftrightarrow$(ii).

To prove (i) $\Rightarrow$ (ii) let X be a anti- Hausdorff U-space. If possible suppose that X is not U-irreducible. Then there exist non- empty U-open sets V and W in X such that $V \cap W = \Phi$. Since V and W are non- empty, there exist $x \in V$ and $y \in W$. Since $V \cap W = \Phi$, $x \neq y$. X being anti-Hausdorff U-space, this is a contradiction. Therefore X is U-irreducible.

We now prove (ii) $\Rightarrow$ (i). Let X be U-irreducible. If possible, let X be not anti-Hausdorff U-space. Then there exist x, y $\in$X with $x \neq y$ and U-open sets V and W in X with $V \cap W = \Phi$ and $x \in V$, $y \in W$. Since V and W are non-empty, this is a contradiction to the fact that X is U-irreducible.

Hence X is anti-Hausdorff U-space.

To prove (iii) $\Rightarrow$(ii), let every U-open set in X be connected U-space. If X is not U-irreducible, then there exist non-empty U-open sets V_1 and V_2 in X, such that $V_1 \cap V_2 = \Phi$. This implies that the U-open set $V = V_1 \cup V_2$ is a disconnected U-open set in X. This is a contradiction to our hypothesis. Hence X is U-irreducible.

We now prove (ii) $\Leftrightarrow$ (iv). Let X be a U-irreducible space. Let V be a non- empty U-open set in X and let $x \in X$. Let W be a U-open set in X such that $x \in W$. Then $W \neq \Phi$. Since X is U-irreducible, $V \cap W \neq \Phi$. So, $x \in \overline{V}$. Thus X = $\overline{V}$. Thus (ii) $\Rightarrow$ (iv).

Conversely, suppose every non-empty U-open set in X is dense in X. Let V and W be two non-empty U-open sets in X and let $x \in V$. Since $\overline{W}$ = X and V is a neighborhood of x, $V \cap W \neq \Phi$. So X is U-irreducible.

Therefore (iv) $\Rightarrow$ (ii). The proof of the theorem is thus complete.

Chapter- 7

I-spaces, CU-spaces, CUI-spaces, FU-spaces and FUI-spaces

Introduction

In this chapter we have introduced the concepts of **I-spaces, CU-spaces, CUI-spaces, FU-spaces and FUI-spaces** as generalization of topological spaces. I-spaces have been called infratopological spaces by some authors [16], [29], [30]. The concepts of limit point of a set, Interior point of a set, closure of a set, three types of continuity, compactness, connectedness, disconnectedness and Heine-Borel Theorem and separation axioms in the topological spaces have been generalized to the case of I-spaces.

These concepts can be defined similarly for CU-spaces, CUI-spaces, FU-spaces and FUI-spaces.

We have constructed many examples and proved a number of theorems involving these concepts in case of I-spaces. For the other types of spaces some of these have been dealt with briefly.

I-spaces

Definition 7.1 Let X be a non- empty set. A collection $\mathscr{I}$ of subsets of X is called an **I- structure** on X if

(i) X, $\Phi \in \mathscr{I}$,

(ii) $G_1, G_2, G_3, G_4, G_5, \cdots\cdots, G_n \in \mathscr{I}$, implies

$G_1 \cap G_2 \cap G_3 \cap G_4 \cap G_5 \cap \cdots\cdots \cap G_n \in \mathscr{I}$.

Then (X, $\mathscr{I}$) is called an **I-space**.

Example 7.1 For a non- empty X, {X, Φ } is an **I-structure.** In fact every topology is an I-structure on X, and so, every topological space is an I-space.

Example 7.2 Let X = $\mathbb{Z}$, and $\mathscr{I} = \{\{m\,\mathbb{Z} \mid m \in \mathbb{N}\} \cup \{\Phi\}\}$.

Then $m\mathbb{Z} \cap m'\mathbb{Z} = l\,\mathbb{Z}$, where m, $m' \in \mathbb{N}$ and l = l.c.m of m and m'.

Then (X, $\mathscr{I}$) is an I-space. However, X is not a U-space.

Definition 7. 2 An I-space which is not a topological space is called a **proper I-space.**

Example 7.3 Let X ={a, b, c, d}, $\mathscr{I}$ = { X, Φ ,{a},{a, b},{a, c},{a, d},{a, b, d}, {a, c, d}} is a **proper I-structure** which is not a **topology**, since {a, b} $\cup$ {a, c} = {a, b, c} $\notin \mathscr{I}$.

Definition 7.3 Let X = $\mathbb{R}$ and $\mathscr{I}$ = {$\mathbb{R}$, Φ , all finite intersection of sets of the form (a, b), a, b $\in \mathbb{R}$}. Then (X, $\mathscr{I}$) is an I-space and is called **the usual I-space R of the first kind.** Thus, $\mathscr{I}$ consists of $\mathbb{R}$, Φ and the intervals (a, b).

Definition 7.4 **The usual I-space $\mathbb{R}$ of the second kind** is the I-space ($\mathbb{R}$, $\mathscr{I}$), where $\mathscr{I}$ = The collection of the finite intersection of all rays (-∞, b) and (a, ∞) together with $\mathbb{R}$ and Φ. Thus, $\mathscr{I}$ consists of the sets of the form $\mathbb{R}$, Φ, (-∞,b), (a, ∞) and (a, b).

We may define the interior points and the interior of a set in an I-space as in the case of a topological space. The limit points and the closure of a subset in an I-space may be defined similarly. But in an I-space the interior and the closure of a subset may not have the properties of those in a topological space.

We consider below the following definitions in this situation.

Let (X, $\mathscr{I}$) be an I-space. Let A$\subseteq$X. We have thus the following definitions.

Definition 7.5 A point x$\in$X such that, for each I- open set G which contains x, G$\cap$A contains an element other than x, is called a **limit point of A**. The set of all limit points of A is called the derived set of A and is denoted by D(A).

Definition 7.6 **The closure of A** written $\overline{A}$, is the subset of X consists of the elements x such that for each an I-open set G containing x, G$\cap$A$\neq \Phi$. i.e., $\overline{A}$= {x$\in$X| for each G$\in \mathscr{I}$ with x$\in$G, G$\cap$A$\neq \Phi$.}. Clearly, $\overline{A}$= A$\cup$D(A)

Definition 7.7 A point x$\in$X is called an **interior point of A**, if there is an I-open set G such that x$\in$G and G$\subseteq$A.

Definition 7.8 The set of all interior points of A is called the interior of A and is denoted by IntA. Thus, IntA = { x$\in$X|$\exists$ G$\in \mathscr{I}$ such that x$\in$G$\subseteq$A }

Comment 7.1

For a subset A of a topological space X,

(i) $\overline{A}$ is an I-closed set and is the intersection of all I-closed supersets of A.

(ii) IntA is an I-open set and is the union of all I-open subsets of A.

But these properties may or may not hold for $\overline{A}$ and IntA in I-spaces. The truth of the comment follows from the following theorems and illustrations;

1. (i) Let X = The usual I-space $\mathbb{R}$. Let A = $\mathbb{Q}$. Then $\overline{A}$ = $\mathbb{R}$ and $\mathbb{R}$ is an **I-closed and is the intersection of all I-closed supersets of** $\mathbb{Q}$.

(ii) Let X ={a, b, c, d} and $\mathscr{I}$ ={X,Φ,{a},{a,b}, {a,c},{a, d},{a, b,d}, {a,c, d}} is proper I-structure on X. Then (X, $\mathscr{I}$) is a proper I-space.

The I-closed sets are {c, d}, {b, d},{b, c, d}, {c}, {b}, {b, c}, X, Φ.

Let A = {b}. Then $\overline{A}$= {b}. $\overline{A}$ **is an I-closed and is the intersection of all I-closed supersets of A.**

2. Let A = {d}. Then $\overline{A}$= {d}. $\overline{A}$ **is not an I-closed, but is the intersection of all I-closed supersets of A.**

3.(i) Let X be the usual I- space $\mathbb{R}$ and let A = $\mathbb{N}$. then $\overline{A}$ **=** $\mathbb{N}$, **and** $\mathbb{N}$ **is neither I-closed nor is the intersection of all I-closed supersets of** $\mathbb{N}$**.**

(ii) Let X ={a, b, c, d} and $\mathscr{I}$ = {X,Φ,{b},{d}, {a, b},{b,d}} is a proper I-structure. Then (X, $\mathscr{I}$) is a proper I-space.

The I-closed sets are {a, c, d}, {a, b, c},{c, d}, {a, c}, X, Φ.

Let A = {b}. Then $\overline{A}$= {a, b}. $\overline{A}$ **is neither I-closed nor is the intersection of all I-closed supersets of A.**

Theorem 7.1 Let (X, $\mathscr{I}$) be an I-space and let $A \subseteq X$. Then, $\overline{A}$ is an I-closed and $\overline{A} = F_0$, the intersection of all I-closed supersets of A.

Proof: Suppose that $\overline{A}$ is an I-closed. Since $\overline{A}$ is an I-closed, $\overline{A} \subseteq F_0$. Let $x \in F_0$ Then, $x \in F_0$, for each I-closed superset of A. Hence $x \in \overline{A}$. So, $F_0 \subseteq \overline{A}$.

4.(i) Let X = The usual I- space $\mathbb{R}$, A = $\mathbb{Q}$. Then IntA = Int $\mathbb{Q}$ = Φ, and so **IntA is an I - open and is the union of all I- open sets $G \subseteq A = \mathbb{Q}$.**

(ii) Let X ={a, b, c, d} and $\mathscr{I}$ = {X,Φ,{a},{a, c}, {a, d},{a, b, d}} is a proper I- structure. The (X, $\mathscr{I}$) is a proper I- space.

Let A = {a}. Then IntA = {a}, and so **IntA is an I-open and is the union of all I- open sets $G \subseteq A$.**

5. (i) Let X = {a,b,c,d} and $\mathscr{I}$ = {X,Φ,{a},{d},{a, c},{a, d},{b, d}} is a proper I-structure. The (X, $\mathscr{I}$) is a proper I-space.

Let A = {a, c, d}. Then IntA = {a, c, d}, and so **IntA is not an I-open and is the union of all I-open sets $G \subseteq A$.**

(ii) Let X be the usual I-space $\mathbb{R}$, and Let A = [a, b]$\cup$[c, d],

where a <b <c < d. I-open sets are of the form $(-\infty, b), (a, \infty), (a, b)$.

IntA = (a, b) $\cup$(c, d) is not an I-open set but is a union of I-open sets.

We shall now show that for an I-space the interior of a subset is the union of all I-open sets contained in the subset.

Theorem 7.2 Let $(X, \mathscr{I})$ be an I- space and let $A \subseteq X$. Then, IntA = V, the union of all I-open sets G in X which are contained in A.

Proof: Let G be an I-open set in X, which is contained in A. Then, by the definition of IntA, $G \subseteq$ IntA. Hence $V \subseteq$ IntA.

Now, let $x \in$ IntA. Then, there exists an I-open set G such that $x \in G \subseteq A$. Hence $x \in V$. Thus, IntA $\subseteq V$.

I-continuity

We define I-continuous, $\bar{I}$ -continuous and I*-continuous similar to U-continuous, $\bar{U}$ -continuous and U*-continuous.

Definition 7.9 If X, Y are I-spaces (resp. X I-space, Y top-space; X top-space, Y I-space) a map $f: X \to Y$ is said to be **I-continuous** (resp. $\bar{I}$ **-continuous, I*-continuous**) if for each I-open set (resp. open, I-open) H in Y, $f^{-1}(H)$ is an I-open (resp. I-open, open) set in X .

Example 7.4 Let X ={a, b, c, d }, $\mathscr{I}$ = {X, Φ, {a}, {b}, {d}, {a, b}, {a, d}, {b, c, d}}

$Y = \{p, q, r, s\}$, $\mathscr{I}'$ = $\{Y, \Phi, \{p\}, \{q\}, \{s\}, \{p, q\},\{p, s\},\{q, r, s\}\}$.

Let f: X $\rightarrow$Y be defined by f(a) = p, f(b) = q, f(c) = r, f(d) = s.

Then f is I-continuous.

Example 7.5 Let X= {a, b, c, d}, $\mathscr{I}$ = {X,Φ,{a},{b},{a, b},{a, c},{b, d}}.

Let Y **=** {p, q, r }, $\mathscr{T}$ = {Y, Φ, {p}, {q}, {p, r}}. Then (X, $\mathscr{I}$) is an I-space and (Y, $\mathscr{I}$) is a topological space. The function f: X $\rightarrow$Y is defined by f(a) = p, f(b) = q, f(c) = r, f(d) = q. Then f is $\bar{I}$ -continuous.

Example 7.6 Let X = {a, b, c, d}, $\mathscr{T}$ = {X, Φ,{b},{c},{b, c},{c, d},{b, c, d}}

Y **= {**p, q, r, s }**,** $\mathscr{I}$ = {Y,Φ, {p}, {q}, {p, q}, {p, r}, {q, s}}. Then **(**Y**,** $\mathscr{I}$ **)** is an I- space. The function f: X $\rightarrow$Y is defined by f(a) = p, f(b) = q, f(c) = q, f(d) = s.

Then f is $\mathscr{I}$*-continuous.

Compactness

Definition 7.10 Let $(X,\mathscr{I})$ be an I- space. An I-**open cover** of K is a collection $\{G_\alpha\}$ of I-open sets such that $K \subseteq \bigcup_\alpha G_\alpha$.

Definition 7.11 An I-space X is said to be **compact** if every I-open cover of X has a finite sub-cover.

A subset K of a I-space X is said to be **compact** if every I-open cover of K has finite sub-cover.

Example 7.7 Let $X = \mathbb{N}$ and let $A_{n_o} = \{n \in \mathbb{N} \mid n \geq n_o\}$, $\mathscr{I} = \{\Phi, \{A_{n_o} \mid n_o \in \mathbb{N}\}\}$. Then $(X, \mathscr{I})$ is an I-space. In this I-space, $\mathbb{N}$ **is compact**, because every I-open cover of $\mathbb{N}$ must contain $A_1 = \mathbb{N}$.

Comment 7. 2 We note however that

(i) **For I-space** $(\mathbb{N}, \mathscr{I})$,

where $\mathscr{I} = \{\mathbb{N}, \Phi\} \cup \{n_o + 1, n_o + 2, \ldots\ldots. n_o + r \mid n_o, r \in \mathbb{N}\}$. $\mathbb{N}$ **is not compact.**

(ii) In the usual I-space $\mathbb{R}$, of the first kind, (and also of the second kind) $\mathbb{N}$ is not compact.

For, $\{(n - \frac{1}{2}, n + \frac{1}{2}) \mid n \in \mathbb{N}\}$ is an I-open cover of $\mathbb{N}$ which does not have a finite subcover.

Theorem 7.3 Every I-continuous image of a compact I-space is compact.

The proof is similar to that in topology.

The Heine-Borel Theorem of topology, 'A subset A of the usual space $\mathbb{R}$ is compact if and only if A is closed and bounded', has the following forms in the case of the usual I-space $\mathbb{R}$ of the first kind:

Theorem 7.4

(1) The compact subsets of $\mathbb{R}$ are precisely the finite subsets of $\mathbb{R}$.

(2) No non-empty compact subset is I-closed.

(3) No non-empty I-closed subset is compact.

Proof :

(1) For, if A is an infinite subset of $\mathbb{R}$, let A = $\{a_n\}_{n\in\mathbb{N}}$ be a countable subset of $\mathbb{R}$, and suppose $a_n < a_{n+1}$, for each n. Consider the intervals

$I_n = \left(a_n - \frac{\epsilon_n}{2}, a_n + \frac{\epsilon_n}{2}\right)$, where $\epsilon_n = a_{n+1} - a_n$. Then, $I_n \cap I_{n'} = \Phi$, if n = n$'$. If $\{I_n\}$ covers A, let $\mathcal{C}$ be this cover. Otherwise, let $\{J_k\}$ be a collection of I-open sets such that (i) $J_k \cap \left(\bigcup_n I_n\right) = \Phi$, for each k, and (ii) $\{I_n\} \cup \{J_k\}$ is a cover of A. In this case, let $\mathfrak{C}$ denote this cover. In both the cases, $\mathfrak{C}$ does not have a finite subcover. Thus, the compact subsets of $\mathbb{R}$ are finite.

(2) For, the definition of the I-structure on $\mathbb{R}$ is shows that every non-empty I-closed set must contain subsets of the form $(-\infty, a]$ and $[b, \infty)$ both of which are infinite. Hence (2) follows.

(3) The discussions in (1) and (2) prove (3).

For the usual I-space $\mathbb{R}$ of the second kind, the theorem corresponding to the Heine-Borel Theorem in topology is the following:

Theorem 7.5

- **(i) a compact subset need not be I-closed,**
- **(ii) a compact subset need not be bounded,**
- **(iii) every I-closed and bounded subset is compact.**

Proof :

(1) Since the I-closed subsets of $\mathbb{R}$ are $\mathbb{R}, \Phi$ the I-closed intervals $[a, b]$ $(a < b)$, and the singleton sets $\{c\}$, the subset $\{1, 2, 3, \ldots, n\}$ of the usual I- space $\mathbb{R}$ of the second kind is compact. But it is not I-closed, since the non-trivial I-closed subsets of $\mathbb{R}$ are of the form $(-\infty, a]$, $[b, \infty)$, $[a, b]$). This proves

(2) Any I-open cover $\mathfrak{C}$ of $\mathbb{N}$ must contain either $\mathbb{R}$, or an I-open subset of the form (b, ∞). Then (i) $\{\mathbb{R}\}$ or (ii) $\{(b, \infty)\}$ together with a finite number of sets in $\mathfrak{C}$ covers $\mathbb{N}$. This is because

(i) there exist at most [b] positive integers preceding b, where [b] is the largest positive integer $\leq b$, and

(ii) there exist at most [b] sets in $\mathfrak{C}$ which cover $\{1,2,3,\cdots,[b]\}$. Thus, $\mathbb{N}$ is compact.

Clearly $\mathbb{N}$ is unbounded.

(3) Let F be an I-closed and bounded subset of $\mathbb{R}$. Then $F = \Phi$ or $F = [a, b]$ or $F = \{c\}$, for some $a, b, c \in \mathbb{R}$, $a < b$. Φ and $\{c\}$ are obviously compact. The proof that [a, b] is compact is exactly similar to corresponding proof in topology.

Definition 7.12 A subset A of an I- space $(X, \mathcal{I})$ is said to be **disconnected** if there exist I- open sets $I_1$and I_2 of X such that $A \cap I_1 \cap I_2 = \Phi$ and $I_1 \cup I_2 \supseteq A$. A said to be **connected** if it is not disconnected.

Example 7.8 Let $X = \{a, b, c, d\}$ and $\mathcal{I} = \{X, \Phi, \{c\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}\}$. Then $(X, \mathcal{I})$ is an I-space. Let $A = \{b, c, d\}$ and $B = \{b, d\}$. Then A is connected and B is disconnected.

Example 7.9 In the usual I-spaces $\mathbb{R}$ of the first and the second kinds, all intervals are connected subsets.

Remark 7.1 As in topological spaces, the closure of a connected subset of I-space is connected too.

Remark 7.2 Although in the usual topological space $\mathbb{R}$ and the usual U-space $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are **disconnected, in the usual I-space $\mathbb{R}$ of the first kind, the above subsets of $\mathbb{R}$ are connected. However, these subsets are again disconnected in the usual I-space $\mathbb{R}$ of the second kind.**

A Housdorff (resp. normal, regular, completely regular) I-space is defined as in topology. The usual I-spaces $\mathbb{R}$ of the first and the second kind are Hausforff.

Remark 7.3 A compact subset of a Hausdorff topological space is closed. But a compact subset of an I-space need not be I-closed.

Its truth follows from (2) of Theorem 7.4 as well as (1) of Theorem 7.5.

Remark 7.4 Unlike the usual topological space $\mathbb{R}$ and the usual U-space $\mathbb{R}$, the usual I-spaces $\mathbb{R}$ of the first kind and the second kind are normal but not regular.

Proof: Let X denote the usual I-space $\mathbb{R}$ of the first kind. The I-closed sets of X are $\mathbb{R}, \Phi$ and sets of the form $(-\infty, a] \cup [b, \infty)$ with $a < b$.

Let $F = (-\infty, a] \cup [b, \infty)$ and $x \notin F$. Then $x \in (a, b)$. But the only I- open set containing F is $\mathbb{R}$ and it also contains x. Hence X is not regular.

The only pairs of disjoint I-closed sets of X are $\{\mathbb{R},\Phi\}$ and $\{F,\Phi\}$. Then the disjoint I-open sets $\mathbb{R}$ and Φ separate each of these pairs of disjoint I-closed sets. Thus, X is normal.

Now, let Y denote the usual I-space $\mathbb{R}$ of the second kind.

Then the I-closed sets of Y are $\mathbb{R},\Phi$, and the sets of the form $(-\infty, a]$, $[b, \infty)$, $(-\infty, c]\cup[d, \infty)$ with $c < d$. As in the case of X, if $F = (-\infty, c]\cup[d, \infty)$ and $x\notin F$, then $x\in(c, d)$. The only I-open set of Y which contains F is $\mathbb{R}$ which also contains x. Hence Y is not regular.

The only pairs of disjoint I-closed sets of Y are $P_1 = \{(-\infty, a], [b, \infty)\}$ $(a < b)$, $P_2 = \{(-\infty, a]\cup[b, \infty), \Phi\}$, $P_3 = \{\mathbb{R},\Phi\}$. Then P_1 is separated by the each of disjoint I-open sets $\left(-\infty,\frac{a+b}{2}\right) and \left(\frac{a+b}{2},\infty\right)$, while P_2 and P_3 is separated by the disjoint I-open sets $\mathbb{R}$ and Φ. Hence Y is normal.

CU – spaces

Definition 7.13 X be a non empty set and let $\mathcal{CU}$ a collection of subsets of X such that

(i) X, $\Phi \in \mathcal{CU}$

(ii) $\mathcal{CU}$ is closed under countable unions.

Then $\mathcal{CU}$ is called a **CU-structure on X** and (X, $\mathcal{CU}$) is called a **CU-space**. [Clearly, every topology $\mathcal{T}$ (resp. every U-structure $\mathcal{U}$) on X is CU-structure on X and (X, $\mathcal{T}$) (resp. (X, $\mathcal{U}$)) is a CU-space.] A CU-space, which is neither a topological space, nor a U-space will be called a **proper CU-space**.

Example 7.10 Let X be an uncountable set and let $\mathcal{CU}$ consists of X, Φ and all countable unions of finite subsets of X. **Then** (X, $\mathcal{CU}$) **is a proper CU-space**.

Example 7.11 The σ algebra $\mathcal{B}$ of Borel sets on $\mathbb{R}$ is a proper CU-structure on $\mathbb{R}$. Hence ($\mathbb{R}, \mathcal{B}$) is a proper CU-space.

To see this, we first note that every singleton subset of $\mathbb{R}$ belongs to $\mathcal{B}$. Let A be a proper uncountable subset of $\mathbb{Q}^c$, the set of irrationals. Then A = $\bigcup_{x \in A}\{x\}$, A $\notin \mathcal{B}$. So, $\mathcal{B}$ is a proper CU-structure.

Example 7.12 let X = $\mathbb{R}$, $\mathcal{TU}$ = { $\mathbb{R}, \Phi$, all countable unions of all closed intervals [a, b]}. **Then** (X, $\mathcal{TU}$) **is a CU-space**. $\mathcal{TU}$ **properly contains the usual topology on** $\mathbb{R}$.

For,

(i) (a, b) = $\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\left[a+\frac{1}{m}, b-\frac{1}{n}\right] \in \mathcal{TU}$ and every proper open set in the usual topology of $\mathbb{R}$ is a countable union of open intervals (a, b).

(ii) [a, b] $\in \mathcal{TU}$, but it does not belong to the usual topology of $\mathbb{R}$.

Definition 7.14 The usual U-space $\mathbb{R}$ is also a CU- space. It is called **the usual CU-space** $\mathbb{R}$.

Definition 7.15 **The closure of A** written $\overline{A}$, is the subset of X consisting of the elements x such that for each CU-open set G containing x, $G \cap A \neq \Phi$. i.e, $\overline{A}$ = {$x \in X$| for each $G \in \mathcal{TU}$ with $x \in G$, $G \cap A \neq \Phi$ }.

CUI-spaces

Definition 7.16 Let X be a non- empty set. A collection $\mathcal{CUI}$ of subsets of X is called a **CUI-structure on X** if X, $\Phi \in \mathcal{CUI}$ and $\mathcal{CUI}$ is closed under countable union, and finite intersection. Then (X, $\mathcal{CUI}$) is a **CUI-space**.

Examples 7.10 and 7.11 of CU-spaces are examples of CUI-spaces too.

Example 7.13 Let X = $\mathbb{R}$ and $\mathcal{CU}$ = {$\mathbb{R}$, Φ, and the infinite countable subsets of $\mathbb{R}$}. Then (X, $\mathcal{CU}$) **is a CU-space**. Let A = {n$\in$ $\mathbb{Z}$ $|-\infty<n<5$} and B = { n$\in$ $\mathbb{Z}$ $|-7<n<\infty$}. Then A, B$\in$ $\mathcal{CU}$. A$\cap$B = {- 6, - 5, - 4, - 3, - 2, - 1, 0, 1, 2, 3, 4}$\notin$ $\mathcal{CU}$. $\mathcal{CU}$ **is a proper CU-space** but not I-space.

Example 7.14 Let X **=** $\mathbb{R}$ and $\mathcal{C}$= {$\mathbb{R}$, Φ, $\cup$ {(n,∞)$|n \in$ $\mathbb{Z}$}, $\cup$ {(-∞, n)$|n \in \mathbb{Z}$}, $\cup\{[(m,\infty)\cup(-\infty,n),\ m,n\in\mathbb{Z}\}$}.

Then ($\mathbb{R}$, $\mathcal{C}$) **is a U- space and so, a CU-space but not an I-space**.

Example 7.15 Let X = $\mathbb{N}$ or, $\mathbb{Z}$, and $\mathcal{I}$ = {X, Φ, all finite subsets of X}.

Then (X, $\mathcal{I}$) is an I-space but not a CU-space, and hence, not a U-space.

Definition 7.17 **The usual topological space** $\mathbb{R}$ **is defined to be the usual CUI- space** $\mathbb{R}$.

FU-spaces

Definition 7.18 Let X be a non-empty set and let $\mathcal{FU}$ be a collection of subsets of X such that

(i) X, $\Phi \in \mathcal{FU}$

(ii) $\mathcal{FU}$ is closed under finite unions.

Then $\mathcal{FU}$ is called an F**U-structure on X** and (X, $\mathcal{FU}$) is called an F**U-space**.

Example 7.16 Topological spaces, U-spaces and CU-spaces are FU-spaces.

Definition 7.19 A FU-space which is not a CU-space (and hence neither a U-space nor a topological space) is called **a proper FU-space.**

Example 7.17 Let X be an infinite set and let $\mathcal{FU}$ be the collection of all finite subsets of X. **Then** (X, $\mathcal{FU}$) **is a proper FU-space.**

Example 7.18 Let X be $\mathbb{R}$ and $\mathcal{FU}$ the collection of all finite union of sets of the form $(-\infty, a)$ and (b, ∞). **Then** (X, $\mathcal{FU}$) **is FU-space.**

Definition 7.20 The usual FU-space $\mathbb{R}$ is $\mathbb{R}$ with the FU-structure consisting of $\mathbb{R}$, Φ, and all finite unions of the sets of the form $(-\infty, a)$, (b, ∞) and (c, d).

We thus note:

Remark 7.5 The FU-structure of the usual FU-space $\mathbb{R}$ consists precisely of the sets $\mathbb{R}$, Φ and sets of the form $(-\infty, a)$, (b, ∞), $(-\infty, a)\cup(b, \infty)$ $(a < b)$ and $(a_1, b_1)\cup(a_2, b_2)\cup\cdots\cup(a_r, b_r)$, for some positive integer r with $a_i < b_i$, $1\leq i\leq r$.

Definition 7.21 Let (X, $\mathcal{FU}$) be an FU-space and let A be a subset of X. For $x\in X$, x is called an interior point of A if $x\in G\subseteq A$, for some FU-open set G in X.

Definition 7.22 The set of all interior points of A is called **the interior of A**, and is denoted by IntA.

Remark 7.6 Unlike in topological spaces, IntA need not be FU-open in an FU-space.

To see this, let us consider the usual FU-space $\mathbb{R}$. Let A = $\bigcup_{n=1}^{\infty}(2n, 2n+1)$.

Then, A = IntA . But A is not FU-open.

Remark 7.7 However, for every FU-open set A in an FU-space, A = IntA. The FU-closed sets of X are the complements of FU-open sets.

Definition 7.23 The FU-closure $\overline{A}$ of a subset A of an FU-space X is defined by $\overline{A} = \{x \in X | x \in G$ for some FU-open set G in X with $G \cap A \neq \Phi\}$.

Theorem 7.6 Let X be an FU-space,

(i) For every FU-closed set F of X, $\overline{F} = F$,

(ii) For a subset A of X, $\overline{A}$ need not be FU-closed.

Proof: (i) Let $x \in \overline{F}$. If $x \notin F$, then $x \in F^c$. Now $x \in \overline{F}$ and since F^c is FU-open, and $x \in F^c$, $F^c \cap F \neq \Phi$, a contradiction. Hence $x \in F$.

(ii) Let X be the usual FU-space $\mathbb{R}$ and A = $(1, 2) \cup (3, 4)$.

Then, $\overline{A} = [1, 2] \cup [3, 4]$. But this is not an FU-closed set in X, since the FU-closed subsets of X are precisely $\mathbb{R}$, Φ and sets of the form [a, b] and $[-\infty, a_1] \cup [a_2, b_1] \cup \cdots \cup [a_r, b_{r-1}] \cup [b_r, \infty]$($a_1 < b_1 < a_2 < b_2 <$ --------$< a_r < b_r$).

Definition 7.24 A subset A of an FU-space X is called **compact** if every FU-open cover has a finite subcover.

Example 7.19 In the usual FU-space $\mathbb{R}$,$\mathbb{N}$ and the intervals [a, b] are compact subsets.

The proof that [a, b] is compact is similar to that in topology.

To see that $\mathbb{N}$ is compact, we note that every FU-open cover of $\mathbb{N}$ must contain a FU-open set of the form (a, ∞). Then, at most [a] more FU-open sets of the cover are needed to cover A, where [a] is the largest positive integer less than or equal to a. Thus, $\mathbb{N}$ is compact.

Theorem 7.7 Every FU-closed subsets of a compact FU-space is compact.

The proof is as in topology.

Remark 7.8 The following is the FU-version of the Heine-Borel Theorem in topology: Let X be the usual FU-space $\mathbb{R}$.

(i) Every FU-closed and bounded set in X is compact,

(ii) A compact set in X may be neither FU-closed nor bounded.

Proof: (i) It follows from the nature of the FU-closed sets in X that every non-empty FU-closed bounded set in X is of the form [a, b] which is obviously compact.

(ii) We have proved above (in Example 7.19) that $\mathbb{N}$ is compact.

However, $\mathbb{N}$ is neither FU-closed nor bounded.

Definition 7.25 A non-empty subset A of an FU-space X is called **disconnected** if there exist FU-open sets G_1 and G_2, such that $A \cap G_1 \neq \Phi \neq A \cap G_2$, $A \cap G_1 \cap G_2 = \Phi$, $A \subseteq G_1 \cup G_2$. A is called **connected** if it is not disconnected.

Example 7.20 In the usual FU-space $\mathbb{R}$, the connected subsets are precisely $\mathbb{R}$, Φ and sets of the form $(-\infty, a)$, (b, ∞) and (c, d).

As in topology, we have every FU- continuous image of a connected set is connected.

FUI-spaces

Definition 7.26 Let X be a non-empty set. A collection $\mathcal{FUI}$ of subsets of X is called an **FUI-structure on X** if

(i) $X, \Phi \in \mathcal{FUI}$

(ii) $\mathcal{UI}$ is closed under finite unions and finite intersections.

Then $\mathcal{FUI}$ is called an F**UI-structure on X** and $(X, \mathcal{FUI})$ is called an F**UI-space**.

Example 7.21 Every topological space and every CUI-space is an FUI-space.

Example 7.22 Let X be an infinite set and $\mathcal{FUI}$ = {$\mathbb{R}$, Φ ,all finite subsets of X}. **Then,** (X, $\mathcal{FUI}$) **is an FUI-space which is neither a CUI-space nor a topological space.**

Example 7.23 Let X = $\mathbb{R}$ and $\mathcal{FUI}$ = The subsets of $\mathbb{R}$ obtained from the sets of the form (-∞, a) and (b, ∞) under finite unions and intersections.

Then, (X, $\mathcal{FUI}$) **is an FUI-space. It is called the usual FUI-space** $\mathbb{R}$**. We note that here** $\mathcal{FUI}$ **consists of** $\mathbb{R}$, Φ and the sets of the form (-∞, a), (b, ∞) and $(a_1, b_1) \cup (a_2, b_2) \cup \cdots \cup (a_r, b_r)$.Thus, the usual FUI-space is exactly the same as the usual FU-space $\mathbb{R}$.

Remark 7.9 Let X be a FUI-space. As in the case FU-spaces,

(i) for each FUI-open subset A of X, A = IntA;

but (ii) InA need not always be FUI-open.

The first part is obvious and the second part follows the example in Remark 7.6.

Remark 7.10 **Example 7.18 is an FU-space but not an FUI-space. Thus, the class of FU-spaces and the class of FUI-spaces are distinct.**

heorem 7.8 Let X be an FUI-space,

(i) For every FUI-closed set F of X, $\overline{F} = F$,

(ii) For a subset A of X, $\overline{A}$ need not be FUI-closed.

The proof is exactly similar to that of Theorem 7.6.

All the statements about the compact sets and the connected sets proved earlier for an FU-space, and in particular the statement corresponding to the Heine-Borel Theorem, hold for an FUI-space.

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