University of Rajshahi **Rajshahi-6205** **Bangladesh.**

RUCL Institutional Repository **http://rulrepository.ru.ac.bd**

Department of Mathematics PhD Thesis

2016

Characterizations of K-Derivations and Bi-Derivations on Lie Ideals of Gamma Rings

Nazneen, Ayesha

University of Rajshahi

http://rulrepository.ru.ac.bd/handle/123456789/382

Copyright to the University of Rajshahi. All rights reserved. Downloaded from RUCL Institutional Repository.

CHARACTERIZATIONS OF k-DERIVATIONS AND BI-DERIVATIONS ON LIE IDEALS OF GAMMA RINGS

A THESIS SUBMITTED TO THE DEPARTMENT OF
MATHEMATICS FOR THE DEGREE OF

DOCTORS OF PHILOSOPHY

SUBMITTED
BY
AYESHA NAZNEEN

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF RAJSHAHI
RAJSHAHI – 6205
BANGLADESH
ROLL NO. 11324,
SESSON: 2011 -2012

JUNE, 2016

Professor Dr. Akhil Chandra Paul
M.Sc. M.Phil (Raj),Ph.D.(Banaras)
Department of Mathematics
Rajshahi University, Rajshahi.
Rajshahi-6205,Bangladesh

Residence : 8/BIHAS, Chiuddapai
Binodpur Bazar, Rajshahi
Phone : 0721-711108(office)
0721-751307(Resi.)
Mobile : 01915813138
Fax : 00880-721750064
e-mail : acpaulrubd_math@yahoo.com

No. .... Dated ....

It is certified that the thesis entitled "CHARACTERIZATIOS OF k-DERIVATIONS AND BI-DERIVATIONS ON LIE IDEALS OF GAMMA RINGS" submitted by Ayesha Nazneen contains the fulfillment of all the requirements for the degree of Doctor of Philosophy in Mathematics ,the University of Rajshahi, has been completed under my supervision. I do believe that this research work is an original one and it has not been submitted elsewhere for any degree.

A.C.Paul
Professor Akhil Chandra Paul

Supervisor

Statement of originality

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University and to the best of my knowledge and belief does not contain any material previously published or written by another person except where due reference is made in the text .

Ayesha Nazneen

Acknowledgement

At first I would like to express my great feelings to Almighty Allah for giving me the ability to appear this thesis.

I acknowledge my heartily thanks to my honorable supervisor Dr. Akhil Chandra Paul . He introduced me to this subject and show me the way to represent what I have gathered from this study. Whenever I went to him in my divided leisure, he spent his valuable time to make me understand what I could not understand and helped me to find out the easy solutions of my different problems. It is a great opportunity for me to have such guidance. I could complete my thesis for his continuous co-operation.

I want to express my cordial gratitude to my dear Sir, Poffessor Subrata majumder . He inspired me, suggest me and all the time he blessed me. His valuable suggestions encouraged me and make me hopeful.

The authority of Rajshahi University have helped me by giving the necessary books and papers relevant to my thesis.

I am also grateful to my colleagues, all my well-wishers who always pray for me.

All of my family members are also deserves the thanks from me for their mental support. I am also grateful to my two young sons with whom I could not spent much time for my study. My thanks are due to my authority for permitting me to be admitted in this course.

Above all I am very much indebted to my husband without his assistance and co-operations it was next to impossible to complete my thesis.

Content

Abstract

The present thesis entitled, "**CHARACTERIZATIONS OF K-DERIVATIONS AND BI-DERIVATIONS ON LIE IDEALS OF GAMMA RINGS**" is the outcome of researches carried out by me under the close supervisions of Dr. Akhil Chandra Paul, Professor, Department of Mathematics, Rajshahi University, Rajshahi.The main goal of this thesis is to characterize k-derivations and then to generalize it.

At the begining we introduce the concept of gamma rings and then we have mentioned the extention works on gamma rings, that means k-derivations, jordan k-derivations, Jordan generalized k-derivations, bi-derivations e.t.c. We have also mentioned the mathemeticians who have worked on these fields.

In the **first** chapter we mainly described our works on Jordan k-derivations. Here we have given the definition of k-derivation, Lie ideal, Jordan k-derivation etc. Some examples are also given. Here we define $\varphi\alpha$ (u, v) for u, v $\in$ U, $\alpha \in \Gamma$; where U is a Lie ideal of a Γ-ring M.When M is a 2-torsion free prime Γ-ring , then we have proved that every Jordan k-derivation is a k-derivation also.

In the **second** chapter, we have discussed about Jordan generalized k-derivation. Here we have define a new additive mapping, which is a generalized form of k-derivation.We have defined it and then defined Jordan generalized k-derivation. For a generalized k-derivation we have defined it with k-derivation. We added some examples. We have defined Jordan generalized k-derivation on Lie ideals. Here we have also defined ψ_α (u, v), for u, v $\in$ U, $\alpha \in \Gamma$; where U is a Lie ideal of a Γ-ring M. At the end of this chapter we have proved that every Jordan generalized k-

derivation is a Jordan k-derivation and so is a k-derivation on a Lie ideal of a 2-torsion free prime Γ-ring M also.

We have worked on semiprime Γ-rings in the **third** chapter. In this chapter we have characterized semiprime Γ-rings and proved that every Jordan k-derivation on a Lie ideal U of a 2-torsion free Γ-ring M is a k-derivation on U of M if M is semiprime.

The conception of Left centralizer is presented in the **forth** chapter. We have denoted it by T. Here we have mentioned an additive mapping $B_\alpha(u, v)$ for all u, v ∈ U; α ∈ Γ.We have also defined Jordan left centralizers. Here we have proved that every Jordan Left centralizer T is a Left centralizer if M is a 2- torsion free semiprime Γ-ring.

In the **fifth** chapter, we have discussed Jordan generalized k-derivations on Lie ideals of semiprime Γ-rings. We have studied Jordan generalized k-derivations earlier. We have worked those on semiprime Γ-rings. With a special condition we have proved first that every Jordan generalized k-derivation on a Lie ideal U of a 2-torsion free semiprime Γ-ring M is also a generalized k-derivation on U of M. Then we have proved the same result without any special condition by using the left centralizer.

In the **sixth** chapter, we have studied bi-derivations. We have defined symmetric mapping, bi-derivation, symmetric bi-derivation etc. Here we have also defined trace, which is associated with a bi-derivation. Using different types of conditions, we have proved that either the Lie ideal U is contained in Z(M) or the trace d is zero, if M is a prime Γ-ring. We have developed these results for a semiprime Γ-ring also.

We have worked on bi-additive mappings on semiprime Γ-rings in chapter **seven**. In this chapter we have studied the commutativity of a Lie ideal of a 2-torsion free semiprime Γ-ring.

In the **eighth** chapter, we have discussed symmetric bi-derivations with symmetric generalized bi-derivations. Here we have worked with two symmetric bi-derivations associated with their respective traces and have found some important results.

We have worked on commutativity with symmetric bi-derivations in the **nineth** chapter. K. K. Dey and A. C. Paul have worked on symmetric bi-derivations. Some of their results are extended here on Lie ideals of prime and semiprime Γ-rings.

At the first stage of this thesis we have discussed about Nobusawa Γ-rings. In the **tenth** chapter we have tried to characterize k-derivations on Lie ideals of Nobusawa Γ-rings. When M is a Γ-ring, then it is clear that Γ is also an M-ring. In the basis of this idea we have found a d-derivation on a Lie ideal Ω of an M-ring Γ. In this chapter, we have tried to find out the same types of results on these two categories of derivations on respective Lie ideals of those rings. Also we have worked on d^2 and d^3.

In the **eleveth** chapter, we have described Left k-derivation. We have defined Jordan left k-derivation also. In this chapter we have used M as a completely prime Γ-ring. We have proved that if M is a 2-torsion free completely prime Γ-ring and U is a Lie ideal of M, then every Jordan left k-derivation on U of M is also a left k-derivation on U of M.

A complete bibliography which have been helped us to finish my total research is also added at the end of this thesis.

CHARACTERIZATIONS OF k-DERIVATIONS AND BI-DERIVATIONS ON LIE IDEALS OF GAMMA RINGS

Introduction

We know that the concept of a Gamma ring is an extensive generalization of a classical ring. A number of renowned mathematician have worked on Gamma rings. They have researched and extended the theories of classical rings to gamma rings .The continuation of their research, the area of gamma ring have been enriched, enlarged and many new area have come in. Throughout the world many famous mathematicians find out very significant results on gamma rings

The notion of a Γ-ring has been introduced by Nobusawa [48]. Furthermore Bernes [7] generaliged the concept of Nobusawa′s Γ-ring. During forty years, many classical ring theories have been developed in Γ-rings by a number of prominent mathematicians.

The notion of derivations and Jordan derivations in Γ-rings have been introduced by Sapanci and Nakajima [58]. Kandamar [40] first introduced k-derivations in Γ-rings and he obtained some remarkable results in Nobusawa Γ-rings. He defined and worked on a k-derivation of a Γ-ring. He studied the commutativity of a Γ-ring. The notion of Jordan k-derivations of a Γ-ring was first introduced by S. Chakraborty and A. C. Paul [15] . They [16], [17], [18], [19], [20], [21], [22]. They have worked on Nobusawa Γ-ring . They define k-derivation and Jordan k-derivation and have developed some important results relating these concepts. They have also worked on completely prime and semiprime Γ-ring and generalized it.

Herstein [30], [31], [32] studied derivations on any ring and also in prime ring and proved that every Jordan derivation of a prime ring is a derivation. He determined the structure of a prime ring R which has a derivation d ≠ 0, such that the values of d commute. A number of mathematicians studied the derivations of Prime rings and Semiprime rings. In [6] Awtar extended some of these results in Lie ideals.We have developed these works on Lie ideals of prime Γ-rings.

Posner [52] initiated the centralizing maps on prime rings. He stated that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative. Mansoor Ahmad [1], [2], Motoshi Hongan and Andrzej Trzepizur [34] studied and developed the theorem of Posner. Vukman [60], [61], [62], [63], [64] worked on commuting and centralizing mappings in prime rings, symmetric bi-derivations in prime and semiprime rings, derivations and centralizers on semiprime rings. Maksa [44], [45] worked on the trace of symmetric bi-derivation and M. Ai Ozturk, M. Sapanci, M. Soyturk, Kyung Ho Kim [49] extended those on Lie ideals of prime **Γ**-ring. I. Jeffrey Bergen, I. N. Herstein and Jeanne Wald Kerr [8] worked on Lie ideals and derivations of prime rings. We have extended those in chapter eleventh on Lie ideals of Nobusawa Γ-rings. Argac and Yenigul [3] and Muthana [47] also worked and got the similar type of results on Lie ideals of R.

Mohammad Ashraf [5] worked on symmetric Bi-derivations in Rings. We have extended these results in eighth chapter on Lie ideals of prime Γ-rings. In [42] P. H. Lee and T. k. Lee worked on Lie ideals of prime rings with derivation. N. Daif and H. E. Bell [24] worked on derivations on semiprime rings. They proved that a semiprime ring R will be commutative if it admits a derivation d such that xy ± d(xy) = yx ± d(yx) for all x, y ∈ R. M. Bresar [10], [11], [12] worked on right ideals and derivations of prime rings, centralizing mapping, Commuting traces. He generalized the results of Herstein . In [26] B. felzenszwalb worked on derivations in prime rings. In [50] , [51], A. C. Paul and Sabur Uddin worked on Lie and Jordan structure in simple Γ-rings and involutions. They developed some properties of these Γ-rings. In [55] I. S. Rakhimov , K. K. Dey and A. C. Paul worked on Commutativity of completely prime Γ-rings. They define inner derivation and studied when the derivation will be the inner derivation.

Asma Ali, V. De Filippis and Faiza Shujat [4] worked on symmetric generalized bi-derivations of prime and semiprime rings. Y. Ceven [13], [14] determined some extensive results of left derivation and Jordan left derivation. He proved that every Jordan left derivation of a 2-torsion free completely prime Γ-ring is a Jordan left derivation. Halder and Paul [28], [29] extended the results of Ceven in Lie ideals.We have developed these works on left k-derivation and Jordan left k-derivation in chapter twelfth. Also M. M. Rahman and A. C. Paul [53], [54], Nadeem Ur Rehman and Abu Zaid Ansari [56] worked on Lie ideals with symmetric bi-additive maps in rings. M. F. Hoque and A. C. Paul [35], [36], [37], [38], [39] worked on left centralizer . June and Kim [58] enlighten Jordan left derivation of a classical ring.

Jordan k- Derivations on Lie Ideals of Prime Γ-rings

Let M be a Γ- ring and U a Lie ideal of M . Let d : M → M and k : Γ → Γ be additive mappings . Then d is a k - derivation on U of M if d(uαv) = d(u)αv + uk(α)v + uαd(v) is satisfied for all u , v ∈ U and α ∈ Γ. Also d is a Jordan k- derivation on U of M if d(uαu) = d(u)αu + uk(α)u + uαd(u) holds for all u ∈ U and α ∈ Γ. It is well-known that every k- derivation on U of M is a Jordan k- derivation on U of M but the converse is not true in general. In this article we prove that every Jordan k- derivation on U of M is a k- derivation on U of M , if M is a 2-torsion free prime Γ- ring and U is a Lie ideal of M such that uαu ∈ U for all u ∈ U and α ∈ Γ.

1. Introduction: The notion of a Γ- ring was introduced as an extensive generalization of the concept of a classical ring . Nobusawa [46] introduced the notion of a Γ- ring and it was generalized by W. E. Bernes [7] as a more broad sense. The definition of a Γ-ring is as follows:

1.1 Definition: Let M and Γ be two abelian groups. Suppose that there is a mapping (composition) from M × Γ × M → M (sending (x,α,y) into xαy) such that

(i) (x+y)αz = xαz + yαz, x(α+β)y = xαy + xβy, xα(y+z) = xαy + xαz;

(ii) (xαy)βz = xα(yβz), where x, y, z ∈ M and α, β ∈ Γ. Then M is called a Γ-ring .

If the conditions of the above definition are strengthened to

(i*) xαy is an element of M ; αxβ is an element of Γ,

(ii*) same as (i)

(iii*) $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$

(iv) $x\alpha y = 0$ for all x, y $\in$ M implies $\alpha = 0$, then M is called a Γ-ring in the sense of Nobusawa and we denote it Γ_N - ring.

1.2 Example: Let X and Y be abelian groups. Let M = Hom (X, Y) and Γ= Hom (Y, X) and $x\alpha y$ is the usual composite map for all x, y $\in$ M and $\alpha \in \Gamma$. Then clearly (i) and (ii) conditions are satisfied and M is a Γ -ring .

1.3 Example : Every associative ring R with unity 1 is a Γ_N –ring with Γ = R.

Note that every ring M is a Γ-ring if we put Γ = M. But the converse is not always true.

Also it is clear that every Γ_N – ring is a Γ-ring , but the converse is not true in general.

1.4 Definition: Let M be a Γ-ring: A subset N of M is called a **Γ-sub ring** if N is itself a Γ-ring. In other words, an additive subgroup N of a Γ-ring M is said to be a Γ- sub ring of M if $N\Gamma N \subseteq N$.

1.5 Example ([15], 1.1.4): Let R be a ring with identity 1 such that M = $M_{1,2}(R)$ and Γ = {(n.1): n is an integer} then M is a Γ-ring under the usual addition and multiplication of matrices. Here if we consider N = $\{(a, a) : a \in R\} \subseteq M$, then N is also a Γ-ring, in which case we say that N is a Γ- sub ring of M.

1.6 Definition: A subset A of a Γ-ring M is a **right (left) ideal** of M, if A is an additive subgroup of M and $A\Gamma M = \{ a\alpha c : a \in A, \alpha \in \Gamma, c \in M \}$ ($M\Gamma A = \{c\alpha a : c \in M, a \in A, \alpha \in \Gamma,\}$) is contained in A.

If A is both a right and a left ideal, then we say that A is an ideal or two sided ideal of M.

1.7 Definition: If A and B are both right (respectively left or two sided) ideals of M . Then A+B = { a+b : a ∈ A , b ∈ B } is clearly a right (respectively left or two sided) ideal , called the **sum of A and B**.We have every finite sum of right (respectively left or two sided) ideals of a Γ-ring is also the same.

Also the intersection of any number of right (left or two sided) ideals of M is again a right (left or two sided) ideal of M.

If A is a left ideal of M, B is a right ideal of M and S is any non empty subset of M, then the set AΓS = { Σ aγs | a ∈ A, γ ∈ Γ, s ∈ S } is a left ideal of M and SΓB is a right ideal of M. AΓB = { Σ aγb | a ∈ A, γ ∈ Γ, b ∈ B } is a two sided ideal of M.

1.8 Definition: An ideal P of a Γ-ring M is said to be **prime** if for any ideals A and B of M, AΓB ⊆ P implies A ⊆ P or B ⊆ P.

1.9 Definition : A Γ-ring M is called a **prime Γ-ring** if for every x , y ∈ M ; xΓMΓy = 0 implies x = 0 or y = 0.

1.10 Definition : A Γ-ring M is called **2-torsion free** if for all x ∈ M , 2x = 0 implies x = 0.

1.11 Definition : A Γ-ring M is said to be commutative if xαy = yαx for every x, y ∈ M and α ∈ Γ.

1.12 Example([]21], Example 1.1.4): Let R be a ring with identity 1 such that M = $M_{1,2}$ (R) and Γ= {$\binom{n.1}{0}$: n is an integer}.Then M is a 2-torsion free commutative Γ-ring under the usual addition and multiplication of matrices. Here if we consider N = {(a, a) : a ∈ R} ⊂ M, then N is also a Γ–ring . In that case we say that N is a Γ–subring.

1.13 Definition : Let M be a Γ-ring. Characteristic of M denoted by char (M) is the least positive integer n such that nx = 0 , for every x ∈ M; if such n exists. Otherwise char (M) = 0.

Brouer's trick: A group cannot be the set theoretic union of two of its proper subgroups. In other words, if H and K are subgroups of a group G such that $G = H \cap K$, then $G = H$ or $G = K$.

1.14 Definition: If $a, b \in M$ and $\alpha \in \Gamma$, then $(a \circ b)_\alpha = a\alpha b + b\alpha a$ is known as the Jordan product or the anti commutator of a and b with respect to α.

The concept of a Jordan derivation of a Γ- ring was first introduced by M. Sapanci and A. Nakajima [56], [57] whereas the notion of k- derivation of a Γ- ring was used and developed by H. Kandamar [38] . The notion of Jordan k- derivation of a Γ- ring was first initiated by S.Chakraborty and A. C. Paul [15] .

We shall use the notation $[x, y]_\alpha$ for the commutator x and y with respect to α defined by $[x, y]_\alpha = x\alpha y - y\alpha x$. If A is a subset of M, by Z(A) we shall mean the centre of A with respect to M, defined by $Z(A) = \{ a \in A : [a, b]_\alpha = 0$ for all $b \in A, \alpha \in \Gamma \}$. The centre of a Γ-ring M is denoted by Z(M) which is defined by $Z(M) = \{ x \in M : [x, y]_\alpha = 0$ for all $y \in M: \alpha \in \Gamma\}$. A Γ- ring M is commutative if and only if M = Z(M).Throughout this paper , we shall use the condition (*) $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$ and $\alpha , \beta \in \Gamma$. By the condition , the commutator identities

$[a\alpha b, x]_\beta = [a, x]_\beta\alpha b + a [\alpha, \beta]_x b + a\alpha[b, x]_\beta$ and

$[x, a\alpha b]_\beta = a\alpha[x, b]_\beta + a[\beta, \alpha]_x b + [x, a]_\beta\alpha\beta$ reduce to

$[a\alpha b, x]_\beta = a\alpha[b, x]_\beta + [a, x]_\beta\alpha b$ and $[x, a\alpha b]_\beta = a\alpha[x, b]_\beta + [x, a]_\beta\alpha b$.

From the definition of commutator of two elements in a Γ-ring, we have the following :

(i) $[a, b]_\alpha + [b, a]_\alpha = 0$

(ii) $[a+b, c]_\alpha = [a, c]_\alpha + [b, c]_\alpha$

(iii) $[a, b+c]_\alpha = [a, b]_\alpha + [a, c]_\alpha$

(iv) $[a, b]_{\alpha+\beta} = [a, b]_\alpha + [a, b]_\beta$.

Note that a Γ-ring M is commutative if and only if $[a, b]_\alpha = 0$ for all a, b ∈ M and α ∈ Γ.

In this chapter, we introduce the concept of Jordan k- derivation on a Lie ideal of a Γ-ring M. We prove that every Jordan k- derivation on a Lie ideal U of a 2-torsion free prime Γ-ring is a k- derivation .

The definition of a k-derivation and a Jordan k- derivation are as follows:

1.15 Definition : Let M be a Γ-ring. Let d: M → M and k: Γ → Γ be an additive mappings. If d(xαy) = d(x)αy + xk(α)y + xαd(y) is satisfied for all x, y ∈ M and α ∈ Γ, then d is said to be a **k-derivation** of M. And if d(xαx) = d(x)αx + xk(α)x + xαd(x) holds for every x ∈ M and α ∈ Γ, then d is said to be a **Jordan k- derivation** of M. Note that every k-derivation is a Jordan k-derivation but the converse is not true always.

1.16 Example : Let R be an associative Ring. Define $M = M_{1,2}(R)$ and $\Gamma = M_{2,1}(R)$. Then M is a Γ- ring. Define d: M → M by d((a, b)) = (0, b) and k : Γ→ Γ by

$k\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right) = \begin{pmatrix}0\\ -\beta\end{pmatrix}$. Then d is a k- derivation of M for,

$$(0\ ,\ b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(x,\ y) + (a,b)\begin{pmatrix}0\\ -\beta\end{pmatrix}(x,\ y) + (a,b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(0,\ y)$$

$$=(b\beta x\ ,\ b\beta y) + (-b\beta x, -b\beta y) + (0, a\alpha y + b\beta y)$$

$$=(b\beta x - b\beta x + 0,\ b\beta y - b\beta y + a\alpha y + b\beta y)$$

$=(0,\ a\alpha y + b\beta y)$

Also $(a,b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(x,y) = (a\alpha x + b\beta x\ ,\ a\alpha y + b\beta y)$

$\Rightarrow\ d((a,b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(x,y)) = d((a\alpha x + b\beta x\ ,\ a\alpha y + b\beta y))$

$= (0,\ a\alpha y + b\beta y)$

$= (0,b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(x,\ y) + (a,\ b)\begin{pmatrix}0\\ -\beta\end{pmatrix}(x,\ y) + (a,\ b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(0,\ y)$

$= d((a,\ b))\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(x,\ y)) + (a,\ b)k\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right)(x,\ y) + (a,\ b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}d((x,\ y)).$

1.18 Definition : Let M be a Γ- ring . An additive subgroup U of M is called a **Lie ideal** of M if $[u, m]_\alpha \in U$ for every $u \in U$, $m \in M$ and $\alpha \in \Gamma$. Note that every ideal of a Γ- ring M is a Lie ideal of M but the converse is not true in general.

1.19 Example : Let R be a commutative ring with unity having characteristic 2. Define $M = M_{1,2}(R)$ and

$\Gamma = \left\{\begin{pmatrix}n.1\\ n.1\end{pmatrix} : n \in \mathbb{Z} \text{ and n is not divisible by } 2\right\}.$

Then M is a Γ- ring. Define $N = \{(a,\ a) : a \in R\}$. It is clear that N is
an additive subgroup of M. Now for $u = (a,\ a) \in N$; $m = (x,\ y) \in M$ and $\alpha = \begin{pmatrix}n\\ n\end{pmatrix} \in \Gamma$, we have

$u\alpha m - m\alpha u = (a,\ a)\begin{pmatrix}n\\ n\end{pmatrix}(x,\ y) - (x,\ y)\begin{pmatrix}n\\ n\end{pmatrix}(a,\ a)$

$= (anx - yna,\ any - xna)$

$= (anx - 2yna + yna,\ any - 2xna + xna)$

$= (anx + yna , any + xna)$

$= (anx + any, anx + any) \in N$

Therefore, $u\alpha m - m\alpha u \in N$ and N is a Lie ideal of M. It is clear that N is not an ideal of M.

In [50] and [51], Paul and Sabur Uddin worked on Lie and Jordan structure of a 2-torsion free simple Γ–ring and they developed a number of significant results of classical ring theories in Γ–rings.

Now we introduce the concepts of a k-derivation,a Jordan k-derivation on Lie ideals in a Γ–ring and then build up a relationship between these two concepts in a concrete manner.

Let M be a Γ–ring and U, a Lie ideal of M. Let $d:M\rightarrow M$ and $k:\Gamma \rightarrow \Gamma$ be additive mappings. If $d(u\alpha v) = d(u)\alpha v + uk(\alpha)v + u\alpha d(v)$ is satisfied for every $u, v \in U$ and $\alpha \in \Gamma$, then d is called a k-derivation on a Lie ideal U of M. And if $d(u\alpha u) = d(u)\alpha u + uk(\alpha)u + u\alpha d(u)$ holds for all $u \in U$ and $\alpha \in \Gamma$, then d is said to be a Jordan k- derivation on a Lie ideal U of M.

It is clear that every k-derivation on a Lie ideal U of a Γ-ring M is a Jordan k-derivation on U of M but the converse may not be true. Now we make an example of a Jordan k-derivation for the case of a Lie ideal which ensures that Jordan k-derivation on a Lie ideal exists and it is evidently not a k-derivation on a Lie ideal.

1.20 Example : Let M be a Γ-ring and let U be a Lie ideal of M. Let $d: M \rightarrow M$ be a k-derivation on a Lie ideal U of M.

Define $M_1 = \{(x, x) : x \in M\}$ and $\Gamma_1 = \{(\alpha, \alpha): \alpha \in \Gamma\}$.Define addition and multiplication on M_1 as follows:

$(x, x) + (y, y) = (x+y, x+y)$ and $(x, x)(\alpha, \alpha)(y, y) = (x\alpha y, x\alpha y)$ Under these addition and multiplication M_1 is a Γ_1–ring. Define $U_1 = \{(u, u): u \in U\}$. Now we show that U_1 is a Lie ideal of M as follows:

For $(u, u) \in U_1$, $(\alpha, \alpha) \in \Gamma_1$, $(x, x) \in M_1$, we have

$(u, u)(\alpha, \alpha)(x, x) - (x, x)(\alpha, \alpha)(u, u) = (u\alpha x, u\alpha x) - (x\alpha u, x\alpha u) = (u\alpha x - x\alpha u, u\alpha x - x\alpha u) \in U_1$, since $u\alpha x - x\alpha u \in U$.

Now let $d_1 : M_1 \to M_1$, $k_1 : \Gamma_1 \to \Gamma_1$ be the mappings defined by $d_1((u, u)) = (d(u), d(u))$ for all $u \in U$ and $k_1((\alpha, \alpha)) = (k(\alpha), k(\alpha))$ for all $\alpha \in \Gamma$.Then d_1 and k_1 are additive mappings. If we say that $(u, u) = u_1 \in U_1$, for all $u \in U$ and $(\alpha, \alpha) = \gamma \in \Gamma_1$ for all $\alpha \in \Gamma$.Then we have

$d_1(u_1\gamma u_1) = d_1((u, u)(\alpha, \alpha)(u, u)) = (d_1(u\alpha u, u\alpha u)) = (d(u\alpha u), d(u\alpha u))$

$= (d(u)\alpha u + uk(\alpha)u + u\alpha d(u), d(u)\alpha u + uk(\alpha)u + u\alpha d(u))$

$= (d(u)\alpha u, d(u)\alpha u) + (uk(\alpha)u, uk(\alpha)u) + (u\alpha d(u), u\alpha d(u))$

$= (d(u), d(u))(\alpha, \alpha)(u, u) + (u, u)(k(\alpha), k(\alpha))(u, u) + (u, u)(\alpha, \alpha)(d(u), d(u))$

$= d_1(u, u)(\alpha, \alpha)(u, u) + (u, u)k_1(\alpha, \alpha)(u, u) + (u, u)(\alpha, \alpha)d_1(u, u)$

$= d_1(u_1)\gamma u_1 + u_1 k_1(\gamma)u_1 + u_1\gamma d_1(u_1)$

Hence it follows that d_1 is a Jordan k_1-derivation on a Lie ideal $U_1$of M_1.It is obvious that d_1 is not a k_1-derivation on a Lie ideal U of M.

1.21 Lemma : Let M be a 2- torsion free Γ- ring satisfying the condition (*) and let U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. Let d : M → M be a Jordan k- derivation on U of M. Then for all u, v, $w \in U$ and $\alpha \in \Gamma$, we have the following :

(i) $d(u\alpha v + v\alpha u) = d(u)\alpha v + uk(\alpha)v + u\alpha d(v) + d(v)\alpha u + vk(\alpha)u + v\alpha d(u)$

(ii) $d(u\alpha v\beta u) = d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u)$

(iii) $d(u\alpha v\beta w + w\alpha v\beta u) = d(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w)+ d(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$.

Proof :

(i) Since $u\alpha v + v\alpha u = (u+v)\alpha(u+v) - u\alpha u - v\alpha v$ and the right side is in U, it is clear that the left side of the identity is in U. Hence

$d(u\alpha v + v\alpha u) = d((u+v)\alpha(u+v) - u\alpha u - v\alpha v)$

$= d(u+v)\alpha(u+v) + (u+v)k(\alpha)(u+v) + (u+v)\alpha d(u+v) - (d(u)\alpha u + uk(\alpha)u + u\alpha d(u) + d(v)\alpha v + vk(\alpha)v + v\alpha d(v))$

$= (d(u) + d(v))\alpha(u+v) + (u+v)k(\alpha)(u+v) +(u+v)\alpha(d(u) + d(v)) - d(u)\alpha u - uk(\alpha)u - u\alpha d(u) - d(v)\alpha v - vk(\alpha)v - v\alpha d(v)$

$= d(u)\alpha u + d(u)\alpha v + d(v)\alpha u + d(v)\alpha v + uk(\alpha)u + uk(\alpha)v + vk(\alpha)u + vk(\alpha)v + u\alpha d(u) + u\alpha d(v) + v\alpha d(u) + v\alpha d(v) - d(u)\alpha u - uk(\alpha)u - u\alpha d(u) - d(v)\alpha v - vk(\alpha)v - v\alpha d(v)$.

$= d(u)\alpha v +uk(\alpha)v +u\alpha d(v) + d(v)\alpha u +vk(\alpha)u +v\alpha d(u)$.

(ii) Replace v by $u\beta v +v\beta u$ in (i) we have,

$d(u\alpha(u\beta v +v\beta u) + (u\beta v +v\beta u)\alpha u) = d(u)\alpha(u\beta v +v\beta u) + uk(\alpha)(u\beta v +v\beta u) + u\alpha d(u\beta v +v\beta u) + d(u\beta v +v\beta u)\alpha u + (u\beta v +v\beta u)k(\alpha)u + (u\beta v +v\beta u)\alpha d(u)$ ……………. (a)

Here, $d((u\alpha u)\beta v + v\beta(u\alpha u)) = d(u\alpha u)\beta v + (u\alpha u)k(\beta)v + (u\alpha u)\beta d(v) + d(v)\beta(u\alpha u) + vk(\beta)(u\alpha u) + v\beta d(u\alpha u)$

$= d(u)\alpha u\beta v + uk(\alpha)u\beta v + u\alpha d(u)\beta v + u\alpha uk(\beta)v + u\alpha u\beta d(v) + d(v)\beta u\alpha u + vk(\beta)u\alpha u + v\beta d(u)\alpha u + v\beta uk(\alpha)u + v\beta u\alpha d(u)$

Then from (a) we have

$d(u\alpha v\beta u + u\beta v\alpha u) + d(u)\alpha u\beta v + uk(\alpha)u\beta v + u\alpha d(u)\beta v + u\alpha uk(\beta)v + u\alpha u\beta d(v) + d(v)\beta u\alpha u + vk(\beta)u\alpha u + v\beta d(u)\alpha u + v\beta uk(\alpha)u + v\beta u\alpha d(u) = d(u)\alpha u\beta v + d(u)\alpha v\beta u + uk(\alpha)u\beta v + uk(\alpha)v\beta u + u\alpha d(u)\beta v + u\alpha uk(\beta)v + u\alpha u\beta d(v) + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u)) + (u)\beta v\alpha u + uk(\beta)v\alpha u + u\beta d(v)\alpha u + d(v)\beta u\alpha u + vk(\beta)u\alpha u + v\beta d(u)\alpha u + u\beta vk(\alpha)u + v\beta uk(\alpha)u + u\beta v\alpha d(u) + v\beta u\alpha d(u)$.

Using the condition (*) we have,

$2d(u\alpha v\beta u) = 2(d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u))$

And hence $d(u\alpha v\beta u) = d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u)$

(iii) Replacing u+w for u in (ii) we have,

$d((u+w)\alpha v\beta(u+w)) = d(u+w)\alpha v\beta(u+w) + (u+w)k(\alpha)v\beta(u+w) + (u+w)\alpha d(v)\beta(u+w) + (u+w)\alpha vk(\beta)(u+w) + (u+w)\alpha v\beta d(u+w)$.

The left side is $= d(u\alpha v\beta u + u\alpha v\beta w + w\alpha v\beta u + w\alpha v\beta w)$

$= d(u\alpha v\beta w + w\alpha v\beta u) + d(u\alpha v\beta u) + d(w\alpha v\beta w)$

$= d(u\alpha v\beta w + w\alpha v\beta u) + d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + d(w)\alpha v\beta w + wk(\alpha)v\beta w + w\alpha d(v)\beta w + w\alpha vk(\beta)w + w\alpha v\beta d(w)$

The right side is $= d(u)\alpha v\beta u + d(u)\alpha v\beta w + d(w)\alpha v\beta u + d(w)\alpha v\beta w + uk(\alpha)v\beta u + uk(\alpha)v\beta w + wk(\alpha)v\beta u + wk(\alpha)v\beta w + u\alpha d(v)\beta u + u\alpha d(v)\beta w + w\alpha d(v)\beta u + w\alpha d(v)\beta w + u\alpha vk(\beta)u + u\alpha vk(\beta)w + w\alpha vk(\beta)u + w\alpha vk(\beta)w + u\alpha v\beta d(u) + u\alpha v\beta d(w) + w\alpha v\beta d(u) + w\alpha v\beta d(w)$.

Hence we have,

$d(u\alpha v\beta w + w\alpha v\beta u) + d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + d(w)\alpha v\beta w + wk(\alpha)v\beta w + w\alpha d(v)\beta w + w\alpha vk(\beta)w +$

$w\alpha v\beta d(w) = d(u)\alpha v\beta u + d(u)\alpha v\beta w + d(w)\alpha v\beta u + d(w)\alpha v\beta w + uk(\alpha)v\beta u + uk(\alpha)v\beta w + wk(\alpha)v\beta u + wk(\alpha)v\beta w + u\alpha d(v)\beta u + u\alpha d(v)\beta w + w\alpha d(v)\beta u + w\alpha d(v)\beta w + u\alpha vk(\beta)u + u\alpha vk(\beta)w + w\alpha vk(\beta)u + w\alpha vk(\beta)w + u\alpha v\beta d(u) + u\alpha v\beta d(w) + w\alpha v\beta d(u) + w\alpha v\beta d(w)$.

That implies

$d(u\alpha v\beta w + w\alpha v\beta u) = d(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + d(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$

1.22 Definition : We define $\varphi_\alpha(u, v) = d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v)$ for every $u, v \in U$ and $\alpha \in \Gamma$.

1.23 Remark : It is clear that d is a k- derivation on U of M if and only if $\varphi_\alpha(u, v) = 0$.

1.24 Lemma : Let M, U and d be as in above . Then for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$, the following relations hold.

(i) $\varphi_\alpha(u, v) + \varphi_\alpha(v, u) = 0$

(ii) $\varphi_\alpha(u+w, v) = \varphi_\alpha(u, v) + \varphi_\alpha(w, v)$

(iii) $\varphi_\alpha(u, v+w) = \varphi_\alpha(u, v) + \varphi_\alpha(u, w)$

(iv) $\varphi_{\alpha+\beta}(u, v) = \varphi_\alpha(u, v) + \varphi_\beta(u, v)$.

1.25 Lemma : Let M , U and d be as in above , then for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$, $\varphi_\alpha(u, v)\beta w\gamma[u, v]_\alpha + [u, v]_\alpha\beta w\gamma\varphi_\alpha(u, v) = 0$.

Proof :

Consider $A = (2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v)$

Then $d(A) = d((2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v))$

$= d(2u\alpha v)\beta w\gamma 2v\alpha u + 2u\alpha vk(\beta)w\gamma(2v\alpha u) + 2u\alpha v\beta d(w)\gamma 2v\alpha u + 2u\alpha v\beta wk(\gamma)2v\alpha u + 2u\alpha v\beta w\gamma d(2v\alpha u) + d(2v\alpha u)\beta w\gamma 2u\alpha v + 2v\alpha uk(\beta)w\gamma 2u\alpha v + 2v\alpha u\beta d(w)\gamma 2u\alpha v + 2v\alpha u\beta wk(\gamma)2u\alpha v + 2v\alpha u\beta w\gamma d(2u\alpha v)$

$= 4(d(u\alpha v)\beta w\gamma v\alpha u + u\alpha v k(\beta)w\gamma v\alpha u + u\alpha v\beta d(w)\gamma v\alpha u + u\alpha v\beta w k(\gamma)v\alpha u + u\alpha v\beta w\gamma d(v\alpha u) + d(v\alpha u)\beta w\gamma u\alpha v + v\alpha u k(\beta)w\gamma u\alpha v + v\alpha u\beta d(w)\gamma u\alpha v + v\alpha u\beta w k(\gamma)u\alpha v + v\alpha u\beta w\gamma d(u\alpha v))$.

Again $A = (2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v)$

$= u\alpha(4v\beta w\gamma v)\alpha u) + v\alpha(4u\beta w\gamma u)\alpha v$

Then $d(A) = d(u\alpha(4v\beta w\gamma v)\alpha u) + v\alpha(4u\beta w\gamma u)\alpha v)$

$= d(u)\alpha(4v\beta w\gamma v)\alpha u + uk(\alpha)(4v\beta w\gamma v)\alpha u + u\alpha d(4v\beta w\gamma v)\alpha u + u\alpha(4v\beta w\gamma v)k(\alpha)u + u\alpha(4v\beta w\gamma v)\alpha d(u) + d(v)\alpha(4u\beta w\gamma u)\alpha v + vk(\alpha)(4u\beta w\gamma u)\alpha v + v\alpha d(4u\beta w\gamma u)\alpha v + v\alpha(4u\beta w\gamma u)k(\alpha)v + v\alpha(4u\beta w\gamma u)\alpha d(v)$.

$= 4(d(u)\alpha v\beta w\gamma v\alpha u + uk(\alpha)v\beta w\gamma v\alpha u + u\alpha d(v)\beta w\gamma v\alpha u + u\alpha v k(\beta)w\gamma v\alpha u + u\alpha\ v\beta d(w)\gamma v\alpha u + u\alpha v\beta w k(\gamma)v\alpha u + u\alpha v\beta w\gamma d(v)\alpha u + u\alpha v\beta w\gamma v k(\alpha)u + u\alpha v\beta w\gamma v\alpha d(u) + d(v)\alpha u\beta w\gamma u\alpha v + vk(\alpha)u\beta w\gamma u\alpha v + v\alpha d(u)\beta w\gamma u\alpha v + v\alpha\ uk(\beta)w\gamma u\alpha v + v\alpha u\beta d(w)\gamma u\alpha v + v\alpha u\beta w k(\gamma)u\alpha v + v\alpha u\beta w\gamma d(u)\alpha v + v\alpha u\beta w\gamma u k(\alpha)v + v\alpha u\beta w\gamma u\alpha d(v))$.

Comparing the two types of expression of d(A) we have

$4(d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v))\beta w\gamma v\alpha u + 4u\alpha v\beta w\gamma(d(v\alpha u) - d(v)\alpha u - vk(\alpha)u - v\alpha d(u)) + 4(d(v\alpha u) - d(v)\alpha u - vk(\alpha)u - v\alpha d(u))\beta w\gamma u\alpha v + 4v\alpha u\beta w\gamma(d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v)) = 0$.

That implies

$4(\varphi_\alpha(u, v)\beta w\gamma v\alpha u - \varphi_\alpha(u, v)\beta w\gamma u\alpha v - u\alpha v\beta w\gamma\ \varphi_\alpha(v, u) + v\alpha u\beta w\gamma\ \varphi_\alpha(u, v)) = 0$.

Since M is 2- torsion free, we have

$- \varphi_\alpha(u, v)\beta w\gamma(u\alpha v - v\alpha u) - (u\alpha v - v\alpha u)\beta w\gamma\varphi_\alpha(u, v) = 0$

Therefore, $\varphi_\alpha(u, v)\beta w\gamma[u, v]_\alpha + [u, v]_\alpha\beta w\gamma\ \varphi_\alpha(u, v) = 0$.

1.26 Lemma : Let $U \not\subset Z(M)$ be a Lie ideal of a 2- torsion free prime Γ-ring M , then Z(U) = Z(M).

Proof : We have Z(U) is both a Γ-subring and a Lie ideal of M. Also we know that Z(U) cannot contain a nonzero ideal of M. So by [51, Lemma 3.7], Z(U) is contained in Z(M). Therefore, Z(U) = Z(M).

1.27 Lemma : let U be a Lie ideal of a 2- torsion free prime Γ- ring M satisfying the condition (*) and $a \in M$. If $a \in Z([U, U]_\Gamma)$, then $a \in Z(U)$. That is $Z([U, U]_\Gamma) = Z(U)$.

Proof : Obviously $Z(U) \subseteq Z([U, U]_\Gamma)$. If $Z([U, U]) \not\subset Z(M)$, then by Lemma 1.26, $a \in Z(M)$ implies $a \in Z(U)$.

On the other hand if $Z([U, U]) \subseteq Z(M)$, then for all $u \in U$; $m \in M$; $\alpha, \beta \in \Gamma$, we have $a = [u, [u, m]_\alpha]_\beta \in Z(M)$.

Using the condition (*) we have $a\gamma u = [u, [u, u\gamma m]_\alpha]_\beta \in Z(M)$.

If $a \neq 0$, we get $u \in Z(M)$ implies $a = 0$.

Thus, $[u, [u, u\gamma m]_\alpha]_\beta = 0$ for all $m \in M$.

By the subLemma 3.8 of [51] , $u \in Z(M)$. Hence $U \subseteq Z(M)$.

In both cases we see that $a \in Z(U)$. This gives that $Z([U, U]_\Gamma) = Z(U)$.

1.28 Lemma : Let $U \not\subset Z(M)$ be a Lie ideal of a 2- torsion free Γ- ring M satisfying the condition (*) such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If $u \in Z(U)$ then $d(u) \in Z(M)$.

Proof :

Let $u \in Z(U) = Z(M)$, then $u\alpha v = v\alpha u$, for every $v \in U$ and $\alpha \in \Gamma$.

From Lemma 1.21 (i) we have,

$d(u\alpha v + v\alpha u) = d(u)\alpha v + uk(\alpha)v + u\alpha d(v) + d(v)\alpha u + vk(\alpha)u + v\alpha d(u)$

which implies

$d(2u\alpha v) = d(u)\alpha v + uk(\alpha)v + u\alpha d(v) + d(v)\alpha u + vk(\alpha)u + v\alpha d(u)$

$= d(u)\alpha v + v\alpha d(u) + 2uk(\alpha)v + 2u\alpha d(v)$

Replace v by $(v\beta w + w\beta v)$, we have

$d(2u\alpha(v\beta w + w\beta v)) = d(u)\alpha(v\beta w + w\beta v) + (v\beta w + w\beta v)\alpha d(u) + 2uk(\alpha)(v\beta w + w\beta v) + 2u\alpha d((v\beta w + w\beta v))$

$= d(u)\alpha v\beta w + d(u)\alpha w\beta v + v\beta w\alpha d(u) + w\beta v\alpha d(u) + 2uk(\alpha)v\beta w + 2uk(\alpha)w\beta v + 2u\alpha d(v)\beta w + 2u\alpha vk(\beta)w + 2u\alpha v\beta d(w) + 2u\alpha d(w)\beta v + 2u\alpha wk(\beta)v + 2u\alpha w\beta d(v)$

Again ,

$d(2u\alpha(v\beta w + w\beta v)) = 2d(u\alpha v\beta w + u\alpha w\beta v) = 2d(u\alpha v\beta w + w\beta v\alpha u)$

$= 2(d(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + d(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u))$.

From these two expressions we have,

$2d(u)\alpha v\beta w + 2uk(\alpha)v\beta w + 2u\alpha d(v)\beta w + 2u\alpha vk(\beta)w + 2u\alpha v\beta d(w) + 2d(w)\alpha v\beta u + 2wk(\alpha)v\beta u + 2w\alpha d(v)\beta u + 2w\alpha vk(\beta)u + 2w\alpha v\beta d(u) = d(u)\alpha v\beta w + d(u)\alpha w\beta v + v\beta w\alpha d(u) + w\beta v\alpha d(u) + 2uk(\alpha)v\beta w + 2uk(\alpha)w\beta v + 2u\alpha d(v)\beta w + 2u\alpha vk(\beta)w + 2u\alpha v\beta d(w) + 2u\alpha d(w)\beta v + 2u\alpha wk(\beta)v + 2u\alpha w\beta d(v)$

That implies

$d(u)\alpha v\beta w + 2d(w)\alpha v\beta u + 2wk(\alpha)v\beta u + 2w\alpha d(v)\beta u + 2w\alpha vk(\beta)u + 2w\alpha v\beta d(u) = d(u)\alpha w\beta v + w\beta v\alpha d(u) + 2uk(\alpha)w\beta v + 2u\alpha d(w)\beta v + 2u\alpha wk(\beta)v + 2u\alpha w\beta d(v)$

Therefore , $d(u)\alpha(v\beta w - w\beta v) = (v\beta w - w\beta v)\alpha d(u)$, for every $v, w \in U$; $\alpha, \beta \in \Gamma$.

i. e., $d(u)\alpha[v, w]_\beta = [v, w]_\beta \alpha d(u)$. That implies $d(u) \in Z([U, U])$.

To prove our main results we have needed the following two Lemmas:

1.29 Lemma [54, Lemma 2.10]: Let U be a Lie ideal of a 2- torsion free prime Γ- ring M satisfying the condition (*) and $U \not\subset Z(M)$. If $a, b \in M$ (res. $b \in U$ and $a \in M$) such that $a\alpha U\beta b = 0$ for all $\alpha, \beta \in \Gamma$, then $a = 0$ or $b = 0$.

1.30 Lemma [54, Lemma 2.11] : Let $U \not\subset Z(M)$ be a 2- torsion free Lie ideal of a prime Γ- ring M . If $a, b \in M$ (res. $a \in M$ and $b \in U$) such that $a\alpha x\beta b + b\alpha x\beta a = 0$ for all $x \in U$ and $\alpha , \beta \in \Gamma$, then $a\alpha x\beta b = b\alpha x\beta a = 0$.

Now we are in the position to prove our main result.

1.31 Theorem : Let M be a 2- torsion free prime Γ- ring satisfying the condition (*) and let U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If $d: M \rightarrow M$ is a Jordan k- derivation on U of M , then d is a k- derivation on U of M .

Proof: If U is commutative Lie ideal of M, then for all $u, v \in U$ and $\alpha \in \Gamma$, $[u, v]_\alpha = 0$. That is, $u\alpha v = v\alpha u$. Then by Lemma 1.21(iii), we have

$d(u\alpha v\beta w + w\alpha v\beta u) = d(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + d(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$.

By using (*) we obtain,

$d(u\alpha v\beta w + w\alpha v\beta u) = d((u\alpha v)\beta w + w\beta(u\alpha v))$

$= d(u\alpha v)\beta w + (u\alpha v)k(\beta)w + u\alpha v\beta d(w) + d(w)\beta u\alpha v + wk(\beta)u\alpha v + w\beta d(u\alpha v)$.

Comparing these two expressions and using (*) we obtain,

$d(u\alpha v)\beta w + (u\alpha v)k(\beta)w + u\alpha v\beta d(w) + d(w)\beta u\alpha v + wk(\beta)u\alpha v + w\beta d(u\alpha v)$
$= d(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + d(w)\beta u\alpha v + wk(\beta)u\alpha v + w\beta d(v)\alpha u + w\beta vk(\alpha)u + w\beta v\alpha d(u)$.

that implies

$(d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v))\beta w + w\beta(d(v\alpha u) - d(v)\alpha u - vk(\alpha)u - v\alpha d(u)) = 0$

That means $0 = \varphi_\alpha(u, v)\beta w + w\beta\varphi_\alpha(v, u) = \varphi_\alpha(u, v)\beta w - w\beta\varphi_\alpha(u, v)$

Then $\varphi_\alpha(u, v)\beta w = w\beta\varphi_\alpha(u, v)$, for all $w \in U$; $\beta \in \Gamma$.

Therefore we have, $\varphi_\alpha(u, v) \in Z(U) = Z(M)$, by Lemma 1.26.

That implies

$d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v) \in Z(M)$.

Since $u\alpha u \in U$ and $u\alpha u\beta v = v\beta u\alpha u$ for all $\beta \in \Gamma$.

Hence $d(u\alpha u\beta v) - d(u\alpha u)\beta v - u\alpha u k(\beta)v - u\alpha u\beta d(v) \in Z(M)$.

That implies $d(u\alpha u\beta v) - d(u)\alpha u\beta v - uk(\alpha)u\beta v - u\alpha d(u)\beta v - u\alpha uk(\beta)v - u\alpha u\beta d(v) \in Z(M)$……….(i)

Also $2u\beta v \in U$ and $u\alpha(2u\beta v) = (2u\beta v)\alpha u$, we get

$d(u\alpha(2u\beta v)) - d(u)\alpha(2u\beta v) - uk(\alpha)(2u\beta v) - u\alpha d(2u\beta v)$

$= 2(d(u\alpha u\beta v) - d(u)\alpha u\beta v - uk(\alpha)u\beta v - u\alpha d(u\beta v)) \in Z(M)$

And hence $d(u\alpha u\beta v) - d(u)\alpha u\beta v - uk(\alpha)u\beta v - u\alpha d(u\beta v) \in Z(M)$ ….(ii)

From (i) and (ii) we have

$d(u\alpha u\beta v) - d(u)\alpha u\beta v - uk(\alpha)u\beta v - u\alpha d(u)\beta v - u\alpha uk(\beta)v - u\alpha u\beta d(v) - d(u\alpha u\beta v) + d(u)\alpha u\beta v + uk(\alpha)u\beta v + u\alpha d(u\beta v)$

$= u\alpha d(u\beta v) - u\alpha uk(\beta)v - u\alpha u\beta d(v) - u\alpha d(u)\beta v$

$= u\alpha(d(u\beta v) - d(u)\beta v - uk(\beta)v - u\beta d(v)) = u\alpha\varphi_\beta(u, v) \in Z(M)$

If $\varphi_\beta(u, v) \neq 0$,since M is prime and $\varphi_\beta(u, v) \in Z(M)$ hence $u \in Z(M)$.

Therefore by Lemma 1.28, $d(u) \in Z(M)$

By Lemma 1.21(i), we have

$d(u\alpha v + v\alpha u) = d(u)\alpha v + uk(\alpha)v + u\alpha d(v) + d(v)\alpha u + vk(\alpha)u + v\alpha d(u)$

That implies $d(2u\alpha v) = 2(d(u)\alpha v + uk(\alpha)v + u\alpha d(v))$

And so $2(d(u\alpha v) - 2(d(u)\alpha v - uk(\alpha)v - u\alpha d(v)) = 0$.

Therefore, $2\varphi_\alpha(u, v) = 0$.

That implies $\varphi_\alpha(u, v) = 0$

Again let U is not commutative . That is $U \not\subset Z(M)$,Then by Lemma 1.25, we have

(a) $\varphi_\alpha(u, v)\beta w\gamma[u, v]_\alpha + [u, v]_\alpha\beta w\gamma\, \varphi_\alpha(u, v) = 0$.

Applying Lemma 1.27 in (a) we obtain

(b) $\varphi_\alpha(u, v)\beta w\gamma[u, v]_\alpha = 0$ or (c) $[u, v]_\alpha\beta w\gamma\, \varphi_\alpha(u, v) = 0$.

In view of Lemma 1.26, we have from (b) that $\varphi_\alpha(u, v) = 0$ or $[u, v]_\alpha = 0$.

The same result follows from (c) by applying Lemma 1.26.

Now, for every $v \in U$, let us define $A = \{u \in U: \varphi_\alpha(u, v) = 0\}$ and $B = \{u \in U : [u, v]_\alpha = 0\}$.

Then A and B are additive subgroup of U such that $A \cup B = U$, Therefore, by Brauer′s trick, either $A = U$ or $B = U$. By using the same argument we have $U = \{v \in U : U = A\}$ and $U = \{ v \in U : U = B\}$. For the latter case, we have $U \subseteq Z(M)$ which is a contradiction.

So we have $\varphi_\alpha(u, v) = 0$, which completes the proof.

Jordan Generalized k- Derivations on Lie Ideals of Prime Γ-rings

Let M be a 2-torsion free prime Γ- ring and U a Lie ideal of M. Let F : M → M be a mapping defined by F (uαv) = F(u)αv + uk(α)v + uαd(v), for all u ,v ∈ U and α ∈ Γ. Then F is a generalized k- derivation on U of M if there exists a k-derivation d on U of M. Also F is a Jordan generalized k-derivation on U of M if there exists a k - derivation d on U of M such that F(uαu) = F(u)αu + uk(α)v + uαd(v), for all u ∈ U and α ∈ Γ. In this article, we prove that every Jordan generalized k - derivation on a Lie ideal U of a 2 - torsion free prime Γ- ring M is a generalized k - derivation on U of M.

2.Introduction : We know that the Γ- ring is a generalized form of a ring. Nobusawa [46] and Barnes [7] developed the concept of a Γ- ring. In the previous chapter we have discussed about Γ- rings and Nobusawa Γ- rings. We know that M is a Γ_N -ring implies that Γ is an M-ring. M is called a semiprime Γ- ring if for all x ∈ M, xΓM Γx = 0 implies x = 0. It is clear that every prime Γ- ring is also semi prime but the converse is not true in general. Also M is called a 2-torsion free if 2x = 0 implies x = 0 for every x ∈ M.

We know that the notion of Jordan k- derivation of a Γ- ring was first introduced by S. Chakraborty and A. C. Paul [15] and they proved that every Jordan k- derivation on a 2-torsion free prime Γ_N - ring M is a k- derivation on M. The generalized derivations of a Γ-ring was introduced by Y.

Ceven and M. A. Ozturk [47] and proved that every Jordan generalized derivation of a Γ-ring M is a generalized derivation of M. Rahman and Paul [7] extended the results of [47] on Lie ideals of prime Γ-rings. In [48], S. Uddin and Paul worked on simple Γ-rings with involutions and extended various results of Herstein [28] in Γ-rings. S. Chakraborty and A. C. Paul [10,11,12,13,14,15] worked on Jordan generalized k-derivations on prime Γ_N - rings, completely prime and completely semiprime Γ_N - rings and developed the various significant results on these fields.

In this chapter, we shall prove that every Jordan generalized k- derivation on a Lie ideal U of M is a generalized k-derivation on U of M.

Generalized and Jordan Generalized k- Derivation

2.1 Definition : Let M be a Γ-ring and let k : Γ → Γ be an additive mapping. An additive mapping F: M → M is called a generalized k-derivation on M if there exists a k-derivation d : M → M such that $F(x\alpha y) = F(x)\alpha y + xk(\alpha)y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$. And if $F(x\alpha x) = F(x)\alpha x + xk(\alpha)x + x\alpha d(x)$ holds for all $x \in M$ and $\alpha \in \Gamma$, then F is said to be a Jordan generalized k-derivation on M.

2.2 Example : Let M be a Γ- ring and let F be a generalized k- derivation of M.Then by definition, there exists a k-derivation d: M→M such that $d(x\alpha y) = d(x)\alpha y + xk(\alpha)y +$

$x\alpha d(y)$ and $F(x\alpha y) = F(x)\alpha y + xk(\alpha)y + x\alpha d(y)$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Let $M_1 = M \times M$ and $\Gamma_1 = \Gamma \times \Gamma$. Define the operations of addition and multiplication of M_1 and Γ_1 by $(x, y) + (z, w) = (x + z, y + w)$ and $(x, y)(\alpha, \beta)(z, w) = (x\alpha z, y\beta w)$, for every $x, y, z, w \in M$ and $\alpha, \beta \in \Gamma$. Then M_1 is obviously a Γ_1-ring under these operations.

Let $F_1 : M_1 \to M_1$, $d_1 : M_1 \to M_1$ and $k_1 : \Gamma_1 \to \Gamma_1$ be the additive mappings defined by $F_1((x, y)) = (F(x)\ F(y))$, $d_1((x,y)) = (d(x),d(y))$ and $k_1((\alpha, \beta)) = (k(\alpha), k(\beta))$ for all $x, y \in M$ and $\alpha ,\beta \in \Gamma$.Then clearly d_1 is a k_1- derivation of M_1.

put $(x,y) = a \in M_1$, $(\alpha, \beta) = \gamma \in \Gamma_1$, for any $x, y \in M$ and $\alpha ,\beta \in \Gamma$; then we have,

$F_1(a\gamma a) = F_1((x, y)(\alpha, \beta)(x , y))$

$= F_1((x\alpha x , y\beta y))$

$= (F(x\alpha x), F(y\beta y))$

$= (F(x)\alpha x + xk(\alpha)x + x\alpha d(x) , F(y)\beta y + yk(\beta)y + y\beta d(y))$

$= (F(x)\alpha x, F(y)\beta y) + (xk(\alpha)x , y k(\beta)y) + (x\alpha d(x) , y\beta d(y))$

$=(F(x), F(y))(\alpha,\beta)(x, y) + (x,y)(k(\alpha) , k(\beta))(x, y) + (x,y)(\alpha, \beta)(d(x),d(y))$

$= F_1(x, y)(\alpha, \beta)(x, y) + (x, y)k_1(\alpha , \beta)(x , y) + (x , y)(\alpha , \beta) d_1(x, y)$

$= F_1(a)\gamma a + ak_1(\gamma)a + a\gamma d_1(a)$,

which follows that F_1 is a Jordan generalized k_1- derivation of M_1 associated with the k_1-derivation d_1 of M_1.

Now we give an example of a Lie ideal which is not an ideal of a Γ-ring M.

2.3 Example : Let R be a ring and U be a Lie ideal of R.

Let M = $M_{1,\,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in \mathbb{Z} \right\}$. Then M is a Γ-ring.

Define N = {(x , x) : x ∈ R } ⊆ M . Then N is a Γ-ring.

Let U_1 = {(u, u) : u ∈ U }.

Now $(u\,,\,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(a,\,a) - (a\,,a)\begin{pmatrix} n \\ 0 \end{pmatrix}(u\,,\,u)$

=(una , una) - (anu , anu)

=(una - anu, una - anu) ∈ U_1.

Then U_1 is a Lie ideal of N. It is clear that U_1 is not an ideal of N.

2.4 Definition : Let M be a Γ-ring and let U be a Lie ideal of M. Let k: Γ → Γ be an additive mapping. An additive mapping F: M → M is called a generalized k- derivation on U of M if there exists a k- derivation d on U of M such that F(uαv) = F(u)αv + uk(α)v + uαd(v) for all u ,v ∈ U and α ∈ Γ. And if F(uαu) = d(u)αu + uk(α)u + uαd(u) , for every u ∈ U and α ∈ Γ. then F is said to be a Jordan generalized k-derivation on U of M.

2.5 Example : Let M be a Γ -ring and let U be a Lie ideal of M. Let f: M → M be a generalized k- derivation on U of M, then there exists a derivation d on U of M such that f(uαv) = f(u)αv + uk(α)v + uαd(v) for all u, v ∈ U and α ∈ Γ.

Let M_1 = {(x, x) : x ∈ M} and Γ_1 = {(α, α) : α ∈ Γ}. Define addition and multiplication of M are as follows:

(x, x) + (y, y) = (x + y, x + y) , (x, x)(α, α)(y, y) = (xαy, xαy) for all (x, x) ∈ M_1 and (α, α) ∈ Γ_1.

Under these operations M_1 is a Γ_1- ring.

Let $U_1 = \{(u, u) : u \in U\}$. Then clearly U_1 is a Lie ideal of M_1. Define $F: M_1 \to M_1$, $D: M_1 \to M_1$ and $k_1: \Gamma_1 \to \Gamma_1$ by $F((x, x)) = (f(x), f(x))$, $D((x, x)) = (d(x), d(x))$ and $k_1((\alpha, \alpha)) = (k(\alpha), k(\alpha))$, for all $x \in U$ and $\alpha \in \Gamma$.

Then $F((x, x)(\alpha, \alpha)(y, y)) = F((x\alpha y, x\alpha y))$

$= (f(x\alpha y), f(x\alpha y))$

$= (f(x)\alpha y + xk(\alpha)y + x\alpha d(y), f(x)\alpha y + xk(\alpha)y + x\alpha d(y))$

$= (f(x)\alpha y, f(x)\alpha y) + (xk(\alpha)y, xk(\alpha)y) + (x\alpha d(y), x\alpha d(y))$

$= (f(x),f(x)(\alpha,\alpha)(y,y) + (x,x)(k(\alpha),k(\alpha))(y, \quad) + (x,x)(\alpha,\alpha)(d(y), d(y))$

$= F((x, x))(\alpha, \alpha)(y, y) + (x, x)k_1(\alpha, \alpha)(y, y) + (x, x)(\alpha, \alpha)D(y, y)$.

Therefore F is a generalized k_1- derivation on U_1 of M_1.

Also $F : M \to M$ is called a Jordan generalized k-derivation on U of M if there exist a k- derivation d on U of M such that $F(u\alpha u) = F(u)\alpha u + uk(\alpha)u + u\alpha d(u)$, for every $u \in U$ and $\alpha \in \Gamma$.

2.6 Example : Let M be a Γ -ring and let U be a Lie ideal of M. Let $f: M \to M$ be a generalized k- derivation on U of M, then there exists a derivation d on U of M such that $f(u\alpha v) = f(u)\alpha v + uk(\alpha)v + u\alpha d(v)$ for all $u, v \in U$ and $\alpha \in \Gamma$.

Let $M_1 = \{(x, x) : x \in M\}$ and $\Gamma_1 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$. Define addition and multiplication of M are as follows:

$(x, x) + (y, y) = (x + y, x + y)$; $(x, x)(\alpha, \alpha)(y, y) = (x\alpha y, x\alpha y)$ for all $(x, x) \in M_1$ and $(\alpha, \alpha) \in \Gamma_1$. Under these operations M_1 is a Γ_1- ring.

Let $U_1 = \{(u, u) : u \in U\}$. Then clearly U_1 is a Lie ideal of M_1. Define $F: M_1 \to M_1$, $D: M_1 \to M_1$ and $k_1: \Gamma_1 \to \Gamma_1$ by $F((x, x)) = (f(x), f(x))$, $D((x, x)) = (d(x), d(x))$ and $k_1((\alpha, \alpha)) = (k(\alpha), k(\alpha))$ for all $x \in U$ and $\alpha \in \Gamma$.

Then $F((x, x)(\alpha, \alpha)(x, x)) = F((x\alpha x, x\alpha x))$

$= (f(x\alpha x), f(x\alpha x))$

$= (f(x)\alpha x + xk(\alpha)x + x\alpha d(x), f(x)\alpha x + xk(\alpha)x + x\alpha d(x))$

$= (f(x)\alpha y, f(x)\alpha x) + (xk(\alpha)x, xk(\alpha)x) + (x\alpha d(x), x\alpha d(x))$

$=(f(x), f(x)(\alpha, \alpha)(x, x) + (x, x)(k(\alpha), k(\alpha))(x, x) + (x, x)(\alpha, \alpha)(d(x), d(x))$

$=F((x, x))(\alpha, \alpha)(x, x)+(x, x)k_1(\alpha, \alpha)(x, x)+(x, x)(\alpha, \alpha)D(x, x)$.

Therefor, F is a Jordan generalized k- derivation on U_1 of M_1.

2.7 Lemma : Let M be a 2- torsion free Γ -ring satisfying (*) and U a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and let $F : M \to M$ be a Jordan generalized k- derivation on U, then

(i) $F(u\alpha v + v\alpha u) = F(u)\alpha v + uk(\alpha)v + u\alpha d(v) + F(v)\alpha u + vk(\alpha)u + v\alpha d(u)$.

(ii) $F(u\alpha v\beta u) = F(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u)$

(iii) $F(u\alpha v\beta w + w\alpha v\beta u) = F(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + F(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$.

Proof. We have $u\alpha v + v\alpha u = (u + v)\alpha(u + v) - u\alpha u - v\alpha v$, and the left side as like as the right side is in U. Hence

$F(u\alpha v + v\alpha u) = F((u+v)\alpha(u+v) - u\alpha u - v\alpha v)$

$= F\ (u+v)\alpha(u+v) + (u+v)k(\alpha)(u + v) + (u+v)\alpha d(u+v) - (F(u)\alpha u + uk(\alpha)u + u\alpha d(u) + F(v)\alpha v + vk(\alpha)v + v\alpha d(v)\ )$

$= F(u)\alpha u + F(u)\alpha v + F(v)\alpha u + F(v)\ \alpha v + uk(\alpha)u + uk(\alpha)v + vk(\alpha)u + vk(\alpha)v + u\alpha d(u) + u\alpha d(v) + v\alpha d(u) + v\alpha d(v) - F(u)\alpha u - uk(\alpha)u - u\alpha d(u) - F(v)\alpha v - vk(\alpha)v - v\alpha d(v).$

That implies

$F(u\alpha v + v\alpha u) = F(u)\alpha v + uk(\alpha)v + u\alpha d(v) + F(v)\alpha\ u + vk(\alpha)u + v\alpha d(u).$

Replacing v by $u\beta v + v\beta u$ we have,

$F(u\alpha(u\beta v + v\beta u) + (u\beta v + v\beta u)\alpha u) = F(u)\alpha(u\beta v+v\beta u) + uk(\alpha)(u\beta v + v\beta u) + u\alpha d(u\beta v + v\beta u) + F(u\beta v + v\beta u)\alpha u + (u\beta v + v\beta u)k(\alpha)u + (u\beta v + v\beta u)\alpha d(u)$…………….(1)

Left side of (1) is equal to

$F(u\alpha u\beta v + u\alpha v\beta u + u\beta v\alpha u + v\beta u\alpha u) = F(u\alpha v\beta u + u\beta v\alpha u) + F((u\alpha u)\beta v + v\beta(u\alpha u))$

$= F(u\alpha v\beta u + u\beta v\alpha u) + F(u\alpha u)\beta v + u\alpha uk(\beta)v + u\alpha u\beta d(v) + F(v)\beta u\alpha u + vk(\beta)u\alpha u + v\beta d(u\alpha u)$

$= F(u\alpha v\beta u + u\beta v\alpha u) + F(u)\alpha u\beta v + uk(\alpha)u\beta v + u\alpha d(u)\beta v + u\alpha uk(\beta)v + u\alpha u\beta d(v) + F(v)\beta u\alpha u + vk(\beta)u\alpha u + v\beta d(u)\alpha u + v\beta uk(\alpha)u + v\beta u\alpha d(u).$

Right side of (1) is equal to

$F(u)\alpha u\beta v + F(u)\alpha v\beta u + uk(\alpha)u\beta v + uk(\alpha)v\beta u + u\alpha d(u)\beta v + u\alpha uk(\beta)v + u\alpha u\beta d(v) + u\alpha d(v)\beta u + u\alpha vk((\beta)u + u\alpha v\beta d(u) + F(u)\beta v\alpha u + uk(\beta)v\alpha u + u\beta d(v)\alpha u + F(v)\beta u\alpha u + vk(\beta)u\alpha u + v\beta d(u)\alpha u + u\beta vk(\alpha)u + v\beta uk(\alpha)u + u\beta v\alpha d(u) + v\beta u\alpha d(u).$

Computing both sides we have,

$F(u\alpha v\beta u + u\beta v\alpha u) = F(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + F(u)\beta v\alpha u + uk(\beta)v\alpha u + u\ d(v)\alpha u + u\beta vk(\alpha)u + u\beta v\alpha d(u)$.

Putting $u\beta v\alpha u = u\alpha v\beta u$ we have ,

$F(2u\alpha v\beta u) = F(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + F(u)\alpha v\beta u + u\alpha vk(\beta)u + u\alpha d(v)\beta u + uk(\alpha)v\beta u + u\alpha v\beta d(u)$

That implies $2F(u\alpha v\beta u) = 2(F(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha\ d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u))$.

Since M is a 2- torsion free, hence we have

$F(u\alpha v\beta u) = F(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u)$.

Replace $u + w$ for u we have,

$F((u + w)\alpha v\beta(u+w)) = F(u+w)\alpha v\beta(u+w) + (u+w)k(\alpha)v\beta(u+w) + (u+w)\alpha d(v)\beta(u+w) + (u+w)\alpha vk(\beta)(u+w) + (u+w)\alpha v\beta d(u+w)$.

…………………….(2)

Left side of (2) is equal to

$F(u\alpha v\beta u + u\alpha v\beta w + w\alpha v\beta u + w\alpha v\beta w) = F(u\alpha v\beta w + w\alpha v\beta u) + F(u\alpha v\beta u) + F(w\alpha v\beta w)$

$= F(u\alpha v\beta w + w\alpha v\beta u) + F(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + F(w)\alpha v\beta w + wk(\alpha)v\beta w + w\alpha d(v)\beta w + w\alpha vk(\beta)w + w\alpha v\beta d(w)$.

Right side of (2) is equal to

$F(u)\alpha v\beta u + F(w)\alpha v\beta u + F(u)\alpha v\beta w + F(w)\alpha v\beta w + uk(\alpha)v\beta u + wk(\alpha)v\beta u + uk(\alpha)v\beta w + wk(\alpha)v\beta w + u\alpha d(v)\beta u + w\alpha d(v)\beta u + u\alpha d(v)\beta w + w\alpha d(v)\beta w + u\alpha vk(\beta)u + w\alpha vk(\beta)u + u\alpha vk(\beta)w + w\alpha vk(\beta)w + u\alpha v\beta d(u) + u\alpha v\beta d(w) + w\alpha v\beta d(u) + w\alpha v\beta d(w)$.

Comparing both sides we get,

$F(u\alpha v\beta w + w\alpha v\beta u) = F(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + F(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$.

2.8 Definition : We define $\psi_\alpha(u, v) = F(u\alpha v) - F(u)\alpha v - uk(\alpha)v - u\alpha d(v)$ for all $u, v \in U$ and $\alpha \in \Gamma$.

2.9 Remark : It is clear that F is a generalized k- derivation if and only if $\psi_\alpha(u, v) = 0$.

2.10 Lemma : Let M, U and F be as in above. Then for all u, v, $w \in U$ and $\alpha, \beta \in \Gamma$, the following relations hold :

(i) $\psi_\alpha(u, v) + \psi_\alpha(v, u) = 0$
(ii) $\psi_\alpha(u + w, v) = \psi_\alpha(u, v) + \psi_\alpha(w, v)$
(iii) $\psi_\alpha(u, v + w) = \psi_\alpha(u, v) + \psi_\alpha(u, w)$
(iv) $\psi_{\alpha+\beta}(u, v) = \psi_\alpha(u, v) + \psi_\beta(u, v)$.

2.11 Lemma : Let M, U, F and d be defined as in above, then for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$, $\psi_\alpha(u, v)\beta w\gamma[u, v]_\alpha = 0$.

Proof : Consider $A = (2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v)$.

From Lemma 2.7 (iii) we have,

$F(A) = F((2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v))$

$=F(2u\alpha v)\beta w\gamma(2v\alpha u) + 2u\alpha vk(\beta)w\gamma(2v\alpha u) + (2u\alpha v)\beta d(w)\gamma(2v\alpha u) + (2u\alpha v)\beta wk(\gamma)(2v\alpha u) + (2u\alpha v)\beta w\gamma d(2v\alpha u) + F(2v\alpha u)\beta w\gamma(2u\alpha v) + (2v\alpha u)k(\beta)w\gamma(2u\alpha v) + (2v\alpha u)\beta d(w)\gamma(2u\alpha v) + (2v\alpha u)\beta wk(\gamma)(2u\alpha v) + (2v\alpha u)\beta w\gamma d(2u\alpha v)$

$=4[F(u\alpha v)\beta w\gamma(v\alpha u) + u\alpha vk(\beta)wv\alpha u + u\alpha v\beta d(w)\gamma v\alpha u + u\alpha v\beta wk(\gamma)v\alpha u + u\alpha v\beta w\gamma d(v\alpha u) + F(v\alpha u)\beta w\gamma u\alpha v +$

$v\alpha uk(\beta)w\gamma u\alpha v + v\alpha u\beta d(w)\gamma u\alpha v + v\alpha u\beta wk(\gamma)u\alpha v + v\alpha u\beta w\gamma d(u\alpha v)]$

Again $A = (2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v) = u\alpha(4v\beta w\gamma v)\alpha u + v\alpha(4u\beta w\gamma u)\alpha v$

That implies

$F(A) = F(u\alpha(4v\beta w\gamma v)\alpha u + v\alpha(4u\beta w\gamma u)\alpha v$

$=4[F(u)\alpha v\beta w\gamma v\alpha u + uk(\alpha)v\beta w\gamma v\alpha u + u\alpha d(v\beta w\gamma v)\alpha u + u\alpha v\beta w\gamma vk(\alpha)u + u\alpha v\beta w\gamma v\alpha d(u) + F(v)\alpha u\beta w\gamma u\alpha v + vk(\alpha)u\beta w\gamma u\alpha v+v\alpha d(u\beta w\gamma u)\alpha v+v\alpha u\beta w\gamma uk(\alpha)v+v\alpha u\beta w\gamma u\alpha d(v)]$ {using Lemma 2.4 (ii)}

$= 4[F(u)\alpha v\beta w\gamma v\alpha u + uk(\alpha)v\beta w\gamma v\alpha u + u\alpha d(v)\beta w\gamma v\alpha u + u\alpha vk(\beta)w\gamma v\alpha u + u\alpha v\beta d(w)\gamma v\alpha u + u\alpha v\beta wk(\gamma)v\alpha u + u\alpha v\beta w\gamma d(v)\alpha u + u\alpha v\beta w\gamma vk(\alpha)u + u\alpha v\beta w\gamma v\alpha d(u) + F(v)\alpha u\beta w\gamma u\alpha v + vk(\alpha)u\beta w\gamma u\alpha v + v\alpha d(u)\beta w\gamma u\alpha v + v\alpha uk(\beta)w\gamma u\alpha v + v\alpha u\beta d(w)\gamma u\alpha v + v\alpha u\beta wk(\gamma)u\alpha v + v\alpha u\beta w\gamma d(u)\alpha v +v\alpha u\beta w\gamma uk(\alpha)v + v\alpha u\,\beta w\gamma u\alpha d(v)]$

Comparing both expressions we have,

$4[F(u\alpha v)\beta w\gamma v\alpha u + F(v\alpha u)\beta w\gamma u\alpha v + u\alpha v\beta w\gamma d(v\alpha u) + v\alpha u\beta w\gamma d(u\alpha v)]$

$= 4[F(u)\alpha v\beta w\gamma v\alpha u + uk(\alpha)v\beta w\gamma v\alpha u + u\alpha d(v)\beta w\gamma v\alpha u + u\alpha v\beta w\gamma d(v)\alpha u + u\alpha v\beta w\gamma vk(\alpha)u + u\alpha v\beta w\gamma v\alpha d(u) + F(v)\alpha u\beta w\gamma u\alpha v + vk(\alpha)u\beta w\gamma u\alpha v + v\alpha d(u)\beta w\gamma u\alpha v + v\alpha u\beta w\gamma d(u)\alpha v + v\alpha u\beta w\gamma uk(\alpha)v + v\alpha u\beta w\gamma u\alpha d(v)]$

Since M is a 2- torsion free , we have

$0 = [F(u\alpha v) - F(u)\alpha v - uk(\alpha)v - u\alpha d(v)]\beta w\gamma v\alpha u + [F(v\alpha u) - F(v)\alpha u - vk(\alpha)u -v\alpha d(u)]\ \beta w\gamma u\alpha v + u\alpha v\beta w\gamma[d(v\alpha u) - (d(v)\alpha u + vk(\alpha)u + v\alpha d(u))] + v\alpha u\beta w\gamma[d(u\alpha v) - (d(u)\alpha v + uk(\alpha)v + u\alpha d(v)]$

$= \psi_\alpha(u, v)\beta w\gamma v\alpha u + \psi_\alpha(v, u)\beta w\gamma u\alpha v + [d(v\alpha u) - d(v\alpha u)] + [d(u\alpha v) - d(u\alpha v)]$

$= \psi_\alpha(u, v)\beta w\gamma v\alpha u - \psi_\alpha(u, v)\beta w\gamma u\alpha v$

$= - \psi_\alpha(u, v)\beta w\gamma(u\alpha v - v\alpha u)$

That implies $\psi_\alpha(u, v)\beta w\gamma[u, v]_\alpha = 0$.

2.12 Lemma: Let $U \not\subset Z(M)$ be a Lie ideal of a 2- torsion free prime Γ- ring M. Then $[u, v]_\alpha\beta w\gamma\psi_\alpha(u, v) = 0$.

Proof. From Lemma 2.11 we have, $\psi_\alpha(u, v)\delta x\mu[u, v]_\alpha = 0$

That implies $[u, v]_\alpha\beta w\gamma\psi_\alpha(u, v)\delta x\mu [u, v]_\alpha\beta w\gamma \psi_\alpha(u, v) = 0$, for all $x \in U$.

In view of Lemma 1.29, we have $[u, v]_\alpha\beta w\gamma \psi_\alpha(u, v) = 0$.

2.13 Lemma: Let $U \not\subset Z(M)$ be a Lie ideal of a 2- torsion free prime Γ - ring M. Then $\psi_\alpha(u, v) \beta w\gamma [x, y]_\delta = 0$ for all u, v, w, x, y $\in$ U and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Proof. From Lemma 2.11 we have,

$0 = \psi_\alpha(u + x, v) \beta w\gamma [u + x, v]_\alpha$

$= \psi_\alpha(u, v)\beta w\gamma[u, v]_\alpha + \psi_\alpha(u, v)\beta w\gamma[x, v]_\alpha + \psi_\alpha(x, v)\beta w\gamma[u, v]_\alpha + \psi_\alpha(x, v)\beta w\gamma[x, v]_\alpha$

$= \psi_\alpha(u, v) \beta w\gamma [x, v]_\alpha + \psi_\alpha(x, v) \beta w\gamma [u, v]_\alpha$

$= \psi_\alpha(u, v) \beta w\gamma [x, v]_\alpha = - \psi_\alpha(x, v) \beta w\gamma [u, v]_\alpha$

$=(\psi_\alpha(u, v) \beta w\gamma [x, v]_\alpha)\delta p\theta \psi_\alpha(u, v) \beta w\gamma [x, v]_\alpha$

$= - \psi_\alpha(x, v) \beta w\gamma [u, v]_\alpha \delta p\theta \psi_\alpha(u, v) \beta w\gamma [x, v]_\alpha$

$= - \psi_\alpha(x, v) \beta w\gamma([u, v]_\alpha \delta p\theta \psi_\alpha(u, v))\beta w\gamma [x, v]_\alpha$

[by Lemma 2.11]

By Lemma 1.29, we have $\psi_\alpha(u, v) \beta w\gamma [x, v]_\alpha = 0$

Likewise by replacing v + y for v we get

$\psi_\alpha(u , v) \beta w\gamma [x , y]_\alpha = 0$……………(i)

Proceeding in the same way as above , by the similar replacement in the result, we have

$[x, y]_\alpha \beta w\gamma \psi_\alpha(u , v) = 0$ ………..(ii)

Now putting $\alpha+\delta$ for α in (i) we have,

$\psi_{\alpha+\delta}(u , v)\beta w\gamma [x, y]_{\alpha+\delta} = 0$

Then $0 = \psi_\alpha(u, v)\beta w\gamma [x, y]_\alpha + \psi_\delta(u, v)\beta w\gamma [x, y]_\alpha + \psi_\alpha(u, v)\beta w\gamma[x, y]_\delta + \psi_\delta(u , v) \beta w\gamma [x ,y]_\delta$

$= \psi_\delta(u , v) \beta w\gamma [x , y]_\alpha + \psi_\alpha(u , v) \beta w\gamma [x , y]_\delta$

That implies $\psi_\alpha(u , v) \beta w\gamma [x , y]_\delta = - \psi_\delta(u , v) \beta w\gamma [x ,y]_\alpha$

Therfore, $\psi_\alpha(u , v) \beta w\gamma [x , y]_\delta \theta q\mu \psi_\alpha(u , v) \beta w\gamma [x , y]_\delta$

$= - \psi_\delta(u , v)\beta w\gamma([x , y]_\alpha \theta q\mu\psi_\alpha(u , v)) \beta w\gamma [x , y]_\delta$

$= 0$ by (ii)

Using Lemma 1.29, $\psi_\alpha(u , v)\beta w\gamma[x, y]_\delta = 0$.

2.14 Lemma : Let $U \not\subset Z(M)$ be a Lie ideal of a 2- torsion free prime Γ- ring M. Then $\psi_\alpha(u ,v) \in Z(U) = Z(M)$ for every $u, v \in U$.

Proof : We have $\psi_\alpha(u , v) \beta w\gamma([x , y]_\delta = 0$

Now $2[\psi_\alpha(u, v) , c]_\delta \beta w\gamma[\psi_\alpha(u , v), c]_\delta$

$= 2(\psi_\alpha(u, v)\delta c - c\delta \psi_\alpha(u , v))\beta w\gamma [\psi_\alpha(u , v), c]_\delta$

$=\psi_\alpha(u, v)\delta(2c\beta w)\gamma[\psi_\alpha(u,v), c]_\delta -2c\delta \psi_\alpha(u, v)\beta w\gamma[\psi_\alpha(u, v), c]_\delta$

$= 0$, for every $c \in U$.

In view of Lemma 1.29, we have $[\psi_\alpha(u , v), c]_\delta = 0$,

Hence $\psi_\alpha(u , v) \in Z(U)$ and that implies $\psi_\alpha(u , v) \in Z(M)$ by Lemma 1.26.

2.15 Lemma : Let M be a 2- torsion free prime Γ- ring satisfying the condition (*) and U a Lie ideal of M. Let $u \in U$ be such that $[u, [u,x]_{\alpha}]_{\alpha} = 0$ for all $x \in M$ and $\alpha \in \Gamma$. Then $[u, x]_{\alpha} = 0$.

Proof. We have $[u, [u,x]_{\alpha}]_{\alpha} = 0$ for all $x \in M$ and $\alpha \in \Gamma$. Let $y \in M$, then $x\alpha y \in M$ for all $\alpha \in \Gamma$.

Replace x by $x\alpha y$ we have, $0 = [u, [u, x\alpha y]_{\alpha}]_{\alpha}$

$= [u, (x\alpha[u,y]_{\alpha} + [u, x]_{\alpha}\alpha y)]_{\alpha}$

$= [u, x\alpha[u,y]_{\alpha}]_{\alpha} + [u, [u, x]_{\alpha}\alpha y]_{\alpha}$

$= x\alpha[u, [u, y]_{\alpha}]_{\alpha} + [u,x]_{\alpha}\alpha[u,y]_{\alpha} + [u,x]_{\alpha}\alpha[u,y]_{\alpha} + [u, [u, x]_{\alpha}]_{\alpha}\alpha y$

Hence $2[u, x]_{\alpha}\alpha[u,y]_{\alpha} = 0$

Since M is 2 - torsion free, we have $[u,x]_{\alpha}\alpha[u,y]_{\alpha} = 0$

Putting $y = u\beta x$, we have $[u, x]_{\alpha}\alpha[u, u\beta x]_{\alpha} = 0$

Then $[u, x]_{\alpha}\alpha u\beta[u,x]_{\alpha} = 0$ by using (*).

Hence by Lemma 1.29, we have $[u, x]_{\alpha} = 0$.

2.16 Lemma : Let M be a 2-torsion free prime Γ- ring satisfying the condition (*) and U be a commutative Lie ideal of M, then $U \subseteq Z(M)$.

Proof : Since U is commutative ,we have $[u,v]_{\alpha} = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$. Also we have $[u, x]_{\alpha} \in U$ for all $u \in U$, $x \in M$ and $\alpha \in \Gamma$.

Replacing v by $[u, x]_{\alpha}$, we obtain $[u, [u, x]_{\alpha}]_{\alpha} = 0$.

By Lemma 2.15 we have $[u, x]_{\alpha} = 0$. Hence $U \subseteq Z(M)$.

2.17 Theorem : Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and let U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and if $F : M \to M$ is a Jordan generalized k-derivation on U of M then, $\psi_\alpha(u, v) = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$.

Proof: Let U be a commutative Lie ideal of M. Then by Lemma 2.16, $U \subseteq Z(M)$.

Since U is commutative, then we have $[v, w]_\beta = 0$.

That implies $v\beta w = w\beta v$, for every $v, w \in U$, $\alpha \in \Gamma$.

From Lemma 2.7 (iii) we have,

$F(u\alpha v\beta w + w\alpha v\beta u) = F(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + F(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$ ……………(1)

Putting $u = 2v\beta w$ in (1), we have

$L.S. = F(2v\beta w\alpha v\beta w + w\alpha v\beta 2v\beta w)$

$= 2F(v\beta w\alpha v\beta w + w\beta v\alpha v\beta w)$

$= 2F(v\beta w\alpha v\beta w + v\beta w\alpha v\beta w)$

$= 4F((v\beta w)\alpha(v\beta w))$

$= 4(F(v\beta w)\alpha(v\beta w) + v\beta wk(\alpha)v\beta w + v\beta w\alpha d(v\beta w))$

Also $R.S. = 2F(v\beta w)\alpha v\beta w + 2v\beta wk(\alpha)v\beta w + 2v\beta w\alpha d(v)\beta w + 2v\beta w\alpha vk(\beta)w + 2v\beta w\alpha v\beta d(w) + F(w)\alpha v\beta 2v\beta w + wk(\alpha)v\beta 2v\beta w + w\alpha d(v)\beta 2v\beta w + w\alpha vk(\beta)2v\beta w + w\alpha v\beta d(2v\beta w)$

$= 2F(v\beta w)\alpha v\beta w + 2v\beta wk(\alpha)v\beta w + 2v\beta w\alpha(d(v)\beta w + vk(\beta)w + v\beta d(w)) + 2F(w)\alpha v\beta v\beta w + 2wk(\alpha)v\beta v\beta w + 2w\alpha d(v)\beta v\beta w + 2w\alpha vk(\beta)v\beta w + 2w\alpha v\beta d(v\beta w)$

$= 2F(v\beta w)\alpha v\beta w + 2v\beta wk(\alpha)v\beta w + 4v\beta w\alpha d(v\beta w) + 2F(w)\beta v\alpha v\beta w + 2wk(\alpha)v\beta v\beta w + 2w\alpha d(v)\beta v\beta w + 2w\alpha vk(\beta)v\beta w$

Comparing both sides we get

$0 = 2F(v\beta w)\alpha v\beta w + 2v\beta wk(\alpha)v\beta w - 2F(w)\beta v\alpha v\beta w - 2wk(\beta)v\alpha v\beta w - 2w\beta d(v)\alpha v\beta w - 2v\beta wk(\alpha)v\beta w$

$= 2(F(w\beta v) - F(w)\beta v - wk(\beta)v - w\beta d(v))\alpha v\beta w$

$= 2\psi_\beta(w, v)\alpha v\beta w$

Since M is 2- torsion free ,we get $\psi_\beta(w , v) \alpha v \beta w = 0$.

Then $0 = \psi_\beta(w , v) \alpha v \beta w\gamma x\delta y$, where $x \in U$, $y \in M$

$= \psi_\beta(w , v) \alpha x\beta y\gamma v\delta w$

$= (\psi_\beta(w , v) \alpha x\gamma y)\beta v\delta w$ using (*)

From Lemma 1.29 either $\psi_\beta(w , v) \alpha x\gamma y = 0$ or $w = 0$.

Since $w \in U$, $w \neq 0$, hence $\psi_\beta(w , v) \alpha x\gamma y = 0$.

That implies $\psi_\beta(w , v)\alpha U\gamma y = 0$

Using Lemma 1.29 we have , $\psi_\beta(w , v) = 0$.

Again if U is not commutative , i.e., $U \not\subset Z(M)$, then from Lemma 2.13 we have, $\psi_\alpha(u , v)\beta w\gamma[x, y]_\delta = 0$.

But $[x, y]_\delta = 0$ implies $U \subseteq Z(M)$, a contradiction .

Hence $\psi_\alpha(u , v) = 0$ for all $u ,v \in U$ and $\alpha \in \Gamma$.

2.18 Corollary : Every Jordan generalized k-derivation of a 2 -torsion free prime Γ- ring M is a generalized k- derivation on M.

Jordan k- Derivations on Lie Ideals of Semiprime Γ- Rings

In this chapter, we obtain some characterizations of semiprime Γ- rings with Lie ideals . By using these results , we prove that every Jordan k-derivation on a Lie ideal U of M is a k- derivation on U of M , where M is a 2- torsion free semiprime Γ- ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all a, b , c ∈ M and α , β ∈ Γ and the Lie ideal U of M is such that $u\alpha u \in U$ for all u ∈ U and α ∈ Γ.

3. Introduction: We know that every k-derivation is a Jordan k-derivation but the converse is not true always. In the first chapter we have proved that every Jordan k-derivation on a Lie ideal U of a 2- torsion free prime Γ- ring M is a k-derivation on U of M. The same results were also proved in the second chapter for the case of generalized k- derivation on U of M.

Let M be a 2-torsion free semiprime Γ- ring and U a Lie ideal of M. First we prove some properties of M with Lie ideal U and then using these, we prove that every Jordan k-derivation on U of M is a k- derivation on U of M where the Lie ideal U is such that $u\alpha u \in U$ for all u ∈U and α ∈ Γ.

Lie Ideals of Semiprime Γ- Rings :

In this section we denote M to be a 2- torsion free semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M .

3.1 Lemma : If $U \neq 0$ is a Γ- sub ring of M , then either $U \subseteq Z(M)$ or U contains a nonzero ideal of M .

Proof: First we assume that U is not commutative , Then for some $u, v \in U$ and $\alpha \in \Gamma$ we have $[u, v]_\alpha \neq 0$ and also $[u, v]_\alpha \in U$.

Therefore, the ideal J of M generated by $[u, v]_\alpha$ is nonzero and $J \subseteq U$.

On the other hand, let us assume that U is commutative .Then for every $u \in U$, we have $[u, [u, x]_\alpha]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$.

Hence by Lemma 2.15 we have $[u, x]_\alpha = 0$.

This shows that $U \subseteq Z(M)$.

3.2 Lemma : $T(U) = \{x \in M: [x, M]_\Gamma \subseteq U\}$ is both a Γ- sub ring and a Lie ideal of M such that $U \subseteq T(U)$.

Proof : We have U is a Lie ideal of M , so $[U, M]_\Gamma \subseteq U$. Thus $U \subseteq T(U)$. Also we have $[T(U), M]_\Gamma \subseteq U \subseteq T(U)$. Hence T(U) is a Lie ideal of M.

Now suppose that $x, y \in T(U)$, then $[x, m]_\alpha \in U$ and $[y, m]_\alpha \in U$ for all $m \in M$ and $\alpha \in \Gamma$.

So that $[x\alpha y, m]_\beta = x\alpha[y, m]_\beta + [x, m]_\beta \alpha y \in U$. Therefore , $[x\alpha y, m]_\beta \in U$ for all $x, y \in T(U)$; $m \in M$ and $\alpha, \beta \in \Gamma$. Hence $x\alpha y \in T(U)$.

3.3 Lemma : Let $U \not\subset Z(M)$, then there exists a nonzero ideal $K = M\Gamma[U,U]_\Gamma \Gamma M$ of M generated by $[U, U]_\Gamma$ such that $[K, M]_\Gamma \subseteq U$.

Proof : First we prove that if $[U, U]_\Gamma = 0$, then $U \subseteq Z(M)$, So let $[U, U]_\Gamma = 0$, then for all $u \in U$ and $\alpha \in \Gamma$, we have $[u, [u, x]_\alpha]_\alpha = 0$ for all $x \in M$. For all $z \in M$ and $\beta \in \Gamma$, we replace x by $x\beta z$ in $[u, [u, x]_\alpha]_\alpha = 0$ and obtain $[u, [u, x\beta z]_\alpha]_\alpha = 0$

That is $0 = [u, x\beta[u, z]_\alpha + [u, x]_\alpha\beta z]_\alpha$

$= [u, x\beta[u, z]_\alpha]_\alpha + [u,[u, x]_\alpha\beta z]_\alpha$

$= x\beta[u, [u, z]_\alpha]_\alpha + [u, x]_\alpha\beta[u,z]_\alpha + [u, [u, x]_\alpha]_\alpha\beta z + [u, x]_\alpha\beta[u, z]_\alpha$

That implies $2[u, x]_\alpha\beta[u, z]_\alpha = 0$

By the 2- torsion freeness of M , we obtain $[u, x]_\alpha\beta[u, z]_\alpha = 0$

Now replacing z by zγx , we get

$$\begin{aligned}0 &= [u, x]_\alpha\beta[u, z\gamma x]_\alpha \\ &=[u, x]_\alpha\beta(z\gamma[u, x]_\alpha + [u, z]_\alpha\gamma x) \\ &= [u, x]_\alpha\beta z\gamma[u, x]_\alpha + [u, x]_\alpha\beta[u, z]_\alpha\gamma x \\ &= [u, x]_\alpha\beta z\gamma[u, x]_\alpha\end{aligned}$$

That is $[u, x]_\alpha\beta M\gamma[u, x]_\alpha = 0$.

Since M is semi prime, $[u, x]_\alpha = 0$. This implies that $u \in Z(M)$. Thus $U \subseteq Z(M)$, a contradiction. So let $[u, U]_\Gamma \neq 0$. Then $K = M\Gamma[U, U]_\Gamma\Gamma M$ is a non zero ideal of M generated by $[U, U]_\Gamma$. Let $x, y \in U$, $m \in M$ and $\alpha, \beta \in \Gamma$, we have

$[x, y\beta m]_\alpha$, y, $[x, x]_\alpha \in U \subseteq T(U)$. Hence by Lemma 3.2,

$[x, y]_\alpha\beta m = [x, y\beta m]_\alpha - y\beta[x, m]_\alpha \in T(U)$.

Also we can show that $m\beta[x,y]_\alpha \in T(U)$.

Therefore , we obtain $[[U, U]_\Gamma, M]_\Gamma \subseteq U$.

That is $[[[x, y]_\alpha , m]_\alpha , s]_\alpha , t]_\alpha \in U$, for all $m, s, t \in M$ and $\alpha \in \Gamma$.

Hence

$[x,y]_\alpha\alpha m\alpha s - m\alpha[x,y]_\alpha\alpha s + [s,m]_\alpha\alpha[x,y]_\alpha - [[s\alpha[x,y]_\alpha,m]_\alpha, t]_\alpha \in U$.

Since, $[x, y]_\alpha \alpha m\alpha s$, $s\alpha[x, y]_\alpha$, $[s, m]_\alpha\alpha[x, y]_\alpha \in T(U)$, we have $[m\alpha[x, y]_\alpha\alpha s, t]_\alpha \in U$ for all $m, s, t \in M$ and $\alpha \in \Gamma$. Hence $[K, M]_\Gamma \subseteq U$.

3.4 Lemma : Let $U \not\subset Z(M)$ and $a \in U$. If $a\alpha U\beta a = \{0\}$ for all $\alpha, \beta \in \Gamma$, then $a\alpha a = 0$ and there exists a non zero ideal $K = M\Gamma[U,U]_\Gamma\Gamma M$ of M generated by $[U,U]_\Gamma$ such that $[K,M]_\Gamma \subseteq U$ and $K\Gamma a = a\Gamma K = \{0\}$.

Proof : If $a\alpha U\beta a = 0$ for all $\alpha, \beta \in \Gamma$, then

$a\alpha[a, a\delta m]_\alpha\beta a = 0$ for all $m \in M$; $\alpha,\beta,\delta \in \Gamma$.

Therefore, $a\alpha(a\alpha a\delta m - a\delta m\alpha a)\beta a = 0$.

That implies $0 = (a\alpha a\alpha a\delta m - a\alpha a\delta m\alpha a)\beta a$

$= a\alpha a\delta a\alpha m\beta a - a\alpha a\delta m\beta a\alpha a$

$= (a\alpha a)\delta m\beta(a\alpha a)$, since $a\alpha a\delta a = 0$

Therefore, $a\alpha a = 0$, since M is semi prime.

Now we obtain

$a\alpha[k\gamma a,m]_\mu\alpha u\beta a = 0$ for all $k \in K$, $m \in M$, $u \in U$ and $\gamma \in \Gamma$.

This implies that

$0 = a\alpha(k\gamma a\mu m - m\mu k\gamma a)\alpha u\beta a$

$= a\alpha k\gamma a\mu m\alpha u\beta a - a\alpha m\mu k\gamma a\alpha u\beta a$

$= a\alpha k\gamma a\mu m\beta u\alpha a$, using (*) and $a\alpha u\beta a = 0$.

Hence we have $0 = a\alpha k\gamma a\mu m\beta[k, a]_\gamma\alpha a$, put $u = [k, a]_\gamma$

$= a\alpha k\gamma a\mu m\beta\,(k\gamma a - a\gamma k)\alpha a$

$= a\alpha k\gamma a\mu m\beta k\gamma a\alpha a - a\alpha k\gamma a\mu m\beta a\gamma k\alpha a$

$= a\alpha k\gamma a\mu m\beta a\gamma k\alpha a$, since $a\alpha a = 0$

$= (a\alpha k\gamma a)\mu m\beta(a\alpha k\gamma a)$ using(*)

Since M is semiprime, $a\alpha k\gamma a = 0$. Thus we find that $(a\alpha k)\gamma M\beta(a\alpha k) = 0$.

Hence $a\alpha k = 0$, for all $k \in K$ i.e., $a\alpha K = \{0\}$.

Similarly we obtain $K\alpha a = \{0\}$.

3.5 Lemma : Let $U \not\subset Z(M)$, $a \in U$ and let α, $\beta \in \Gamma$, then the following:

(i) If $a\alpha U\beta a = \{0\}$, then $a = 0$

(ii) If $a\alpha U = \{0\}$(or $U\alpha a = \{0\}$), then $a = 0$

(iv) If $u\alpha u \in U$ for all $u \in U$ and $a\alpha U\beta b = \{0\}$, then $a\alpha b = 0$ and $b\alpha a = 0$ for all $\alpha \in \Gamma$.

Proof : (i) By Lemma 3.4, we have $K\alpha a = M\Gamma[U,U]_\Gamma\Gamma M\alpha a = \{0\}$ and $a\alpha a = 0$ for all $\alpha \in \Gamma$. Therefore for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, we get

$$0 = [[a, x]_\alpha, a]_\gamma\beta y\alpha a$$

$$= [a\alpha x - x\alpha a, a]_\gamma\beta y\alpha a$$

$$= a\alpha[x,a]_\gamma\beta y\alpha a - [x,a]_\gamma\alpha a\beta y\alpha a$$

$$= a\alpha x\gamma a\beta y\alpha a - a\alpha a\gamma x\beta y\alpha a - x\gamma a\alpha a\beta y\alpha a + a\gamma x\alpha a\beta y\alpha a$$

$$= a\alpha x\gamma a\beta y\alpha a + a\gamma x\alpha a\beta y\alpha a$$

$$= a\alpha x\gamma a\beta y\alpha a + a\alpha x\gamma a\beta y\alpha a \quad \text{using } (*)$$

$$= 2a\alpha x\gamma a\beta y\alpha a$$

By the 2- torsion freeness of M we have $a\alpha x\gamma a\beta y\alpha a = 0$

Therefore we obtain that, $a\alpha x\gamma a\beta y\alpha a\delta x\gamma a = 0$

That implies $(a\alpha x\gamma a)\beta y\delta(a\alpha x\gamma a) = 0$, using (*)

and we have $(a\alpha x\gamma a)\beta M\delta(a\alpha x\gamma a) = 0$

Since M is semi prime $a\alpha x\gamma a = 0$ for all $x \in M$ and $\alpha, \gamma \in \Gamma$.

Again using the semi primeness of M we get $a = 0$.

(ii) If $a\alpha U = \{0\}$, then $a\alpha U\beta a = \{0\}$ for all $\beta \in \Gamma$. Therefore by (i) we obtain $a = 0$. Similarly if $U\alpha a = \{0\}$, then $a = 0$.

(iii) If $a\alpha U\beta b = \{0\}$, then we have $(b\gamma a)\alpha U\beta(b\gamma a) = 0$ and hence by (i) $b\gamma a = 0$ for all $\gamma \in \Gamma$. Also $(a\gamma b)\alpha U\beta(a\gamma b) \subseteq a\alpha U\beta b = 0$ and hence $a\gamma b = 0$.

Jordan k- Derivations on Lie Ideals of Semiprime Γ-Rings

From Lemma 1.21 we have the following :

Let d : M → M be a Jordan k-derivation on U of M, then

(i) $d(u\alpha v + v\alpha u) = d(u)\alpha v + uk(\alpha)v + u\alpha d(v) + d(v)\alpha u + vk(\alpha)u + v\alpha d(u)$

(ii) $d(u\alpha v\beta u) = d(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u)$

(iii) $d(u\alpha v\beta w + w\alpha v\beta u) = d(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + d(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$.

3.6 Remark : If U is commutative , then by Lemma 2.16 , $U \subseteq Z(M)$. Therefore, from Lemma 1.21 (i) , $\varphi_\alpha(u,v) = 0$. So we consider $U \not\subset Z(M)$.

3.7 Lemma : If $U \subseteq Z(M)$ and d is a Jordan k-derivation on U of M, then $[v, w]_\alpha \beta \varphi_\alpha(v, w) = 0$.

Proof : For any v, w ∈ U , vαw + wαv ∈ U and vαw - wαv ∈ U, as U is a Lie ideal . Hence we have 2vαw ∈ U. From lemma 1.16(iii) we have,

$d(u\alpha v\beta w + w\alpha v\beta u) = d(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + d(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$.

Putting 2vαw for u we get ,

$d((2v\alpha w)\alpha v\beta w + w\alpha v\beta(2v\alpha w)) = d(2v\alpha w)\alpha v\beta w + 2v\alpha wk(\alpha)v\beta w + 2v\alpha w\alpha d(v)\beta w + 2v\alpha w\alpha vk(\beta)w + 2v\alpha w\alpha v\beta d(w) + d(w)\alpha v\beta 2v\alpha w + wk(\alpha)v\beta 2v\alpha w + w\alpha d(v)\beta 2v\alpha w + w\alpha vk(\beta)2v\alpha w + w\alpha v\beta d(2v\alpha w)$

L.S. $= d(2(v\alpha w\beta v\alpha w + w\alpha v\beta v\alpha w))$

$= 2d(v\alpha w\beta v\alpha w) + 2d(w\alpha(v\beta v)\alpha w)$ (using (*))

$= 2[d(v\alpha w)\beta(v\alpha w)+(v\alpha w)k(\beta)(v\alpha w) + (v\alpha w)\beta d(v\alpha w) + d(w)\alpha(v\beta v)\alpha w + wk(\alpha)(v\beta v)\alpha w + w\alpha\,(d(v)\,\beta v + vk(\beta)v + v\beta d(v))\alpha w + w\alpha(v\beta v)k(\alpha)w + w\alpha(v\beta v)\alpha d(w)$

$=2[d(v\alpha w)\beta v\alpha w+v\alpha wk(\beta)v\alpha w + v\alpha w\beta d(v\alpha w) + d(w)\alpha v\beta v\alpha w+wk(\alpha)v\beta v\alpha w+ w\alpha d(v)\beta v\alpha w + w\alpha vk(\beta)v\alpha w + w\alpha v\beta d(v)\alpha w + w\alpha v\beta vk(\alpha)w + w\alpha v\beta v\alpha d(w)]$

Also R. S. = 2 $[d(v\alpha w)\alpha v\beta w + v\alpha wk(\alpha)v\beta w + v\alpha w\alpha d(v)\beta w + v\alpha w\alpha vk(\beta)w + v\alpha w\alpha v\beta d(w) + d(w)\alpha v\beta v\alpha w + wk(\alpha)v\beta v\alpha w + w\alpha d(v)\beta v\alpha w + w\alpha vk(\beta) v\alpha w + w\alpha v\beta d(v\alpha w)]$

Comparing both sides and using the condition (*) we get

$2[d(v\alpha w)\beta v\alpha w - d(v\alpha w)\beta v\alpha w + v\alpha w\beta d(v\alpha w) - w\alpha v\beta d(v\alpha w) + v\alpha wk(\beta)v\alpha w + w\alpha v\beta d(v)\alpha w + w\alpha v\beta vk(\alpha)w + w\alpha v\beta v\alpha d(w) - v\alpha wk(\alpha)v\beta w - v\alpha w\alpha d(v)\beta w - v\alpha w\alpha vk(\beta)w - v\alpha w\alpha v\beta d(w)] = 0.$

Since M is 2- torsion free , we have

$(v\alpha w - w\alpha v)\beta d(v\alpha w) + (w\alpha v - v\alpha w)\beta d(v)\alpha w + (w\alpha v - v\alpha w)k(\alpha)v\beta w + (w\alpha v - v\alpha w)\beta v\alpha d(w) = 0$

That implies $(v\alpha w - w\alpha v)\, \beta(d(v\alpha w) - d(v)\alpha w - vk(\alpha)w - v\alpha d(w)) = 0$

Therefore , $[v, w]_\alpha \beta \varphi_\alpha(v, w) = 0$.

Similarly we can show that $\varphi_\alpha(v, w)\beta[v, w]_\alpha = 0$

3.8 Lemma : If M is a semiprime Γ-ring , then $\varphi_\alpha(u, v) \in Z(U)$.

Proof : We have from Lemma 3.7

$[\varphi_\alpha(u, v),[u, v]_\alpha]_\beta = \varphi_\alpha(u, v)\beta[u, v]_\alpha - [u, v]_\alpha\beta\varphi_\alpha(u, v) = 0$

Therefore , $\varphi_\alpha(u, v) \in Z([U, U]_\Gamma) = Z(U)$. [by Lemma 1.27]

3.9 Lemma : If M is a semiprime Γ-ring**,** then

$\varphi_\alpha(x, y)\beta w\gamma[u,v]_\delta = 0$ for all $x, y, w, u, v \in U$ and $\alpha,\beta,\gamma,\delta \in \Gamma$

Proof: We have $\varphi_\alpha(u,v)\beta[u,v]_\alpha = 0$

That is $0 = w\beta\varphi_\alpha(u, v)\gamma[u, v]_\alpha$, for all $w \in U$

$= \varphi_\alpha(u, v)\beta w\gamma[u, v]_\alpha$, since $\varphi_\alpha(u, v) \in Z(U) = Z(M)$.

Replacing u by u+x we have

$0 = \varphi_\alpha(u+x, v)\beta w\gamma[u+x, v]_\alpha$

$= \varphi_\alpha(u, v)\beta w\gamma[u, v]_\alpha + \varphi_\alpha(x,v)\beta w\gamma[u,v]_\alpha + \varphi_\alpha(u, v)\beta w\gamma[x, v]_\alpha + \varphi_\alpha(x, v)\beta w\gamma[x, v]_\alpha$

Then we have $\varphi_\alpha(x, v)\beta w\gamma[u, v]_\alpha = -\varphi_\alpha(u, v)\beta w\gamma[x, v]_\alpha$

Now $\varphi_\alpha(x,v)\beta w\gamma[u,v]_\alpha\gamma p\gamma\varphi_\alpha(x,v)\beta w\gamma[u,v]_\alpha$

$$= -\varphi_\alpha(u,v)\beta w\gamma[x,v]_\alpha\gamma p\gamma\varphi_\alpha(x,v)\beta w\gamma[u,v]_\alpha$$
$$= -\varphi_\alpha(u,v)\beta w\gamma[x,v]_\alpha\gamma\varphi_\alpha(x,v)\gamma p\beta w\gamma[u,v]_\alpha$$
$$= 0 \text{ for } p \in U, \gamma \in \Gamma.$$

Hence $\varphi_\alpha(x,v)\beta w\gamma[u,v]_\alpha = 0$

Similarly replacing v by v+y we can show that $\varphi_\alpha(x, y)\beta w\gamma[u,v]_\alpha = 0$

Again replacing α by α+δ we get

$\varphi_{\alpha+\delta}(x, y)\beta w\gamma[u, v]_{\alpha+\delta} = 0$

Then

$$0 = \varphi_\alpha(x, y)\beta w\gamma[u, v]_\alpha + \varphi_\alpha(x, y)\beta w\gamma[u, v]_\delta + \varphi_\delta(x,y)\beta w\gamma[u,v]_\alpha + \varphi_\delta(x, y)\beta w\gamma[u, v]_\delta$$
$$= \varphi_\alpha(x, y)\beta w\gamma[u, v]_\delta + \varphi_\delta(x, y)\beta w\gamma[u, v]_\alpha$$

Hence $\varphi_\alpha(x, y)\beta w\gamma[u, v]_\delta = -\varphi_\delta(x, y)\beta w\gamma[u, v]_\alpha$

Now $\varphi_\alpha(x,y)\beta w\gamma[u,v]_\delta\mu m\eta\varphi_\alpha(x,y)\beta w\gamma[u,v]_\delta$

$$= -\varphi_\delta(x,y)\beta w\gamma[u,v]_\alpha\mu m\eta\varphi_\alpha(x,y)\beta w\gamma[u,v]_\delta$$
$$= 0, m \in U; \mu, \eta \in \Gamma.$$

Therefore, $\varphi_\alpha(x,y)\beta w\gamma[u,v]_\delta = 0$.

Similarly we can show that $[u,v]_\alpha\beta w\gamma\varphi_\alpha(x, y) = 0$.

3.10 Lemma : If M is a semiprime Γ-ring then $[u.v]_\delta\beta w\gamma\varphi_\alpha(x,y) = 0$.

Proof : We have from Lemma 3.7, $[u,v]_\alpha\beta\varphi_\alpha(u,v)=0$

That is $0 = [u, v]_\alpha\beta\varphi_\alpha(u,v)\gamma w$, for all $w \in U$

$=[u,v]_\alpha\beta w\gamma\varphi_\alpha(u,v)$, since $\varphi_\alpha(u,v) \in Z(M)$

replacing u by u+x, we get $[u+x,v]\beta w\gamma\varphi_\alpha(u+x,v) = 0$

That implies

$$0 = [u,v]_\alpha\beta w\gamma\varphi_\alpha(u,v) + [x,v]_\alpha\beta w\gamma\varphi_\alpha(u,v) + [u,v]_\alpha\beta w\gamma\varphi_\alpha(x,v) + [x,v]_\alpha\beta w\gamma\varphi_\alpha(x,v)$$
$$= [x,v]_\alpha\beta w\gamma\varphi_\alpha(u,v) + [u,v]_\alpha\beta w\gamma\varphi_\alpha(x,v)$$

Hence we have, $[x,v]_\alpha \beta w\gamma\varphi_\alpha(u,v) = -[u,v]_\alpha \beta w\gamma\varphi_\alpha(x,v)$

Now, $[x,v]_\alpha \beta w\gamma\varphi_\alpha(u,v)\mu p\eta[x,v]_\alpha \beta w\gamma\varphi_\alpha(u,v)$

$= -[u,v]_\alpha \beta w\gamma\varphi_\alpha(x,v)\mu p\eta[x,v]_\alpha \beta w\gamma\varphi_\alpha(u,v)$

$= -[u,v]_\alpha \beta w\gamma p\mu\varphi_\alpha(x,v)\eta[x,v]_\alpha \beta w\gamma\varphi_\alpha(u,v)$

$= 0$

Therefore , $[x,v]_\alpha \beta w\gamma\varphi_\alpha(u,v) = 0$

Similarly replacing w by w+y we have $[v,w]_\alpha \beta\varphi_\alpha(x,y) = 0$

Proceeding in the same way replacing α by α+δ we have $[v,w]_\delta \beta\varphi_\alpha(x,y) = 0$.

3.11 Lemma : If M is a semiprime Γ-ring , then

(i) $\varphi_\alpha(x, y)\beta[u, v]_\delta = 0$ (ii) $[u, v]_\alpha\beta\, \varphi_\alpha(x, y)$.

Proof : We have , $[u, v]_\delta\beta w\gamma\, [u, v]_\delta\beta w\gamma\, \varphi_\alpha(x, y) = 0$**.**

Hence, $\varphi_\alpha(x, y)\beta[u, v]_\delta\gamma w\mu\varphi_\alpha(x, y)\beta[u, v]_\delta = 0$**.**

Therefore, $\varphi_\alpha(x,y)\beta[u,v]_\delta = 0$ **.**

Similarly we can prove that $[u, v]_\delta\beta\varphi_\alpha(x,y) = 0$.

3.12 Definition : Let M be a Γ-ring . An element x of M is called nilpotent if for some $\gamma\in\Gamma$, there exists a positive integer n such that $(x\gamma)^n x = (x\gamma x\gamma \ldots\ldots\gamma x\gamma)x = 0$.

3.13 Definition: An ideal A of a Γ- ring M is called nilpotent if $(A\Gamma)^n A = (A\Gamma\ldots\Gamma A\Gamma)A = 0$, where n is the least positive integer.

3.14 Lemma ([21], Lemma 3.1) Let M be a semiprime Γ -ring. Then M contains no nonzero nilpotent ideal.

3.15 Theorem : Let $U \not\subset Z(M)$ be a Lie ideal of a semiprime Γ-ring M and $d : M \rightarrow M$ a Jordan k- derivation on U .Then d is a k- derivation on U of M .

Proof : Let $U \not\subset Z(M)$.

Now $\varphi_\alpha(v,w)\beta 2\varphi_\alpha(v,w) = \varphi_\alpha(v,w)\beta(\varphi_\alpha(v,w)+\varphi_\alpha(v,w))$

$= \varphi_\alpha(v,w)\beta(\varphi_\alpha(v,w) - \varphi_\alpha(w,v))$

$= \varphi_\alpha(v,w)\beta(d(v\alpha w)-d(v)\alpha w-vk(\alpha)w - v\alpha d(w) - d(w\alpha v) + d(w)\alpha v + wk(\alpha)v + w\alpha d(v))$

$= \varphi_\alpha(v,w)\beta[d(v\alpha w - w\alpha v)+(w\alpha d(v)-d(v)\alpha w)+(wk(\alpha)v - vk(\alpha)w)+(d(w)\alpha v - v\alpha d(w)]$

$= \varphi_\alpha(v,w)\beta(d([v,w]_\alpha) + [w,d(v)]_\alpha + [w,v]_{k(\alpha)} + [d(w),v]_\alpha)$

$= \varphi_\alpha(v,w)\beta d([v,w]_\alpha)+\varphi_\alpha(v,w)\beta[w,d(v)]_\alpha+\varphi_\alpha(v,w)\beta[w,v]_{k(\alpha)}+\varphi_\alpha(v,w)\beta[d(w),v]_\alpha$

$= \varphi_\alpha(v,w)\beta d([v,w]_\alpha)-\varphi_\alpha(v,w)\beta[d(v),w]_\alpha - \varphi_\alpha(v,w)\beta[v,w]_{k(\alpha)} - \varphi_\alpha(v,w)\beta[v,d(w)]_\alpha$

$= \varphi_\alpha(v,w)\beta d([v,w]_\alpha)$ by Lemma 3.11. Hence we have

$2\varphi_\alpha(v,w)\beta\varphi_\alpha(v,w) = \varphi_\alpha(v,w)\beta d([v,w]_\alpha)$ …………(i)

Also

$\varphi_\alpha(v,w)\beta[p,q]_\gamma+[p,q]_\gamma\beta\varphi_\alpha(v,w) = 0$

That means $0 = d(\varphi_\alpha(v,w)\beta[p,q]_\gamma+[p,q]_\gamma\beta\varphi_\alpha(v,w))$

$= d(\varphi_\alpha(v,w)\beta[p,q]_\gamma+\varphi_\alpha(v,w)k(\beta)[p,q]_\gamma+\varphi_\alpha(v,w)\beta d([p,q]_\gamma)+d([p,q]_\gamma)\beta\varphi_\alpha(v,w)+$
$[p,q]_\gamma k(\beta)\varphi_\alpha(v,w)+[p,q]_\gamma\beta d(\varphi_\alpha(v,w))$

$= d(\varphi_\alpha(v,w))\beta[p,q]_\gamma + \varphi_\alpha(v,w)\beta d([p,q]_\gamma)+d([p,q]_\gamma)\beta\varphi_\alpha(v,w)+[p,q]_\gamma\beta d(\varphi_\alpha(v,w))$

Now $\varphi_\alpha(v,w) \in Z(M)$ implies $\varphi_\alpha(v,w)\beta d([p,q]_\gamma) = d([p,q]_\gamma)\beta\varphi_\alpha(v,w)$

Hence we have,

$d(\varphi_\alpha(v,w))\beta[p,q]_\gamma+2\varphi_\alpha(v,w)\beta d([p,q]_\gamma)+[p,q]_\gamma\beta d(\varphi_\alpha(v,w))=0$

That implies $2\varphi_\alpha(v,w)\beta d([p,q]_\gamma) = - d(\varphi_\alpha(v,w))\beta[p,q]_\gamma - [p,q]_\gamma\beta d(\varphi_\alpha(v,w))$ ......(ii)

Hence we obtain $4\varphi_\alpha(v,w)\beta\varphi_\alpha(v,w) = 2.2\varphi_\alpha(v,w)\beta\varphi_\alpha(v,w)$

$=2\varphi_\alpha(v,w)\beta d([v,w]_\alpha)$

$= - d(\varphi_\alpha(v,w))\beta[v,w]_\alpha-[v,w]_\alpha\beta d(\varphi_\alpha(v,w))$

That is $4\varphi_\alpha(v,w)\beta\varphi_\alpha(v,w)\beta\varphi_\alpha(v,w)$

$= - d(\varphi_\alpha(v,w))\beta[v,w]_\alpha\beta\varphi_\alpha(v,w) - [v,w]_\alpha\beta d(\varphi_\alpha(v,w))\beta\varphi_\alpha(v,w)$

But $[v, w]_\alpha\beta\varphi_\alpha(v, w) = 0$ implies $d(\varphi_\alpha(v, w))\beta[v, w]_\alpha\beta\varphi_\alpha(v, w) = 0$

Also $[v,w]_\alpha\beta d(\varphi_\alpha(v,w))\beta\varphi_\alpha(v,w) = 0$,

since $d(\varphi_\alpha(v,w)) \in M$, for all $v,w \in M, \alpha \in \Gamma$.

That implies $0 = 4\varphi_\alpha(v,w)\beta\varphi_\alpha(v,w)\beta\varphi_\alpha(v,w)$

Therefore,

$\varphi_\alpha(v,w)\beta\varphi_\alpha(v,w)\beta\varphi_\alpha(v,w) = 0$, since M is 2- torsion free .

Since semiprime Γ- ring contains no nonzero central nilpotent element, we obtain $\varphi_\alpha(v, w) = 0$. Therefore, d is a k- derivation.

Left centralizer on Lie ideals in prime and Semiprime Gamma rings

Let U be a Lie ideal of a 2-torsion free prime gamma ring M such that $u\alpha u \in u$ for all $u \in U$ and $\alpha \in \Gamma$. If $T : M \to M$ is an additive mapping satisfying the relation $T(u\alpha u) = T(u)\alpha u$, then we prove that $T(u\alpha v) = T(u)\alpha v$ for all $u, v \in U$ and $\alpha \in \Gamma$. Also this result is extended to semiprime Γ-rings.

4. Introduction : An extensive generalized concept of classical ring set forth the notion of a gamma ring theory. As an emerging field of research, the research work of classical ring theory to the gamma ring theory has been drawn interest of many algebraists and prominent mathematicians over the world to determine many basic properties of gamma ring and to enrich the world of algebra. The different researchers on this field have been doing significant contributions to this field from its inception. In recent years, a large number of researchers are engaged to increase the efficacy of the results of gamma ring theory over the world.

Nobusawa [48] has shown that Γ- ring is more general than a ring. Bernes[7] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. Bernes, kyuno [41] , Luh [43], Ceven[13, 14] , Haque and Paul [35, 36, 37, 38, 39] and others were obtained a large numbers of important basic properties of Γ-rings in various ways and determined some more remarkable results of Γ-rings. Note that during the last some decades many authors have studied Lie ideals in the context of prime and semiprime rings and Γ-rings. We start with the following necessary definitions.

4.1 Definition : An additive mapping T : M → M is called a left (right) centralizer if T(aαb) = T(a)αb (resp. T(aαb) = aαT(b)) for all a, b ∈ M and α ∈ Γ.

The goal of this chapter is to extend the results of [33] on Lie ideals in prime and semiprime Γ- rings.

Left Centralizers of Prime Gamma Rings:

4.2 Lemma : Let M be a Γ-ring and U be a Lie ideal of M such that uαu ∈ U for all u ∈ U and α ∈ Γ . If T : M → M is an additive mapping satisfying the relation T(uαu) = T(u)αu for all u ∈ U and α ∈ Γ , then

(a) T(uαv + vαu) = T(u)αv + T(v)αu

(b) T(uαvβu + uβvαu) = T(u)αvβu + T(u)βvαu

(c) T(uαvβu) = T(u)αvβu

(d) T(uαvβw + wβvαu) = T(u)αvβw + T(w)βvαu, for all u, v, w ∈ U and α ∈ Γ.

Proof : By the definition of Lie ideal U , uαu ∈ U for all u ∈ U ; α ∈ Γ.

Thus we have uβv + vβu = (u+v)β(u+v) – uβu – vβv ∈ U for all u, v ∈ U and β ∈ Γ. Therefore ,

T(uαv + vβu) = T((u+v)α(u+v)) – T(uαu) – T(vαv)

= T(u+v)α(u+v) – T(u)αu – T(v)αv

= T(u)αu + T(u)αv + T(v)αu + T(v)αv – T(u)αu – T(v)αv

= T(u)αv + T(v)αu

Hence T(uαv + vαu) = T(u)αv + T(v)αu ……(i)

Since uβv + vβu ∈ U for all u, v ∈ U and β ∈ Γ, we replace v by (uβv + vβu) in relation (i) , we obtain,

T(uα(uβv + vβu) + (uβv + vβu) αu) = T(u)α(uβv + vβu) + T(uβv + vβu)αu

Left side implies T(uαuβv + uαvβu + uβvαu + vβuαu)

$= T(u\alpha v\beta u + u\beta v\alpha u) + T(u\alpha u)\beta v + T(v)\beta u\alpha u$

$= T(u\alpha v\beta u + u\beta v\alpha u) + T(u)\alpha u\beta v + T(v)\beta u\alpha u$

And Right side implies $T(u)\alpha u\beta v + T(u)\alpha v\beta u + T(u)\beta v\alpha u + T(v)\beta u\alpha u$

Hence we have

$T(u\alpha v\beta u + u\beta v\alpha u) = T(u)\alpha v\beta u + T(u)\beta v\alpha u$ ……..(ii)

By using the condition (*) we have

$2T(u\alpha v\beta u) = 2T(u)\alpha v\beta u$ which implies $T(u\alpha v\beta u) = T(u)\alpha v\beta u$ ……..(iii)

Putting $u = u+w$ in the relation (iii) we obtain,

$T((u+w)\alpha v\beta(u+w)) = T(u+w)\alpha v\beta(u+w)$

Left side implies $T(u\alpha v\beta u + u\alpha v\beta w + w\alpha v\beta u + w\alpha v\beta w)$

$= T(u\alpha v\beta w + w\beta v\alpha u) + T(u\alpha v\beta u) + T(w\alpha v\beta w)$

$= T(u\alpha v\beta w + w\beta v\alpha u) + T(u)\alpha v\beta u + T(w)\alpha v\beta w$

Right side implies $T(u)\alpha v\beta u + T(u)\alpha v\beta w + T(w)\alpha v\beta u + T(w)\alpha v\beta w$

$= T(u)\alpha v\beta u + T(u)\alpha v\beta w + T(w)\alpha v\beta u + T(w)\alpha v\beta w$

Therefore , $T(u\alpha v\beta w + w\beta v\alpha u) = T(u)\alpha v\beta w + T(w)\alpha v\beta u$.

4.3 Definition: We define $B_\alpha(u, v) = T(u\alpha v) - T(u)\alpha v$, for all $u, v \in U$ and $\alpha \in \Gamma$.

4.4 Remark : It is clear that $B_\alpha(u, v)$ is an additive mapping such that $B_\alpha(u, v) + B_\alpha(v, u) = 0$.

4.5 Remark : It is also clear that T is a left centralizer if and only if $B_\alpha(u, v) = 0$.

4.6 Lemma : Let M be a 2-torsion free Γ-ring and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If $T : M \to M$ is an additive mapping satisfying the relation $T(u\alpha u) = T(u)\alpha u$ for all $u \in U$ and $\alpha \in \Gamma$, then $B_\alpha(u, v)\, \beta w\gamma[u, v]_\delta = 0$ and $[u, v]_\delta\beta w\gamma\, B_\alpha(u, v) = 0$.

Proof : First we shall compute $A = T(u\alpha v\beta w\gamma v\delta u + v\alpha u\beta w\gamma u\delta v)$ in two different ways.

$A = T(u\alpha(v\beta w\gamma v)\delta u + v\alpha(u\beta w\gamma u)\delta v)$

$= T(u)\alpha v\beta w\gamma v\delta u + T(v)\alpha u\beta w\gamma u\delta v$ ....(4) [using 4.2(c)]

Again $A = T(u\alpha v\beta w\gamma v\delta u + v\alpha u\beta w\gamma u\delta v)$

$= T(u\alpha v\beta w\gamma v\delta u + v\delta u\beta w\gamma u\alpha v)$

$= T((u\alpha v)\beta w\gamma(v\delta u) + (v\alpha u)\beta w\gamma(u\delta v))$

$= T(u\alpha v)\beta w\gamma v\delta u + T(v\alpha u)\beta w\gamma u\delta v$ .....(5) [using 4.2(d)]

Comparing (4) and (5) we obtain,

$0 = T(u\alpha v)\beta w\gamma v\delta u + T(v\alpha u)\beta w\gamma u\delta v - T(u)\alpha v\beta w\gamma v\delta u - T(v)\alpha u\beta w\gamma u\delta v$

$= (T(u\alpha v) - T(u)\alpha v)\beta w\gamma v\delta u + (T(v\alpha u) - T(v)\alpha u)\beta w\gamma u\delta v$

$= B_\alpha(u, v)\beta w\gamma v\delta u + B_\alpha(v, u)\beta w\gamma u\delta v$

$= B_\alpha(u, v)\beta w\gamma v\delta u - B_\alpha(u, v)\beta w\gamma u\delta v$

$= B_\alpha(u, v)\beta w\gamma (v\delta u - u\delta v)$

$= B_\alpha(u, v)\beta w\gamma[v, u]_\delta$

Hence we have $B_\alpha(u, v)\beta w\gamma[u, v]_\delta = 0$

Similarly we can easily prove that $[u, v]_\delta\beta w\gamma B_\alpha(u, v) = 0$.

4.7 Theorem : Let U be a Lie ideal of a 2-torsion free semiprime Γ- ring such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If T: M→M be an additive mapping satisfying the relation $T(u\alpha u) = T(u)\alpha u$ for all $u \in U$, $\alpha \in \Gamma$, then $T(u\alpha v) = T(u)\alpha v$ for all $u, v \in U$ and $\alpha \in \Gamma$.

Proof : If U is a commutative Lie ideal of M , then by Lemma 2.16, $U \subseteq Z(M)$. Therefore by Lemma 4.2 (a) we have $2T(u\alpha v) = 2T(u)\alpha v$.

Thus by 2-torsion freeness of M we have $T(u\alpha v) = T(u)\alpha v$.

If U is not commutative, then $U \not\subset Z(M)$. In this case we have from Lemma 4.6 $B_\alpha(u, v)\beta w\gamma[u, v]_\delta = 0$

Putting $u = u+x$ for all $u \in U$, we have

$0 = B_\alpha(u+x, v)\beta w\gamma[u+x, v]_\delta$

$=B_\alpha(u, v)\beta w\gamma[u, v]_\delta + B_\alpha(u, v)\beta w\gamma[x, v]_\delta + B_\alpha(x, v)\beta w\gamma[u, v]_\delta + B_\alpha(x, v)\beta w\gamma[x, v]_\delta$

$= B_\alpha(u, v)\beta w\gamma[x, v]_\delta + B_\alpha(x, v)\beta w\gamma[u, v]_\delta$

Hence $B_\alpha(u, v)\beta w\gamma[x, v]_\delta = - B_\alpha(x, v)\beta w\gamma[u, v]_\delta$

Now $B_\alpha(u, v)\beta w\gamma[x, v]_\delta \mu z\gamma B_\alpha(u, v)\beta w\gamma[x, v]_\delta$

$= - B_\alpha(u, v)\beta w\gamma ([x, v]_\delta\mu z\gamma B_\alpha(x, v)) \beta w\gamma[u, v]_\delta$

$= 0$ [using the 2nd part of Lemma 4.6]

Therefore by Lemma 1.29 we have

$B_\alpha(u, v)\beta w\gamma[x, v]_\delta = 0$ for all $x \in U$.

Similarly using $v = v+y$, we obtain $B_\alpha(u, v)\beta w\gamma[x, y]_\delta = 0$, for all $v \in U$.

Againg using Lemma 1.29 we have $B_\alpha(u, v) = 0$ or $[x, v]_\delta = 0$.

If $[x, v]_\delta = 0$, then U is commutative , a contradiction. Therefore $B_\alpha(u, v) = 0$.

4.8 Corollary: Let M be a 2-torsion free prime Γ-ring and T: M → M be a Jordan left centralizer , then T is a left centralizer.

Left Centralizers of Semiprime Gamma Rings

4.9 Lemma: Let U be a commutative Lie ideal of a 2-torsion free semiprime Γ-ring M. Then $U \subseteq Z(M)$.

Proof : For $u \in U$ and $x \in M$, we have $[u, [u, x]_\alpha]_\beta = 0$.

Replacing x by $x\gamma y$ we have $0 = [u, [u, x\gamma y]_\alpha]_\beta$

$= [u, x\gamma[u, y]_\alpha + [u, x]_\alpha\gamma y]_\beta$

$= [u, x\gamma[u, y]_\alpha]_\beta + [u, [u, x]_\alpha\gamma y]_\beta$

$= x\gamma [u, [u, y]_\alpha]_\beta + [u, x]_\beta\gamma [u, y]_\alpha + [u, [u, x]_\alpha]_\beta \gamma y + [u, x]_\alpha\gamma[u, y]_\beta$

$= [u, x]_\beta\gamma [u, y]_\alpha + [u, x]_\alpha\gamma[u, y]_\beta$

$= 2[u, x]_\alpha\gamma[u, y]_\beta$ [using condition (*)

Then $[u, x]_\alpha\gamma[u, y]_\beta = 0$.

Now replacing y by $y\delta x$ we have $0 = [u, x]_\alpha\gamma[u, y\delta x]_\beta$

$= [u, x]_\alpha \gamma y \delta [u, x]_\beta + [u, x]_\alpha \gamma [u, y]_\beta \delta x$

$= [u, x]_\alpha \gamma y \delta [u, x]_\beta$, for all y in M.

Since M is semiprime, $[u, x]_\alpha = 0$. Therefore, $U \subseteq Z(M)$.

4.10 Lemma : Let U be a Lie ideal M satisfying the condition (*) . Then $T(U) = \{x \in M : [x, M]_\Gamma \subseteq U\}$ is both a Γ- sub ring and a Lie ideal of M such that $U \subseteq T(U)$.

Proof: Since U is a Lie ideal of M, we have $[U, M]_\Gamma \subseteq U$. Thus $U \subseteq T(U)$. Also we have $[T(U), M]_\Gamma \subseteq U \subseteq T(U)$. Hence T(U) is a Lie ideal of M. Suppose that $x, y \in T(U)$, then $[x, m]_\alpha$, $[y, m]_\alpha \in U$, for all $m \in M$ and $\alpha \in \Gamma$.

Now $[x\alpha y, m]_\beta = x\alpha[y, m]_\beta + [x, m]_\beta \alpha y \in U$ implies $x\alpha y \in T(U)$.

4.11 Lemma : Let $U \not\subset Z(M)$ be a Lie ideal of M .Then there exists a nonzero ideal $K = M\Gamma[U, U]_\Gamma \Gamma M$ of M generated by $[U, U]_\Gamma$ such that $[K, M]_\Gamma \subseteq U$.

Proof : First we have to prove that if $[U, U]_\Gamma = 0$, then for all $a \in U$; $\alpha \in \Gamma$ we have $[u, [u, x]_\alpha]_\alpha = 0$ for all $x \in M$. Then using the proof of Lemma 4.9 we obtain $U \subseteq Z(M)$, a contradiction. Thus let $[U, U]_\Gamma \neq 0$. Then $K = M\Gamma[U, U]_\Gamma \Gamma M$ is a nonzero ideal of M generated by $[U, U]_\Gamma$. Let $x, y \in U$; $m \in M$ and $\alpha, \beta \in \Gamma$, we have $[x, y\beta m]_\alpha$, y, $[x, m]_\alpha \in U \subseteq T(U)$. Hence by Lemma 4.10, we have

$[x, y]_\alpha \beta m = [x, y\beta m]_\alpha - y\beta[x, m]_\alpha \in T(U)$.

Also we can show that $m\beta[x, y]_\alpha \in T(U)$ and therefore , we obtain $[[U, U_\Gamma], M]_\Gamma \subseteq U$. That is , $[[[[x, y]_\alpha, m]_\alpha, s]_\alpha, t]_\alpha \in U$, for all $m, s, t \in M$ and $\alpha \in \Gamma$.

Hence $[[x, y]_\alpha \alpha m s - m\alpha[x, y]_\alpha \alpha s + [s, m]_\alpha \alpha[x, y]_\alpha - [s\alpha[x, y]_\alpha, m]_\alpha, t]_\alpha \in T(U)$. Since $[x, y]_\alpha \alpha m \alpha s$, $s\alpha[x, y]_\alpha$, $[s, m]_\alpha \alpha [x, y]_\alpha \in T(U)$.Thus we have $[m\alpha[x, y]_\alpha \alpha s, t]_\alpha \in U$, for all m, s, t $\in$ M and $\alpha \in \Gamma$. Hence $[K, M]_\Gamma \subseteq U$.

4.12 Lemma : Let $U \not\subset Z(M)$ be a Lie ideal of a 2-torsion free semiprime Γ- ring and $a \in U$. If $a\alpha U\beta a = \{0\}$ for all $\alpha, \beta \in \Gamma$, then $a\alpha a = 0$ and there exists a nonzero ideal $K = M\Gamma[U, U]_\Gamma\Gamma M$ of M generated by $[U, U]_\Gamma$ such that $[K, M]_\Gamma \subseteq U$ and $K\Gamma a = a\Gamma K = \{0\}$.

Proof : If $a\alpha U\beta a = \{0\}$ for all $\alpha, \beta \in \Gamma$, then $a\alpha[a, a\delta m]_\alpha\beta a = 0$ for all $m \in M$ and $\delta \in \Gamma$. Therefore,

$0 = a\alpha(a\alpha a\delta m - a\delta m\alpha a)\beta a$

$= a\alpha a\alpha a\delta m\beta a - a\alpha a\delta m\alpha a\beta a$

$= a\alpha a\delta a\alpha m\beta a - a\alpha a\delta m\beta a\alpha a$

Since $a\alpha a\delta a = 0$, we have $(a\alpha a)\delta m\beta(a\alpha a) = 0$ and hence $a\alpha a = 0$ for semiprimeness of M. Now we obtain $a\alpha[k\gamma a, m]_\mu\beta u\alpha a = 0$ for all $k \in K$; $m \in M$; $u \in U$ and $\alpha, \beta, \mu \in \Gamma$. Therefore,

$0 = a\alpha(k\gamma a\mu m - m\mu k\gamma a)\beta u\alpha a$

$= a\alpha k\gamma a\mu m\beta u\alpha a$

Thus we have $0 = a\alpha k\gamma a\mu m\beta[k, a]_\gamma\alpha a$

$= a\alpha k\gamma a\mu m\beta(k\gamma a - a\gamma k)\alpha a$

$= a\alpha k\gamma a\mu m\beta k\gamma a\alpha a - a\alpha k\gamma a\mu m\beta a\gamma k\alpha a$

$= (a\alpha k\gamma a)\mu m\beta(a\gamma k\alpha a)$ [by using $a\alpha a = 0$]

$= (a\alpha k\gamma a)\mu m\beta(a\alpha k\gamma a)$ [using (*)]

That implies $(a\alpha k\gamma a) = 0$, Since M is semiprime.

Thus we find that $(a\alpha k)\Gamma M\Gamma(a\alpha k) = 0$.

Hence $a\alpha k = 0$ for all $k \in K$. That is $a\alpha k = \{0\}$.

Similarly we have $ka\alpha = \{0\}$.

4.13 Lemma : Let $U \not\subset Z(M)$ be a Lie ideal of M and $a, b \in U$; $\alpha, \beta \in \Gamma$.

(i) If $a\alpha U\beta a = \{0\}$, then $a = 0$.

(ii) If $a\alpha U = \{0\}$ $(U\alpha a = 0)$, then $a = 0$

(iii) If $u\alpha u \in U$ for all $u \in U$ and $a\alpha U\beta b = \{0\}$, then $a\alpha b = 0$ and $b\alpha a = 0$.

Proof (i): By Lemma 4.12, we have $K\alpha a = M\Gamma[U, U]_\Gamma\Gamma M\alpha a = \{0\}$ and $a\alpha a = 0$, for all $x, y \in M$ and $\alpha \in \Gamma$ We have,

$0 = [[a, x]_\alpha, a]_\gamma \beta y\alpha a$

$= [a\alpha x - x\alpha a, a]_\gamma \beta y\alpha a$

$= [a\alpha x, a]_\gamma \beta y\alpha a - [x\alpha a, a]_\gamma \beta y\alpha a$

$= a\alpha[x, a]_\gamma \beta y\alpha a + [a, a]_\gamma \alpha x\beta y\alpha a - x\alpha[a, a]_\gamma \beta y\alpha a - [x, a]_\gamma \alpha a\beta y\alpha a$

$= a\alpha[x, a]_\gamma \beta y\alpha a - [x, a]_\gamma \alpha a\beta y\alpha a$

$= a\alpha(x\gamma a - a\gamma x) \beta y\alpha a + (x\gamma a - a\gamma x)\alpha a\beta y\alpha a$

$= a\alpha x\gamma a\beta y\alpha a - a\alpha a\gamma x\beta y\alpha a + x\gamma a\alpha a\beta y\alpha a - a\gamma x\alpha a\beta y\alpha a$

$= 2a\alpha x\gamma a\beta y\alpha a$ [since $a\alpha a = 0$]

Thus $a\alpha x\gamma a\beta y\alpha a = 0$

Then we have $a\alpha x\gamma a\beta y\alpha a\delta x\gamma a = 0$

Using (*) we have $(a\alpha x\gamma a)\beta y\delta(a\alpha x\gamma a) = 0$.

Since M is semiprime , we have $a\alpha x\gamma a = 0$.

And then we have $a = 0$ for semiprimeness of M.

(ii) If $a\alpha U = \{0\}$, then $a\alpha U\beta a = \{0\}$ for all $\beta \in \Gamma$.Thus by (i) we have $a = 0$. Similarly if $U\alpha a = \{0\}$ then $a = 0$.

(iii) If $a\alpha U\beta b = \{0\}$, then we have $(b\gamma a)\alpha U\beta(b\gamma a) = \{0\}$.

Hence by (i)

$b\gamma a = 0$, for all $\gamma \in \Gamma$. Also $a\gamma b\alpha U\beta a\gamma b = \{0\}$ implies $a\gamma b = 0$.

4.14 Theorem : Let U be a lie ideal of M such that $u\alpha u \in U$ for every $u \in U$ and $\alpha \in \Gamma$. If $T: M \to M$ is an additive mapping satisfying the relation $T(u\alpha u) = T(u)\alpha u$ for all $u \in U$ and then, $T(u\alpha v) = T(u)\alpha v$ for all $u, v \in U$ and $\alpha \in \Gamma$.

Proof: If U is a commutative Lie ideal of M then by Lemma 4.9 we have $U \subseteq Z(M)$. Thus from Lemma 4.2(a), $2T(u\alpha v) = 2T(u)\alpha v$ and hence $T(u\alpha v) = T(u)\alpha v$. If U is not commutative , then $U \not\subset Z(M)$. In this case, we have from Lemma 4.6

$B_\alpha(u, v)\beta w\gamma[u, v]_\delta = 0$

Linearising we obtain,

$0 = B_\alpha(u+x, v)\beta w\gamma[u+x, v]_\delta$

$= B_\alpha(u, v)\beta w\gamma[u, v]_\delta + B_\alpha(u, v)\beta w\gamma[x, v]_\delta + B_\alpha(x, v)\beta w\gamma[u, v]_\delta + B_\alpha(x, v)\beta w\gamma[x, v]_\delta$

$= B_\alpha(u, v)\beta w\gamma[x, v]_\delta + B_\alpha(x, v)\beta w\gamma[u, v]_\delta$

Then $B_\alpha(u, v)\beta w\gamma[x, v]_\delta = - B_\alpha(x, v)\beta w\gamma[u, v]_\delta$

Now $B_\alpha(u, v)\beta w\gamma[x, v]_\delta \mu z\lambda B_\alpha(u, v)\beta w\gamma[x, v]_\delta$

$= - B_\alpha(u, v)\beta w\gamma([x, v]_\delta \mu z\lambda B_\alpha(x, v))\beta w\gamma[u, v]_\delta$

$= 0$, for all $z \in U$. Hence $B_\alpha(u, v)\beta w\gamma[x, v]_\delta = 0$.

Similarly linearizing v we obtain $B_\alpha(u, v)\beta w\gamma[x, y]_\delta = 0$, for all $y \in U$.

Hence the similar proof of the Theorem 2.1 in [35], we obtain the required result.

4.15 Corollary : Let M be a 2-torsion free semiprime Γ- ring and T: M → M be a Jordan left centralizer , then T is a left centralizer.

Jordan Generalized k-Derivation on Lie Ideals of Semiprime Gamma Rings

In view of k-derivations and Jordan k-derivations, Jordan generalized k-derivation has been introduced. In this paper we have worked on Jordan generalized k-derivations on Lie ideals of semiprime gamma rings.We have characterized Left centralizers and Jordan left centralizers and going through this way we have proved that every Jordan generalized k-derivation on a Lie ideal U of M is a generalized k-derivation on U of M.

5. Introduction: The notion of a Γ-ring has been developed by Nobusawa [48] as a generalized form of a ring and then Bernes generalized Γ-rings as a new type, which is known as Γ_N-ring.

The notion of derivation and Jordan derivation in Γ-rings have been introduced by Sapanci and Nakajima [58]. Ceven [13] , [14] has studied left derivations and Jordan derivations. Haldar and Paul [28], [29] extended the results of Ceven in Lie ideals . Awtar [6] extended some results on derivations in Lie ideals. In this chapter we extend some results on Lie ideals of semiprime Γ- rings.

5.1 Definition : Let M be a gamma ring , U a Lie ideal of M and let k : M → M be an additive mapping . An additive mapping F : M → M is called a generalized k-derivation if there exists a k-derivation d : M → M such that $F(u\alpha v) = F(u)\alpha v + uk(\alpha)v + u\alpha d(v)$, for all u, v ∈ U and α ∈ Γ. And if $F(u\alpha u) = F(u)\alpha u + uk(\alpha)u + u\alpha d(u)$ holds for all u ∈ U and α ∈ Γ , then F is said to be a Jordan generalized k-derivation on U of M.

5.2 Theorem : Let M be a 2-torsion free semiprime Γ-ring and U be a Lie ideal of M such that $u\alpha u \in U$ for all u ∈ U and α ∈ Γ. If U is commutative and $F(u)\alpha v = F(v)\alpha u$; $u\alpha d(v) = d(u)\alpha v$, then $\psi_\alpha(u, v) = 0$, for all $u, v \in U$; $\alpha \in \Gamma$.

Proof : Since U is commutative and $F(u)\alpha v = F(v)\alpha u$, $u\alpha d(v) = d(u)\alpha v$. From lemma 2.7, we have

$F(u\alpha v + v\alpha u) = F(u)\alpha v + uk(\alpha)v + u\alpha d(v) + F(v)\alpha u + vk(\alpha)u + v\alpha d(u)$

Then $F(2u\alpha v) = 2F(u)\alpha v + 2uk(\alpha)v + 2u\alpha d(v)$ [since $v\alpha d(u) = d(u)\alpha v = u\alpha d(v)$]

$= 2(F(u)\alpha v + uk(\alpha)v + u\alpha d(v))$

That is, $F(u\alpha v) - F(u)\alpha v - uk(\alpha)v - u\alpha d(v) = 0$

And hence we have $\psi_\alpha(u, v) = 0$.

5.3 Lemma : If M is semiprime Γ- ring and $U \not\subset Z(M)$ is a Lie ideal of M , then $[u, v]_\alpha\beta w\gamma\psi_\alpha(u, v) = 0$

Proof : We have from Lemma 2.11, if M is a Γ-ring and U is a Lie ideal of M then for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$, $\psi_\alpha(u, v)\beta w\gamma[u, v]_\alpha = 0$.

Then $[u, v]_\alpha\beta w\gamma\psi_\alpha(u, v)\delta x\mu[u, v]\alpha\beta w\gamma\psi_\alpha(u, v) = 0$, for all $x \in U$.

Then from Lemma 3.5(i) we have $[u, v]_\alpha\beta w\gamma\psi_\alpha(u, v) = 0$.

5.4 Lemma : If M is semiprime, then $\psi_\alpha(u, v)\beta w\gamma[x, y]_\delta = 0$.

Proof : From Lemma 2.11, we have

$\psi_\alpha(u, v)\beta w\gamma[u, v]_\alpha = 0$, for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$

Then, $\psi_\alpha(u+x, v)\beta w\gamma[u+x, v]_\alpha = 0$, for all $x \in U$

That implies

$0 = \psi_\alpha(u, v)\beta w\gamma[u, v]_\alpha + \psi_\alpha(x, v)\beta w\gamma[u, v]_\alpha + \psi_\alpha(u, v)\beta w\gamma[x, v]_\alpha + \psi_\alpha(x, v)\beta w\gamma[x, v]_\alpha$

$= \psi_\alpha(x, v)\beta w\gamma[u, v]_\alpha + \psi_\alpha(u, v)\beta w\gamma[x, v]_\alpha$

Then $\psi_\alpha(u, v)\beta w\gamma[x, v]_\alpha = - \psi_\alpha(x, v)\beta w\gamma[u, v]_\alpha$

Hence $\psi_\alpha(u, v)\beta w\gamma[x, v]_\alpha \mu\rho\eta\, \psi_\alpha(u, v)\beta w\gamma[x, v]_\alpha$

$= - \psi_\alpha(x, v)\beta w\gamma([u, v]_\alpha \mu p\eta \psi_\alpha(u, v))\beta w\gamma[x, v]_\alpha$

$= 0$, for all $p \in U$; $\mu, \eta \in \Gamma$.

From the semiprimeness of M, we have

$\psi_\alpha(u, v)\beta w\gamma[x, v]_\alpha = 0$.

By similar replacement of v by v+y , $y \in U$ we have

$\psi_\alpha(u, v)\beta w\gamma[x, y]_\alpha = 0$.

Again replacing α by $\alpha+\delta$ we have

$0 = \psi_{\alpha+\delta}(u, v) \beta w\gamma [x, y]_{\alpha+\delta}$

$= \psi_\alpha(u, v)\beta w\gamma[x, y]_\alpha + \psi_\delta(u, v)\beta w\gamma[x, y]_\alpha + \psi_\alpha(u, v)\beta w\gamma[x, y]_\delta + \psi_\delta(u, v)\beta w\gamma[x, y]_\delta$

$= \psi_\delta(u, v)\beta w\gamma[x, y]_\alpha + \psi_\alpha(u, v)\beta w\gamma[x, y]_\delta$

Then $\psi_\alpha(u, v)\beta w\gamma[x, y]_\delta = - \psi_\delta(u, v)\beta w\gamma[x, y]_\alpha$

Now $\psi_\alpha(u, v)\beta w\gamma[x, y]_\delta \mu u\eta \psi_\alpha(u, v)\beta w\gamma[x, y]_\delta$

$= - \psi_\delta(u, v)\beta w\gamma([x, y]_\alpha \mu u\eta \psi_\alpha(u, v)) \beta w\gamma[x, y]_\delta$

$= 0$, for all $u \in U$ and $\mu, \eta \in \Gamma$.

Since M is semiprime, $\psi_\alpha(u, v)\beta w\gamma[x, y]_\delta = 0$.

5.5 Lemma ([21], Lemma 3.2) The center of a semiprime Γ-ring does not contain any nonzero nilpotent element.

5.6 Theorem : Let M be a Jordan generalized k- derivation on a Lie ideal U of a 2- torsion free semiprime Γ- ring M . If $F(u)\alpha v = F(v)\alpha u$ and $u\alpha d(v) = d(u)\alpha v$ hold for all $u, v \in U$ and $\alpha \in \Gamma$, then F is a generalized k- derivation on U of M.

Proof : Let F be a Jordan generalized k- derivation on U of M. Suppose that $F(u)\alpha v = F(v)\alpha u$ and $u\alpha d(v) = d(u)\alpha v$.

From Lemma 2.11, we have $\psi_\alpha(u, v)\beta w\gamma[u, v]_\alpha = 0$, for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$.

Now $2[\psi_\alpha(u, v), y]_\gamma \beta w\beta [\psi_\alpha(u, v), y]_\gamma$

$= 2 (\psi_\alpha(u, v) \gamma y - y\gamma\psi_\alpha(u, v)) \beta w\beta [\psi_\alpha(u, v), y]_\gamma$

$= \psi_\alpha (u, v)\gamma(2y\beta w)\beta [\psi_\alpha(u, v), y]_\gamma - 2y\gamma\psi_\alpha(u, v)\beta w\beta[\psi_\alpha(u, v), y]_\gamma$

$= 0$, since $2y\beta w \in U$; $\psi_\alpha(u, v) \in M$ for all u, v, y, w $\in$ U; $\alpha, \beta, \gamma \in \Gamma$.

Since M is semiprime and 2- torsion free, $[\psi_\alpha(u, v), y]_\gamma = 0$, for all u, v, y $\in$ U and $\alpha,\gamma \in \Gamma$.

That implies $\psi_\alpha(u, v) \in Z(U) = Z(M)$, the centre of M.

Now let $\delta \in \Gamma$. Then we have

$\psi_\alpha(u, v)\delta[x, y]_\gamma\beta w\beta\psi_\alpha(u, v)\delta[x, y]_\gamma = 0$ [from Lemma 5.3]

Since M is semiprime , we have $\psi_\alpha(u, v)\delta[x, y]_\gamma = 0$…………..(i)

Similarly we can prove that $[x, y]_\gamma\delta\psi_\alpha(u, v) = 0$ ……..(ii)

Again $2\psi_\alpha(u, v)\delta\psi_\alpha(u, v) = \psi_\alpha(u, v)\delta(\psi_\alpha(u, v) + \psi_\alpha(u, v))$

$= \psi_\alpha(u, v)\delta(\psi_\alpha(u, v) - \psi_\alpha(v, u))$

$= \psi_\alpha(u, v)\delta(F(u\alpha v) - F(u)\alpha v - uk(\alpha)v - u\alpha d(v) - F(v\alpha u) + F(v)\alpha u + vk(\alpha)u + v\alpha d(u))$

$= \psi_\alpha(u, v)\delta (F([u, v]_\alpha) - [u, v]_{k(\alpha)})$

$= \psi_\alpha(u, v)\delta (F([u, v]_\alpha) - \psi_\alpha(u, v) \delta [u, v]_{k(\alpha)})$

Here $k(\alpha) \in \Gamma$ implies $\psi_\alpha(u, v)\delta F([u, v]_\alpha)$

Hence $2\psi_\alpha(u, v)\delta\psi_\alpha(u, v) = \psi_\alpha(u, v)\delta (F([u, v]_\alpha)$ ……(iii)

From (i) and (ii) we have

$0 = \psi_\alpha(u, v)\delta[x, y]_\gamma + [x, y]_\gamma\delta\psi_\alpha(u, v)$

$= F(\psi_\alpha(u, v)\delta[x, y]_\gamma + [x, y]_\gamma\delta\psi_\alpha(u, v))$

$= F(\psi_\alpha(u, v)) \delta[x, y]_\gamma + \psi_\alpha(u, v)k(\delta)[x, y]\gamma + \psi_\alpha(u, v)\delta d([x, y]\gamma) + F([x, y]_\gamma)\delta\psi_\alpha(u, v) + [x, y]_\gamma k(\delta)\psi_\alpha(u, v) + [x, y]_\gamma\delta d(\psi_\alpha(u, v))$

$= F(\psi_\alpha(u, v))\delta[x, y]_\gamma + \psi_\alpha(u, v)\delta d([x, y]_\gamma) + F([x, y]_\gamma)\delta\psi_\alpha(u, v) + [x, y]_\gamma\delta d(\psi_\alpha(u, v))$

$= F (\psi_\alpha(u, v))\delta[x, y]_\gamma + \psi_\alpha(u, v)\delta F([x, y]_\gamma) + d([x, y]_\gamma)\delta \psi_\alpha(u, v) + [x, y]_\gamma\delta d(\psi_\alpha(u, v))$

$= F([x, y]_\gamma)\delta \psi_\alpha(u, v) + F([x, y]\alpha) \psi_\alpha(u, v) + [x, y]\gamma\delta d(\psi_\alpha(u, v)) + [x, y]\gamma\delta d(\psi_\alpha (u, v))$ [using the given condition]

$= 2F([x, y]_\gamma)\delta \psi_\alpha (u, v) + 2[x, y]_\gamma\delta d(\psi_\alpha(u, v))$

Since M is 2- torsion free , we have

$F([x, y]_\gamma)\delta\psi_\alpha(u, v) + [x, y]_\gamma\delta d(\psi_\alpha(u, v)) = 0$

Then, $F([x, y]_\gamma)\delta\psi_\alpha(u, v) = - [x, y]_\gamma\delta d(\psi_\alpha(u, v))$ .....(iv)

From (iii) and (iv) we obtain

$2\psi_\alpha(u, v)\ \delta\psi_\alpha(u, v)\ \delta\psi_\alpha(u, v) = \psi_\alpha(u, v)\delta F([u, v]\alpha)\ \delta\psi_\alpha(u, v)$

$= - \psi_\alpha(u, v)\delta[x, y]_\gamma\delta d(\psi_\alpha(u, v))$

That implies $\psi_\alpha(u, v)\delta\ \psi_\alpha(u, v)\delta\ \psi_\alpha(u, v) = 0$, since M is 2-torsion free. Hence it follows that $\psi_\alpha(u, v)$ is a nilpotent element of the Γ-ring M. But we know that the centre of a semiprime Γ- ring does not contain any nonzero nilpotent element.Therefore $\psi_\alpha(u, v) = 0$, for all u, v ∈ U ; α ∈ Γ.

For removing the condition F(u)αv = F(v)αu and uαd(v) = d(u)αv from the theorem 5.6, we start the following:

We know that an additive mapping T : M → M is called a left (right) centralizer of M if T(uαv) = T(u)αv (= uαT(v)) for all u, v ∈ M ; and α ∈ Γ. Also T is called a left (right) centralizer on U of M if T(uαv) = T(u)αv (= uαT(v)) for every u, v ∈ U and α ∈ Γ.

5.7 Theorem : Let M be a 2-torsion free semiprime Γ-ring satisfying the condition (*) and let U be a Lie ideal of M such that uαu ∈ U for every u ∈ U and α ∈ Γ.If F is a Jordan generalized k-derivation on U of M with an associated k-derivation d on U of M, then F is a generalized k-derivation on U of M.

Proof : We have F is a Jordan generalized k-derivation on U of M, then there exists a k- derivation d on U of M such that F(uαu) = F(u)αu + uk(α)u + uαd(u) , for every u ∈ U and α ∈ Γ.

Let T = F- d

Then T(uαu) = F(uαu) – d(uαu)

= F(u)αu + uk(α)u + uαd(u) – d(u)αu – uk(α)u – uαd(u)

$= (F(u) - d(u))\alpha u$

$= T(u)\alpha u$

Hence T is a Jordan left centralizer on U of M. By theorem 4.14 ,T is a left centralizer on U of M.

That is $T(u\alpha v) = T(u)\alpha v$ for all $u, v \in U$ and $\alpha \in \Gamma$.

Therefore $F(u\alpha v) = T(u\alpha v) + d(u\alpha v)$

$= T(u)\alpha v + d(u)\alpha v + uk(\alpha)v + ud(\alpha)v$

$= F(u)\alpha v + uk(\alpha)v + u\alpha d(v)$.

Bi-Derivations in Lie ideals of Gamma Rings

In this chapter we have studied the trace of symmetric bi-derivations on Lie ideals of prime and semiprime gamma rings. Using some conditions on the trace d of a bi-derivation D on a Lie ideal U of a gamma ring M,we have proved that either U is commutative or d is zero.

6. Introduction : We know that a subset U of M is a Lie ideal of M if $[u, m]_\alpha \in U$ for every $u \in U$, $m \in M$ and $\alpha \in \Gamma$. If U is a Lie ideal of M such that $u\alpha u \in U$ for every $u \in U$, $\alpha \in \Gamma$, then we have $u\alpha v + v\alpha u \in U$. Also $u\alpha v - v\alpha u \in U$. Hence we conclude that $2u\alpha v \in U$ for every u, v $\in$ U and $\alpha \in \Gamma$.

An additive mapping $d : M \to M$ is called a derivation if $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for every $x, y \in M$ and $\alpha, \beta \in \Gamma$.

A mapping D(,) : $M\times M \to M$ is called symmetric bi- additive, if it is additive in both the arguments and $D(x, y) = D(y, x)$ for all $x, y \in M$. Then the mapping $d: M \to M$ defined by $d(x) = D(x, x)$ is called the trace of D. A symmetric bi-additive mapping is called symmetric bi-derivation if $D(x\alpha y, z) = D(x, z)\alpha y + x\alpha D(y, z)$ for all $x, y, z \in M$ and $\alpha \in \Gamma$. We denote Z(M) as the centre of a Γ-ring M.

In [61] ,Vukman proved that the existance of a nonzero symmetric bi-derivation D in R, a prime ring of characteristic not two, with the property $D(x, x)\alpha x = x\alpha D(x, x)$, x in R, forces R to be commutative. In [3] Argac and Yenigul obtained the similar results on Lie ideals of R. In [49], Mehmet Ali, Ozturk, M .Sapanci ,M. Soyturk and Kyung Ho Kim extended all of [61] to the ideal of prime gamma rings. In [44], Maksa worked on the trace of symmetric bi-derivation. In [9], J.Bergen, I. N. Herstein worked on Lie ideals and derivations of prime rings. In [31], [32] I. N. Herstein worked on Lie structure of an associative ring. In [52], Posner has worked on the commutativity of a ring with derivation. In [33], Nadeem ur Rehman and Abu Jaid Ansari has worked on the trace of bi-derivation on a ring.

The objective of this paper is to obtain some general results of prime and semiprime gamma rings M considering various conditions on Lie ideals of M involving the trace of the symmetric bi-derivation D.

6.1 Definition : Let M be a Γ- ring . Then a mapping $D: M\times M \to M$ is called a bi- additive mapping if it is additive in both the arguments. It is called symmetric if $D(x, y) = D(y, x)$ for all $x, y \in M$.

6.2 Definition : Let M be a Γ- ring and $D : M\times M \to M$ be an additive mapping and U be a Lie ideal of M.Then D is called a bi - derivation on U of M if one of the following relations hold.

(i) $D(u\alpha v, w) = D(u, w)\alpha v + u\alpha D(v, w)$

(ii) $D(u, v\alpha w) = D(u, v)\alpha w + v\alpha D(u, w)$ for all $u, v, w \in U$; $\alpha \in \Gamma$.

Note : If D is symmetric , then these two relations are equivalent .

We need the following Lemmas for the next results.

6.3 Lemma : Let M be a 2 torsion free prime Γ- ring and U be a non zero Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$. Let $D : M \times M \to M$ be a symmetric bi-derivation and d be the trace of D such that

(i) $d(U) = 0$, then $U \subseteq Z(U)$ or $d = 0$.

(ii) $d(U) \subseteq Z(U)$, then $U \subseteq Z(U)$ or $d = 0$.

6.4 Lemma : Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If $D: M\times M\to M$ is a symmetric bi-derivation with trace d such that $[d(u), v]_\alpha \in Z(M)$ for all $u, v \in U$, $\alpha \in \Gamma$,then $U \subseteq Z(M)$ or $d = 0$.

Proof : Suppose that $U \not\subset Z(M)$. We have given that $[d(u), v]_\alpha \in Z(M)$ for all $u, v \in U$ and $\alpha \in \Gamma$.

Replacing v by $2v\beta w$; $w \in U$, $\beta \in \Gamma$, we get

$[d(u), 2v\beta w]_\alpha = 2v\beta[d(u), w]_\alpha + 2[d(u), v]_\alpha\beta w \in Z(M)$

That implies $v\beta[d(u), w]_\alpha + [d(u), v]_\alpha\beta w \in Z(M)$

Then we have $0 = [\,[d(u), v]_\alpha\beta w + v\beta[d(u), w]_\alpha , m]_\gamma$, for every $m \in M$.

$= [d(u), v]_\alpha\beta[w, m]_\gamma + [[d(u), v]_\alpha, m]_\gamma\beta w + v\beta[[d(u), w]_\alpha, m]_\gamma + [v, m]_\gamma\beta[d(u), w]_\alpha$

$= [d(u), v]_\alpha\beta[w, m]_\gamma + [v, m]_\gamma\beta[d(u), w]_\alpha$

Now in particular replacing m by w we obtain

$[v, w]_\gamma\beta[d(u), w]_\alpha = 0$

Replacing v by $2v\delta x$; $x \in U$,$\delta \in \Gamma$ we get

$0 = [2v\delta x, w]_\gamma\beta[d(u), w]_\alpha$

$= (2v\delta[x, w]_\gamma + 2[v, w]_\gamma\delta x)\,\beta[d(u), w]_\alpha$

$= 2v\delta[x, w]_\gamma\beta[d(u), w]_\alpha + 2[v, w]_\gamma\delta x\beta[d(u), w]_\alpha$

$= 2[v, w]_\gamma\delta x\beta[d(u), w]_\alpha$

And hence $[v, w]_\gamma\delta x\beta[d(u), w]_\alpha = 0$

Since M is prime , we have

$[v, w]_\gamma = 0$ or $[d(u), w]_\alpha = 0$

If $[v, w]\gamma = 0$, then $[U, U]_\Gamma = 0$ implies $U \subset Z(M)$, contradicts our assumption.

Hence $[d(u), w]_\alpha = 0$.

Now let $A = \{v \in U : [v, w]_\gamma = 0\}$ and $B = \{u \in U : [d(u), w]_\alpha = 0\}$.

Hence A and B are the additive subgroup of U such that $A \cup B = U$. By Brauer′s trick, we have either $U = A$ or $U = B$. If $A = U$, then $[v, w]_\gamma = 0$ for all $v, w \in U$ and then $U \subset Z(M)$, a contradiction. On the other hand if $U = B$, then $[d(u), w]_\alpha = 0$ for all $u, w \in U$ implies $d(u) \subseteq C_M(U) = Z(M)$. Then by Lemma 6.3 we get $d = 0$.

6.5 Theorem : Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and U be a Lie ideal of M such that $u \in U$ implies $u\alpha u \in U$ for all $\alpha \in \Gamma$. If $D: M\times M \rightarrow M$ is a symmetric bi-derivation with trace d such that $[d(u), u]_\alpha = 0$ for all $u \in U$, $\alpha \in \Gamma$. Then either $U \subseteq Z(M)$ or d = 0.

Proof : Suppose that $U \not\subset Z(M)$.

Here we have $[d(u), u]_\alpha = 0$, for all $u \in U$ and $\alpha \in \Gamma$.

Linearizing we have

$0 = [d(u+v), u+v]_\alpha$

$= [d(u) + d(v) + 2D(u, v) , u+v]_\alpha$

$= [d(u), u]_\alpha + [d(v), u]_\alpha + 2[D(u, v), u]_\alpha + [d(u), v]_\alpha + [d)(v), v]_\alpha + 2[D(u, v), v]_\alpha$

$= [d(v), u]_\alpha + 2[D(u, v), u]_\alpha + [d(u), v]_\alpha + 2[D(u, v), v]_\alpha$

Replacing u by – u we get

$0 = [d(v), -u]_\alpha + 2[D(-u, v), -u]_\alpha + [d(-u), v]_\alpha + 2[D(-u, v), v]_\alpha$

$= [d(u), v]_\alpha - [d(v), u]_\alpha + 2[D(u, v), u]_\alpha - 2[D(u, v), v]_\alpha$

Combining these two expressions we have

$0 = 2[d(u), v]_\alpha + 4 [D(u, v), u]_\alpha$

Using 2-torsion freeness of M we get

$[d(u), v]_\alpha + 2 [D(u, v), u]_\alpha = 0$

Replacing v by $2v\beta w$ we have

$0 = [d(u), 2v\beta w]_\alpha + 2 [D(u, 2v\beta w), u]_\alpha$

$= 2v\beta[d(u), w]_\alpha + 2[d(u), v]_\alpha\beta w + 4[D(u, v)\beta w, u]_\alpha + 4[v\beta D(u, w), u]_\alpha$

$= 2v\beta[d(u), w]_\alpha + 2[d(u), v]_\alpha\beta w + 4[D(u, v), u]_\alpha\beta w + 4D(u, v)\beta[w, u]_\alpha + 4v\beta[D(u, w), u]_\alpha + 4[v, u]_\alpha\beta D(u, w)$

$= 2v\beta([d(u), w]_\alpha +2[D(u, w), u]_\alpha) + 2([d(u), v]_\alpha +2[D(u, v), u]_\alpha)\beta w + 4D(u, v)\beta[w, u]_\alpha + 4[v, u]_\alpha\beta D(u, w)$

$= 4D(u, v)\beta[w, u]_\alpha + 4[v, u]_\alpha\beta D(u, w)$

Then $D(u, v)\beta[w, u]_\alpha + [v, u]_\alpha\beta D(u, w) = 0$

In particular putting w = u we obtain

$0 = [v, u]_\alpha \beta D(u, u) = [v, u]_\alpha \beta\, d(u)$

Replacing v by 2vγw, w ∈ U ; we have

$0 = [2v\gamma w, u]_\alpha \beta\, d(u)$

$= 2v\gamma[w, u]_\alpha \beta d(u) + 2[v, u]_\alpha \gamma w \beta d(u)$

$= 2[v, u]_\alpha \gamma w \beta d(u)$

And hence $[v, u]_\alpha \gamma w \beta d(u) = 0$

Since M is prime, w ∈ U, we have $[v, u]_\alpha = 0$ or d(u) = 0.

If $[v, u]_\alpha = 0$, then $[U, U]_\Gamma = 0$ implies U ⊆ Z(M), a contradiction.

Hence d(u) = 0. Then D(u, u) = 0.

If u ∈ Z(M) and w ∉ Z(M) then u+w, u-w ∉ Z(M).

Thus D(u+w, u+w) = 0 and D(u-w, u-w) = 0.

Adding these two relations we have,

0 = D(u+w, u+w) + D(u-w, u-w)

= D(u, u) + D(u, w) + D(w, u) + D(w, w) + D(u, u) - D(u, w) – D(w, u) + D(w, w)

= 2d(u) + 2d(w). Then

0 = 2d(U) + 2d(U) = 4d(U). Therefore, d(U) = 0.

Since U ⊄ Z(M) , hence d = 0.

6.6 Theorem : Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and U be a Lie ideal of M such that u ∈ U implies uαu ∈ U for all α ∈ Γ. Suppose that D: M×M → M is a symmetric bi-derivation with trace d such that $d([u, v]_\alpha) - [d(u), v]_\alpha \in Z(M)$ for all u, v ∈ U ; α ∈ Γ. Then either U ⊆ Z(M) or d = 0.

Proof: Suppose that U ⊄ Z(M) . We have

$d([u, v]_\alpha) – [d(u), v]_\alpha \in Z(M)$ for all u, v ∈ U ; α ∈ Γ.

Replacing v by v+w in the above expression, we obtain

$d([u, v+w]_\alpha) – [d(u), v+w]_\alpha$

$= d[u, v]_\alpha + d[u, w]_\alpha + 2D([u, v]_\alpha , [u, w]_\alpha) – [d(u), v]_\alpha – [d(u), w]_\alpha$

$= d([u, v]_\alpha) - [d(u), v]_\alpha + d([u, w]_\alpha) - [d(u), w]_\alpha + 2D([u, v]_\alpha, [u, w]_\alpha) \in Z(M)$

This implies that $2D([u, v]_\alpha, [u, w]_\alpha \in Z(M)$.

Hence $D([u, v]_\alpha, [u, w]_\alpha) \in Z(M)$.

In particular putting w = v we find that

$D([u, v]_\alpha, [u, v]_\alpha) \in Z(M)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

That means $d([u, v]_\alpha) \in Z(M)$ and so $[d(u), v]_\alpha \in Z(M)$.

By Lemma 6.4, since $U \not\subset Z(M)$, we have d = 0.

6.7 Theorem : Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$. Suppose that $D: M\times M \rightarrow M$ is a symmetric bi-derivation with trace d such that $d((uov)_\alpha) - [d(u), v]_\alpha \in Z(M)$ for all $u, v \in U$, $\alpha \in \Gamma$.Then either $U \subseteq Z(M)$ or d = 0.

Proof : Suppose that $U \not\subset Z(M)$. We have

$d((uov)_\alpha) - [d(u), v]_\alpha \in Z(M)$ for all $u, v \in U$, $\alpha \in \Gamma$.

Replacing v by v+w we get

$d((uo(v+w))_\alpha) - [d(u), v+w]_\alpha$

$= d(uov)_\alpha + d(uow)_\alpha + 2D(uov, uow)_\alpha - [d(u), v]_\alpha - [d(u), w]_\alpha \in Z(M)$

$= d(uov)_\alpha - [d(u), v]_\alpha + d(uow)_\alpha - [d(u), w]_\alpha + 2D(uov, uow)_\alpha \in Z(M)$

That implies $2D(uov, uow)_\alpha \in Z(M)$ and hence $D(uov, uow)_\alpha \in Z(M)$.

In particular putting w = v we get

$D(uov, uov) \in Z(M)$.

That means $d(uov) \in Z(M)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

In view of our hypothesis we have $[d(u), v]_\alpha \in Z(M)$.

Then by Lemma 6.4, since $U \not\subset Z(M)$, we have d = 0.

6.8Theorem : Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$,

$\alpha \in \Gamma$. Suppose that $D: M\times M\rightarrow M$ is a symmetric bi-derivation with trace d such that $(d(u)ov)_\alpha - [d(u), v]_\alpha \in Z(M)$ for all $u, v \in U$, $\alpha \in \Gamma$. Then either $U \subseteq Z(M)$ or $d = 0$.

Proof : Suppose that $U \not\subset Z(M)$.

We have $(d(u) \text{ o } v)_\alpha - [d(u), v] \in Z(M)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

This implies that $(d(u)ov)_\alpha - [d(u), v]_\alpha$

$= d(u)\alpha v + v\alpha d(u) - d(u)\alpha v + v\alpha d(u)$

$= 2v\alpha d(u) \in Z(M)$.

And then $v\alpha d(u) \in Z(M)$.

Hence $[v\alpha d(u), m]_\beta = 0$, for all $m \in M$; $u, v \in U$ and $\alpha, \beta \in \Gamma$.

Then $v\alpha[d(u), m]_\beta + [v, m]_\beta\alpha d(u) = 0$ ……….(i)

Replacing v by $2w\gamma v$; $w \in U$; $\gamma \in \Gamma$, we have,

$0 = 2w\gamma v\alpha[d(u), m]_\beta + [2w\gamma v, m]_\beta\alpha d(u)$

$= 2w\gamma v\alpha[d(u), m]_\beta + 2w\gamma[v, m]_\beta\alpha d(u) + 2[w, m]_\beta\gamma v\alpha d(u)$

$= 2w\gamma(v\alpha[d(u), m]_\beta + [v, m]_\beta\alpha d(u)) + 2[w, m]_\beta\gamma v\alpha d(u)$

$= 2[w, m]_\beta\gamma v\alpha d(u)$ [From (i)]

Hence $[w, m]_\beta\gamma v\alpha d(u) = 0$

Putting $m\delta x$ for m ($x \in M$), we get

$0 = [w, m\delta x]_\beta\gamma v\alpha d(u) = [w, m]_\beta\delta x\gamma v\alpha d(u) + m\delta[w, x]_\beta\gamma v\alpha d(u)$

$= [w, m]_\beta\delta x\gamma v\alpha d(u)$

Since M is prime, we have $[w, m]_\beta = 0$ or $d(u) = 0$

If $[w, m]_\beta = 0$ then $U \subseteq Z(M)$, a contradiction.

Hence we have $d(u) = 0$, for all $u \in U$.

Since $U \not\subset Z(M)$, we have $d = 0$.

6.9 Theorem : Let M be a 2-torsion free prime Γ -ring satisfying the condition (*) and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$. Suppose that $D: M\times M\rightarrow M$ is a symmetric bi-derivation with trace d and g : M→M is any mapping such that $[d(u),v]_\alpha$ - $[u, g(v)]_\alpha \in$ Z(M) for all u ,v ∈ U, α ∈ Γ.Then either U ⊆ Z(M) or d = 0.

Proof : Suppose that U ⊄ Z(M).

Given that $[d(u), v]_\alpha - [u, g(v)]_\alpha \in Z(M)$, for all u, v ∈ U; α ∈ Γ.

Replacing u by u + w in the above expression, we get

$[d(u+w), v]_\alpha - [u+w, g(v)]_\alpha \in Z(M)$

$[d(u) + d(w) + 2D(u, w), v]_\alpha - [u+w, g(v)]_\alpha$

$=[d(u), v]_\alpha + [d(w), v]_\alpha + 2[D(u, w), v]_\alpha - [u, g(v)]_\alpha - [w, g(v)]_\alpha \in Z(M)$

Using the hypothesis we have

$[2D(u, w), v]_\alpha \in Z(M)$. Then $[D(u, w), v]_\alpha \in Z(M)$.

In particular putting w = u, we find that

$[D(u, u), v]_\alpha = [d(u), v]_\alpha \in Z(M)$.

From Lemma 6.4 since U ⊄ Z(M) , we have d = 0.

6.10 Theorem : Let M be a 2-torsion free prime Γ -ring satisfying the condition (*) and U be a Lie ideal of M such that uαu ∈ U for all u ∈ U, α ∈ Γ.Suppose that D : M×M→M is a symmetric bi- derivation with trace d such that $(d(u)od(v))_\alpha - [d(u), v]_\alpha \in Z(M)$ for all u, v ∈ U, α ∈ Γ. Then either U ⊆ Z(M) or d = 0.

Proof : Suppose that U ⊄ Z(M) .

Given that $(d(u)od(v))_\alpha - [d(u),v]_\alpha \in Z(M)$, for all u, v ∈ U , α ∈ Γ.

Replacing v by v+w in the above expression, we have

$(d(u)\ od(v+w))_\alpha - [d(u),v+w]_\alpha$

$= (d(u)\ o\ d(v))_\alpha + (d(u)\ o\ d(w))_\alpha + (d(u)\ o\ 2D(v, w))_\alpha - [d(u), v]_\alpha - [d(u), w]_\alpha \in Z(M)$

That implies 2(d(u) o D(v, w)) and hence (d(u) o D(v, w)) ∈ Z(M).

In particular putting w = v we find that (d(u) o D(v, v)) ∈ Z(M) .

So (d(u) od(v)) ∈ Z(M).

Using the hypothesis we have $[d(u), v]_\alpha \in Z(M)$.

From Lemma 6.4, since $U \not\subset Z(M)$, we have $d = 0$.

6.11 Theorem : Let M be a 2-torsion free prime Γ -ring satisfying the condition (*) and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$.Suppose that $D: M\times M \to M$ is a symmetric bi-derivation with trace d and $g : M \to M$ be any mapping such that $d(u)\alpha v - u\alpha g(v) \in Z(M)$ for all $u,v \in U$; $\alpha \in \Gamma$. Then either $U \subseteq Z(M)$ or $d = 0$.

Proof : Suppose that $U \not\subset Z(M)$ .Given that

$d(u)\alpha v - u\alpha g(v) \in Z(M)$ for all $u,v \in U$, $\alpha \in \Gamma$.

Replacing u by u+w ; $w \in U$, we get,

$d(u+w)\alpha v - (u+w)\alpha g(v)$

$= d(u)\alpha v + d(w)\alpha v + 2D(u, w)\alpha v - u\alpha g(v) - w\alpha g(v) \in Z(M)$.

That implies $2D(u, w)\alpha v \in Z(M)$ and so $D(u, w)\alpha v \in Z(M)$.

In particular putting $w = u$ we get $d(u)\alpha v \in Z(M)$.

Then $[d(u)\alpha v, m]\beta = 0$ for all $u, v \in U$; $\alpha, \beta \in \Gamma$; $m \in M$.

That is $d(u)\alpha[v, m]_\beta + [d(u), m]_\beta\alpha v = 0$

Replacing v by $2v\gamma t$ we get

$0 = d(u)\alpha[2v\gamma t, m]_\beta + [d(u), m]_\beta\alpha 2v\gamma t$

$= 2d(u)\alpha v\gamma[t, m]_\beta + 2d(u)\alpha[v, m]_\beta\gamma t + 2[d(u), m]_\beta\alpha v\gamma t$

$= 2(d(u)\alpha v\gamma[t, m]_\beta + (d(u)\alpha[v, m]_\beta + [d(u), m]_\beta\alpha v)\gamma t)$

$= 2\, d(u)\alpha v\gamma[t, m]_\beta$

And hence $d(u)\alpha v\gamma[t, m]_\beta = 0$, for all $u, v, t \in U$; $m \in M$; $\alpha, \beta, \gamma \in \Gamma$.

Since M is prime $d(u) \in M$, from [52, Lemma 2.10], we have $d(u) = 0$ or $[t, m]_\beta = 0$

If $[t, m]_\beta = 0$, for all $t \in U$; $m \in M$, then $U \subseteq Z(M)$, a contradiction. Hence we have $d(u) = 0$ .Then from Lemma 6.3 since $U \not\subset Z(M)$, $d = 0$.

6.12 Theorem : Let M be a. 2-torsion free prime Γ -ring satisfying the condition (*) and U be a Lie ideal of M , such that $u\alpha u \in U$ for all $u \in U$,

$\alpha \in \Gamma$. Suppose that $D: M \times M \to M$ is a symmetric bi- derivation with trace d such that $d(u\alpha v) - d(u)\alpha v - u\alpha d(v) \in Z(M)$ for all $u, v \in U$, $\alpha \in \Gamma$. Then either $U \subseteq Z(M)$ or $d = 0$.

Proof : Suppose that $U \not\subset Z(M)$. Given that $d(u\alpha v) - d(u)\alpha v - u\alpha d(v) \in Z(M)$ for all $u, v \in U$; $\alpha \in \Gamma$. Replacing u by u+w we get

$d((u+w)\alpha v) - d(u+w)\alpha v - (u+w)\alpha d(v)$

$= d(u\alpha v) + d(w\alpha v) + 2D(u\alpha v, w\alpha v) - d(u)\alpha v - d(w)\alpha v - 2D(u, w)\alpha v - u\alpha d(v) - w\alpha d(v) \in Z(M)$

That implies $2D(u\alpha v, w\alpha v) - 2D(u, w)\alpha v \in Z(M)$

Hence $D(u\alpha v, w\alpha v) - D(u, w)\alpha v \in Z(M)$.

In particular putting $w = u$ we get

$D(u\alpha v, u\alpha v) - D(u, u)\alpha v = d(u\alpha v) - d(u)\alpha v \in Z(M)$ …….(i)

Again replacing v by v+w we have

$d(u\alpha(v+w)) - d(u)\alpha(v+w)$

$= d(u\alpha v) + d(u\alpha w) + 2D(u\alpha v, u\alpha w) - d(u)\alpha v - d(u)\alpha w \in Z(M)$

That implies $2D(u\alpha v, u\alpha w) \in Z(M)$ and so $D(u\alpha v, u\alpha w) \in Z(M)$

In particular putting $w = v$ we get $D(u\alpha v, u\alpha v) = d(u\alpha v) \in Z(M)$.

Using relation (i) we have $d(u)\alpha v \in Z(M)$ for all $u, v \in U$; $\alpha \in \Gamma$.

Then $[d(u)\alpha v, m]_\beta = 0$, for all $u, v \in U$; $m \in M$; $\alpha, \beta \in \Gamma$.

i.e., $d(u)\alpha[v, m]_\beta + [d(u), m]_\beta\alpha v = 0$

In particular putting $m = d(u)$, we get $d(u)\alpha[v, d(u)]_\beta = 0$ for all $u, v \in U$; $m \in M$; $\alpha, \beta \in \Gamma$.

Now replacing v by $2v\gamma w$ we get

$0 = d(u)\alpha[2v\gamma w, d(u)]_\beta$

$= 2d(u)\alpha[v, d(u)]_\beta\gamma w + 2d(u)\alpha v\gamma[w, d(u)]_\beta$

$= 2d(u)\alpha v\gamma[w, d(u)]_\beta$

Then $d(u)\alpha v\gamma[w, d(u)]_\beta = 0$

Also $d(u)\alpha v\gamma[d(u), w]_\beta = 0$ …….(2)

Then $z\delta d(u)\alpha v\gamma[w, d(u)]_\beta = 0$; $z \in U$, $\delta \in \Gamma$.

Again replacing v by $2z\delta v$ in (2) , we get

$0 = d(u)\alpha 2z\delta v\gamma[d(u), w]_\beta$

That implies $d(u)\alpha z\delta v\gamma[d(u), w]_\beta = 0$ ….(3)

Combining (2) and (3) and using (*) we have

$0 = (d(u)\alpha z - z\alpha d(u))\delta v\gamma[d(u), w]\beta$

$= [d(u), z]_\alpha\delta v\gamma[d(u), w]_\beta$, for all $u, v, w \in U$; $\alpha, \beta, \gamma, \delta \in \Gamma$.

Since M is prime , we have $[d(u), z]_\alpha = 0$ or $[d(u), w]_\beta = 0$

In both cases we have $d(u) \subseteq Z(U) = Z(M)$.

Then from Lemma 6.3(ii) we have $d = 0$.

6.13 Theorem : Let M be a 2-torsion free prime Γ -ring satisfying the condition (*) and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$. Suppose that $D : M \times M \to M$ is a symmetric bi- derivation with trace d such that $d(u\alpha v) - v\alpha d(u) - d(v)\alpha u \in Z(M)$ for all $u, v \in U$; $\alpha \in \Gamma$. Then either $U \subseteq Z(M)$ or $d = 0$.

Proof : Suppose that $U \not\subset Z(M)$. Here we have

$d(u\alpha v) - v\alpha d(u) - d(v)\alpha u \in Z(M)$, for all $u, v \in U$; $\alpha \in \Gamma$.

Replacing u by u+w we get

$d((u + w)\alpha\, v) - v\alpha d(u+w) - d(v)\alpha(u + w)$

$= d(u\alpha v) + d(w\alpha v) + 2D(u\alpha v, w\alpha v) - v\alpha d(u) - v\alpha d(w) - 2v\alpha D(u, w) - d(v)\alpha u - d(v)\alpha w \in Z(M)$

Then $2D(u\alpha v, w\alpha v) - 2v\alpha D(u, w) \in Z(M)$

That means $D(u\alpha v, w\alpha v) - v\alpha D(u, w) \in Z(M)$

In particular putting $w = u$, we obtain that $D(u\alpha v, u\alpha v) - v\alpha D(u, u) \in Z(M)$.

That implies $d(u\alpha v) - v\alpha d(u) \in Z(M)$

Using hypothesis we have $d(v)\alpha u \in Z(M)$ for all $u, v \in U$; $\alpha \in \Gamma$.

Then $[d(v)\alpha u, m]_\beta = 0$ for all $m \in M$.

That implies $d(v)\alpha[u, m]_\beta + [d(v), m]_\beta\alpha u = 0$

Replacing u by 2uγw we find that

$0 = d(v)\alpha[2u\gamma w, m]_\beta + [d(v), m]_\beta\alpha 2u\gamma w$

$= 2d(v)\alpha u\gamma[w, m]_\beta + 2d(v)\alpha[u, m]_\beta\gamma w + 2[d(v), m]_\beta\alpha u\gamma w$

$= 2(d(v)\alpha[u, m]_\beta + [d(v), m]_\beta u)\gamma w + 2d(v)\alpha u\gamma[w, m]_\beta$

$= 2d(v)\alpha u\gamma[w, m]_\beta$

Hence $d(v)\alpha u\gamma[w, m]_\beta = 0$

Since M is prime and $d(v) \in M$, we have $d(v) = 0$ or $[w, m]_\beta = 0$

If $[w, m]_\beta = 0$, then $U \subseteq Z(M)$; a contradiction . Hence we have d = 0.

6.14 Theorem : Let M be a 2-torsion free prime Γ -ring satisfying the condition (*) and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$. Suppose that $D: M \times M \to M$ is a symmetric bi-derivation with trace d such that $d(u\alpha v) - u\alpha d(v) - v\alpha d(u) \in Z(M)$ for all $u, v \in U$; $\alpha \in \Gamma$.Then either $U \subseteq Z(M)$ or d=0.

Proof: Suppose that $U \not\subset Z(M)$. We have

$d(u\alpha v) - u\alpha d(v) - v\alpha d(u) \in Z(M)$ for all $u, v \in U$; $\alpha \in \Gamma$.

Replacing u by u+w, we have

$d((u+w)\alpha v) - (u+w)\alpha d(v) - v\alpha d(u+w)$

$= d(u)\alpha v + d(w\alpha v) + 2D(u\alpha v, w\alpha v) - u\alpha d(v) - w\alpha d(v) - v\alpha d(u) - v\alpha d(w)$

$- 2v\alpha D(u, w) \in Z(M)$

Then $2D(u\alpha v, w\alpha v) - 2v\alpha D(u, w) \in Z(M)$

And so $D(u\alpha v, w\alpha v) - v\alpha D(u, w) \in Z(M)$

In particular putting w = u we have

$D(u\alpha v, u\alpha v) - v\alpha D(u, u) = d(u\alpha v) - v\alpha d(u) \in Z(M)$

Using the hypothesis we have $u\alpha d(v) \in Z(M)$

Then $[u\alpha d(v), m]_\beta = 0$ for all $u, v \in U$; $m \in M$; $\alpha, \beta \in \Gamma$.

Using similar process of the proof of last theorem we get d = 0.

6.15 Theorem : Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$. Suppose that $D: M \times M \to M$ is a symmetric bi- derivation with trace d such that $d([u, v]_\alpha) - [d(u), v]_\alpha - [u, d(v)]_\alpha \in Z(M)$ for all $u, v \in U$, $\alpha \in \Gamma$. Then either $U \subseteq Z(M)$ or $d = 0$.

Proof : Suppose that $U \not\subset Z(M)$. We have given that

$d([u, v]_\alpha) - [d(u), v]_\alpha - [u, d(v)]_\alpha \in Z(M)$ for all $u, v \in U$, $\alpha \in \Gamma$.

Replacing u by u+w we get

$d([u+w, v]_\alpha) - [d(u+w), v]_\alpha - [u+w, d(v)]_\alpha$

$= d([u, v]_\alpha) + d([w, v]_\alpha) + 2D([u, v]_\alpha, [w, v]_\alpha) - [d(u), v]_\alpha - [d(w), v]_\alpha - 2[D(u, w), v]_\alpha - [u, d(v)]_\alpha - [w, d(v)]_\alpha \in Z(M)$

That implies $2D([u, v]_\alpha, [w, v]_\alpha) - 2[D(u, w), v]_\alpha \in Z(M)$

Using the hypothesis we have $[u, d(v)]_\alpha \in Z(M)$.

Therefore $[d(v), u]_\alpha \in Z(M)$.

Hence from Lemma 6.4, we have $d = 0$.

Results on Semiprime Γ- rings

6.16 Theorem : Let M be a 2-torsion free semiprime Γ-ring and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$. Suppose that $D: M \times M \to M$ is a symmetric bi-additive mapping with trace d such that $d([u, v]_\alpha) - [u, v]_\alpha \in Z(M)$ for all $u, v \in U$, $\alpha \in \Gamma$.Then either $U \subseteq Z(M)$ or $d = 0$.

Proof : We have $d([u, v]_\alpha) - [u, v]_\alpha \in Z(M)$ for all $u, v \in U$; $\alpha \in \Gamma$.

Replacing u by u+w we get ,

$d([u+w, v]_\alpha) - [u+w, v]_\alpha$

$= d([u, v]_\alpha) + d([w, v]_\alpha) + 2D([u, v]_\alpha, [w, v]_\alpha) - [u, v]_\alpha - [w, v]_\alpha \in Z(M)$.

That implies $2D([u, v]_\alpha, [w, v]_\alpha) \in Z(M)$. And hence $D([u, v]_\alpha, [w, v]_\alpha) \in Z(M)$.

In particular putting $w = u$ we get $D([u, v]_\alpha, [u, v]_\alpha) = d([u, v]_\alpha) \in Z(M)$.
Hence from the hypothesis $[u, v]_\alpha \in Z(M)$
That implies $[u, v]_\Gamma \subseteq Z(M)$. Therefore, $U \subseteq Z(M)$.

6.17 Theorem : Let M be a 2-torsion free semiprime Γ-ring and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$. Suppose that $D: M\times M \to M$ is a symmetric bi- additive mapping with trace d such that $d((uov)_\alpha) - (uov)_\alpha \in Z(M)$ for all $u, v \in U$; $\alpha \in \Gamma$. Then $U \subseteq Z(M)$.

Proof : Suppose that $U \not\subset Z(M)$. We have

$d(uov)_\alpha - (uov)_\alpha \in Z(M)$, for all $u, v \in U$; $\alpha \in \Gamma$.

Replacing u by u+w we get

$d(u+w \text{ o } v)_\alpha - (u+w \text{ o } v)_\alpha$

$= d(uov)_\alpha + d(wov)_\alpha + 2D(uov, wov)_\alpha - (uov) + wov)_\alpha \in Z(M)$

That implies $2D(uov, wov)_\alpha \in Z(M)$ and hence $D(uov, wov)_\alpha \in Z(M)$.

In particular putting $w = u$ we obtain, $D(uov, uov)_\alpha = d(uov)_\alpha \in Z(M)$.

Using hypothesis we have $(uov)_\alpha \in Z(M)$, for all $u, v \in U$; $\alpha \in \Gamma$.

Replacing u by $2v\beta u$ we have

$(2v\beta u \text{ o } v)_\alpha = 2v\beta(uov)_\alpha + 2(vov)_\alpha\beta u = 2v\beta(uov)_\alpha \in Z(M)$

Then $v\beta(uov)_\alpha \in Z(M) = Z(U)$, for all $u, v \in U$; $\alpha, \beta \in \Gamma$.

Therefore, $[v\beta(uov)_\alpha , w]_\gamma = 0$, for all $u, v, w \in U$; $\alpha, \beta \in \Gamma$.

$0 = v\beta[(uov)_\alpha, w]_\gamma + [v, w]_\gamma\beta(uov)_\alpha$

$= [v, w]_\gamma\beta(uov)_\alpha$ [since $(uov)_\alpha \in Z(M) = Z(U)$]

Again replacing u by $2u\delta w$ we obtain

$0 = [v, w]_\gamma\beta 2(u\delta wov)_\alpha$

$= 2[v, w]_\gamma\beta((uov)_\delta w + u\delta[w, v]_\alpha)$

$= 2[v, w]_\gamma\beta(uov)_\delta w + 2[v, w]_\gamma\beta u\delta[w, v]_\alpha$

$= 2[v, w]_\gamma\beta u\delta[w, v]_\alpha$

That implies $[v, w]_\gamma\beta u\delta[w, v]_\alpha = 0$. And then $[v, w]_\gamma\beta U\delta[v, w]_\alpha = 0$

By Lemma 4.13 we get $U \subseteq Z(M)$, a contradiction.

6.18 Theorem : Let M be a 2-torsion free semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$. Suppose that $D: M \times M \to M$ is a symmetric bi-additive mapping with trace d such that $d([u, v]_\alpha) - (uov)_\alpha \in Z(M)$ for all $u, v \in U$, $\alpha \in \Gamma$.Then $U \subseteq Z(M)$.

Proof : Suppose that $U \not\subset Z(M)$. Given that

$d([u,v]_\alpha) - (uov)_\alpha \in Z(M)$ for all $u,v \in U$, $\alpha \in \Gamma$.

Replacing u by u+w we get

$d([u+w, v]_\alpha) - (u+w \text{ o } v)_\alpha$

$= d([u, v]_\alpha) + d([w, v]_\alpha) + 2D([u, v]_\alpha, [w, v]_\alpha) - (uov)_\alpha - (wov)_\alpha \in Z(M)$

Using the hypothesis we have $2D([u, v]_\alpha, [w, v]_\alpha) \in Z(M)$

In particular putting w = u we find that $2D([u, v]_\alpha, [u, v]_\alpha) = 2d([u, v]_\alpha) \in Z(M)$.

Then $d([u, v]_\alpha) \in Z(M)$.

Again using hypothesis $(uov)_\alpha \in Z(M)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

From the last steps of proof of theorem 6.17 we have $U \subseteq Z(M)$.

6.19 Theorem : Let M be a 2-torsion free semiprime Γ -ring and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$. Suppose that $D: M\times M \to M$ is a symmetric bi- additive mapping with trace d such that $d((uov)_\alpha) - [u, v]_\alpha \in Z(M)$ for all $u, v \in U$, $\alpha \in \Gamma$.Then either $U \subseteq Z(M)$ or $d = 0$.

Proof : We have given that $d((uov)_\alpha) - [u, v]_\alpha \in Z(M)$ for all $u, v \in U$; $\alpha \in \Gamma$.

Replacing u by u+w we have $d((u+w \text{ o } v)_\alpha) - [u+w, v]_\alpha$

$= d((uov)_\alpha) + d((wov)_\alpha) + 2D(uov, wov)_\alpha - [u, v]_\alpha - [w, v]_\alpha \in Z(M)$

That implies $2D(uov, wov)_\alpha \in Z(M)$ and hence $D(uov, wov)_\alpha \in Z(M)$.

In particular putting w = u we get

$D(uov, uov)_\alpha = d(uov)_\alpha \in Z(M)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

Using hypothesis we have $[u, v]\alpha \in Z(M)$.

That means $[U, U]_{\Gamma} \subseteq Z(M)$. Therefore by Lemma 2.15 we get $U \subseteq Z(M)$.

6.19 Theorem : Let M be a 2-torsion free semiprime Γ-ring and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$.Suppose that $D: M{\times}M \to M$ is a symmetric bi- additive mapping with trace d such that $2(uov)_{\alpha} = d(u) - d(v)$ for all $u,v \in U$, $\alpha \in \Gamma$.Then $U \subseteq Z(M)$.

Proof : Suppose that $U \not\subset Z(M)$.

We have $2(uov)_{\alpha} = d(u) - d(v)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

Replacing u by u+v in the above expression , we obtain

$2(u+v \text{ o } v)_{\alpha} = d(u+v) - d(v)$

Computing both sides we have $4v\alpha v = 2D(u, v) + d(v)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

Replacing u by – u we get $4v\alpha v = 2D(-u, v) + d(v)$

Combining these two expressions , we have $8v\alpha v = 2d(v)$

That implies $4v\alpha v = d(v)$.

Putting $u = v$ in our hypothesis we have $4v\alpha v = 0$.Then $2(uov)_{\alpha} = 0$.

And so $(uov)_{\alpha} = 0$, for all $u, v \in U$; $\alpha \in \Gamma$.

Replacing u by $2u\beta w$ we have ,

$0 = (2u\beta w \text{ o } v)_{\alpha} = 2(uov)_{\alpha}\beta w + 2u\beta[w, v]_{\alpha} = 2u\beta[w, v]_{\alpha}$.

Hence $u\beta[w, v]_{\alpha} = 0$. That implies

$0 = (uov)_{\alpha}\beta w + u\beta[w, v]_{\alpha} = u\beta[w, v]_{\alpha}$.

Hence $[w, v]_{\alpha}\gamma u\beta[w, v]_{\alpha} = 0$ for all $u, v, w \in U$ and $\alpha, \beta, \gamma \in \Gamma$.

Since M is semiprime , we have $[w, v]_{\alpha} = 0$ .That is $[U, U]_{\Gamma} = 0$. Therefore $U \subseteq Z(M)$, a contradiction.

6.21 Theorem : Let M be a 2-torsion free semiprime Γ-ring and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$.Suppose that D:

$M \times M \to M$ is a symmetric bi-additive mapping with trace d such that $(d(u)od(v)_{\alpha}) - (uov)_{\alpha} \in Z(M)$ for all $u,v \in U$, $\alpha \in \Gamma$.Then $U \subseteq Z(M)$.

Proof: Suppose that $U \not\subset Z(M)$ .We have

$(d(u)od(v))_{\alpha} - (uov)_{\alpha} \in Z(M)$, for all $u,v \in U$, $\alpha \in \Gamma$.

Replacing v by v+w in the above expression we get

$(d(u)od(v+w))_{\alpha} - (uov+w)_{\alpha}$

$= (d(u)od(v))_{\alpha} + (d(u)od(w)_{\alpha} + 2(d(u)oD(v, w))_{\alpha} - (uov)_{\alpha} - (uow)_{\alpha} \in Z(M)$.

From the hypothesis we have $(d(u)o2D(v, w))_{\alpha} \in Z(M)$.

That implies $(d(u)oD(v, w))_{\alpha} \in Z(M)$.

In particular putting $w = v$ we have $(d(u)od(v))_{\alpha} \in Z(M)$.

From the hypothesis we find that $(uov)_{\alpha} \in Z(M)$.

Then using same technique used in Theorem 6.17 we have $U \subseteq Z(M)$, a contradiction.

6.22 Theorem : Let M be a 2-torsion free semiprime Γ -ring and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$. Suppose that D: $M \times M \to M$ is a symmetric bi-additive mapping with trace d such that $(d(u)od(v))_{\alpha} - [u, v]_{\alpha} \in Z(M)$ for all $u, v \in U$, $\alpha \in \Gamma$.Then $U \subseteq Z(M)$.

Proof : Given that $(d(u)od(v))_{\alpha} - [u, v]_{\alpha} \in Z(M)$ for all $u, v \in U$; $\alpha \in \Gamma$.

Replacing v by v+w in the above expression we have

$(d(u)od(v+w))_{\alpha} - [u, v+w]_{\alpha}$

$= (d(u) \text{ o } d(v))_{\alpha} + (d(u)od(w))_{\alpha} + (d(u)o2D(v, w))_{\alpha} - [u, v]_{\alpha} - [u, w]_{\alpha} \in Z(M)$

Using hypothesis we have $2(d(u)oD(v, w))_{\alpha} \in Z(M)$

and so $(d(u)oD(v, w))_{\alpha} \in Z(M)$.

In particular putting $w = v$ we have

$(d(u)oD(v, v))_{\alpha} = (d(u)od(v))_{\alpha} \in Z(M)$, for all $u, v \in U$ and $\alpha \in \Gamma$.

That implies $[u, v]_{\alpha} \subseteq Z(M)$,for all $u, v \in U$ and $\alpha \in \Gamma$.

That is $[U, U]_\Gamma \subseteq Z(M)$ which means $U \subseteq Z(M)$.

6.23 Theorem : Let M be a 2-torsion free semiprime Γ -ring and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$.Suppose that $D: M\times M \to M$ is a symmetric bi- additive mapping with trace d such that $u\alpha v - d(u) = v\alpha u - d(v)$ for all $u, v \in U$; $\alpha \in \Gamma$. Then $U \subseteq Z(M)$.

Proof: We have $u\alpha v - d(u) = v\alpha u - d(v)$ for all $u, v \in U$; $\alpha \in \Gamma$.

We can write $[u, v]_\alpha = d(u) - d(v)$

Replacing u by u+v we have $[u+v, v]_\alpha = d(u+v) - d(v)$

That implies $[u, v]_\alpha = d(u) + 2D(u, v)$

Putting – u for u we have $-[u, v]_\alpha = d(u) - 2D(u, v)$

Combining these two expressions we have $d(u) = 0$ for $u \in U$.

From the hypothesis we have $u\alpha v = v\alpha u$ for all $u, v \in U$;$\alpha \in \Gamma$. i.e., $[u,v]_\alpha = 0$.

Hence from Lemma 2.15 we get $U \subseteq Z(M)$.

6.24 Theorem : Let M be a 2-torsion free semiprime Γ -ring and U be a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$, $\alpha \in \Gamma$.Suppose that $D : M\times M\to M$ is a symmetric bi-derivation with trace d such that $[u, v]_\alpha = d(u\alpha v) - d(v\alpha u)$ for all $u, v \in U$; $\alpha \in \Gamma$. Then $U \subseteq Z(M)$.

Proof : Given that $[u, v]_\alpha = d(u\alpha v) - d(v\alpha u)$ for all $u, v \in U$; $\alpha \in \Gamma$.

This can be written as $[u, v]_\alpha = D(u\alpha v, u\alpha v) - D(v\alpha u, v\alpha u)$

$= u\alpha D(v, u\alpha v) + D(u, u\alpha v)\alpha v - v\alpha D(u, v\alpha u) - D(v, v\alpha u)\alpha u$

$= u\alpha u\alpha D(v, v) + u\alpha D(v, u)\alpha v + u\alpha D(u, v)\alpha v + D(u, u)\alpha v\alpha v - v\alpha v\alpha D(u, u) - v\alpha D(u, v)\alpha u - v\alpha D(v, u)\alpha u - D(v, v)\alpha u\alpha u$

$= u\alpha u\alpha d(v) + u\alpha D(v, u)\alpha v + u\alpha D(u, v)\alpha v + d(u)\alpha v\alpha v - v\alpha v\alpha d(u) - v\alpha D(u, v)\alpha u - v\alpha D(v,u)\alpha u - d(v)\alpha u\alpha u$

$= u\alpha u\alpha d(v) + 2u\alpha D(v, u)\alpha v + d(u)\alpha v\alpha v - v\alpha v\alpha d(u) -2 v\alpha D(u, v)\alpha u - d(v)\alpha u\alpha u$

[since $D(u, v) = D(v, u)$]

$= [u\alpha u, d(v)]_\alpha + [d(u), v\alpha v]_\alpha + 2u\alpha D(u, v)\alpha v - 2v\alpha D(u, v)\alpha u$ ....(i)

Replacing u by u+v in (i) we have

$[u+v, v]_\alpha = [(u+v)\alpha(u+v), d(v)]_\alpha + [d(u+v), v\alpha v]_\alpha + 2(u+v)\alpha D(u+v, v)\alpha v - 2v\alpha D(u+v, v)\alpha(u+v)$

$= [u\alpha u, d(v)]_\alpha + [u\alpha v, d(v)]_\alpha + [v\alpha u, d(v)]_\alpha + [v\alpha v, d(v)]_\alpha + [d(u), v\alpha v]_\alpha + d(v), v\alpha v]_\alpha + 2[D(u, v), v\alpha v]_\alpha + 2u\alpha D(u, v)\alpha v + 2v\alpha D(u, v)\alpha v + 2u\alpha D(v, v)\alpha v + 2v\alpha D(v, v)\alpha v - 2v\alpha D(u, v)\alpha u + 2v\alpha D(u, v)\alpha v - 2v\alpha D(v, v)\alpha u - 2v\alpha D(v, v)\alpha v$

That implies

$[u, v]_\alpha = [u\alpha u, d(v)]_\alpha + [u\alpha v, d(v)]_\alpha + [v\alpha u, d(v)]_\alpha + [d(u), v\alpha v]_\alpha + 2[D(u, v), v\alpha v]_\alpha + 2u\alpha D(u, v)\alpha v + 2u\alpha d(v)\alpha v - 2v\alpha D(u, v)\alpha u - 2v\alpha d(v)\alpha u$ ......(ii)

Again replacing u by u+v in (ii) we have

$[u, v]_\alpha = [(u+v)\alpha(u+v), d(v)]_\alpha + [(u+v)\alpha v, d(v)]_\alpha + [v\alpha(u+v), d(v)]_\alpha + [d(u+v), v\alpha v]_\alpha + 2[D(u+v, v), v\alpha v]_\alpha + 2(u+v)\alpha D(u+v, v)\alpha v + 2(u+v)\alpha d(v)\alpha v - 2v\alpha D(u+v, v)\alpha(u+v) - 2v\alpha d(v)\alpha(u+v)$

That implies

$[u, v]_\alpha = [u\alpha u + u\alpha v + v\alpha u + v\alpha v, d(v)]_\alpha + [u\alpha v + v\alpha v, d(v)]_\alpha + [v\alpha u + v\alpha v, d(v)]_\alpha + [d(u) + d(v) + 2D(u, v), v\alpha v]_\alpha + 2[D(u, v) + D(v, v), v\alpha v]_\alpha + 2u\alpha D(u, v)\alpha v + 2v\alpha D(u, v)\alpha v + 2u\alpha D(v, v)\alpha v + 2v\alpha D(v, v)\alpha v + 2u\alpha d(v)\alpha v + 2v\alpha d(v)\alpha v - 2v\alpha D(u, v)\alpha u - 2v\alpha D(v, v)\alpha u - 2v\alpha D(u, v)\alpha v - 2v\alpha D(v, v)\alpha v - 2v\alpha d(v)\alpha u - 2v\alpha d(v)\alpha v$

$= [u\alpha u + 3v\alpha v + 2u\alpha v + 2v\alpha u, d(v)]\alpha + [d(u) + 3d(v), v\alpha v]_\alpha + 4[D(u, v), v\alpha v]_\alpha + 2u\alpha D(u, v)\alpha v - 4v\alpha d(v)\alpha u + 4u\alpha d(v)\alpha v - 2v\alpha D(u, v)\alpha u$

$= [u\alpha u, d(v)]_\alpha + 2[u\alpha v, d(v)]_\alpha + 2[v\alpha u, d(v)]_\alpha + [d(u), v\alpha v]_\alpha + 4[D(u, v), v\alpha v]_\alpha + 2u\alpha D(u, v)\alpha v + 4u\alpha d(v)\alpha v - 4v\alpha d(v)\alpha u - 2v\alpha D(u, v)\alpha u$

$= 2([u\alpha v, d(v)]_\alpha + [v\alpha u, d(v)]_\alpha + 2u\alpha d(v)\alpha v + 2[D(u, v), v\alpha v]_\alpha - 2v\alpha d(v)\alpha u) + [u\alpha u, d(v)]_\alpha + [d(u), v\alpha v]_\alpha + 2u\alpha D(u, v)\alpha v - 2v\alpha D(u, v)\alpha u.$

$= 2([u\alpha u, d(v)]_\alpha + [u\alpha v, d(v)]_\alpha + [v\alpha u, d(v)]_\alpha + [d(u), v\alpha v]_\alpha + 2[D(u, v), v\alpha v]_\alpha +$

$2u\alpha D(u, v)\alpha v + 2u\alpha d(v)\alpha v - 2v\alpha D(u, v)\alpha u - 2v\alpha d(v)\alpha u) - ([u\alpha u, d(v)]_\alpha + [d(u), v\alpha v]_\alpha + 2u\alpha D(u, v)\alpha v - 2v\alpha D(u, v)\alpha u)$

i.e., $[u, v]_\alpha = 2[u, v]_\alpha$

That implies $[u, v]_\alpha = 0$ for all $u, v \in U$; $\alpha \in \Gamma$ and hence $U \subseteq Z(M)$.

Symmetric bi-derivations on Lie ideals of Semiprime Γ-rings

We know that a bi-additive mapping with the condition of symmetricity is a symmetric bi-derivation. For a bi-derivation D we have a mapping d: $M \to M$ defined by $d(u) = D(u, u)$, which is known as the trace of D. In this chapter we have tried to show the different activity of D according to the application of different condition. Also we have proved that for the particular case the Lie ideal will be commutative.

7. Introduction : Throughout this chapter M represents a 2-torsion free semiprime Γ-ring. We know that a Γ-ring is called semiprime if $a\Gamma M\Gamma a = 0$ implies $a = 0$ for every $a \in M$. A mapping $F : M \to M$ is said to be centralizing on M if $[F(m), m]_\alpha \in Z(M)$ for all $m \in M$, $\alpha \in \Gamma$. For the special case $[F(m), m]_\alpha = 0$, for all $m \in M$, $\alpha \in \Gamma$, the mapping F is called commuting on M. A mapping $D(\ , \ ) : M\times M \to M$ is called symmetric if $D(x, y) = D(y, x)$ for all pairs $x, y \in M$. A mapping $d: M \to M$ defined by $d(x) = D(x, x)$, where D is symmetric is called the trace of D. A symmetric bi-additive mapping is called symmetric bi-derivation if $D(x\alpha y, z) = x\alpha D(y, z) + D(x, z)\alpha y$ holds for every $x, y, z \in M$ and $\alpha \in \Gamma$.

7.1 Definition : A mapping $F : M \to M$ is said to be centralizing on U if $[F(u), u]_\alpha \in Z(U)$ for all $u \in U$, $\alpha \in \Gamma$. For the special case $[F(u), u]_\alpha = 0$, for all $u \in U$, $\alpha \in \Gamma$, the mapping F is called commuting on U.

7.2 Theorem : Let M be a 2, 3-torsion free semiprime Γ-ring satisfying the condition (*) and U a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$. Suppose that there exists a symmetric bi-derivation $D : M\times M \to M$ such that the mapping $u \rightarrow [d(u), u]_\alpha$ is centralizing on U, where d denotes the trace of D. Then d is commuting on U.

Proof : First we show that the mapping $u \rightarrow [d(u), u]_\alpha$ is commuting on U . By our hypothesis we have $[[d(u), u]_\alpha , u]_\beta \in Z(U)$ for all $u \in U$........(I)

Putting u+v for u we get

$[[d(u+v), u+v]_\alpha , u+v]_\beta = [d(u) + d(v) + 2D(u, v) , u+v]_\alpha , u+v]_\beta$

$= [[d(u), u]_\alpha+[d(v), u]_\alpha+[d(u), v]_\alpha + [d(v), v]_\alpha +[2D(u, v), u]_\alpha +2D(u, v), v]_\alpha, u+v]_\beta \in Z(U)$.

Hence, $[[d(v), u]_\alpha , u]_\beta + [[d(u), v]_\alpha, u]_\beta +[[d(v), v]_\alpha, u]_\beta + [[2D(u, v), u]_\alpha, u]_\beta + 2 [[D(u, v), v]_\alpha, u]_\beta+[[d(u), u]_\alpha, v]_\beta +[[d(v), u]_\alpha , v]_\beta + [[d(u), v]_\alpha, v]_\beta +[[2D(u, v), u]_\alpha, v]_\beta + [[2D(u, v), v]_\alpha, v]_\beta \in Z(U)$.........(ii)

since $[[d(u), u]_\alpha, u]_\beta , [[d(v), v]_\alpha, v]_\beta \in Z(U)$.

Putting – u for u we have

$[[d(v), u]_\alpha , u]_\beta -[[d(u), v]_\alpha, u]_\beta - [[d(v), v]_\alpha, u]_\beta - [[2D(u, v), u]_\alpha, u]_\beta + [[2D(u, v), v]_\alpha, u]_\beta - [[d(u), u]_\alpha, v]_\beta - [[d(v), u]_\alpha , v]_\beta +[[d(u), v]_\alpha, v]_\beta +[[2D(u, v), u]_\alpha, v]_\beta - [[2D(u, v), v]_\alpha, v]_\beta \in Z(U)$

and hence

$- [[d(v), u]_\alpha , u]_\beta +[[d(u), v]_\alpha, u]_\beta +[[d(v), v]_\alpha, u]_\beta +[[2D(u, v), u]_\alpha, u]_\beta - [[2D(u, v), v]_\alpha, u]_\beta + [[d(u), u]_\alpha, v]_\beta +[[d(v), u]_\alpha , v]_\beta -[[d(u), v]_\alpha, v]_\beta -[[2D(u, v), u]_\alpha, v]_\beta +[[2D(u, v), v]_\alpha, v]_\beta \in Z(U)$ ....(iii)

Adding (ii) and (iii) we get

$2[[d(u), v]_\alpha, u]_\beta +2[[d(v), v]_\alpha, u]_\beta + 2[[2D(u, v), u]_\alpha, u]_\beta +2[[d(u), u]_\alpha, v]_\beta + 2[[d(v), u]_\alpha , v]_\beta +2[[2D(u, v), v]_\alpha, v]_\beta \in Z(U)$ ....(iv)

Since M is 2-torsion free, we have

$[[d(u), v]_\alpha, u]_\beta +[[d(v), v]_\alpha, u]_\beta + 2[[D(u, v), u]_\alpha, u]_\beta + [[d(u), u]_\alpha, v]_\beta +[[d(v), u]_\alpha , v]_\beta +2[[D(u, v), v]_\alpha, v]_\beta \in Z(U)$ ....(v)

Putting 2u for u, we have

$[[d(2u), v]_\alpha, u]_\beta + [[d(v), v]_\alpha, 2u]_\beta + 2[[D(2u, v),2u]_\alpha, 2u]_\beta + [[d(2u), 2u]_\alpha, v]_\beta + [[d(v), 2u]_\alpha , v]_\beta +2[[D(2u, v), v]_\alpha, v]_\beta \in Z(U)$

That means

$8[[d(u), v]_\alpha, u]_\beta + 2[[d(v), v]_\alpha, u]_\beta + 8[[D(u, v), u]_\alpha, u]_\beta + 2[[d(u), u]_\alpha, v]_\beta + 16[[d(v), u]_\alpha , v]_\beta + 4[[D(u, v), v]_\alpha, v]_\beta \in Z(U)$ ....(vi)

Subtracting (iv) from (vi)

$6[[d(u), v]_\alpha, u]_\beta +6[[d(u), u]_\alpha, v]_\beta + 12[[D(u, v), u]_\alpha, u]_\beta \in Z(U)$

Hence

$[d(u), v]_\alpha, u]_\beta+ [[d(u), u]_\alpha, v]_\beta + 2[[D(u, v), u]_\alpha, u]_\beta \in Z(U)$ ……..(vii)

Replacing v by $u\alpha u$ in (vii) we have

$[[d(u), u\alpha u]_\alpha, u]_\beta + [[d(u), u]_\alpha, u\alpha u]_\beta + 2[[D(u, u\alpha u), u]_\alpha, u]_\beta \in Z(U)$

This yields that

$[u\alpha[d(u), u]_\alpha, u]_\beta + [[d(u), u]_\alpha\alpha u, u]_\beta + [u\alpha[d(u), u]_\alpha, u]_\beta + [[d(u), u]_\alpha, u]_\beta\alpha u + 2[[D(u, u)\alpha u, u]_\alpha, u]_\beta + 2[[u\alpha D(u, u), u]_\alpha, u]_\beta$

$= u\alpha[[d(u), u]_\alpha, u]_\beta + [u, u]_\beta\alpha[d(u), u]_\alpha + [d(u), u]_\alpha\alpha[u, u]_\beta+ [[d(u), u]_\alpha, u]_\beta\alpha u + u\alpha[[d(u), u]_\alpha, u]_\beta + [u, u]_\beta\alpha[d(u), u]_\alpha+[[d(u), u]_\alpha, u]_\beta\alpha u + 2[[d(u)\alpha u, u]_\alpha, u]_\beta + 2[[u\alpha d(u), u]_\alpha, u]_\beta$

$=2u\alpha[[d(u), u]_\alpha, u]_\beta +2[[d(u), u]_\alpha, u]_\beta\alpha u + 2[d(u)\alpha[u, u]_\alpha+ [d(u), u]_\alpha\alpha u, u]_\beta +2[u\alpha[d(u), u]_\beta + 2[u, u]_\beta\alpha d(u), u]_\beta$

$= 2u\alpha[[d(u), u]_\alpha, u]_\beta +2[[d(u), u]_\alpha, u]_\beta\alpha u + 2[[d(u), u]_\alpha\alpha u, u]_\beta +2[u\alpha[d(u), u]_\alpha, u]_\beta$

$= 2u\alpha[[d(u), u]_\alpha, u]_\beta +2[[d(u), u]_\alpha, u]_\beta\alpha u +2[[d(u), u]_\alpha , u]_\beta\alpha u+ 2[d(u), u]_\alpha\alpha[u, u]_\beta +2u\alpha[[d(u), u]_\alpha, u]_\beta +2[u, u]_\beta\alpha[d(u), u]_\alpha$

$= 4(u\alpha[[d(u), u]_\alpha, u]_\beta +[[d(u), u]_\alpha, u]_\beta\alpha u)$

$= 4 [[d(u), u]_\alpha, u]_\beta\alpha u \in Z(U)$ [from (i)]

Since M is 2-torsion free we have $[[d(u), u]_\alpha, u]_\beta\alpha u \in Z(U)$

Then $[[d(u), u]_\alpha, u]_\beta\alpha[v, u]_\beta = 0$, for all $u, v \in U; \alpha, \beta \in \Gamma$ ......(viii)

Replacing v by $v\alpha[d(u), u]_\alpha$, we get

$0 = [[d(u), u]_\alpha, u]_\beta\ \alpha[v\alpha\ [d(u), u]_\alpha, u]_\beta$

$= [[d(u), u]_\alpha, u]_\beta\ \alpha(v\alpha[[d(u), u]_\alpha, u]_\beta + [v, u]_\beta\alpha[d(u), u]_\alpha)$

$= [[d(u), u]_\alpha, u]_\beta\ \alpha v\alpha[[d(u), u]_\alpha, u]_\beta + [[d(u), u]_\alpha, u]_\beta\ \alpha\ [v, u]_\beta\alpha[d(u), u]_\alpha$

$= [[d(u), u]_\alpha, u]_\beta\ \alpha v\alpha[[d(u), u]_\alpha, u]_\beta$

From the semiprimeness of M we have

$[[d(u), u]_\alpha, u]_\beta = 0$, for all $u \in U$.

Again linearize this and applying as in above we have

$0 = [[d(u+v), u+v]_\alpha, u+v]_\beta$ which implies

$0 = [[d(u), u]_\alpha, v]_\beta + [d(u), v]_\alpha, u]_\beta + [[2D(u, v), u]_\alpha, u]_\beta$,

for all $u, v \in U$ and $\alpha, \beta \in \Gamma$……………….(ix)

Substituting $2v\delta w$ for v we get

$0 = [[d(u), u]_\alpha, 2v\delta w]_\beta + [d(u), 2v\delta w]_\alpha, u]_\beta + [[2D(u, 2v\delta w), u]_\alpha, u]_\beta$

Now $[[d(u), u]_\alpha, 2v\delta w]_\beta = 2v\delta[[d(u), u]_\alpha, w]_\beta + 2[[d(u), u]_\alpha, v]_\beta\delta w$ ....(a)

$[d(u), 2v\delta w]_\alpha, u]_\beta = 2v\delta[[d(u),w]_\alpha, u]_\beta + 2[v, u]_\beta\delta[d(u), w]_\alpha + 2[d(u), v]_\alpha\delta[w,u]_\beta + 2[[d(u), v]_\alpha,u]_\beta\delta w$ ....(b) , and

$2[[D(u, 2v\delta w), u]_\alpha, u]_\beta = 2[[2v\delta D(u, w) + 2D(u, v)\delta w, u]_\alpha,u]_\beta$

$= 4([[v\delta D(u, w), u]_\alpha + [D(u, v)\delta w, u]_\alpha, u]_\beta)$

$= 4([v\delta[D(u, w), u]_\alpha,u]_\beta + [[v, u]_\alpha\delta D(u, w), u]_\beta + [[D(u, v), u]_\alpha\delta w, u]_\beta + [D(u, v)\delta[w, u]_\alpha, u]_\beta)$

$= 4(v\delta[D(u, w), u]_\alpha, u]_\beta + [v, u]_\beta\delta[D(u, w), u]_\alpha + [v, u]_\alpha\delta[D(u, w), u]_\beta + [[v, u]_\alpha, u]_\beta\delta D(u, w) + [D(u, v), u]_\alpha\delta\ [w, u]_\beta + [[D(u, v), u]_\alpha, u]_\beta\delta w + D(u, v)\delta[[w, u]_\alpha, u]_\beta + [D(u, v), u]_\beta\delta[w, u]_\alpha)$ ....(c)

From (a), (b) and (c) we have

$0 = 2v\delta[[d(u), u]_\alpha, w]_\beta + 2[[d(u), u]_\alpha, v]_\beta\delta w + 2v\delta[[d(u),w]_\alpha, u]_\beta + 2[v, u]_\beta\delta[d(u), w]_\alpha + 2[d(u), v]_\alpha\delta[w,u]_\beta + 2[[d(u), v]_\alpha,u]_\beta\delta w + 4v\delta[D(u, w), u]_\alpha, u]_\beta + 4[v, u]_\beta\delta[D(u, w), u]_\alpha + 4[v, u]_\alpha\delta[D(u, w), u]_\beta + 4[[v, u]_\alpha, u]_\beta\delta D(u, w) + 4[D(u, v), u]_\alpha\delta\ [w, u]_\beta + 4[[D(u, v), u]_\alpha, u]_\beta\delta w + 4D(u, v)\delta[[w, u]_\alpha, u]_\beta + 4[D(u, v), u]_\beta\delta[w, u]_\alpha$

$= 2v\delta([[d(u), u]_\alpha, w]_\beta + [[d(u),w]_\alpha, u]_\beta + 2[D(u, w), u]_\alpha, u]_\beta) + 2[d(u), v]_\alpha\delta[w, u]_\beta + 2[v, u]_\beta\delta[d(u), w]_\alpha + 8[D(u, v), u]_\alpha\delta[w, u]_\beta + 4D(u, v)\delta[[w, u]_\alpha, u]_\beta + 4[[v, u]_\beta\delta, u]_\beta\delta D(u, w) + 8[v, u]_\beta\delta[D(u, w), u]_\alpha + 2([[d(u), u]_\alpha, v]\beta + [[d(u), v]_\alpha, u]_\beta + 2[[D(u, v), u]_\alpha, u]_\beta\delta w)$

$= 2[d(u), v]_\alpha\delta[w, u]_\beta + 2[v, u]_\beta\delta[d(u), w]_\alpha + 8[D(u, v), u]_\alpha\delta[w, u]_\beta + 4D(u, v)\delta[[w, u]_\alpha, u]_\beta + 4[[v, u]\alpha, u]_\beta\delta D(u, w) + 8[v, u]_\beta\delta[D(u, w), u]_\alpha$

Using 2- torsion freeness we have

$0 = [d(u), v]_\alpha\delta[w, u]_\beta + [v, u]_\beta\delta[d(u), w]_\alpha + 4[D(u, v), u]_\alpha\delta[w, u]_\beta + 2D(u, v)\delta[[w, u]_\alpha, u]_\beta + 2[[v, u]_\alpha, u]_\beta\delta D(u, w) + 4[v, u]_\beta\delta[D(u, w), u]_\alpha$ ....(x)

Replacing w by u in (x) w get

$0 = [d(u), v]_\alpha\delta[u, u]_\beta + [v, u]_\beta\delta[d(u), u]_\alpha + 4[D(u, v), u]_\alpha\delta[u, u]_\beta + 2D(u, v)\delta[[u, u]_\alpha, u]_\beta + 2[[v, u]_\alpha, u]_\beta\delta D(u, u) + 4[v, u]_\beta\delta[D(u, u), u]_\alpha$

$= [v, u]_\beta\delta[d(u), u]_\alpha + 2[[v, u]_\alpha, u]_\beta\delta d(u) + 4[v, u]_\beta\delta[d(u), u]_\alpha$

$= 5[v, u]_\beta \delta [d(u), u]_\alpha + 2[[v, u]_\alpha, u]_\beta \delta d(u)$ ......(xi)

Again replacing v by u and w by v in (x) and using the condition (*) we get

$0 = [d(u), u]_\alpha\delta[v, u]_\beta + [u, u]_\beta\delta[d(u), v]_\alpha + 4[D(u, u), u]_\alpha\delta[v, u]_\beta + 2D(u, u)\delta[[v, u]_\alpha, u]_\beta + 2[[u, u]_\alpha, u]_\beta\delta D(u, v) + 4[u, u]_\beta\delta[D(u, v), u]_\alpha$

$= [d(u), u]_\alpha\delta[v, u]_\beta + 4[d(u), u]_\alpha\delta[v, u]_\beta + 2d(u)\delta[[v, u]_\alpha, u]_\beta$

$= 5[d(u), u]_\alpha\delta[v, u]_\beta + 2d(u)\delta[[v, u]_\alpha, u]_\beta$, for all $u, v \in U$ and $\alpha \in \Gamma$ ....(xii)

Now replacing v by $2v\gamma w$ in (xi)

$0 = 5[2v\gamma w, u]_\beta \delta [d(u), u]_\alpha + 2[[2v\gamma w, u]_\alpha, u]_\beta \delta d(u)$

$= 10v\gamma[w, u]_\beta\delta[d(u), u]_\alpha + 10[v, u]_\beta\gamma w\delta[d(u), u]_\alpha + 4[v\gamma[w, u]_\alpha, u]_\beta\delta d(u) + 4[[v, u]_\alpha\gamma w, u]_\beta\delta d(u)$

$= 10v\gamma[w, u]_\beta \delta [d(u), u]_\alpha + 10[v, u]_\beta\gamma w \delta[d(u), u]_\alpha + 4v\gamma[[w, u]_\alpha, u]_\beta \delta d(u) + 4[v,u]_\beta \gamma [w, u]_\alpha \delta d(u) + 4[v, u]_\alpha\gamma[w, u]_\beta \delta d(u) + 4[[v, u]_\alpha, u]_\beta\gamma w\delta d(u)$

Since M is 2-torsion free

$0 = 5v\gamma[w, u]_\beta\ \delta\ [d(u), u]_\alpha + 5[v, u]_\beta\gamma\ w\ \delta[d(u), u]_\alpha + 2v\gamma[[w, u]_\alpha, u]_\beta\ \delta$ $d(u) + 4[v,u]_\beta\ \gamma\ [w, u]_\alpha\ \delta\ d(u) + 2[[v, u]_\alpha, u]_\beta\gamma w\delta\ d(u)$

$= v\gamma(5[w, u]_\beta\ \delta\ [d(u), u]_\alpha +2\ [[w, u]_\alpha, u]_\beta\ \delta\ d(u)) +5[v, u]_\beta\gamma\ w\ \delta[d(u), u]_\alpha$ $+ 4[v,u]_\beta\ \gamma\ [w, u]_\alpha\ \delta\ d(u) + 2[[v, u]_\alpha, u]_\beta\gamma w\delta\ d(u)$

$= 5[v, u]_\beta\gamma\ w\ \delta[d(u), u]_\alpha + 4[v,u]_\beta\ \gamma\ [w, u]_\alpha\ \delta\ d(u) + 2[[v, u]_\alpha, u]_\beta\gamma w\delta$ $d(u)$, for all $u, v, w \in U$; $\alpha, \beta, \gamma, \delta \in \Gamma$ ……(xiii)

Now putting d(u) for w we get

$0 = 5[v, u]_\beta\gamma d(u)\delta[d(u), u]_\alpha + 4[v,u]_\beta\ \gamma[d(u), u]_\alpha\ \delta d(u) + 2[[v, u]_\alpha,$ $u]_\beta\gamma d(u)\delta d(u)$ ....(xiii)(a)

From (xi)we have,

$0 = 5[v, u]_\beta\ \delta\ [d(u), u]_\alpha\gamma d(u) + 2[[v, u]_\alpha, u]_\beta\ \delta\ d(u)\gamma d(u)$ ………..(xiv)

From (xiii) and (xiv) we get

$5[v, u]_\beta\gamma d(u)\delta[d(u), u]_\alpha + 4[v,u]_\beta\ \gamma[d(u), u]_\alpha\ \delta d(u) + 2[[v, u]_\alpha,$ $u]_\beta\gamma d(u)\delta d(u) - 5[v, u]_\beta\ \delta\ [d(u), u]_\alpha\gamma d(u) - 2[[v, u]_\alpha, u]_\beta\ \delta\ d(u)\gamma d(u) = 0$

Using (*) we have

$5[v, u]_\beta\gamma d(u)\delta[d(u), u]_\alpha - [v, u]_\beta\ \gamma\ [d(u), u]_\alpha\delta d(u) = 0$

That implies $[v, u]_\beta\gamma(5\ d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u)) = 0$ ....(xv)

Replacing v by $2w\mu v$; $\mu \in \Gamma$ we obtain

$0 = [2w\mu v, u]_\beta\gamma(5\ d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u))$

$= (2w\mu[v, u]_\beta + 2[w, u]_\beta\ \mu v)\ \gamma(5\ d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u))$

$= 2w\mu[v, u]_\beta\gamma(5\ d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u)) + 2[w, u]_\beta\ \mu v\ \gamma(5$ $d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u))$

$= 2[w, u]_\beta\ \mu v\ \gamma(5\ d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u))$

By 2-torsion freeness of M we have

$[w, u]_\beta\ \mu v\gamma(5\ d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u)) = 0$, for all $u, v, w \in U$ and $\alpha, \beta, \gamma, \delta, \mu \in \Gamma$.

In particular if we put $w = d(u)$ we obtain

$0 = [d(u), u]_\beta\ \mu v\ \gamma(5\ d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u))$ ....(xvi)

Left multiplying by 5d(u), we have

$0 = 5d(u)\alpha\,[d(u), u]_\beta\,\mu v\gamma\,(5d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u))$; for all u, v $\in U$; $\alpha, \beta, \gamma, \delta, \mu \in \Gamma$.............(xvii)

If we put $d(u)\alpha v$ for v in (xvi) then we have

$0 = [d(u), u]_\beta\,\mu d(u)\alpha v\gamma(5\,d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u))$ .....(xviii)

Subtracting (xviii) from (xvii) we get

$0 = 5d(u)\alpha[d(u), u]_\beta\mu v\gamma\,(5d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u)) - [d(u), u]_\beta\,\mu d(u)\alpha v\gamma(5\,d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u))$

$= (5d(u)\alpha[d(u), u]_\beta - [d(u), u]_\beta\,\alpha d(u))\mu v\gamma(5d(u)\delta[d(u), u]_\alpha - [d(u), u]_\alpha\delta d(u))$

From the semiprimeness of M we have

$5d(u)\alpha[d(u), u]_\beta - [d(u), u]_\beta\,\alpha d(u) = 0$

Or $5d(u)\beta[d(u), u]_\alpha - [d(u), u]_\alpha\,\beta d(u) = 0$ ....(xix)

Now replace v by $2w\beta v$ in (xii)

$0 = 5[d(u), u]_\alpha\delta[2w\beta v, u]_\beta + 2d(u)\delta[[2w\beta v, u]_\alpha, u]_\beta$

Using the same techniques used to get (xv) from (xi) we get

$(5[d(u), u]_\alpha\beta d(u) – d(u)\beta[d(u), u]_\alpha)[v, u]_\alpha = 0$

Again repetition of the way which gives (xix) from(xv) , we have

$5[d(u), u]\alpha\beta d(u) – d(u)\beta[d(u), u]\alpha = 0$ ….(xx)

Combining (xix) and (xx) we get $24d(u)\beta[d(u), u]_\alpha = 0$

Since M is 2, 3 torsion free ,

$d(u)\beta[d(u), u]_\alpha = 0$ for all $u \in U$; $\alpha, \beta \in \Gamma$…………….(xxi)

Again linearizing we have

$0 = d(u+v)\beta[d(u+v), u+v]_\alpha = 0$

$= (d(u) + d(v) + 2D(u, v))\beta\,[d(u) + d(v) + 2D(u, v) , u+ v]_\alpha$

$= (d(u) + d(v) + 2D(u, v))\beta([d(u), u]_\alpha + [d(v), v]_\alpha + [d(u), v]_\alpha + [d(v), u]_\alpha + [2D(u, v), u]\alpha + [2D(u, v) ,v]_\alpha$

$= d(u)\,\beta[d(u), v]_\alpha + d(u)\,\beta[d(v), u]_\alpha + d(u)\,\beta\,[d(v),v]_\alpha + 2d(u)\,\beta\,[D(u, v),u]_\alpha + 2d(u)\,\beta\,[D(u, v),v]_\alpha + d(v)\,\beta\,[d(u), u]_\alpha + d(v)\,\beta[d(u), v]_\alpha + d(v)\,\beta\,[d(v), u]_\alpha + 2d(v)\beta[D(u, v),u]_\alpha + 2d(v)\,\beta[D(u,v),v]_\alpha + 2D(u, v)\,\beta[d(u), u]_\alpha$

$+ 2D(u,v)\ \beta[d(u), v]_\alpha + 2D(u, v)\ \beta[(d(v),u]_\alpha + 2D(u, v)\beta[d(v), v]_\alpha +$
$4D(u,v)[D(u,v),u]_\alpha + 4D(U,v)\ \beta\ [D(u,v),v]_\alpha$

Replacing v by –v we have

$0 = - d(u)\ \beta[d(u), v]_\alpha + d(u)\ \beta[d(v), u]_\alpha - d(u)\ \beta\ [d(v),v]_\alpha - 2d(u)\ \beta\ [D(u, v),u]_\alpha + 2d(u)\ \beta\ [D(u, v),v]_\alpha + d(v)\ \beta\ [d(u), u]_\alpha - d(v)\ \beta[d(u), v]_\alpha + d(v)\ \beta\ [d(v), u]_\alpha - 2d(v)\beta[D(u, v),u]_\alpha + 2d(v)\ \beta[D(u,v),v]_\alpha - 2D(u, v)\ \beta[d(u), u]_\alpha + 2D(u,v)\ \beta[d(u), v]_\alpha - 2D(u, v)\ \beta[(d(v),u]_\alpha + 2D(u, v)\beta[d(v), v]_\alpha + 4D(u,v)[D(u,v),u]_\alpha - 4D(u,v)\ \beta\ [D(u,v),v]_\alpha$

Adding these two expressions we have

$0 = 2d(u)\ \beta[d(v), u]_\alpha + 4d(u)\ \beta\ [D(u, v),v]_\alpha + 2d(v)\ \beta\ [d(u), u]_\alpha + 2d(v)\ \beta\ [d(v), u]_\alpha + 4d(v)\beta[D(u,v),v]_\alpha + 4D(u,v)\beta[d(u),v]_\alpha + 4D(u, v)\beta[d(v), v]_\alpha + 8D(u,v)\beta[D(u,v),u]_\alpha$ ………………(xxii)

Since M is 2-torsion free

$0 = d(u)\ \beta[d(v), u]_\alpha + 2d(u)\ \beta\ [D(u, v),v]_\alpha + d(v)\ \beta\ [d(u), u]_\alpha + d(v)\ \beta\ [d(v), u]_\alpha + 2d(v)\beta[D(u,v),v]_\alpha + 2D(u,v)\beta[d(u),v]_\alpha + 2D(u, v)\beta[d(v), v]_\alpha + 4D(u,v)\beta[D(u,v),u]_\alpha$

Now replacing u by 2u we have

$0 = d(2u)\ \beta[d(v), 2u]_\alpha + 2d(2u)\ \beta\ [D(2u, v),v]_\alpha + d(v)\ \beta\ [d(2u), 2u]_\alpha + d(v)\ \beta\ [d(v), 2u]_\alpha + 2d(v)\beta[D(2u,v),v]_\alpha + 2D(2u,v)\beta[d(2u),v]_\alpha + 2D(2u, v)\beta[d(v), v]_\alpha + 4D(2u,v)\beta[D(2u,v),2u]_\alpha$

$= 8d(u)\ \beta[d(v), u]_\alpha + 16d(u)\ \beta\ [D(u, v),v]_\alpha + 8d(v)\ \beta\ [d(u), u]_\alpha + 2d(v)\ \beta\ [d(v), u]_\alpha + 4d(v)\beta[D(u,v),v]_\alpha + 4D(u,v)\beta[d(u),v]_\alpha + 2D(u, v)\beta[d(v), v]_\alpha + 32D(u,v)\beta[D(u,v),u]_\alpha$ …………………(xxiii)

Comparing (xxi) and (xxiii) we have

$0 = 6d(v)\beta[d(v), u]_\alpha + 12d(v)\ \beta[D(u,v), v]_\alpha + 12D(u, v)\ \beta[d(v), v]_\alpha$

Since M is 2 and 3 torsion free

$0 = d(v)\beta[d(v), u]_\alpha + 2d(v)\ \beta[D(u,v), v]_\alpha + 2D(u, v)\ \beta[d(v), v]_\alpha$ …..(xxiv)

Substituting $2u\alpha v$ for u we have

$0 = d(v)\beta[d(v), 2u\alpha v]_\alpha + 2d(v)\beta[D(2u\alpha v,v), v]_\alpha + 2D(2u\alpha v, v)\ \beta[d(v), v]_\alpha$

$= 2\ d(v)\beta u\alpha\ [d(v),v]_{\alpha} + 2d(v)\beta[d(v),\ u]_{\alpha}\ \alpha v + 4d(v)\beta[u\alpha D(v,\ v),\ v]_{\alpha} + 4d(v)\beta[D(u,\ v)\alpha v,\ v]_{\alpha} + 4D(u,\ v)\alpha v\beta[d(v),\ v]_{\alpha} + 4u\alpha D(v,v)\beta[d(v),\ v]_{\alpha}$

$= 2d(v)\ \beta u\alpha\ [d(v),v]_{\alpha} + 2d(v)\beta[d(v),\ u]_{\alpha}\ \alpha v + 4d(v)\beta u\alpha[d(v),\ v]_{\alpha} + 4d(v)\beta[u,\ v]_{\alpha}\alpha d(v) + 4d(v)\beta[D(u,\ v),\ v]_{\alpha}\ \alpha v + 4D(u,\ v)\alpha v\beta[d(v),\ v]_{\alpha} + 4u\alpha d(v)\beta[d(v),\ v]_{\alpha}$

Since M is 2- torsion free

$0 = d(v)\beta u\alpha[d(v),v]_{\alpha} + d(v)\beta[d(v),\ u]_{\alpha}\ \alpha v + 2d(v)\beta u\alpha[d(v),\ v]_{\alpha} + 2d(v)\beta[u,\ v]_{\alpha}\alpha d(v) + 2d(v)\beta[D(u,\ v),\ v]_{\alpha}\ \alpha v + 2D(u,\ v)\alpha v\beta[d(v),\ v]_{\alpha} + 2u\alpha d(v)\beta[d(v),\ v]_{\alpha}$

$= 3d(v)\beta u\alpha\ [d(v),v]_{\alpha} + d(v)\beta[d(v),\ u]_{\alpha}\ \alpha v + 2d(v)\beta[u,\ v]_{\alpha}\alpha d(v) + 2d(v)\beta[D(u,\ v),\ v]_{\alpha}\ \alpha v + 2D(u,\ v)\alpha v\beta[d(v),\ v]_{\alpha} + 2u\alpha d(v)\beta[d(v),\ v]_{\alpha}$

Comparing with (xxiv) we have

$0 = 3d(v)\beta u\alpha[d(v),\ v]_{\alpha} - 2D(u,\ v)\beta[[d(v),\ v]_{\alpha},\ v]_{\alpha} + 2d(v)\beta[u,\ v]_{\alpha}\ \alpha d(v)$

$= 3d(v)\beta u\alpha[d(v),\ v]_{\alpha} + 2d(v)\beta[u,\ v]_{\alpha}\ \alpha d(v) + 2u\alpha d(v)\beta[d(v),\ v]_{\alpha}$ ……………….(xxv)

Replace u by $2v\alpha u$ we have

$0 = 3d(v)\beta 2v\alpha u\ \alpha[d(v),\ v]_{\alpha} + 2d(v)\beta[2v\alpha u,\ v]_{\alpha}\ \alpha d(v) + 2.2v\alpha u\alpha d(v)\beta[d(v),\ v]_{\alpha}$

$= 6d(v)\beta v\alpha u\ \alpha[d(v),\ v]_{\alpha} + 4d(v)\beta v\alpha[u,\ v]_{\alpha}\ \alpha d(v) + 4d(v)\beta[v,\ v]_{\alpha}\ \alpha u\alpha d(v) + 4v\alpha u\alpha d(v)\beta[d(v),\ v]_{\alpha}$

$= 6d(v)\beta v\alpha u\ \alpha[d(v),\ v]_{\alpha} + 4d(v)\beta v\alpha[u,\ v]_{\alpha}\ \alpha d(v) + 4v\alpha u\alpha d(v)\beta[d(v),\ v]_{\alpha}$

Since M is 2-torsion free

$0 = 3d(v)\beta v\alpha u\alpha[d(v),v]_{\alpha} + 2d(v)\beta v\alpha[u,v]_{\alpha}\alpha d(v) + 2v\alpha u\alpha d(v)\beta[d(v),\ v]_{\alpha}$ …………….(xxvi)

Left multiplication of (xxv) by v we get

$0 = 3v\alpha d(v)\beta u\alpha[d(v),\ v]_{\alpha} + 2v\alpha d(v)\beta[u,\ v]_{\alpha}\ \alpha d(v) + 2v\alpha u\alpha d(v)\beta[d(v),v]_{\alpha}$ ……………….(xxvii)

From (xxvi) and (xxvii) using (*) we have

$0 = 3[d(v), v]_\alpha \beta u \alpha [d(v),v]_\alpha + 2[d(v), v]_\alpha \beta\, [u, v]_\alpha \alpha d(v)$.

Then $0 = 6[d(v), v]_\alpha \beta u \alpha [d(v),v]_\alpha + 4[d(v), v]_\alpha \beta\, [u, v]_\alpha \alpha d(v)$

From (xiii) we have

$0 = 5[v, u]_\beta \gamma w \delta [d(u), u]_\alpha + 4[v,u]_\beta \gamma [w, u]_\alpha\, \delta\, d(u) + 2[[v, u]_\alpha, u]_\beta \gamma w \delta\, d(u)$ ………………(xxviii)

In particular putting v = d(v) and changing u by v and w by u we get

$0 = 5[d(v), v]_\beta \gamma u \delta [d(u), v]_\alpha + 4[d(v),v]_\beta \gamma [u, v]_\alpha\, \delta\, d(u) + 2[[d(v), v]_\alpha, v]_\beta \gamma u \delta\, d(u)$

$= 5[d(v), v]_\beta \gamma u \delta [d(v), v]_\alpha + 4[d(v),v]_\beta \gamma [u, v]_\alpha\, \delta\, d(u)$ ………(xxix)

From (xxviii) and (xxix) we have $[d(v), v]_\beta \gamma u \delta [d(v), v]_\alpha = 0$

Since M is semiprime we have $[d(v), v]_\beta = 0$ for all $v \in U$.

Therefore, d is commutating on U .

7.3 Remark : If d(u) is centralizing on M . i. e., $[d(u), u]_\alpha \in Z(U)$ for all $u \in U$; $\alpha \in \Gamma$, then an important result has been obtained by Bresar [10]. In there it has been shown that if M is a semiprime ring with 2 and 3-torsion freeness and D: M×M → M, a symmetric bi-additive mapping such that $[d(u), u]_\alpha \in Z(U)$ for all $u \in U$, then $[d(u), u]_\alpha = 0$ for all $u \in U$; $\alpha \in \Gamma$.

7.4 Corollary : Let M be 2 and 3 torsion free prime gamma ring and U, a Lie ideal of M. If there exists a non zero symmetric bi-derivation D: M×M → M such that the mapping u → $[d(u), u]_\alpha$ is centralizing on U, where d denotes the trace of D, then U is commutative.

Daif and Bell [24] proved that if a semiprime ring R admits a derivation d such that either $u\alpha v - d(u\alpha v) = v\alpha u - d(v\alpha u)$ for all $u, v \in U$ and $\alpha \in \Gamma$

or, $u\alpha v + d(u\alpha v) = v\alpha u + d(v\alpha u)$ for all $u, v \in U$ and $\alpha \in \Gamma$, then U is commutative.

In this sense of view we prove the following :

7.5 Theorem : Let M be a 2-torsion free gamma ring with a Lie ideal U. Suppose that there exists a symmetric bi-derivation $D: M\times M \rightarrow M$ such that $u\alpha v - d(u\alpha v) = v\alpha u - d(v\alpha u)$ for all $u, v \in U$ and $\alpha \in \Gamma$, where d is the trace of D. Then U is commutative.

Proof : Given that $u\alpha v - d(u\alpha v) = v\alpha u - d(v\alpha u)$

That is $u\alpha v - v\alpha u = d(u\alpha v) - d(v\alpha u)$

$$= D(u\alpha v, u\alpha v) - D(v\alpha u, v\alpha u)$$

$$= u\alpha D(v, u\alpha v) + D(u,u\alpha v)\alpha v - v\alpha D(u, v\alpha u) - D(v,v\alpha u)\alpha u$$

$= u\alpha D(v, u)\alpha v + u\alpha u\alpha D(v, v) + D(u, u)\alpha v\alpha v + u\alpha D(u, v)\alpha v - v\alpha D(u, v)\alpha u - v\alpha v D(u, u) - v\alpha D(v, u)\alpha u - D(v, v)\alpha u\alpha u$

$= u\alpha D(v, u)\alpha v + u\alpha u\alpha d(v) + d(u)\alpha v\alpha v + u\alpha D(u, v)\alpha v - v\alpha D(u, v)\alpha u - v\alpha v d(u) - v\alpha D(v, u)\alpha u - d(v)\alpha u\alpha u$

$= u\alpha u\alpha d(v) - d(v)\alpha u\alpha u + d(u)\alpha v\alpha v - v\alpha v d(u) + 2u\alpha D(u, v)\alpha v - 2v\alpha D(u, v)\alpha u$

$$= [u\alpha u, d(v)]_\alpha + [d(u), v\alpha v]_\alpha + 2u\alpha D(u, v)\alpha v - 2v\alpha D(u, v)\alpha u \quad \ldots\ldots(1)$$

Replace u by u+v we have,

$[u+v, v]_\alpha = [(u+v)\alpha(u+v), d(v)]_\alpha + [d(u+v), v\alpha v]_\alpha + 2(u+v)\alpha D(u+v, v)\alpha v - 2v\alpha D(u+v, v)\alpha(u+v)$

That implies

$[u, v]_\alpha = [u\alpha u+u\alpha v+ v\alpha u+ v\alpha v,\ d(v)]_\alpha + [d(u)+d(v)+2D(u, v), v\alpha v]_\alpha + 2(u+v)\alpha(D(u, v)+D(v, v))\alpha v - 2v\alpha(D(u, v) + D(v, v))\alpha(u+v)$

$= [u\alpha u, d(v)]_\alpha + [u\alpha v, d(v)]_\alpha +[v\alpha u,d(v)]_\alpha + [v\alpha v, d(v)]_\alpha +[d(u), v\alpha v] + [d(v),v\alpha v]\alpha + 2[D(u, v), v\alpha v] + 2u\alpha D(u, v)\alpha v + 2v\alpha D(u, v)\alpha v + 2u\alpha d(v)\alpha v + 2v\alpha d(v)\alpha v - 2v\alpha D(u, v)\alpha u - 2v\alpha D(u, v)\alpha v - 2v\alpha d(v)\alpha u - 2v\alpha d(v)\alpha v$

$= [u\alpha u, d(v)]_\alpha + [u\alpha v, d(v)]_\alpha +[v\alpha u,d(v)]_\alpha + [d(u), v\alpha v] + 2[D(u, v), v\alpha v] + 2u\alpha D(u, v)\alpha v + 2u\alpha d(v)\alpha v - 2v\alpha D(u, v)\alpha u - 2v\alpha d(v)\alpha u$ ….(2)

From (1) and (2) we have

$0 = [u\alpha v, d(v)]_\alpha +[v\alpha u,d(v)]_\alpha +2[D(u, v), v\alpha v] +2u\alpha d(v)\alpha v - 2v\alpha d(v)\alpha u$……(3), for all $u, v \in U$ and $\alpha \in \Gamma$.

Again replacing v by u+v we get

$0 = [u\alpha(u+v), d(u+v)]_\alpha +[(u+v)\alpha u,d((u+v)]_\alpha +2[D(u, u+v), (u+v)\alpha(u+v)] +2u\alpha d(u+v)\alpha(u+v) -2(u+v)\alpha d(u+v)\alpha u$

$= [u\alpha u+u\alpha v, d(u)+d(v)+2D(u, v)]\alpha + [u\alpha u+v\alpha u, d(u)+d(v)+2D(u, v)]\alpha +2[D(u, u) + D(u, v),\ u\alpha u+v\alpha u+u\alpha v+v\alpha v]_\alpha + 2u\alpha(d(u) + d(v) + 2D(u,v))\alpha(u+v) - 2u\alpha(d(u) + d(v) + 2D(u, v))\alpha u - 2v\alpha(d(u) + d(v) + 2D(u, v))\alpha u$

$= [u\alpha u, d(u)]_\alpha +[u\alpha v,d(u)]_\alpha +[u\alpha u, d(v)]_\alpha +[u\alpha v,d(v)]_\alpha + 2[u\alpha u,D(u, v)]_\alpha +2[u\alpha v, D(u, v)]_\alpha +[u\alpha u,d(u)]_\alpha + [v\alpha u, d(u)]_\alpha +[u\alpha u, d(v)]_\alpha + [v\alpha u, d(v)]_\alpha + 2[u\alpha u, D(u, v)]_\alpha+2[v\alpha u, D(u, v)]_\alpha +2[d(u), u\alpha u]_\alpha +2[D(u, v), u\alpha u]_\alpha + 2[d(u),v\alpha u]_\alpha +2[D(u, v),v\alpha u]_\alpha +2[d(u), u\alpha v]_\alpha +2[D(u, v), u\alpha v]_\alpha +2[d(u), v\alpha v]_\alpha +2[D(u, v),v\alpha v]_\alpha +2u\alpha d(u)\alpha u +2u\alpha d(v)\alpha u+4u\alpha D(u, v)\alpha u +2u\alpha d(u)\alpha v+2u\alpha d(v)\alpha v+4u\alpha D(u, v)\alpha v-2u\alpha d(u)\alpha u - 2u\alpha d(v)\alpha u - 4u\alpha D(u, v)\alpha u - 2v\alpha d(u)\alpha u -2v\alpha d(v)\alpha u - 4v\alpha D(u, v)\alpha u$

$= 2[u\alpha u,d(v)]_\alpha + [u\alpha v,d(v)_\alpha]-[u\alpha v,d(u)]_\alpha - [v\alpha u,d(u)]_\alpha + [v\alpha u,d(v)]_\alpha$ $+2[d(u),v\alpha v]_\alpha +2[D(u,v),v\alpha v]_\alpha +2u\alpha d(u)\alpha v+2u\alpha d(v)\alpha v+ 4u\alpha D(u,v)\alpha v-$ $2v\alpha d(u)\alpha u-2v\alpha d(v)\alpha u-4v\alpha D(u,v)\alpha u$

$= 2([u\alpha u,d(v)]_\alpha +[d(u),v\alpha v]_\alpha+ 2u\alpha D(u,v)\alpha v) - 2v\alpha D(u,v)\alpha u)$ $+([u\alpha v,d(v)_\alpha] + [v\alpha u,d(v)]_\alpha+2[D(u,v),v\alpha v]_\alpha +2u\alpha d(v)\alpha v - 2v\alpha d(v)\alpha u) –$ $([u\alpha v,d(u)]_\alpha + [v\alpha u,d(u)]_\alpha+2[D(u, v),u\alpha u]_\alpha - 2u\alpha d(u)\alpha v + 2v\alpha d(u)\alpha u)$

$= 2([u\alpha u,d(v)]_\alpha +[d(u),v\alpha v]_\alpha+ 2u\alpha D(u,v)\alpha v) - 2v\alpha D(u,v)\alpha u))$

Since M is 2-torsion free

$[u, v]_\alpha = [u\alpha u,d(v)]_\alpha +[d(u),v\alpha v]_\alpha+ 2u\alpha D(u,v)\alpha v) - 2v\alpha D(u,v)\alpha u) = 0$, for all $u, v \in U$; $\alpha \in \Gamma$.

And hence U is commutative.

Similarly, we have proved the following theorem.

7.6 Theorem : Let M be a 2-torsion free Γ-ring with a Lie ideal U. Suppose that there exists a symmetric bi- derivation $D: M \times M \rightarrow M$ such that $u\alpha v + d(u\alpha v) = v\alpha u + d(v\alpha u)$ for all $u, v \in U$ where d is the trace of D . Then U is commutative.

7.7 Theorem : Let M be a 2-torsion free Γ-ring and U, a Lie ideal of M . Suppose that there exists a symmetric bi-additive mapping $B: M \times M \rightarrow M$ for which either $u\alpha v - B(u, u) = v\alpha u - B(v,v)$ or $u\alpha v + B(u, u) = v\alpha u + B(v, v)$ for all $u, v \in U$ and $\alpha \in \Gamma$. Then U is commutative .

Proof : Suppose that $u\alpha v – B(u, u) = v\alpha u – B(v, v)$, for all $u, v \in U$; $\alpha \in \Gamma$.

This can be written as $[u, v]_\alpha = B(u, u) – B(v, v)$

$= d(u) – d(v)$, where d is the trace of B.

Replacing u by u+v we have

$[u+v, v]_\alpha = d(u+v) – d(v)$

That implies, $[u, v]_\alpha = d(u) + d(v) + 2B(u, v) – d(v)$

$= d(u) +2B(u, v)$ …….(i) , for all $u, v \in U; \alpha \in \Gamma$.

Now substituting – u for u in (i), we have

$[-u, v]_\alpha = d(-u) +2B(-u, v)$

That is $– [u, v] = d(u) – 2B(u, v)$

Comparing this with (i) we obtain, $2d(u) = 0$

Since M is 2-torsion free, we have $d(u) = 0$, for all $u \in U$.

Then $d(u+v) = 0$ and hence $B(u, v) = 0$, for all $u, v \in U$ and then from (i) we have the required result.

Again if M satisfies the condition $u\alpha v + B(u, u) = v\alpha u + B(v, v)$ for all $u, v \in U$ and $\alpha \in \Gamma$, then proceeding the same way we can prove the result.

7.8 Theorem : Let M be a 2-torsion free semiprime Γ- ring with a Lie ideal U. Suppose that there exists a symmetric bi-derivation $D : M \times M \to M$ such that either $[u, v]_\alpha – d(u\alpha v) + d(v\alpha u) \in Z(U)$ or $[u, v]_\alpha + d(u\alpha v) – d(v\alpha u) \in Z(U)$ for all $u, v \in U$ and $\alpha \in \Gamma$,where d is the trace of D. Then U is commutative.

Proof : Let U satisfy $[u, v]_\alpha – d(u\alpha v) + d(v\alpha u) \in Z(U)$ for all $u \in U$; $\alpha \in \Gamma$.

From (3) of Theorem 7.5 we have

$[u\alpha v, d(v)]_\alpha +[v\alpha u,d(v)]_\alpha +2[D(u, v), v\alpha v] +2u\alpha d(v)\alpha v -2v\alpha d(v)\alpha u \in Z(U)$, for all $u, v \in U$; $\alpha \in \Gamma$.

Again replacing v by u+v in the last relation

$[u\alpha(u+v), d(u+v)]_\alpha +[(u+v)\alpha u, d(u+v)]_\alpha +2[D(u, u+v), (u+v)\alpha(u+v)] +2u\alpha d(u+v)\alpha(u+v) -2(u+v)\alpha d(u+v)\alpha u = 2([u\alpha u,d(v)]_\alpha +[d(u),v\alpha v]_\alpha + 2u\alpha D(u,v)\alpha v) - 2v\alpha D(u,v)\alpha u) \in Z(U)$ and hence

$[u\alpha u,d(v)]_\alpha +[d(u),v\alpha v]_\alpha + 2u\alpha D(u,v)\alpha v) - 2v\alpha D(u,v)\alpha u) \in Z(U)$.

Therefore, $[u, v]_\alpha \in Z(U)$

Now replace v by vαu we get

$[u,v\alpha u]_\alpha = v\alpha[u, u]_\alpha + [u, v]_\alpha \alpha u = [u, v]_\alpha \alpha u \in Z(U)$

Hence, $[\,[u, v]_\alpha \alpha u, w]_\beta = 0$, for every $w \in U$; $\beta \in \Gamma$.

Then $0 = [[u, v]_\alpha \alpha u, w]_\beta$

$= [u, v]_\alpha \alpha[u, w]_\beta + [[u, v]_\alpha, w]_\beta \alpha u$

$= [u, v]_\alpha \alpha[u, w]_\alpha$

Substituting 2wγv for w we have

$0 = [u, v]_\alpha \alpha[u, 2w\gamma v]_\alpha$

$= 2[u, v]_\alpha \alpha w\gamma[u, v]_\alpha + [u, v]_\alpha \alpha[u, w]_\alpha \gamma v$

$= 2[u, v]_\alpha \alpha w\gamma[u, v]_\alpha$

Since M is 2-torsion free hence $0 = [u, v]_\alpha w\gamma[u, v]_\alpha$

Also M is semiprime implies $[u, v]_\alpha = 0$.

Using similar arguments if M satisfies the property $[u, v]_\alpha + d(u\alpha v) - d(v\alpha u) \in Z(U)$ we can prove the same.

Similarly we can prove the following :

7.9 Theorem : Let M be a 2-torsion free semiprime Γ-ring and U be a Lie ideal of M .Suppose that there exists a symmetric bi-additive mapping $B : M \times M \rightarrow M$ such that either $[u, v]_\alpha - B(u, u) + B(v, v) \in Z(U)$ or $[u, v]_\alpha + B(u, u) - B(v, v) \in Z(U)$, for all $u, v \in U$; $\alpha \in \Gamma$ .Then U is commutative.

7.10 Remark : It is equally easy to prove the commutativity of a 2-torsion free (Γ-ring M (respectively 2-torsion free semiprime Γ-ring) satisfying the property $[u, v]_\alpha = B(u, v)$, for all $u, v \in U$ (respectively $[u, v]_\alpha - B(u, v) \in Z(U)$, for all $u, v \in U$) , where $B : M \times M \rightarrow M$ is a symmetric bi-additive mapping.

Commutativity of Lie ideals of Prime gamma rings with symmetric bi-derivations

Let M be a Γ-ring and U be a Lie ideal of M. Let D : M×M → M be a symmetric bi- derivation on U of M with the trace d : M → M defined by d(u) = D(u, u) for all u ∈ U. In this paper we have proved some results of symmetric bi- derivation on Lie ideals of prime and semiprime Γ-rings . If M is a 2-torsion free prime Γ-ring and D ≠ 0 be a symmetric bi-derivation on U with the trace d, d is commutative, then U ⊆ Z(M). We also proved some other results in Γ-rings.

8. Introduction: In view of the concept of the trace of symmetric bi-derivations developed by Ozturk, Sapanci, Soyturk, Kim [47] , some important results in Γ-rings have been extended by K. K. Dey and A. C. Paul [23(i)] . These results are extended here on Lie ideals of prime and semiprime Γ-rings.

We know that a mapping f : M → M is said to be commuting on U if $[f(u), u]_{\alpha} = 0$, for all u ∈ U and α ∈ Γ and also a mapping f : M → M is centralizing on U if $[f(u), u]_{\alpha} \in Z(U)$ for all u ∈ U and α ∈ Γ.

8.1 Lemma [46, Lemma 1] : Let d : M→ M be a derivation, where M is a prime Γ-ring . Suppose that either (i) aΓd(x) = 0 for all x ∈ M or (ii) d(x)Γa = 0 for all x ∈ M holds , then we have (i) a = 0 or d = 0.

We have easily can show the following :

8.2 Lemma: Let M be a 2-torsion free prime Γ - ring , U a Lie ideal of M and d : M → M be a derivation on U of M such that aΓd(u) = 0 or d(u)Γa = 0, then a = 0 or d(u) = 0 , for all u ∈ U.

Proof : We have aαd(u) = 0 for all u ∈ U and α ∈ Γ.

Then $0 = a\alpha d(2u\beta v) = 2(a\alpha d(u)\beta v + a\alpha u\beta d(v))$

$= 2a\alpha u\beta d(v)$

Hence $a\alpha u\beta d(v) = 0$.

From Lemma 1.29 we have $a = 0$ or $d(v) = 0$.

8.3 Theorem : Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and U be a Lie ideal of M. Let $D : M\times M \rightarrow M$ be a symmetric bi-derivation with the trace d . Suppose that d is commuting on U , then $U \subseteq Z(M)$ or $D = 0$ on U.

Proof : We have $[d(u), u]_\alpha = 0$, for all $u \in U$, $\alpha \in \Gamma$.

Linearizing , we get $[d(u) + d(v) + 2D(u,v), u+v]_\alpha = 0$

That is

$0 = [d(u), u]_\alpha + [d(u), v]_\alpha + [d(v), u]_\alpha + [d(v), v]_\alpha + 2[D(u, v), u]_\alpha + 2[D(u, v), v]_\alpha$

$=[d(u), v]_\alpha + [d(v), u] + 2[D(u, v), u]_\alpha + 2[D(u, v),v]_\alpha$ ....(1)

for all $u, v \in U$ and $\alpha \in \Gamma$.

Substituting $- u$ for u we have

$0 = [d(u), v]_\alpha - [d(v), u]_\alpha + 2[D(u, v), u]_\alpha - 2[D(u, v), v]_\alpha$ ....(2)

Adding these we have ,

$0 = 2[d(u), v]_\alpha + 4[D(u, v), u]_\alpha$

Since M is 2-torsion free , $0 = [d(u), v]_\alpha + 2[D(u, v), u]_\alpha$ ....(3)

Replacing v by $2u\beta v$ we get ,

$0 = [d(u) , 2u\beta v]_\alpha + 2[D(u, 2u\beta v), u]_\alpha$

$= 2([d(u), u]_\alpha\beta v + u\beta[d(u), v]_\alpha + 2[u\beta D(u, v) + d(u)\beta v , u]_\alpha)$

$= 2(u\beta[d(u), v]_\alpha + 2(u\beta[D(u, v) , u]_\alpha + [u, u]_\alpha\beta D(u, v) + d(u)\beta[v, u]_\alpha +$

$[d(u), u]_\alpha\beta v)$

$= 2(u\beta[d(u), v]_\alpha + 2u\beta[D(u, v) , u]_\alpha + 2d(u)\beta[v, u]_\alpha)$

Since M is 2-torsion free ,we get

$0 = u\beta([d(u), v]_\alpha + 2[D(u, v), u]_\alpha) + 2d(u)\beta[v, u]_\alpha$ using (3)

$= 2d(u)\beta[v, u]_\alpha$

Using 2-torsion freeness of M , we obtain

$d(u)\beta[v, u]_\alpha = 0$,for all $u, v \in U$ and $\alpha \in \Gamma$ ....(4)

From Lemma 8.2 we can write $d(u) = 0$ or $[v, u]_\alpha = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$. Then U is commutative if $[u, v]_\alpha = 0$.

On the other hand for any $u \notin Z(U)$, we have $[u, v]_\alpha \neq 0$, then $d(u) = 0$.

let $u \in Z(U)$ and $v \notin Z(U)$. Then $u + v \notin Z(U)$ and $u - v \notin Z(U)$.

Then $d(u+v) = 0$ implies $0 = d(u) + d(v) + 2D(u, v) = d(u) + 2D(u, v)$ ……………………..(5)

and also $0 = d(u) + d(v) - 2D(u, v) = d(u) - 2D(u, v)$ ....(6)

From these two relations we have $4D(u, v) = 0$.

By the 2-torsion freeness of M , we have $D(u,v) = 0$ for all $u, v \in U$.

8.4 Theorem : Let M be a 2 and 3 torsion free prime Γ-ring satisfying the condition (*). Let $D : M \times M \to M$ and d be a symmetric bi-derivation and the trace of D respectively . Suppose that d is centralizing on U , then $U \subseteq Z(M)$ or $D = 0$ on U.

Proof : We have $[d(u), u]_\alpha \in Z(U)$ for all $u \in U, \alpha \in \Gamma$.

By linearizing we get,

$[d(u + v), u + v]_\alpha = [d(u) + d(v) + 2D(u, v) , u + v]_\alpha$

$= [d(u), u]_\alpha + [d(v), u]_\alpha + 2[D(u, v), u]_\alpha + [d(u), v]_\alpha + [d(v), v]_\alpha + 2[D(u, v), v]_\alpha \in Z(U)$.

Then $[d(v), u]_\alpha + [d(u), v]_\alpha + 2[D(u, v), u]_\alpha + 2[D(u,v), v]_\alpha \in Z(U)$ for all $u, v \in U$ and $\alpha \in \Gamma$. .... (7)

Rplacing u by – u we have

$- [d(v), u]_\alpha + [d(u), v]_\alpha + 2[D(u,v), u]_\alpha - 2[D(u, v), v]_\alpha \in Z(U)$ ……………….(8)

From (7) and (8) we have $2[d(u), v]_\alpha + 4[D(u, v), u]_\alpha \in Z(U)$.

Since M is 2-torsion free , $[d(u), v]_\alpha + 2[D(u, v), u]_\alpha \in Z(U)$

Replacing v by $u\beta u$ we get

$[d(u), u\beta u]_\alpha + 2[D(u, u\beta u), u]_\alpha$

$= u\beta[d(u), u]_\alpha + [d(u), u]_\alpha\beta u + 2[u\beta D(u, u) + D(u, u)\beta u, u]_\alpha$

$= u\beta[d(u), u]_\alpha + [d(u), u]_\alpha\beta u + 2[u\beta d(u) + d(u)\beta u, u]_\alpha$

$= u\beta[d(u), u]_\alpha + [d(u), u]_\alpha\beta u + 2u\beta[d(u), u]_\alpha + 2[u, u]_\alpha\beta d(u) + 2[d(u),u]_\alpha\beta u + 2d(u)\beta[u, u]_\alpha$

$= 3u\beta[d(u), u]_\alpha + 3[d(u), u]_\alpha\beta u = 6[d(u), u]_\alpha\beta u \in Z(U)$,

since $[d(u), u]_\alpha \in Z(U)$.

For the 2 and 3 torsion freeness of M , we have $[d(u), u]_\alpha\beta u \in Z(U)$. Then for any $v \in U$ and $\alpha \in \Gamma$, $[d(u), u]_\alpha\beta[u, v]_\alpha = 0$.

Then using Lemma 8.2 we have either $[d(u), u]_\alpha = 0$ or $[u, v]_\alpha = 0$.

If $[d(u), u]_\alpha = 0$, then from theorem 8.3 we can conclude the theorem. On the other hand if $[u, v]_\alpha = 0$ then by same argument using in theorem 8.3 the theorem will be proved.

8.5 Theorem : Let M be a 2-torsion free prime Γ–ring satisfying the condition (*) and let U be a Lie ideal of M. Suppose that there exist symmetric bi-derivations $D_1: M \times M \to M$ and $D_2: M \times M \to M$ such that $D_1(d_2(u), u) = 0$ holds for all $u \in U$,where d_2 denotes the trace of D_2. Then $D_1 = 0$ or $D_2 = 0$ on U.

Proof : Given that $D_1(d_2(u), u) = 0$ for all $u \in U$ ....(11)

Linearizing we have $D_1(d_2(u + v), u + v) = 0$ for all $u, v \in U$

Then $0 = D_1(d_2(u) + d_2(v) + 2D_2(u, v), u + v)$

$= D_1(d_2(u), u) + D_1(d_2(v), u) + 2D_1(D_2(u, v), u) + D_1(d_2(u), v) + D_1(d_2(v), v) + 2D_1(D_2(u, v), v)$

$= D_1(d_2(v), u) + 2D_1(D_2(u, v), u) + D_1(d_2(u), v) + 2D_1(D_2(u, v), v)$

for all $u, v \in U$. ....(12)

Replacing u by – u we have ,

$0 = D_1(d_2(-u),- u) + D_1(d_2(v), -u) + 2D_1(D_2(-u, v), -u) + D_1(d_2(-u), v) + D_1(d_2(v), v) + 2D_1(D_2(-u, v), v)$

$= - D_1(d_2(v), u) + 2D_1(D_2(u, v), u) + D_1(d_2(u), v) - 2D_1(D_2(u, v), v)$ ....(13)

Adding (12) and 13() we get, $0 = 4D_1(D_2(u, v), u) + 2D_1(d_2(u), v)$

Since M is 2-torsion free , $0 = 2D_1(D_2(u, v), u) + D_1(d_2(u), v)$

Replacing v by $2u\alpha v$ we obtain,

$0 = 2D_1(D_2(u, 2u\alpha v), u) + D_1(d_2(u), 2u\alpha v)$

$= 2D_1(2u\alpha D_2(u, v) + 2D_2(u, u)\alpha v, u) + 2u\alpha D_1(d_2(u), v) + 2D_1(d_2(u), u)\alpha v$

$= 4D_1(u\alpha D_2(u, v) + 4d_2(u)\alpha v, u) + 2u\alpha D_1(d_2(u), v)$

$= 4u\alpha D_1(D_2(u, v), u) + 4D_1(u, u)\alpha D_2(u, v) + 4d_2(u)\alpha D_1(u, v) + 4D_1(d_2(u),u)\alpha v + 2u\alpha D_1(d_2(u), v)$

$= 4u\alpha D_1(D_2(u, v), u)+ 4d_1(u)\alpha D_2(u, v) +4d_2(u)\alpha D_1(u, v) +2u\alpha D_1(d_2(u), v)$

$= 2u\alpha D_1(d_2(u), v)+4u\alpha D_1(D_2(u, v), u) + 4(d_1(u)\alpha D_2(u, v) +d_2(u)\alpha D_1(u, v))$

$= u\alpha(4D_1(D_2(u, v), u) +2D_1(d_2(u), v)) + 4(d_1(u)\alpha D_2(u, v) +d_2(u)\alpha D_1(u, v))$

$= 4(d_1(u)\alpha D_2(u, v) +d_2(u)\alpha D_1(u, v))$

Since M is 2- torsion free , we have

$d_1(u)\alpha D_2(u, v) +d_2(u)\alpha D_1(u, v) = 0$, for all $u, v \in U$ . ...(14)

Again replacing v by $2v\beta u$ we get,

$0 = d_1(u)\alpha D_2(u, 2v\beta u) + d_2(u)\alpha D_1(u, 2v\beta u)$

$= 2d_1(u)\alpha v\beta D_2(u, u) +2d_1(u)\alpha D_2(u, v)\beta u +2d_2(u)\alpha v\beta D_1(u, u) + 2d_2(u)\alpha D_1(u, v)\beta u$

$= 2d_1(u)\alpha v\beta d_2(u) + 2d_1(u)\alpha D_2(u, v)\beta u + 2d_2(u)\alpha v\beta d_1(u) + 2d_2(u)\alpha D_1(u, v)\beta u$

Since M is 2-torsion free, we have ,

$0 = d_1(u)\alpha v\beta d_2(u) + d_2(u)\alpha v\beta d_1(u) + (d_1(u)\alpha D_2(u, v) + d_2(u)\alpha D_1(u, v))\beta u$

$= d_1(u)\alpha v\beta d_2(u) + d_2(u)\alpha v\beta d_1(u)$, for all $u, v \in U$ and $\alpha \in \Gamma$. ....(15)

Let us assume that d_1 and d_2 are both different from zero. In other words, there exist elements $u_1, u_2 \in U$ such that $d_1(u_1) \neq 0$ and $d_2(u_2) \neq 0$. Then using Lemma 1.24 and Lemma 1.25 , it follows that $d_1(u_2) = 0$ or $d_2(u_1)= 0$. Since $d_1(u_2) = 0$, the relation (13) becomes $d_2(u_2)\alpha D_2(u_2, v) = 0$.

Using this with Lemma 8.2 we obtain $D_1(u_2, v) = 0$ for all $v \in U$, since $d_2(u_2) \neq 0$.

In particular we have $D_1(u_2, u_1) = 0$.

Similarly we write $u_1 + u_2$ for v,

then $d_1(v) = d_1(u_1 + u_2)$

$= d_1(u_1) + d_1(u_2) + 2D_1(u_1, u2)$

$= d_1(u_1) \neq 0$

Similarly we have $d_2(v) \neq 0$.

But according to (14) with Lemma 1.29 and Lemma 1.30, d_1 and d_2 cannot be both different from zero. Therefore, we have $d_1 = 0$ or $d_2 = 0$.

Then $d_1(u) = 0$ implies $D_1(u, u) = 0$.

Linearizing of one variable we have $D_1(u, u+v) = 0$.

Then $0 = D_1(u,u) + D_1(u, v)$

$= D_1(u, v)$ for all $u, v \in U$.

Therefore $D_1 = 0$.

Similarly we can show that $D_2 = 0$,when $d_2 = 0$.

In case of $D_1 = D_2$, this can be proved for semiprime Γ–rings.

8.6 Theorem : Let M be a 2-torsion free semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M . Suppose that there exists a symmetric bi- derivation $D: M\times M \to M$ that $D(d(u), u) = 0$ holds for all $u \in U$ where d denotes the trace of D . Then D = 0.

Proof : In this case (15) reduces to $d(u)\alpha v\beta d(u) = 0$ for $u, v \in U$; $\alpha \in \Gamma$. That implies $d(u) = 0$ for all $u \in U$. Therefore, D = 0.

8.7 Theorem : Let M be a 2 and 3- torsion free prime Γ-ring satisfying (*)and U be a Lie ideal of M. Let $D_1 : M\times M \to M$ and $D_2 : M\times M \to M$

be symmetric bi-derivations . Suppose further that there exist a symmetric bi-additive mapping $B : M \times M \to M$ such that $d_1(d_2(u)) = f(u)$ holds for all $u \in U$, where d_1 and d_2 are the traces of D_1 and D_2 respectively and f is the trace of B. Then $D_1 = 0$ or $D_2 = 0$.

Proof : From the Linearization of the relation $d_1(d_2(u)) = f(u)$ ....(16) ,we have

$d_1(d_2(u + v)) = f(u+v)$

That implies $d_1(d_2(u) + d_2(v) + 2D_2(u, v)) = f(u) + f(v) + 2B(u, v)$

That is,

$f(u) + f(v) + 2B(u, v) = d_1(d_2(u) + d_2(v)) + d_1(2D_2(u, v)) + 2D_1(d_2(u) + d_2(v), 2D_2(u, v)$

$= d_1(d_2(u)) + d_1(d_2(v) + 2D_1(d_2(u), d_2(v)) + 2d_1(D_2(u, v)) + 4D_1(d_2(u), D_2(u,v)) \quad + \quad 4D_1(d_2(v), D_2(u, v))$

$= f(u) + f(v) + 2D_1(d_2(u), d_2(v)) + 2d_1(D_2(u, v) + 4D_1(d_2(u), D_2(u, v)) + 4D_1(d_2(v), D_2(u, v))$

That implies $2B(u, v) = 2D_1(d_2(u), d_2(v)) + 2d_1(D_2(u, v) + 4D_1(d_2(u), D_2(u, v)) + 4D_1(d_2(v), D_2(u, v))$.

And hence , $B(u, v) = D_1(d_2(u), d_2(v)) + d_1(D_2(u, v) + 2D_1(d_2(u), D_2(u, v)) + 2D_1(d_2(v), D_2(u, v))$, for all $u, v \in U$.

Substituting – u for u we have

$B(-u, v) = D_1(d_2(-u), d_2(v)) + d_1(D_2(-u, v) + 2D_1(d_2(-u), D_2(-u, v)) + 2D_1(d_2(v), D_2(-u, v))$

That is $- B(u, v) = D_1(d_2(u), d_2(v)) + d_1(D_2(u, v)) - 2D_1(d_2(u), D_2(u, v)) – 2D_1(d_2(v), D_2(u, v))$, for all $u, v \in U$

Comparing these two relations we have ,

$2B(u, v) = 4D_1(d_2(u), D_2(u, v)) + 4D_1(d_2(v), D_2(u, v))$

Then, $B(u, v) = 2D_1(d_2(u), D_2(u, v)) + 2D_1(d_2(v), D_2(u, v))$ ....(17)

replacing u by 2u we get,

$B(2u, v) = 2D_1(d_2(2u), D_2(2u, v)) + 2D_1(d_2(v), D_2(2u, v))$

That is $2B(u, v) = 2D_1(4d_2(u), 2D_2(u, v)) + 2D_1(d_2(v), 2D_2(u, v))$

$= 16D_1(d_2(u), D_2(u, v)) + 4D_1(d_2(v), D_2(u, v))$

[Since $d(2u) = 4d(u)$, $d(-u) = d(u)$, $D(-u, v) = -D(u, v)$]

Using 2-torsion freeness we have,

$B(u, v) = 8D_1(d_2(u), D_2(u, v)) + 2D_1(d_2(v), D_2(u, v))$ ....(18)

comparing (16) and (17) we obtain,

$2D_1(d_2(u), D_2(u, v)) + 2D_1(d_2(v), D_2(u, v)) = 8D_1(d_2(u), D_2(u, v)) + 2D_1(d_2(v), D_2(u, v))$

That implies , $6D_1(d_2(u), D_2(u, v)) = 0$

Since M is 2 and 3-torsion free , $D_1(d_2(u), D_2(u, v)) = 0$ ....(19)

Also we can write $D_1(d_2(v), D_2(u, v)) = 0$

Then from (16) we have , B = 0. and then from (15) we get $d_1(d_2(u)) = 0$ .... (20)

Again replacing v by $2v\alpha u$ on (19) we have ,

$0 = D_1(d_2(u), D_2(u, 2v\alpha u))$

$= 2(D_1(d_2(u), v\alpha D_2(u, u) + D_2(u, v)\alpha u))$

Then $0 = D_1(d_2(u), v\alpha d_2(u)) + D_1(d_2(u), D_2(u, v)\alpha u)$

$= v\alpha D_1(d_2(u), d_2(u)) + D_1(d_2(u), v)\alpha d_2(u) + D_2(u, v)\alpha D_1(d_2(u), u) + D_1(d_2(u), D_2(u, v))\alpha u$

$= v\alpha d_1(d_2(u)) + D_1(d_2(u), v)\alpha d_2(u) + D_2(u, v)\alpha D_1(d_2(u), u) + D_1(d_2(u), D_2(u, v))\alpha u$

$= D_1(d_2(u), v)\alpha d_2(u) + D_2(u, v)\alpha D_1(d_2(u), u)$ .....(21) [using (19) and (20)]

Replacing v by $2u\beta v$ we have,

$0 = D_1(d_2(u), 2u\beta v)\alpha d_2(u) + D_2(u, 2u\beta v)\alpha D_1(d_2(u), u)$

$= 2(u\beta D_1(d_2(u), v)\alpha d_2(u) + D_1(d_2(u), u)\beta v\alpha d_2(u) + u\beta D_2(u, v)\alpha D_1(d_2(u), u) + D_2(u, u)\beta v\alpha D_1(d_2(u), u))$

$= 2(u\beta(D_1(d_2(u), v)\alpha d_2(u) + D_2(u, v)\alpha D_1(d_2(u), u)) + D_1(d_2(u), u)\beta v\alpha d_2(u) + d_2(u)\beta v\alpha D_1(d_2(u), u))$

Hence

$0 = D_1(d_2(u), u)\beta v\alpha d_2(u) + d_2(u)\beta v\alpha D_1(d_2(u), u)$ ....(22) [using (21)]

Using Lemma 1.29 and Lemma 1.30, we can conclude that

$D_1(d_2(u), u) = 0$ or $d_2(u) = 0$ for all $u \in U$

If $D_1(d_2(u), u) \neq 0$ for some $u \in U$, then $d_2(u) = 0$, a contradiction.

Hence $D_1(d_2(u), u) = 0$ for all $u \in U$.

Using theorem 8.5 we conclude that the theorem is proved.

Note : In case $D_1 = D_2$, the theorem can be proved for semiprime Γ-rings.

8.8 Theorem : Let M be a 2, 3-torsion free semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M. Let D: M×M → M and B : M×M →M be a symmetric bi-derivation and a symmetric bi-additive mapping respectively. Suppose that d(d(u) = f(u) holds for all $u \in U$, where d and f are the trace of D and B respectively, then D = 0.

Proof : From theorem 8.4 and from (18) we can write

$D(d(u), D(u, v)) = 0$ for all $u, v \in U$ ....(23)

and $d(d(u)) = 0$ ....(24)

In (23) put $2v\alpha w$ for v, then

$0 = D(d(u), D(u, 2v\alpha w))$

$= 2(D(d(u), v\alpha D(u, w) + D(u, v)\alpha w))$

$= 2(v\alpha D(d(u), D(u, w)) + D(d(u), v)\alpha D(u, w) + D(u, v)\alpha D(d(u), w) + D(d(u), D(u, v))\alpha w)$

Hence, $0 = D(d(u), v)\alpha D(u, w) + D(u, v)\alpha D(d(u), w)$

In particular , for $w = d(u)$ we obtain

$0 = D(d(u), v)\alpha D(u, d(u)) + D(u, v)\alpha D(d(u), d(u))$

$= D(d(u), v)\alpha D(u, d(u)) + D(u, v)\alpha d(d(u))$

$= D(d(u), v)\alpha D(u, d(u))$,for all $u, v \in U, \alpha \in \Gamma$ ....(25)

Replacing v by $2u\beta v$ we get

$0 = D(d(u), 2u\beta v)\alpha D(u, d(u))$

$= 2((u\beta D(d(u), v) + D(d(u), u)\beta v)\alpha D(u, d(u)))$

$= 2(u\beta D(d(u), v)\alpha D(u, d(u)) + D(d(u), u)\beta v\alpha D(u, d(u))$

Then $0 = D(d(u), u)\beta v\alpha D(u, d(u))$, for all $u, v \in U$; $\alpha, \beta \in \Gamma$.

That implies $D(d(u), u)\beta v\alpha D(d(u), u) = 0$.

And Hence $D(d(u), u) = 0$ for all $u \in U$.

Thus by theorem 8.6 the proof is complete.

Symmetric bi-derivations with symmetric generalized bi-derivations on Lie ideals of Prime Gamma rings

In this research we have studied derivation, bi-derivation, symmetric bi-derivation and then we have mentioned symmetric generalized bi-derivation. We have worked these on Lie ideals of prime gamma rings. If $U \neq 0$ is a Lie ideal of M and M admits symmetric bi-derivations D and G with trace d and g respectively then we get some important results.

9. Introduction : Maksa [44] first introduced the notion of symmetric bi-derivation. He developed the trace of symmetric bi-derivations. In [10] Bresar studied generalized bi-derivations on rings. Ozturk, Sapanci, Soyturk, and kim [49] have introduced this symmetric bi- derivations in Γ-rings and they also have extended their study on a ideal of a prime Γ-ring. In this chapter we generalized symmetric bi-derivation on Lie ideals of prime Γ-ring. Here we have used the commutator and the anti-commutator identities both at a time .

9.1 Definition: Let M be a Γ-ring and U be a Lie ideal of M . Suppose that $f : M \times M \to M$ is an additive mapping. f is called a generalized bi-derivation on U of M , if there exists a bi-derivation $d : M \times M \to M$ such that $f(u\alpha v, w) = f(u, w)\alpha v + u\alpha d(v, w)$ for all $u, v, w \in U$, $\alpha \in \Gamma$. Also $f(u, v\alpha w) = f(u, v)\alpha w + v\alpha d(u, w)$.

Here f is symmetric if $f(u, v) = f(v, u)$ for all $u, v \in U$.

9.2 Definition: Let M be a Γ-ring, U a Lie ideal of M and d be a bi-derivation on U of M. If $s\delta d(u, v) = 0$ for some $u, v, s \in U$, then we say , d act as a left bi-multiplier.

9.3 Theorem : Let M be a 2-torsion free prime Γ-ring and U be a Lie ideal of M . If M admits a symmetric generalized bi- derivation f with an associated bi-derivation d such that $[f(u, u), u]_\alpha = 0$ for all $u \in U$,$\alpha \in \Gamma$ then U is contained in Z(M) or d acts as a left bi-multiplier .

Proof : Assume that $U \not\subset Z(M)$.

Suppose that $[f(u, u), u]_\alpha = 0$, for all $u \in U$, $\alpha \in \Gamma$..........(1)

Linearizing (i) on u, we obtain

$0 = [f(u+v, u+v) , (u+v)]_\alpha$

$= [f(u, u), u]_\alpha + [f(u, v), u]_\alpha + [f(v, u), u]_\alpha + [f(v, v), u]_\alpha + [f(u, u), v]_\alpha + [f(u, v), v]_\alpha + [f(v, u), v]_\alpha + [f(v, v), v]_\alpha$

$= 2[f(u, v), u]_\alpha +2[f(u, v), v]_\alpha + [f(v, v), u]_\alpha+ [f(u, u), v]_\alpha$ ....(2)

[using (1) and symmetricity]

Putting –v for v in (2) , we get

$0 = 2[f(u, -v), u]_\alpha +2[f(u, -v), -v]_\alpha + [f(-v, -v), u]_\alpha+ [f(u, u), -v]_\alpha$

$= -2[f(u, v), u]_\alpha +2[f(u, v), v]_\alpha + [f(v, v), u]_\alpha - [f(u, u), v]_\alpha$ ....(3),

[since $f(u, -v) = -f(u, v)$]

Adding (2) and (3) and using 2-torsion freeness of M we have ,

$0 = 2[f(u, v), v]_\alpha + [f(v, v), u]_\alpha$ ....(4)

Replacing u by $2u\beta w$; $w \in U$, $\beta \in \Gamma$ in (iv), we have

$0 = 2[f(2u\beta w, v), v]_\alpha + [f(v, v), 2u\beta w]_\alpha$

$= 4 [f(u, v) \beta w + u\beta d(w, v) , v]_\alpha + 2[f(v, v), u\beta w]_\alpha$

$= 4f(u, v) \beta[w, v]_\alpha + 4[f(u, v), v]_\alpha\beta w + 4u\beta[d(w, v), v]_\alpha + 4[u, v]_\alpha \beta d(w, v) + 2u\beta[f(v, v), w]_\alpha + 2[f(v, v), u]_\alpha\beta w.$

Since M is 2-torsion free , we have

$0 = 2f(u, v) \beta[w, v]_\alpha + 2[f(u, v), v]_\alpha\beta w + 2u\beta[d(w, v), v]_\alpha + 2[u, v]_\alpha \beta d(w, v) + u\beta[f(v, v), w]_\alpha + [f(v, v), u]_\alpha\beta w.$

$= 2f(u, v) \beta[w, v]_\alpha + (2[f(u, v), v]_\alpha +[f(v, v), u]_\alpha) \beta w+ 2u\beta[d(w, v), v]_\alpha + 2[u, v]_\alpha \beta d(w, v) + u\beta[f(v, v), w]_\alpha$.

$= 2f(u, v) \beta[w, v]_\alpha + 2u\beta[d(w, v), v]_\alpha + 2[u, v]_\alpha \beta d(w, v) + u\beta[f(v, v), w]_\alpha$ .....(5), for all $u, v, w \in U$ and $\alpha, \beta \in \Gamma$.

Putting v for w in (5) we find that

$0 = 2f(u, v)\, \beta[v, v]_\alpha + 2u\beta[d(v, v), v]_\alpha + 2[u, v]_\alpha\, \beta d(v, v) + u\beta[f(v, v), v]_\alpha$

$= 2u\beta[d(v, v), v]_\alpha + 2[u, v]_\alpha\, \beta d(v, v)$, using (1)

Since M is 2- torsion free , we have

$u\beta[d(v, v), v]_\alpha + [u, v]_\alpha\, \beta d(v, v) = 0$ ....(6)

Replacing u by $2w\gamma u$ we get

$0 = 2w\gamma u\, \beta[d(v, v), v]_\alpha + [2w\gamma u, v]_\alpha\, \beta d(v, v)$

$= 2w\gamma u\, \beta[d(v, v), v]_\alpha + 2w\gamma[u, v]_\alpha\, \beta d(v, v) +2\, [w, v]_\alpha \gamma u\, \beta d(v, v)$

$= 2w\gamma(u\, \beta[d(v, v), v]_\alpha + [u, v]_\alpha\, \beta d(v, v)) + 2[w, v]_\alpha \gamma u\, \beta d(v, v)$

$= 2[w, v]_\alpha \gamma u\, \beta d(v, v)$, and then

$[w, v]_\alpha \gamma u\, \beta d(v, v) = 0$ for all $u, v, w \in U$; $\alpha, \beta, \gamma \in \Gamma$.........(7)

Substituting $2\, u\delta s$ for u ; $s \in U$, $\delta \in \Gamma$ we find that

$0 = [w, v]_\alpha \gamma\, 2\, u\delta s\, \beta d(v, v)$

$= 2[w, v]_\alpha \gamma\, u\delta s\, \beta d(v, v)$

Hence $[w, v]_\alpha \gamma\, u\delta s\, \beta d(v, v) = 0$ …………(8)

That implies $[w, v]_\alpha \gamma\, U\delta s\, \beta d(v, v) = 0$

Here $[w, v] \in U$ and $s\beta d(v, v) \in M$.

Now , from Lemma 1.29, we have $[w, v]_\alpha = 0$ or $s\beta d(v, v) = 0$

If $[w, v]_\alpha = 0$, for every $w, v \in U$ and $\alpha \in \Gamma$,

then $U \subset Z(U) = Z(M)$ [Lemma 1.26], a contradiction

Hence $s\beta d(v, v) = 0$, for all $s, v \in U$; $\beta \in \Gamma$.

putting $v = v + w$, we obtain

$0 = s\beta d(v + w, v + w)$

$= s\beta d(v, v) + s\beta d(v, w) + s\beta d(w, v) + s\beta d(w, w)$

$= s\beta d(v, w) + s\beta d(w, v)$, for all $u, v, w \in U$; $\beta \in \Gamma$.

$= 2s\beta d(v, w)$, since d is symmetric .

That implies $s\beta d(v, w) = 0$; $s \in U$, $\beta \in \Gamma$.

Therefore, d acts as a left bi-multiplier.

9.4 Theorem : Let M be a 2-torsion free prime Γ-ring and U ≠ 0 be a Lie ideal of M. If M admits a symmetric generalized bi-derivation f with an associated bi-derivation d such that $f(u, v) \pm [u, v]_\alpha \in Z(M)$, for all u, v ∈ U ,α ∈ Γ .Then M is commutative or d acts as a left bi-multiplier.

Proof : Assume that

$f(u, v) + [u, v]_\alpha \in Z(M)$, for all $u, v \in U$, $\alpha \in \Gamma$........(9)

Substituting $2v\beta w$ for v in (9), we find that

$f(u, 2v\beta w) + [u, 2v\beta w]_\alpha = 2(f(u, v)\beta w + v\beta d(u, w) + [u, v]_\alpha\beta w + v\beta[u, w]_\alpha)$

$= 2((f(u, v) + [u, v]_\alpha)\beta w + v\beta([u, w]_\alpha + d(u, w))$

Then from (9) we have

$v\beta(d(u, w) + [u, w]_\alpha) \in Z(M)$, for all u, v, w ∈ U and α, β ∈ Γ.

Then $[v\beta d(u, w) + v\beta[u, w]_\alpha , m]_\gamma = 0$; for all u, v, w ∈ U, m ∈ M and α, β, γ ∈ Γ.

Putting w for u we have

$0 = [v\beta d(w, w) + v\beta[w, w]_\alpha , m]_\gamma$

$= [v\beta d(w, w) , m]_\gamma$

$= v\beta[d(w, w), m]_\gamma + [v, m]_\gamma\beta d(w, w)$ …….(10) for all v, w ∈ U, m ∈ M and β, γ ∈ Γ.

Replacing v by $2v\alpha u$; u ∈ U, α ∈ Γ, we get

$0 = 2v\alpha u\beta[d(w, w), m]_\gamma + [2v\alpha u, m]_\gamma\beta d(w, w)$

$= 2v\alpha u\beta[d(w, w), m]_\gamma + 2v\alpha[u, m]_\gamma\beta d(w, w) + 2[v, m]_\gamma\alpha u\beta d(w, w)$

$= 2v\alpha (u\beta[d(w, w), m]_\gamma + [u, m]_\gamma\beta d(w, w)) + 2[v, m]_\gamma\alpha u\beta d(w, w)$

$= 2[v, m]_\gamma\alpha u\beta d(w, w)$ [using (10)]

Hence $[v, m]_\gamma\alpha u\beta d(w, w) = 0$ ……..(11)

Now replacing m by $m\delta x$, x ∈ M, δ ∈ Γ we have

$0 = [v, m\delta x]_\gamma\alpha u\beta d(w, w)$

$= [v, m]_\gamma\delta x\alpha u\beta d(w, w) + m\delta([v, x]_\gamma\alpha u\beta d(w, w))$

$= [v, m]_\gamma\delta x\alpha u\beta d(w, w)$, for every u, v, w ∈ U ; m, x ∈ M ; α, β,δ, γ ∈ Γ

Hence $[v, m]_\gamma\Gamma M\Gamma u\beta d(w, w) = 0$.

Since M is prime , we have $[v, m]_\gamma = 0$ or $u\beta d(w, w) = 0$.

If $[v, m]_\gamma = 0$, then $U \subseteq Z(M)$ and for the case of $u\beta d(w, w) = 0$, using the same argument used in Theorem 9.3 , we have the result.

The case $f(u, v) - [u, v]_\alpha \in Z(M)$ for all $u, v \in U$ and $\alpha \in \Gamma$ can be proved in the similar manner.

9.5 Theorem : Let M be a 2-torsion free prime Γ-ring satisfying condition (*) and $U \neq 0$ be a Lie ideal of M . If M admits a symmetric generalized bi-derivation f with an associated bi-derivation d such that $(f(u, u), u)_\alpha = 0$ for all $u \in U$; $\alpha \in \Gamma$ .Then M is commutative or f acts as a left bi-multiplier.

Proof : By hypothesis $(f(u, u), u)_\alpha = 0$

Putting u+v for u, we have

$0 = (f(u+v, u+v), u+v)_\alpha$

$= (f(u, u), u)_\alpha + (f(u, u), v)_\alpha +(f(u, v), u)_\alpha +(f(u, v), v)_\alpha +(f(v, u), u)_\alpha +(f(v, u), v)_\alpha +(f(v, v), u)_\alpha +(f(v, v), v)_\alpha$

$= (f(u, u), v)_\alpha+2(f(u, v), u)_\alpha+2(f(u, v), v)_\alpha+(f(v, v), u)_\alpha$ ....(12)

Putting –v for v in the above, we get

$(f(u, u), v)_\alpha -2(f(u, v), u)_\alpha+2(f(u, v), v)_\alpha+(f(v, v), u)_\alpha = 0$ ....(13)

Adding (12) and (13) and using 2-torsion freeness of M , we get

$2(f(u, v), v)_\alpha+ (f(v, v), u)_\alpha = 0$

i.e., $2f(u, v)\alpha v + 2v\alpha f(u, v) + f(v, v)\alpha u + u\alpha f(v, v) = 0$ ....(14)

Replacing u by $2u\beta w$ we have

$0 = 2f(2u\beta w, v)\alpha v + 2v\alpha f(2u\beta w, v) + f(v, v)\alpha 2u\beta w + 2u\beta w\alpha f(v, v)$

Using torsion freeness of M we get

$0 = f(2u\beta w, v)\alpha v + v\alpha f(2u\beta w, v) + f(v, v)\alpha u\beta w + u\beta w\alpha f(v, v)$

$= 2u\beta f(w, v)\alpha v + 2f(u, v)\beta w\alpha v + 2v\alpha f(u, v)\beta w + 2v\alpha u\beta f(w, v) + f(v, v)\alpha u\beta w + u\beta w\alpha f(v, v)$..............(15)

Right multiplying (14) by w we get

$0 = 2f(u, v)\alpha v\beta w + 2v\alpha f(u, v)\beta w + f(v, v)\alpha u\beta w + u\alpha f(v, v)\beta w$

Subtracting (15) from (14) we have

$0 = 2u\beta f(w, v)\alpha v + 2f(u, v)\beta w\alpha v + 2v\alpha u\beta f(w, v) - 2f(u, v)\alpha v\beta w$

$- \ u\alpha f(v, v)\beta w$

Using (*) we obtain,

$2f(u, v)\beta[w, v]_\alpha - u\beta[f(v, v), w]_\alpha + 2u\beta d(w, v)\alpha v + 2v\alpha u\beta d(w, v) = 0$

Replacing w by v we get

$u\beta[v, f(v, v)]_\alpha + 2u\beta d(v, v)\alpha v + 2v\alpha u\beta d(v, v) = 0$ ....(16)

Putting $2x\gamma u$, $x \in U$ for u in (16) we have

$2x\gamma u\beta[v, f(v, v)] + 4x\gamma u\beta d(v, v)\alpha v + 4v\alpha x\gamma u\beta d(v, v) = 0$

Then $x\gamma u\beta[v, f(v, v)] + 2x\gamma u\beta d(v, v)\alpha v + 2v\alpha x\gamma u\beta d(v, v) = 0$ ....(17)

for all $u, v, x \in U$; $\alpha, \beta, \gamma \in \Gamma$.

Left multiplying (xvii) by x we get

$x\gamma u\beta[v, f(v, v)]_\alpha + 2x\gamma u\beta d(v, v)\alpha v + 2x\gamma v\alpha u\beta d(v, v) = 0$ ....(18)

Subtracting (18) from (17) and using 2-torsion freeness of M we get

$v\alpha x\gamma u\beta d(v, v) - x\gamma v\alpha u\beta d(v, v) = 0$

Using the condition (*) $v\alpha x\gamma u\beta d(v, v) - x\alpha v\gamma u\beta d(v, v) = 0$

And then we get $[v, x]_\alpha\gamma u\beta d(v, v) = 0$

This is the same type of relation (8).

So proceeding same way we have the required result.

9.6 Theorem : Let M be a 2-torsion free prime Γ-ring and U be a nonzero Lie ideal of M . If M admits a symmetric generalized bi-derivation f with an associated bi-derivation d such that $f(a, b) \pm (a, b)_\alpha \in Z(M)$ for all u, v $\in U$ and $\alpha \in \Gamma$.

Proof : We have $f(u, v) + (u, v)_\alpha \in Z(M)$ for all $u, v \in U$; $\alpha \in \Gamma$.

Putting $2v\beta w$, $w \in U$; $\beta \in \Gamma$ for v we obtain

$f(u, 2v\beta w) + (u, 2v\beta w)_\alpha = 2f(u, v)\beta w + 2v\beta d(u, w) + 2(u, v)\beta w$

$- 2v\beta[u, w]_\alpha$

$= 2(f(u, v) + (u, v))\beta w + 2v\beta d(u, w) - 2v[u, w]_\alpha \in Z(M)$.

Hence we have $v\beta d(u, w) - \beta[u, w]_\alpha \in Z(M)$ for all u, v , w $\in$ U and $\alpha, \beta \in \Gamma$.

Thus we can write

$[v\beta d(u,w) - v\beta[u, w]_\alpha, m]_\gamma = 0$ ………..(19) for all u, v $\in$ U ; m $\in$ M ; $\alpha, \beta, \gamma \in \Gamma$.

This is the same relation as in Theorem 9.4, The required result follows from the similar manner as in the proof of the Theorem 9.4.

The other case $f(u, v) - (u, v)_\alpha \in Z(M)$ for all u, v $\in$ U and $\alpha \in \Gamma$can be proved similarly.

9.7 Theorem : Let M be a 2-torsion free semiprime Γ-ring and U be a Lie ideal of M such that $u\alpha u \in U$ for every $u \in U$. If M admits a symmetric generalized bi-derivation f with an associated bi-derivation d with the trace F of f . If F is centralizing on U, then F is commuting on U.

Proof : Suppose that $u \in U$. Let $D = [F(u), u]_\alpha$, where $F(u) = f(u, u)$. Then we have $D \in Z(M)$. By hypothesis we get

$[F(u), u]_\alpha \in Z(M)$ for all $u \in U$ and $\alpha \in \Gamma$ ....(20)

Putting u+v for u we get

$[F(u+v), u+v]_\alpha = [F(u), u]_\alpha + [F(u), v]_\alpha + [F(v), u]_\alpha + [F(v), v]_\alpha + 2[f(u, v), u]_\alpha + 2[f(u, v), v]_\alpha \in Z(M)$ ....(21)

That implies

$[F(u), v]_\alpha + [F(v), u]_\alpha + 2[f(u, v), u]_\alpha + 2[f(u, v), v]_\alpha \in Z(M)$ .....(22)

Substituting - u for u we get

$[F(u), v]_\alpha - [F(v), u]_\alpha + 2[f(u, v), u]_\alpha - 2[f(u, v), v]_\alpha \in Z(M)$ .....(23)

Combining (22) and (23) and using 2-torsion freeness of M , we get

$[F(u), v]_\alpha + 2[f(u, v), u]_\alpha \in Z(M)$, for all $u, v \in U; \alpha \in \Gamma$ .....(24)

Putting $2u\beta u$ for v in (24) we have

$[F(u), 2u\beta u]_\alpha + 2[f(u, 2u\beta u), u]_\alpha$

$= 2(u\beta [F(u), u]_\alpha + [F(u), u]_\alpha\beta u + 2[f(u, u)\beta u, u]_\alpha + 2[u\beta d(u, u), u]_\alpha)$

$= 2(u\beta D + D\beta u + 2[f(u, u), u]_\alpha\beta u + 2f(u, u)\beta[u, u]_\alpha + 2u\beta[d(u, u), u]_\alpha$

$+ 2[u, u]_\alpha\beta d(u, u))$

$= 4(u\beta D + [F(u), u]_\alpha\beta u + u\beta[d(u, u), u]_\alpha)$

$= 4(u\beta D + D\beta u + u\beta[d(u, u), u]_\alpha)$

$= 4(2u\beta D + u\beta[d(u, u), u]_\alpha) \in Z(M)$

And so $2u\beta D + u\beta[d(u, u), u]_\alpha \in Z(M)$ for every $u \in U$; $\alpha \in \Gamma$..........(25)

Let $m = 2u\beta D + u\beta[d(u, u), u]_\alpha$

Then $m - 2u\beta D = u\beta[d(u, u), u]_\alpha$

Now replacing $u\beta u$ in (xxii) we have,

$F(u\beta u, u\beta u) = [f(u\beta u, u\beta u), u\beta u]_\alpha$

$= [f(u\beta u, u)\beta u + u\beta d(u\beta u, u), u\beta u]_\alpha$

$= [f(u\beta u, u), u\beta u]_\alpha\beta u + f(u\beta u, u)\beta[u, u\beta u]_\alpha + u\beta[d(u\beta u, u), u\beta u]_\alpha$

$+ [u, u\beta u]_\alpha\beta d(u\beta u, u)$

$= u\beta[f(u, u)\beta u + u\beta d(u, u), u]_\alpha\beta u + [f(u, u)\beta u + u\beta d(u, u), u]_\alpha\beta u\beta u +$ $(f(u\beta u, u) \beta[u, u]_\alpha\beta u + f(u\beta u, u)\beta u\beta[u, u]_\alpha) + u\beta u\beta[d(u, u) \beta u + u\beta d(u, u),$ $u]_\alpha + u\beta[d(u, u) \beta u + u\beta d(u, u), u]_\alpha\beta u + ([u, u]_\alpha\beta u + u\beta[u, u]_\alpha)\beta d(u\beta u, u).$

$= u\beta[f(u, u), u]_\alpha\beta u\beta u + u\beta f(u, u)\beta[u, u]_\alpha \beta u + u\beta u\beta[d(u, u), u]_\alpha\beta u + u\beta[u,$ $u]_\alpha\beta d(u, u) \beta u) + ([f(u, u), u]_\alpha\beta u\beta u\beta u + u\beta[d(u, u), u]_\alpha\beta u\beta u) + u\beta u\beta [d(u,$ $u), u]_\alpha \beta u + u\beta u\beta u\beta[d(u, u),u]_\alpha + u\beta[d(u, u), u]_\alpha \beta u\beta u + u\beta u\beta [d(u, u), u]_\alpha$ $\beta u + u\beta d(u, u), u]_\alpha\beta u + ([u, u]_\alpha\beta u + u\beta[u, u]_\alpha)\beta d(u\beta u, u).$

$= u\beta[F(u), u]_\alpha \beta u\beta u + u\beta u\beta[d(u, u), u]_\alpha\beta u + [F(u), u]_\alpha\beta u\beta u\beta u + u\beta[d(u,$ $u), u]_\alpha \beta u\beta u + u\beta u\beta[d(u, u), u]_\alpha\beta u + u\beta u\beta u\beta[d(u, u), u]_\alpha + u\beta[d(u, u),$ $u]_\alpha\beta u\beta u + u\beta u\beta[d(u, u),u]_\alpha\beta u.$

$= u\beta D\beta u\beta u + u\beta(m - 2u\beta D)\beta u + D\beta u\beta u\beta u + (m - 2u\beta D)\beta u\beta u + u\beta(m - 2u\beta D)\beta u + u\beta u\beta(m - 2u\beta D) + (m - 2u\beta D)\beta u\beta u + u\beta(m - 2u\beta D)\beta u.$

$= u\beta D\beta u\beta u + 3u\beta m\beta u - 6u\beta u\beta D\beta u + D\beta u\beta u\beta u + 2m\beta u\beta u - 4u\beta D\beta u\beta u + u\beta u\beta m - 2u\beta u\beta u\beta D$

$= 2D\beta u\beta u\beta u + 3u\beta m\beta u - 12D\beta u\beta u\beta u + 2m\beta u\beta u + u\beta u\beta m$

$= -10D\beta u\beta u\beta u + 6m\beta u\beta u$

This shows that $-10D\beta u\beta u\beta u + 6m\beta u\beta u \in Z(M)$.

Commuting both sides with F(u), we have

$[F(u), -10D\beta u\beta u\beta u + 6m\beta u\beta u]_\alpha = 0$

That is $0 = -10D\beta[F(u), u\beta u\beta u]_\alpha + 6m\beta[F(u), u\beta u]_\alpha$

$= -10D\beta[F(u), u]_\alpha\beta u\beta u - 10D\beta u\beta[F(u), u\beta u]_\alpha + 6m\beta u\beta[F(u), u]_\alpha + 6m\beta[F(u), u]_\alpha \beta u$

$= -10D\beta D\beta u\beta u - 10D\beta u\beta[F(u), u]_\alpha\beta u - 10D\beta u\beta u\beta[F(u), u]_\alpha + 6m\beta u\beta D + 6m\beta D\beta u$

$= -10D\beta D\beta u\beta u - 10D\beta u\beta D\beta u - 10D\beta u\beta u\beta D + 6m\beta u\beta D + 6m\beta D\beta u$

$= -30D\beta D\beta u\beta u + 12m\beta D\beta u$

Again commuting with F(u) , we get

$0 = [F(u), -30D\beta D\beta u\beta u + 12m\beta D\beta u]_\alpha$

$= -30D\beta D\beta[F(u), u\beta u]_\alpha + [12m\beta D\beta[F(u), u]_\alpha$

$= -30D\beta D\beta u\beta[F(u), u] - 30D\beta D\beta[F(u), u]\beta u + 12m\beta D\beta D$

$= -60D\beta D\beta u\beta D + 12m\beta D\beta D$

Repeating the same argument we finally conclude that

$-60D\beta D\beta D\beta u = 0$

Since M is 2, 3 torsion free ,we have $D\beta D\beta D\beta u = 0$. But we know that the centre of a semiprime Γ–ring contains no non-zero nilpotent element ([21], Lemma 3.2) , so we come to the conclusion that D = 0 . Thus the proof of the theorem is completed.

9.8 Theorem : Let M be a 2-torsion free prime Γ-ring and U be a Lie ideal of M . If M admits symmetric bi-derivation F and G with traces f and g respectively such that $f(u)\alpha u + u\alpha g(u) = 0$ for all $u \in U$ and $\alpha \in \Gamma$, then one of the following holds:

(a) U is commutative

(b) G acts as a left bi-multiplier

(c) $U\Gamma[U, U]_{\Gamma} = 0$ unless $D \neq 0$.

Proof : We have $f(u)\alpha u + u\alpha g(u) = 0$, for all $u \in U$ and $\alpha \in \Gamma$ .....(26)

Putting u+v for u we have

$0 = f(u+v)\alpha(u+v) + (u+v)\alpha g(u+v)$

$= f(u)\alpha u + f(v)\alpha u +2F(u, v)\alpha u + f(u)\alpha v + f(v)\alpha v + 2F(u, v)\alpha v + u\alpha g(u) + u\alpha g(v) + 2u\alpha G(u, v) +v\alpha g(u) + v\alpha g(v) +2v\alpha G(u, v)$

$= f(v)\alpha u + f(u)\alpha v +2F(u, v)\alpha u +2F(u, v)\alpha v + u\alpha g(v) + v\alpha g(u) +2u\alpha G(u, v) +2v\alpha G(u, v)$ ....(27)

Replacing v by - v in (27), we obtain

$0 = f(-v)\alpha u + f(u)\alpha(-v) +2F(u, -v)\alpha u +2F(u, -v)\alpha(-v) + u\alpha g(-v) + (-v)\alpha g(u) +2u\alpha G(u, -v) +2(-v)\alpha G(u, -v)$

$= f(v)\alpha u - f(u)\alpha v - 2F(u, v)\alpha u + 2F(u, v)\alpha v + u\alpha g(v) - v\alpha g(u) -2u\alpha G(u, v) +2v\alpha G(u, v)$ ....(28)

Adding (27) and (28) and using 2-torsion freeness of M we get

$0 = f(v)\alpha u + 2F(u, v)\alpha v + u\alpha g(v) + 2v\alpha G(u, v)$ ....(29)

Substituting $2u\beta w$; $w \in U$, for u in (29) we get

$0 = f(v)\alpha\, 2u\beta w + 2F(2u\beta w, v)\alpha v + 2u\beta w\, \alpha g(v) + 2v\alpha G(2u\beta w, v)$

$= 2f(v)\alpha u\beta w + 4F(u, v)\beta w\alpha v +4u\beta F(w, v)\alpha v + 2u\beta w\alpha g(v) + 4v\alpha G(u, v)\beta w +4v\alpha u\beta G(w, v)$

Since M is 2-torsion free we get,

$0 = f(v)\alpha u\beta w + 2F(u, v)\beta w\alpha v +2u\beta F(w, v)\alpha v + u\beta w\alpha g(v) + 2v\alpha G(u, v)\beta w +2v\alpha u\beta G(w, v)$ ....(30)

Multiplying (29) by w on the right we have

$0 = f(v)\alpha u\beta w + 2F(u, v)\alpha v\beta w + u\alpha g(v)\beta w + 2v\alpha G(u, v)\beta w$ ....(31)

Subtrating (31) from (30) we get

$0 = 2F(u, v)\beta w\alpha v - 2F(u, v)\alpha v\beta w + 2u\beta F(w, v)\alpha v + u\beta w\alpha g(v) +2v\alpha u\beta G(w, v) - u\alpha g(v)\beta w$

Using (*) we have

$0 = 2F(u, v)\beta(w\alpha v - v\alpha w) + 2u\alpha F(w, v)\beta v + u\beta(w\alpha g(v) - g(v)\alpha w) + 2v\alpha u\beta G(w, v)$

$= 2F(u, v)\beta[w, v]_\alpha + 2u\alpha F(w, v)\beta v + 2v\alpha u\beta G(w, v) + u\beta[w, g(v)]_\alpha$ ....(32)

Replacing u by $2x\gamma u$ ($x\in U$) in (32) and using 2-torsion freeness of M,we get

$0 = F(2x\gamma u, v)\beta[w, v]_\alpha + 2x\gamma u\alpha F(w, v)\beta v + 2v\alpha x\gamma u\, \beta G(w, v) + x\gamma u\beta[w, g(v)]_\alpha$

$= 2x\gamma F(u, v)\beta[w, v]_\alpha + 2F(x, v)\gamma u\beta[w, v]_\alpha + 2x\gamma u\alpha F(w, v)\beta v + 2v\alpha x\gamma u\beta G(w, v) + x\gamma u\beta[w, g(v)]_\alpha$

$= 2F(x, v)\gamma u\beta[w, v]_\alpha + x\gamma(2F(u, v)\beta[w, v]_\alpha + u\beta[w, g(v)]_\alpha + 2u\alpha F(w, v)\beta v)+ 2v\alpha x\gamma u\beta G(w, v)$ ....(33)

$= 2F(x, v)\gamma u\beta[w, v]_\alpha + 2v\alpha x\gamma u\beta G(w, v) - 2x\gamma v\alpha u\beta G(w, v)$

Since M is 2-torsion free

$F(x, v)\gamma u\beta[w, v]_\alpha + v\alpha x\gamma u\beta G(w, v) - x\gamma v\alpha u\beta G(w, v) = 0$ ....(34)

Using (*) we obtain

$F(x, v)\gamma u\beta[w, v]_\alpha + (v\alpha x - x\alpha v)\gamma u\beta G(w, v) = 0$

That implies $F(x, v)\gamma u\beta[w, v]_\alpha + [v, x]_\alpha\gamma u\beta G(w, v) = 0$...........(35)

Substituting w for v we get $[w, x]_\alpha\gamma u\beta G(w, w) = 0$ ....(36), for all $u, w, x \in U$; $\alpha, \beta, \gamma \in \Gamma$.

The relation is same as (7) in Theorem 9.3.

Hence proceeding in the same way as in Theorem 9.3, we have the result (a) and (b)

Again if $u\beta G(w, w) = 0$, then from (35) ,

$0 = F(x, v)\gamma u\beta[w, v]_\alpha$

Putting $2x\delta s$ for x ,we have

$0 = F(2x\delta s, v)\gamma u\beta[w, v]_\alpha$

$= 2F(x, v)\delta s\gamma u\beta[w, v]_\alpha + 2\, x\delta F(s, v)\gamma u\beta[w, v]_\alpha$

$= 2F(x, v)\delta s\gamma u\beta[w, v]_\alpha$

This implies that

$F(x, v)\Gamma U\Gamma u\beta[w, v]_\alpha = 0$, for all $u, v, w, x \in U$; $\alpha, \beta \in \Gamma$.......(37)

From Lemma 1.29 we have either $F(x, v) = 0$ or $u\beta[w, v]_\alpha = 0$, for all $u, v, w, x \in U$; $\alpha, \beta \in \Gamma$.

Later yields that $U\Gamma[U, U]_\Gamma = 0$ as $F \neq 0$.

Using the same parallel arguments, we can prove the following :

9.9 Theorem : Let M be a 2-torsion free prime Γ–ring and $U \neq 0$, a right ideal of M . If M admits symmetric bi-derivations D and G with trace f and g respectively such that $f(u)\alpha u + u\alpha g(u) = 0$ for all $u \in U$; $\alpha \in \Gamma$, then one of the following holds :

(a) U is commutative

(b) D acts as a right bi-multiplier

(c) $[U, U]_\Gamma \Gamma U = 0$ unless $G \neq 0$.

k- Derivations on Lie ideals of Nobusawa Gamma Rings

We know that every ring M is a Γ-ring if we put Γ = M. Also M is a Γ-ring in the sense of Nobusawa implies that Γ is an M-ring . Let M be a Γ_N-ring and U be a Lie ideal of M. Then Γ be an M-ring. Let Ω be a Lie ideal of Γ. In that case for some results on a lie ideal U of M we have the same type of result on Ω of Γ.We define a k-derivation d on U . At the same time we have a d-derivation k on Ω. In this chapter we extend some results on d^2 and d^3 also.

10. Introduction : In the first chapter we have mentioned Γ-ring as follows:

Let M and Γ be two additive abelian groups. Suppose that there is a mapping (composition) from $M \times \Gamma \times M \rightarrow M$(sending (x, α, y) into xαy) such that

(i) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)y = x\alpha y + x\beta y$, $x\alpha(y+z) = x\alpha y + x\alpha z$;

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$, where x, y, z ∈ M and α, β ∈ Γ. Then M is called a gamma ring in the sense of Bernes . We simply say gamma ring and denote it by Γ.

In addition, if there is another mapping $\Gamma \times M \times \Gamma \rightarrow \Gamma$ (sending (α,x,β) → αxβ such that the properties

(i*) $(\alpha+\beta)x\gamma = \alpha x\gamma + \beta x\gamma$; $\alpha(x+y)\beta = \alpha x\beta + \alpha y\beta$; $\alpha x(\beta+\gamma) = \alpha x\beta + \alpha x\gamma$,

(ii*) $(x\alpha y)\beta z = x(\alpha y\beta)z = x\alpha(y\beta z)$ and

(iii*) $x\alpha y = 0$ implies $\alpha = 0$ hold for all x, y, z ∈ M and α, β, γ ∈ Γ, then M is called a gamma ring in the sense of Nobusawa or simply a Nobusawa gamma ring and we denote it by Γ_N- ring.

In this chapter we mean M as a Γ_N- ring. So that we can define a Lie ideal Ω of Γ along with the Lie ideal U of M. And also we can define the d-derivation k on Ω of Γ.

Here we also use the notation $[\alpha, \beta]_x$ defined by $[\alpha, \beta]_x = \alpha x\beta - \beta x\alpha$.We define the centre of Γ by $Z(\Gamma) = \{\alpha \in \Gamma , [\alpha, \beta]_x = 0$ for all $\beta \in \Gamma\}$. Here we use the condition $\alpha x\beta y\gamma = \alpha y\beta x\gamma$ …………(**)for all $\alpha, \beta, \gamma \in \Gamma$ and $x, y \in M$ along with the condition (*).By using (**) the commutator identities

$[\alpha x\beta, \gamma]_y = \alpha x[\beta, \gamma]_y + \alpha[x, y]_\gamma\beta + [\alpha, \gamma]_y x\beta$ and

$[\alpha, \beta x\gamma]_y = \beta x[\alpha, \gamma]_y + \beta[y, x]_\alpha\gamma + [\alpha, \beta]_y x\gamma$ reduces to

$[\alpha x\beta, \gamma]_y = \alpha x[\beta, \gamma]_y + [\alpha, \gamma]_y x\beta$ and

$[\alpha, \beta x\gamma]_y = \beta x[\alpha, \gamma]_y + [\alpha, \beta]_y x\gamma$.

10.1 Definition : Let M be a Γ-ring . An additive subgroup Ω of Γ is called a Lie ideal of Γ if $[\alpha, \gamma]_x \in \Omega$ for all $\alpha \in \Omega$, $\gamma \in \Gamma$and $x \in M$.

10.2 Definition : Let Ω be a Lie ideal of Γ. Suppose that $d: M \rightarrow M$ and $k: \Gamma \rightarrow \Gamma$ are additive mappings. If $k(\alpha x\beta) = k(\alpha)x\beta + \alpha d(x)\beta + \alpha xk(\beta)$ holds for all $\alpha, \beta \in \Omega$ and $x \in M$, then k is called a d-derivation on Ω of Γ.

10.3 Lemma : Let M be a 2-torsion free Γ_N–ring , then Γ is a 2-torsion free M-ring.

Proof: Let $2\alpha = 0$ for all $\alpha \in \Gamma$.

Now, $a2\alpha b = 0$. That is $2a\alpha b = 0$

Since $a\alpha b \in M$ and M is 2-torsion free , we have $a\alpha b = 0$.

Therefore, $\alpha = 0$.

10.4 Example : Let M be a Γ_N - ring , $a \in M$ be fixed . Let $d : M \to M$ and $k : \Gamma \to \Gamma$ be two additive mappings defined by $d(u) = -(a\alpha u + u\alpha a)$ and $k(\beta) = (\alpha a\beta + \beta a\alpha)$; $u \in M$, $\beta \in \Gamma$. Then for $v \in M$, we have

$d(u\beta v) = - a\alpha u\beta v - u\beta v\alpha a$

$= - a\alpha u\beta v - u\alpha a\beta v + u\alpha a\beta v + u\beta a\alpha v - u\beta a\alpha v - u\beta v\alpha a$

$= - (a\alpha u - u\alpha a)\beta v + u(\alpha a\beta + \beta a\alpha)v - u\beta(a\alpha v - v\alpha a)$

$= d(u)\beta v + uk(\beta)v + u\beta d(v)$

Therefore, d is a k- derivation of M.

10.5 Example : Let M be a Γ_N – ring and let $a \in M$, $\alpha \in \Gamma$ be fixed . Define two additive mappings $d : M \to M$ and $k : \Gamma \to \Gamma$ by $d(u) = - u\alpha a$, $k(\beta) = \alpha a\beta$ respectively, then

$d(u\beta v) = - u\beta v\alpha a$

$= - u\beta v\alpha a + u\alpha a\beta v - u\alpha a\beta v$

$= - (u\alpha a)\beta v + u(\alpha a\beta)v + u\beta(- v\alpha a)$

$= d(u)\beta v + uk(\beta)v + u\beta d(v).$

Hence d is a k- derivation.

10.6 ([22], Lemma 3.2.1): If d is a k-derivation of a Γ_N –ring M, then k is a d-derivation of the associated M- ring Γ.

10.7 Lemma : If k is a d-derivation of a Γ_N –ring M, then $k(Z) \subseteq Z(\Gamma)$

Proof : Let $a \in M$, $\alpha \in \Gamma$, $\gamma \in Z(\Gamma)$. Then

$k(\alpha a\gamma) = k(\alpha)a\gamma + \alpha d(a)\gamma + \alpha ak(\gamma)$ and

$k(\gamma a\alpha) = k(\gamma)a\alpha + \gamma d(a)\alpha + \gamma ak(\alpha)$

Now since $\alpha a\gamma = \gamma a\alpha$, hence $k(\alpha a\gamma) = k(\gamma a\alpha)$

Then $k(\alpha)a\gamma + \alpha d(a)\gamma + \alpha ak(\gamma) = k(\gamma)a\alpha + \gamma d(a)\alpha + \gamma ak(\alpha)$

That implies $k(\alpha)a\gamma + \alpha d(a)\gamma + \alpha ak(\gamma) = k(\gamma)a\alpha + \alpha d(a)\gamma + k(\alpha)a\gamma$

That means $\alpha ak(\gamma) = k(\gamma)a\alpha$. i.e, $k(\gamma) \in Z(\Gamma)$.

10.8 Lemma : Let M be a 2-torsion free prime Γ_N –ring satisfying the condition (*) . Suppose that U is a Lie ideal of M . If $[U, U]_\Gamma = 0$, then U $\subseteq Z(M)$.

Proof : Here $[u, [u, x]_\alpha]_\beta = 0$, for every $u \in U$; $x \in M$; $\alpha, \beta \in \Gamma$. Let $y \in M$, then $x\gamma y \in M$ for all $\gamma \in \Gamma$. Replacing x by $x\gamma y$ we have,

$0 = [u, [u, x\gamma y]_\alpha]_\beta$

$= [u, x\gamma[u, y]_\alpha + [u, x]_\alpha\gamma y]_\beta$

$= [u, x\gamma[u, y]_\alpha]_\beta + [u, [u, x]_\alpha\gamma y]_\beta$

$= x\gamma [u, [u, y]_\alpha]_\beta + [u, x]_\beta\gamma[u, y]_\alpha + [u, [u, x]_\alpha]_\beta\gamma y + [u, x]_\alpha\gamma[u, y]_\beta$

$= [u, x]_\beta\gamma[u, y]_\alpha + [u, x]_\alpha\gamma[u, y]_\beta$

$= (u\beta x - x\beta u)\gamma(u\alpha y - y\alpha u) + (u\alpha x - x\alpha u)\gamma(u\beta y - y\beta u)$

$= u\beta x\gamma u\alpha y - x\beta u\gamma u\alpha y - u\beta x\gamma y\alpha u + x\beta u\gamma y\alpha u + u\alpha x\gamma u\beta y - x\alpha u\gamma u\beta y - u\alpha x\gamma y\beta u + x\alpha u\gamma y\beta u$

$= 2(u\alpha x\gamma u\beta y - u\alpha x\gamma y\beta u - x\alpha u\gamma u\beta y + x\alpha u\gamma y\beta u)$

[since $u\beta x\gamma u\alpha y = u\gamma x\beta u\alpha y = u\gamma x\alpha u\beta y = u\alpha x\gamma u\beta y$]

$= 2((u\alpha x - x\alpha u)\gamma u\beta y - (u\alpha x - x\alpha u)\gamma y\beta u)$

$= 2(u\alpha x - x\alpha u)\gamma(u\beta y - y\beta u)$

$= 2[u, x]_\alpha\gamma[u, y]_\beta$

Since M is 2-torsion free , we have $[u, x]_\alpha\gamma[u, y]_\beta = 0$

Replacing y by $m\delta y$, $m \in M$, we have

$0 = [u, x]_\alpha\gamma[u, m\delta y]_\beta$

$= [u, x]_\alpha\gamma(m\delta[u, y]_\beta + [u, m]_\beta\delta y)$

$= [u, x]_\alpha\gamma m\delta[u, y]_\beta + [u, x]_\alpha\gamma [u, m]_\beta\delta y$

$= [u, x]_\alpha\gamma m\delta[u, y]_\beta$

Since M is prime ,we have either $[u, x]_\alpha = 0$ or $[u, y]_\beta = 0$.

10.9 Lemma : Let Γ be a prime M- ring satisfying the condition (**). Suppose that Ω is a Lie ideal of Γ . If $[\Omega, \Omega]_M = 0$, then $\Omega \subseteq Z(\Gamma)$.

Proof : Here $[\alpha, [\alpha, \beta]_m]_n = 0$; m , n ∈ M ; α ∈ Ω ; β ∈ Γ .

Let γ ∈ Γ , then βxγ ∈ Γ for all x ∈ M.

Replacing β by βxγ we have,

$0 = [\alpha, [\alpha, \beta x\gamma]_m]_n = [\alpha, \beta x[\alpha, \gamma]_m + [\alpha, \beta]_m x\gamma]_n$

$= [\alpha, \beta x[\alpha, \gamma]_m]_n + [\alpha, [\alpha, \beta]_m x\gamma]_n$

$= \beta x[\alpha, [\alpha, \gamma]_m]_n + [\alpha, \beta]_n x[\alpha, \gamma]_m + [\alpha, [\alpha, \beta]_m]_n x\gamma + [\alpha, \gamma]_n x[\alpha, \beta]_m$

$= [\alpha, \beta]_n x[\alpha, \gamma]_m + [\alpha, \gamma]_n x[\alpha, \beta]_m$

$= 2[\alpha, \beta]_m x[\alpha, \gamma]_n$ using (**)

Since Γ is 2-torsion free, we obtain $[\alpha, \beta]_m x[\alpha, \gamma]_n = 0$.

Now replacing γ by δyγ we get

$0 = [\alpha, \beta]_m x[\alpha, \delta y\gamma]_n$

$= [\alpha, \beta]_m x (\delta y[\alpha, \gamma]_n + [\alpha, \delta]_n y\gamma)$

$= [\alpha, \beta]_m x\delta y[\alpha, \gamma]_n + [\alpha, \beta]_m x [\alpha, \delta]_n y\gamma)$

$= [\alpha, \beta]_m x\delta y[\alpha, \gamma]_n$

Since Γ is prime , either $[\alpha, \beta]_m = 0$ or $[\alpha, \gamma]_n = 0$.

10.10 Lemma : If U ⊄ Z(M) is a Lie ideal of a 2-torsion free prime Γ_N–ring M, then there exists an ideal N of M such that $[N, M]_\Gamma \subseteq U$ but $[N, M]_\Gamma \not\subset Z(M)$.

Proof : Here we have , U ⊄ Z(M). Hence $[U, U]_\Gamma \neq 0$. Let $N = M\Gamma[U, U]_\Gamma\Gamma M \neq 0$ is an ideal of M generated by $[U, U]_\Gamma$. Clearly $[N, M]_\Gamma \not\subset Z(M)$. For, if [N, M] ⊆ Z(M), then $[N, [N, M]_\Gamma]_\Gamma = 0$.

That implies $[N, M]_\Gamma = 0$, a contradiction.

10.11 Lemma : Let U ⊄ Z(M) be a Lie ideal of a 2-torsion free prime Γ_N –ring M Satisfying the condition (*) and let a, b ∈ M. If aΓUΓb = 0 , then a = 0 or b = 0.

Proof : From Lemma 10.10, there exist an ideal N of M such that $[N, M]_\Gamma \not\subset Z(M)$ but $[N, M]_\Gamma \subseteq U$.

Let $u \in U$; $x \in N$; $y \in M$; $\beta, \delta, \gamma \in \Gamma$. Then $[x\beta a\gamma u, y]_\delta \in [N, M]_\Gamma \subseteq U$.

Thus $a\alpha[x\beta a\gamma u, y]_\delta\mu b = 0$; $\alpha, \beta, \gamma, \delta, \mu \in \Gamma$.

i.e., $0 = a\alpha[x\beta a, y]_\delta\gamma u\mu b + a\alpha x\beta a\gamma[u, y]_\delta\mu b$

$= a\alpha[x\beta a, y]_\delta\gamma u\mu b$ [Since $[u, y]_\delta \in U$, $a\gamma[u, y]_\delta\mu b = 0$]

$= a\alpha x\beta a\delta y\gamma u\mu b - a\alpha y\delta x\beta a\gamma u\mu b$

$= (a\alpha x\beta a)\, \delta y\gamma\, (u\mu b)$.

Now if $a \neq 0$, then $a\alpha x\beta a \neq 0$.

Since M is prime , we have $u\mu b = 0$.

So, $[u, y]_\alpha\mu b = 0$

i.e, $0 = u\alpha y\mu b - y\alpha u\mu b = u\alpha y\mu b$.

Since $u \neq 0$, we have $b = 0$.

10.12 Lemma : If $\Omega \not\subset Z(\Gamma)$ is a Lie ideal of a 2-torsion free prime M-ring Γ satisfying the condition (**) , then there exists an ideal φ of Γ such that $[\Phi, \Gamma]_M \subseteq \Omega$ but $[\Phi, \Gamma]_M \not\subset Z(\Gamma)$.

Proof : Since $\Omega \not\subset Z(\Gamma)$, $[\Omega, \Omega]_M \neq 0$.

Let $\Delta = \Gamma M [\Omega, \Omega]_M M\Gamma \neq 0$ is an ideal of Γ generated by $[\Omega, \Omega]_M$. Clearly $[\Omega, \Gamma]_M \not\subset Z(\Gamma)$. For, if $[\Omega, \Gamma]_M \subseteq Z(\Gamma)$, then $[\Omega, [\Omega, \Gamma]_M]_M = 0$, which implies $[\Omega, \Gamma]_M = 0$, a contradiction.

10.13 Lemma : Let Γ be a 2-torsion free prime M-ring satisfying the condition (**). Suppose that $\Omega \not\subset Z(\Gamma)$ be a Lie ideal of Γ and let $\alpha, \beta \in \Gamma$. If $\alpha M\Omega M\beta = 0$,then $\alpha = 0$ or $\beta = 0$.

Proof : From Lemma 10.12, there exists an ideal Φ of Γ such that $[\Phi, \Gamma]_M \subseteq \Omega$ but $[\varphi, \Gamma]_M \not\subset Z(\Gamma)$.

Let $\delta \in \Omega, \varphi \in \Phi, \gamma \in \Gamma$; x, y, p $\in$ M.

Then $[\varphi x\alpha y\delta, \gamma]_p \in [\Phi, \Gamma]_M \subseteq \Omega$.

Thus $0 = \alpha a [\varphi x\alpha y\delta, \gamma]_p b\beta$; x, y, p, a, b $\in$ M

$= \alpha a [\varphi x\alpha, \gamma]_p y\delta b\beta + \alpha a\varphi x\alpha y [\delta, \gamma]_p b\beta$

$= \alpha a [\varphi x\alpha, \gamma]_p y\delta b\beta$ [$\alpha y [\delta, \gamma]_p b\beta = 0$, since $[\delta, \gamma]_p \in \Omega$]

$= \alpha a\varphi x\alpha p\gamma y\delta b\beta - \alpha a\gamma p\varphi x\alpha y\delta b\beta$

$= (\alpha a\varphi x\alpha)p\gamma y(\delta b\beta)$

Now if $\alpha \neq 0$, then $\alpha a\varphi x\alpha \neq 0$

Since Γ is prime , we have $\delta b\beta = 0$.

Hence $[\delta, \varphi]_a b\beta = 0$, a $\in$ M

That is $0 = \delta a\varphi b\beta - \varphi a\delta b\beta$

$= \delta a\varphi b\beta$

Since $\delta \neq 0$, we have $\beta = 0$.

10.14 Lemma : Let U and Ω be Lie ideals of M and Γ respectively where M is a 2-torsion free prime Γ_N-ring satisfying the condition (*) and let d $\neq$ 0 be a k-derivation on U . If $d(U) = 0 = k(\Omega)$, then $U \subseteq Z(M)$ and $\Omega \subseteq Z(\Gamma)$.

Proof : Let u $\in$ U , x $\in$ M and $\alpha \in \Omega$.

Now $d(U) = 0$ implies $d([u, x]_\alpha) = 0$.

Define $d(x) = [u, x]_\alpha$ and $k(\beta) = [\alpha, \beta]_u$ for all u $\in$ U , x $\in$ M, $\beta \in \Omega$, $\alpha \in \Gamma$.

Then $d(d(x)) = [u, d(x)]_\alpha = 0$

That is $d^2(x) = 0$

Now replacing x by $x\gamma y$ we have

$0 = d^2(x\gamma y) = d(d(x\gamma y))$

$= d(d(x)\gamma y + xk(\gamma)y + x\gamma d(y))$

$= d(d(x)\gamma y + x\gamma d(y))$

$= d^2(x)\gamma y + d(x)k(\gamma)y + d(x)\gamma d(y) + d(x)\gamma d(y) + xk(\gamma)d(y) + x\gamma d^2(y)$

$= 2d(x)\gamma d(y)$, for every $x, y \in M$; $\gamma \in \Omega$.

Since M is 2-torsion free , $d(x)\gamma d(y) = 0$.

Now replacing y by $y\delta x$, we have

$0 = d(x)\gamma d(y\delta x)$

$=d(x)\gamma d(y)\delta x + d(x)\gamma y k(\delta)x + d(x)\gamma y\delta d(x)$

$= d(x)\gamma y\delta d(x)$.

Since M is prime , we have $d(x) = 0$.

Therefore , $[u, x]_\alpha = 0$ for every $u \in U, x \in M, \alpha \in \Omega$.

Hence $u \in Z(M)$.

Thus $U \subseteq Z(M)$.

10.15 Lemma : Let $a \in M$ and $\alpha \in \Gamma$. Define $d(x) = [a, x]_\alpha$ and $k(\beta) = [\alpha, \beta]_a$ for all $x \in U$ and $\beta \in \Gamma$. Then d is a k-derivation on U of M and k is a d- derivation Ω of Γ.

Proof : Given $d(x) = [a, x]_\alpha$

Replacing x by $2x\gamma y$ we have

$d(2x\gamma y) = [a, 2x\gamma y]_\alpha = 2x\gamma[a, y]_\alpha + 2x[\alpha, \gamma]_a y +2 [a, x]_\alpha\gamma y$

$= 2(x\gamma d(y) + xk(\gamma)y + d(x)\gamma y)$

Then $d(x\gamma y) = x\gamma d(y) + xk(\gamma)y + d(x)\gamma y$.

and also $k(\beta b\delta) = [\alpha, \beta b\delta]_a$

$= \beta b[\alpha, \delta]_a + \beta[a, b]_\alpha\delta + [\alpha, \beta]_a b\delta = \beta bk(\delta) + \beta d(b)\delta + k(\beta)b\delta$.

10.16 Lemma : Let M, Γ, U, Ω , $k \neq 0$, $d \neq 0$ be defined as in above . Suppose that $U \not\subset Z(M)$, $\Omega \not\subset Z(\Gamma)$. Then $p\alpha d(U) = 0$ implies $p = 0$ and also $\beta a k(\Omega) = 0$ implies $\beta = 0$; $\alpha \in \Gamma$, $p \in M$, $\beta \in \Gamma$.

Proof : Let $u \in U, x \in M$. then $2[u, x]_\beta\gamma u \in U$ and $2u\gamma[u, x]_\beta \in U$, $\gamma \in \Omega$.

Thus $0 = p\alpha d(2[u, x]_\beta\gamma u)$

Then $0 = p\alpha d([u, x]_\beta\gamma u)$; since M is 2- torsion free.

$= p\alpha d([u, x]_\beta)\gamma u + p(\alpha [u, x]_\beta k(\gamma))u + p\alpha[u, x]_\beta\gamma d(u)$

$= p\alpha[u, x]_\beta\gamma d(u)$; since $[u, x]_\beta \in M$ implies $p\alpha [u, x]_\beta k(\gamma) = 0$

Replacing x by $d(v)\delta y$, $v \in U$; $y \in M$ $\delta \in \Omega$, we get

$0 = p\alpha[u, d(v)\delta y]_\beta\gamma d(u)$

$= p\alpha d(v)\delta[u, y]_\beta\gamma d(u) + p\alpha[u, d(v)]_\beta\delta y\gamma d(u) + p\alpha d(v)[\beta, \delta]_u y\gamma d(u)$

$= p\alpha[u, d(v)]_\beta\delta y\gamma d(u)$

$= p\alpha u\beta d(v)\delta y\gamma d(u) - p\alpha d(v)\beta u\delta y\gamma d(u)$

$= p\alpha u\beta d(v)\delta y\gamma d(u)$

That implies $p\alpha u\beta d(U)\delta M\gamma d(u) = 0$

Since M is prime and $d(U) \neq 0$, we get $p\alpha u\beta d(U) = 0$.

Again $d(U) \neq 0$ and $p\alpha u\beta d(U) = 0$ implies $p = 0$.

Now let $\alpha \in \Omega$; $\gamma \in \Gamma$, $m \in M$, Then $[\alpha, \gamma]_m \in \Omega$ implies $2[\alpha, \gamma]_m u\alpha \in \Omega$, $u \in U$.

That implies $0 = \beta ak(2[\alpha, \gamma]_m u\alpha)$

$= \beta ak([\alpha, \gamma]_m u\alpha)$; since M is 2- torsion free.

$= \beta a(k([\alpha, \gamma]_m)u\alpha + [\alpha, \gamma]_m d(u)\alpha + [\alpha, \gamma]_m uk(\alpha))$

$= \beta ak([\alpha, \gamma]_m)u\alpha + \beta a[\alpha, \gamma]_m d(u)\alpha + \beta a[\alpha, \gamma]_m uk(\alpha)$

$= \beta ak([\alpha, \gamma]_m)u\alpha + \beta(a[\alpha, \gamma]_m d(u))\alpha + \beta a[\alpha, \gamma]_m uk(\alpha)$

$= \beta a[\alpha, \gamma]_m uk(\alpha)$,

since $\beta ak([\alpha, \gamma]_m) = 0$; $[\alpha, \gamma]_m \in \Omega \subseteq \Gamma$, $a \in M$, $a[\alpha, \gamma]_m d(u) = 0$

Replacing γ by $k(\delta)v\mu$, $\delta \in \Omega$; $\mu \in \Gamma$; $u \in U$, we get

$0 = \beta a[\alpha, k(\delta)v\mu]_m uk(\alpha)$

$= \beta a\alpha mk(\delta)v\mu uk(\alpha) - \beta ak(\delta)v\mu m\alpha uk(\alpha)$

$= \beta a\alpha mk(\delta)v\mu uk(\alpha)$

That implies $(\beta a\alpha mk(\Omega))v\Gamma uk(\Omega) = 0$

Since $k(\alpha) \neq 0$, Γ is prime , $\beta a\alpha mk(\Omega) = 0$.

i.e. $\beta a\Omega mk(\Omega) = 0$.

Again $k(\Omega) \neq 0$ implies $\beta = 0$.

10.17 Lemma : Let M be a 2- torsion free prime Γ-ring and U be a Lie ideal of M. If $d \neq 0$ is a k- derivation on U of M such that $d(U) \subseteq Z(M)$, $k(\Omega) = 0$, Ω is a Lie ideal of Γ . Then $U \subseteq Z(M)$.

Proof : If possible , let $U \not\subset Z(M)$. Then $V = [U, U]_\Gamma \not\subset Z(M)$.

Also if $u, w \in U$; $\alpha \in \Omega$, then

$d([u, w]_\alpha) = [d(u), w]_\alpha + [u, w]_{k(\alpha)} + [u, d(w)]_\alpha$

$= [u, w]_{k(\alpha)} = 0$.

Thus $d(V) = 0$, a contradiction that $V \subseteq Z(M)$ from Lemma 10.14.

10.18 Lemma: Let M, Γ, U, Ω be defined as in above . If $k \neq 0$ is a d-derivation on Ω of Γ such that $k(\Omega) \subseteq Z(\Gamma)$, $d(U) = 0$, then $\Omega \subseteq Z(\Gamma)$.

Proof : If possible let $\Omega \not\subset Z(\Gamma)$. Then $\Phi = [\Omega, \Omega]_M \not\subset Z(\Gamma)$.

Let $\alpha , \beta \in \Omega$, $m \in M$; then

$k([\alpha, \beta]_m) = [k(\alpha), \beta]_m + [\alpha, \beta]_{d(m)} + [\alpha, k(\beta)]_m = 0$

Thus $k(\Phi) = 0$. That implies $\Phi \subseteq Z(\Gamma)$ from lemma 10.14, a contradiction.

10.19 Theorem : Let M be a 2- torsion free prime Γ_N –ring and U be a Lie ideal of M . If $d \neq 0$ is a k- derivation on U of M such that $d^2(U) = 0$, $k(\Omega) = 0$. Then $U \subseteq Z(M)$.

Proof : Suppose that $U \not\subset Z(M)$. Then $V = [U, U]_\Gamma \not\subset Z(M)$.

Let X be an ideal of M such that $[X, M]_\Gamma \not\subset Z(M)$, but $[X, M]_\Gamma \subseteq U$.

Also $[X, M]_\Gamma \subseteq X$.

Now let $x \in [X, M]_\Gamma$ and $u \in [U, U]_\Gamma = V$. Then $d(u)\alpha d[U, U]_\Gamma \subseteq U$.

Hence $x\alpha d(u) \in X$ and so $[x\alpha d(u), y]_\gamma \in [X, M]_\Gamma \subseteq U$.

Then, $0 = d^2([x\alpha d(u), y]_\gamma)$

$= d^2([x, y]_\gamma \alpha d(u) + x[\alpha, \gamma]_y d(u) + x\alpha[d(u), y]_\gamma)$

$= d(d[x, y]_\gamma \alpha d(u) + [x, y]_\gamma k(\alpha)d(u) + [x, y]_\gamma \alpha d2(u) + d(x)[\alpha, \gamma]_y d(u) + xk[\alpha, \gamma]_y d(u) + x[\alpha, \gamma]_y d^2(u) + d(x)\alpha[d(u), y]_\gamma + xk(\alpha)[d(u), y]_\gamma + x\alpha d[d(u), y]_\gamma$.

$= d(d[x, y]_\gamma \alpha d(u) + d(x)[\alpha, \gamma]_y d(u) + d(x)\alpha[d(u), y]_\gamma + x\alpha d[d(u), y]_\gamma)$.

Since $[\alpha, \gamma]_y \in \Omega$ implies $k[\alpha, \gamma]_y = 0$.

$= d^{2(}[x, y]_\gamma)\alpha d(u) + d[x, y]_\gamma k(\alpha)d(u) + d[x, y]_\gamma \alpha d^2(u) + d^2(x)[\alpha,\gamma]_y d(u) + d(x)k[\alpha, \gamma]_y d(u) + d(x)[\alpha, \gamma]_y d^2(u) + d^2(x)\alpha[d(u), y]_\gamma + d(x)k(\alpha)[d(u), y]_\gamma + d(x)\alpha d[d(u), y]_\gamma + d(x)\alpha d[d(u), y]_\gamma + xk(\alpha)[d(u), y]_\gamma + x\alpha d^2[d(u), y]_\gamma$

$=2d(x)\alpha d[d(u), y]_\gamma$

Since $x \in U, y \in M$ implies $[x, y]_\gamma \in U$, $d^2[x, y]_\gamma = 0, d(u) \in X$; $y \in M$ implies $[d(u), y]_\gamma \in [X, M]_\Gamma \subseteq U$; $d^2[d(u), y]_\gamma = 0]$.

Since M is 2- torsion free, we have $d(x)\alpha d[d(u), y]_\gamma = 0$.

That is $d ([X, M])_\Gamma \alpha d([d(V), M]_\Omega) = 0$.

But $[X, M]_\Gamma \not\subset Z(M)$. Hence from Lemma 10.16, if $u \in V$; $m \in M$, $\beta \in \Omega$; then

$0 = d(d(u)\beta m – m\beta d(u))$

$= d^2(u)\beta m + d(u)k(\beta)m + d(u)\beta d(m) – d(m)\beta d(u) – mk(\beta)d(u) – m\beta d^2(u)$

$= d(u)\beta d(m) – d(m)\beta d(u)$.

Therefore, d(V) centralizes d(M). From Lemma 10.14, d(V) $\subseteq$ Z(M) and hence from Lemma 10.17, V $\subseteq$ Z(M). Then the theorem is proved.

10.20 Theorem : Let M be a 2- torsion free prime Γ_N - ring and U be a Lie ideal of M . If $k \neq 0$ is a d- derivation on Ω of Γ such that $k^2(\Omega) = 0 = d(u)$; $u \in U$ .Then $\Omega \subseteq Z(\Gamma)$.

Proof : Suppose that $\Omega \not\subset Z(\Gamma)$. Then $\Phi = [\Omega, \Omega]_M \not\subset Z(\Gamma)$.

Let Ψ be an ideal of Γ such that $[\Psi, \Gamma]_M \not\subset Z(\Gamma)$, but $[\Psi, \Gamma]_M \subseteq \Omega$, from Lemma 6.10.

Also $[\Psi, \Gamma]_M \subseteq \Psi$.

Now let $\alpha \in [\Psi, \Gamma]_M$ and $\beta \in [\Omega, \Omega]_M = \Phi$.

Then $k(\beta) \in k[\Omega, \Omega]_M \in \Omega$.

Hence $\alpha uk(\beta) \in \Psi$ and so

$[\alpha uk(\beta), \gamma]_m \in [\Psi, \Gamma]_M \subseteq \Omega$; $\gamma \in \Gamma$; $m \in M$; $u \in U$.

Therefore , $[\alpha uk(\beta), \gamma]_m \in \Omega$.

Thus we have $0 = k^2[\alpha uk(\beta), \gamma]_m$.

Then $0 = k^2([\alpha, \gamma]_m uk(\beta) + \alpha[u, m]_\gamma k(\beta) + \alpha u[k(\beta), \gamma]_m)$

$= k(k[\alpha, \gamma]_m uk(\beta) + [\alpha, \gamma]_m d(u)k(\beta) + [\alpha, \gamma]_m uk^2(\beta) + k(\alpha)[u, m]_\gamma k(\beta) + \alpha d([u, m]_\gamma)k(\beta) + \alpha [u, m]_\gamma k^2(\beta) + k(\alpha)u[k(\beta), \gamma]_m + \alpha d(u)[k(\beta), \gamma]_m + \alpha uk[k(\beta), \gamma]_m)$

$= k(k[\alpha, \gamma]_m uk(\beta) + k(\alpha)[u, m]_\gamma k(\beta) + k(\alpha)u[k(\beta), \gamma]_m + \alpha uk[k(\beta), \gamma]_m)$

[Since $d(u) = 0$; $k^2(\beta) = 0$; $[u, m]_\gamma \in U$ implies $d([u, m]_\gamma) = 0$]

$= k^2[\alpha, \gamma]_m uk(\beta) + k[\alpha, \gamma]_m d(u)k(\beta) + k[\alpha, \gamma]_m uk^2(\beta) + k^2(\alpha)[u, m]_\gamma k(\beta) + k(\alpha)d[u, m]_\gamma k(\beta) + k(\alpha)[u, m]_\gamma k^2(\beta) + k^2(\alpha)u[k(\beta), \gamma]_m + k(\alpha)d(u)[k(\beta), \gamma]_m + k(\alpha)uk[k(\beta), \gamma]_m + k(\alpha)uk[k(\beta), \gamma]_m + \alpha d(u)k[k(\beta), \gamma]_m + \alpha uk^2[k(\beta), \gamma]_m$.

$= 2k(\alpha)uk([k(\beta), \gamma]_m)$.

Since Γ is 2- torsion free , $k(\alpha)uk([k(\beta), \gamma]_m) = 0$.

That is $k ([\Psi, \Gamma]_M)uk ([k(\Phi), \Gamma]_M) = 0$.

But $[\Psi, \Gamma]_M \not\subset Z(\Gamma)$. Hence from Lemma 10.16, $k([k(\Phi), \Gamma]_M = 0$.

Hence if $\alpha \in \Phi$; $\gamma \in \Gamma$; $u \in U$; then

$0 = k(k(\alpha)u\gamma - \gamma uk(\alpha))$

$= k^2(\alpha)u\gamma + k(\alpha)d(u)\gamma + k(\alpha)uk(\gamma) - k(\gamma)uk(\alpha) - \gamma d(u)k(\alpha) - \gamma uk^2(\alpha)$.

$= k(\alpha)uk(\gamma) - k(\gamma)uk(\alpha)$.

That implies $k(\Phi)$ centralizes $k(\Gamma)$. From Lemma 10.14, $k(\Phi) \subseteq Z(\Gamma)$ and from Lemma 10.15, $\Phi \subseteq Z(\Gamma)$.

10.21 Theorem : If $U \not\subset Z(M)$ is a Lie ideal of a 2- torsion free prime Γ_N-ring M and $k(\Omega) = 0$, where Ω is a Lie ideal of Γ and $d \neq 0$ is a k-derivation of M, then $C_M(d(U) = Z(M)$.

Proof : Let $a \in C_M(d(U))$ and $a \notin Z(M)$.

Since $U \not\subset Z(M)$, $V = [U, U]_\Gamma \not\subset Z(M)$. Also $d(V) \subseteq U$.

Then for all $u \in V$; $d(u) \in U$ implies $d^2(u) \in d(U)$.

Hence $a\alpha d^2(u) = d^2(u)\alpha a$ for some $\alpha \in \Omega$.

Again $u \in [U, U]_\Gamma \subseteq U$ implies $a\alpha d(u) = d(u)\alpha a$, for some $\alpha \in \Omega$.

Then $d(a\alpha d(u)) = d(d(u)\alpha a)$

That is $d(a)\alpha d(u) + ak(\alpha)d(u) + a\alpha d^2(u) = d^2(u)\alpha a + d(u)k(\alpha)a + d(u)\alpha d(a)$

That implies $d(a)\alpha d(u) = d(u)\alpha d(a)$.

So both a and d(a) centralize d(V).

But $d(a\alpha u - u\alpha a) = d(a)\alpha u + ak(\alpha)u + a\alpha d(u) - d(u)\alpha a - uk(\alpha)a - u\alpha d(a)$

$= d(a)\alpha u - u\alpha d(a) \in d(V)$.

That means $[d(a), u]_\alpha \in d(V)$, and so

$[d(a), [d(a), v]_\alpha]_\alpha = 0$. Then $[d(a), v]_\alpha = 0$.

$C_M(V)$ is both a Γ-subring and a Lie ideal of M. Since $C_M(V)$ cannot contain a nonzero ideal of M we conclude that

$C_M(V) = Z(M)$, Since $V \not\subset Z(M)$. from [51], Lemma 3.7.

Therefore , $d(a) \in Z(M)$. By the same way since $a \in C_M(d(U))$, $a\alpha a \in C_M(d(U))$

$d(a\alpha a) = d(a)\alpha a + ak(\alpha)a + a\alpha d(a)$

$= a\alpha d(a) + a\alpha d(a)$

$= 2a\alpha d(a)$.

$d(a\alpha a) \in Z(M)$ implies $2a\alpha d(a) \in Z(M)$.

$a \in Z(M)$, $d(a) \in Z(M)$

Hence $a\alpha d(a) \in Z(M)$, which implies $d(a) = 0$

Hence $d(a) = 0$ for all $a \in C_M(d(U))$ which are not in $Z(M)$.

Let $d(b) \neq 0$ for some $b \in C_M(d(U))$.

Then $b \in Z(M)$. Furthermore, if $a \in C_M(d(U))$; $a \notin Z(M)$, then $d(a) = 0$.

Hence $d(a + b) = d(a) + d(b) = d(b) \neq 0$.

Consequently, $a + b \in Z(M)$ together with $b \in Z(M)$.

Then $a \in Z(M)$, a contradiction . Hence $C_M(d(U)) \not\subset Z(M)$ implies $d(a) = 0$ for all $a \in C_M(d(U))$.

Let $W = \{ x \in M \mid d(x) = 0 \}$. Then $C_M(d(U)) \subseteq W$.

Moreover if $a \in C_M(d(U))$ and $u \in U$,$\alpha \in \Omega$; then,

$d(a\alpha u - u\alpha a) = d(a)\alpha u + ak(\alpha)u + a\alpha d(u) - d(u)\alpha a - uk(\alpha)a - u\alpha d(a)$.

$= a\alpha d(u) - d(u)\alpha a = 0$, since $d(a) = 0$.

Thus $[a, u]_\alpha \in W$. That is $[a, U]_\alpha \subseteq W$.

Now since $U \not\subset Z(M)$, $[X, M]_\Gamma \subseteq U$ for some ideal X of M such that $[X, M]_\Gamma \not\subset Z(M)$.

If $p \in [X, M]_\Gamma \subseteq U \cap X$, then $p\alpha a \in X$, hence for $u \in U$; $[p\alpha a, u]_\alpha \in U$.

i.e., $[p, u]_\alpha\alpha a + p\alpha[a, u]_\alpha \in U$.

Therefore, a centralizes $d([p, u]_\alpha\alpha a + p\alpha[a, u]_\alpha)$

$$= d([p, u]_\alpha)\alpha a + [p, u]_\alpha k(\alpha)a + [p, u]_\alpha\alpha d(a) + d(p)\alpha[a, u]_\alpha + pk(\alpha)[a, u]_\alpha + p\alpha d([a, u]_\alpha)$$

$$= d([p, u]_\alpha)\alpha a + d(p)\alpha[a, u]_\alpha$$

since $k(\Omega) = 0$, $d(a) = 0 = d([a, u]_\alpha)$; $a, [a, u]_\alpha \in W$.

again a centralizes $d([p, u]_\alpha)$ and $d(p)$, we get

$d(p)\alpha[a, [a, u]_\alpha]_\alpha = d(p)\alpha[a, (a\alpha u - u\alpha a)]_\alpha$

$= d(p)(a\alpha a\alpha u - a\alpha u\alpha a - a\alpha u\alpha a - u\alpha a\alpha a)$

$= 0$, for all $p \in [X, M]_\Gamma$, $u \in U$, $\alpha \in \Omega$.

Thus $d([X, M]_\Gamma)$ $[a, [a, U]_\Gamma]_\Gamma = 0$ is a noncentral Lie ideal of M , by Lemma 10.16, we have that $[a, [a, U]_\Gamma]_\Gamma = 0$.

Therefore, by theorem 10.19, we get $a \in Z(M)$. Since $U \not\subset Z(M)$, with this the theorem is proved.

10.22 Lemma : Let M be a prime Γ_N –ring and U be a Lie ideal of M. Let d be a k-derivation on U of M such that $k(\Omega) = 0$. If $d^3 \neq 0$ and if $d(\bar{V})$ contains a nonzero left ideal A of M and a right ideal B of M , then $d(\bar{U})$ contains a nonzero ideal of M, where $d(\bar{V})$ is a subring generated by d(V).

Proof : Here $V = [U, U]_\Gamma$ and $d(V) \subseteq U$. We know that $d(d(V)) \subseteq d(\bar{U})$

Let $a \in A \subseteq d(\bar{V})$ and $m \in M, \alpha \in \Omega$; $m\alpha a \in A$. Then,

$d(m\alpha a) \in d(A) \subseteq d(\overline{d(v)}) \subseteq d(\bar{U})$.

Hence $d(m)\alpha a + mk(\alpha)a + m\alpha d(a) \in d(\bar{U})$.

i.e., $d(m)\alpha a + m\alpha d(a) \in d(\bar{U})$

Now $d(m)\alpha a$ is in A and so in $d(\bar{V}) \subseteq d(\bar{U})$.

We get $m\alpha d(a) \in d(\bar{U})$. Thus $M\alpha d(A) \subseteq d(\bar{U})$.

Similarly $d(B)\alpha M \subseteq d(\bar{U})$.

If $a \in A$, $u \in U$, $\alpha \in \Omega$, then $d(u\alpha a - a\alpha u) \in d(\bar{v})$.

Hence $d(u\alpha a - a\alpha u) = d(u)\alpha a + uk(\alpha)a + u\alpha d(a) - d(a)\alpha u - ak(\alpha)u - a\alpha d(u)$

$= d(u)\alpha a + u\alpha d(a) - d(a)\alpha u - a\alpha d(u) \in d(\bar{v})$.

But $d(u)\alpha a \in A \subseteq d(\bar{v})$, $d(a) \in d(A)$ implies $u\alpha d(a) \in d(\bar{U})$

and $a\alpha d(u) \in A\alpha d(V) \subseteq d(\bar{V})$

Also $d(A)\alpha V \subseteq d(\bar{U})$.

Similarly $V\alpha d(B) \subseteq d(\bar{U})$.

Let $I = A\alpha V\alpha B$; I is an ideal of M and by Lemma 10.10 , $I \neq 0$. Moreover $d(I) = d(A\alpha V\alpha B) \subseteq d(A)\alpha V\alpha B + A\alpha d(V)\alpha B + A\alpha V\alpha d(B)$ lies in $d(\bar{U})$, since $d(A)\alpha V$, $V\alpha d(B)$, A , B are all in $d(\bar{U})$.

Thus $d(I^-) \subseteq d(U^-)$. But if $d^3 \neq 0$, it is easy to see as in [29] that because I is an ideal of the prime Γ_N- ring M , $d(I^-)$ contains a nonzero ideal of M . Therefore $d(U^-)$ contains a nonzero ideal of M.

10.23 Lemma : Let U be a Lie ideal of a prime Γ_N- ring and d be a k-derivation on U of M such that $k(\Omega) = 0$. If $I \neq 0$ is an ideal of M and if $d(U^-)$ does not contain both a nonzero left ideal and a nonzero right ideal of M , then if $[c, I] \subseteq d(U^-)$, c must be in Z(M).

Proof : Let $u \in d(U)$ and $i \in I$; $\alpha \in \Gamma$.

Then $[c, u\alpha i]_\alpha = u\alpha[c, i]_\alpha + u[\alpha, \alpha]_c i + [c, u]_\alpha \alpha i$

$= u\alpha[c, i]_\alpha + [c, u]_\alpha \alpha i \in d(U)$.

Because $[c, i]_\alpha \in d(U^-)$, $u \in d(U^-)$, we have $u\alpha[c, i]_\alpha \in d(U^-)$.

Hence $[c, u]_\alpha \alpha i \in d(U^-)$; i. e., the right ideal of M . Then $[c, d(U)]_\alpha \alpha I \subseteq d(U^-)$.

Similarly $I\alpha[c, d(U)]_\alpha \subseteq d(U^-)$ is a left ideal of M lying in $d(U^-)$. By our hypothesis $I\alpha[c, d(U)]_\Gamma = 0$ or $[c, d(U)]_\Gamma \alpha I = 0$.

Therefore $[c, d(U)]_\Gamma = 0$.

By theorem 10.21, we conclude that $c \in Z(M)$.

10.24 Lemma : Let U be a Lie ideal of a 2- torsion free prime Γ_N-ring M and d be a k- derivation on U of M . If $d^3(U) = 0$, then $d^3 = 0$.

Proof : Let $u \in U$ and $m \in M$; then for any $\alpha \in \Omega$;

$0 = d^3[u, m]\alpha$ [where Ω is a Lie ideal of Γ]

$= [d^3(u), m]_\alpha + 3[d^2(u), d(m)]_\alpha + 3 [d(u), d^2(m)]_\alpha + [u, d^3(m)]_\alpha$

$= 3[d^2(u), d(m)]_\alpha + 3[d(u), d^2(m)]\alpha + [u, d^3(m)]\alpha$ ....(i), since $d^3(u) = 0$.

Replacing u by $d^2(w)$, $w \in W = [V, V]_\Gamma$, $V = [U, U]_\Gamma$, we have

$0 = 2(3[d^2(d^2(w)), d(m)]_\alpha + 3[d(d^2(w)), d^2(m)]_\alpha + [d^2(w), d^3(m)]_\alpha)$

$= 2(3[d(d^3(w)), d(m)]_\alpha + 3[d^3(w), d^2(m)]_\alpha + [d^2(w), d^3(m)]_\alpha)$

$= 2([d^2(w), d^3(m)]_\alpha)$

That implies $([d^2(w), d^3(m)]_\alpha) = 0$ ....(ii)

Now replace u by d(w) ; m by d(m) in (i) , we have

$0 = 3\,[d^3(w), d^2(m)]_\alpha + 3[d^2(w), d^3(m)]_\alpha + [d(w), d^4(m)]_\alpha$

$= [d(w)\,, d^4(m)]_\alpha$, for all $m \in M$; $w \in W$; $\alpha \in \Omega$.

Since $W \not\subset Z(M)$, hence $d^4(m) \in Z(M)$, for all $m \in M$.

Then

$0 = d^4([u, m]_\alpha) = d(d^3([u, m]_\alpha)$

$= d(\,3[d^2(u), d(m)]_\alpha + 3[d(u), d^2(m)]_\alpha + [u, d^3(m)]_\alpha)$

$= 3[d^3(u), d(m)]_\alpha + 6[d^2(u), d^2(m)]_\alpha + 4[d(u), d^3(m)]_\alpha + [u, d^4(m)]_\alpha$

$=6[d^2(u), d^2(m)]\alpha + 4[d(u), d^3(m)]_\alpha$ …………(iii)

We also have

$0 = d^3([u, d(m)]_\alpha$

$= [d^3(u), d(m)]_\alpha + 3[d^2(u), d^2(m)]_\alpha + 3[d(u), d^3(m)]_\alpha + [u, d^4(m)]_\alpha$

$= 3[d^2(u), d^2(m)]_\alpha + 3[d(u), d^3(m)]_\alpha$ ……….(iv)

From (iii) and (iv) we have , $2[d(u), d^3(m)]_\alpha = 0$

That implies $[d(u), d^3(m)] = 0$ for all $u \in U$, $m \in M$ and $\alpha \in \Omega$.

Then $d^3(m) \in Z(M)$.

since $U \subseteq Z(M)$, Hence $d^3(M) \subseteq Z(M)$.

Thus if $m \in M$, $u \in U$, then $d^3(m\alpha d^2(u) \in Z(M)$.

Now,

$d^3(m\alpha d^2(u)) = d^2(d(m\alpha d^2(u)))$

$= d^2(d(m)\alpha d^2(u))$

$= d(d^2(m)\alpha d^2(u))$

$= d^3(m)\alpha d^2(u) \in Z(M)$.

That implies $d^3(M)\alpha d^2(U) \subseteq Z(M)$.

However $d^3(M) \subseteq Z(M)$, so since $d^3(M)\alpha d^2(U) \subseteq Z(M)$, if $d^3(M) \neq 0$ then $d^2(U)$ must contained in Z(M).

Then suppose that $d^3(M) \neq 0$.

If $m \in M$, $u \in U$ then

$d^4(m\alpha d(u)) = d^3(d(m)\alpha d(u) + m\alpha d^2(u))$

$= d^2(d^2(m)\alpha d(u) + 2d(m)\alpha d^2(u))$

$= d(d^3(m)\alpha d(u) + 3d^2(m)\alpha d^2(u))$

$= d^4(m)\alpha d(u) + 4d^3(m)\alpha d^2(u) \in Z(M)$

And since $d^3(m)$, $d^2(u) \in Z(M)$, we see that $d^4(m)\alpha d(u) \in Z(M)$.

That is $d^4(M)\alpha d(U) \subseteq Z(M)$.

By Lemma 10.17, we know that $d(U) \not\subset Z(M)$ and $d^4(M) \not\subset Z(M)$ and $d^4(M)\alpha d(U) \subseteq Z(M)$ forced that $d(M) = 0$.

Again if $m \in M$, $u \in U$, then

$0 = d^4(m\alpha d(u)) = 4d^3(m)\alpha d^2(u)$.

That is $d^3(M)\alpha d^2(U) = 0$. But $d^2(U) \neq 0 \subset Z(M)$ (by thorem 10.19).

So we conclude that $d(M) = 0$.

Jordan Left k-derivation

Let M be a gamma ring and U be a Lie ideal of M .We have seen that every left k-derivation on U of M is a Jordan left k-derivation on U of M, but the converse is not always true. In this chapter we have proved that every Jordan left k-derivation on U of M is a left k-derivation on U of M if M is a completely prime gamma ring.

11. Introduction : Y. Ceven [14] defined Jordan left derivation on gamma rings and he showed that a Jordan left derivation on completely prime gamma rings is also a left derivation.A. C. Paul and M. M. Rahman worked on Jordan left derivations on semiprime gamma rings. They prove that d(M) = 0, where d and G be Jordan left derivations on M such that $d^2(M) = G(M)$. Haldar and Paul worked on Jordan left derivations on Lie ideals of prime gamma rings. M. Soyturk worked on the commutativity of prime gamma rings with left and right derivations.M. Asci and S. Ceran worked on prime gamma rings with left derivations. In this chapter we got some results which are the extension of left derivations to left k-derivations on Lie ideals of 2-torsion free gamma rings.

11.1 Definition: Let U be a Lie ideal of a Γ- ring M. Suppose that d: M → M and k : Γ → Γ are additive mappings. If $d(u\alpha v) = u\alpha d(v) + uk(\alpha)v + v\alpha d(u)$ is satisfied for every u, v ∈ U and then d is said to be a left k-derivation on U of M. Also d is a Jordan left k-derivation on U of M if $d(u\alpha u) = 2u\alpha d(u) + uk(\alpha)u$ holds for every u ∈ U and α ∈ Γ.

11.2 Definition : A Γ-ring M is said to be a completely prime Γ-ring if for every x, y ∈ M, xΓy = 0 implies x = 0 or y = 0.

11.2 Lemma: Let U be a Lie ideal of a 2-torsion free Γ-ring M satisfying the condition (*) and d is a Jordan left k-derivation on U of M. Then the following hold :

(i) $d(u\alpha v + v\alpha u) = 2u\alpha d(v) + 2v\alpha d(u) + uk(\alpha)v + vk(\alpha)u$

(ii) $d(u\alpha v\alpha u) = u\alpha u\alpha d(v) + (3u\alpha v - v\alpha u)\alpha d(u) + u\alpha(2vk(\alpha)u + uk(\alpha)v - v\alpha uk(\alpha)u)$

(iii) $d(u\alpha v\alpha w + w\alpha v\alpha u) = (u\alpha w + w\alpha u)d(v) + (3w\alpha v - v\alpha w)\alpha d(u) + (3u\alpha v - v\alpha u)\alpha d(w) + (2w\alpha v - v\alpha w)k(\alpha)u + (2u\alpha v - v\alpha u)k(\alpha)w + (u\alpha w + w\alpha u)k(\alpha)v$

(iv) $(u\alpha v - v\alpha u)\alpha u\alpha d(u) = u\alpha(u\alpha v - v\alpha u)\alpha d(u) + u\alpha(u\alpha v - v\alpha u)k(\alpha)u - (u\alpha v - v\alpha u)\alpha uk(\alpha)u$

(v) $(u\alpha v - v\alpha u)\alpha(d(u\alpha v) - u\alpha d(v) - v\alpha d(u) - vk(\alpha)u) = 0$

(vi) $d(u\alpha u\alpha v) = u\alpha u\alpha d(v) + (u\alpha v + v\alpha u)\alpha d(u) + u\alpha d(u\alpha v - v\alpha u) + uk(\alpha)u\alpha v + vk(\alpha)u\alpha u$

(vii) $d(v\alpha u\alpha u) = u\alpha u\alpha d(v) + (3v\alpha u - u\alpha v)\alpha d(u) - u\alpha d(u\alpha v - v\alpha u) + 2vk(\alpha)u\alpha u$

(viii) $(u\alpha v - v\alpha u)(d(u\alpha v - v\alpha u) + uk(\alpha)v - vk(\alpha)u = 0$

(ix) $(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\alpha d(v) - uk(\alpha) (u\alpha v\alpha u - 2v\alpha u\alpha v) + vk(\alpha)u\alpha u\alpha v = 0$

(x) $(v\alpha v\alpha u - 2v\alpha u\alpha v + u\alpha v\alpha v)\alpha d(u) + uk(\alpha)v\alpha v\alpha u + vk(\alpha)(v\alpha u\alpha u - 2u\alpha v\alpha u) = 0$

Proof (i) : We have $u\alpha v + v\alpha u = (u+v)\alpha(u+v) - u\alpha u - v\alpha v$ and since right side is in U , hence left side is also in U.

Now $d(u\alpha v + v\alpha u) = d((u+v)\alpha(u+v) - u\alpha u - v\alpha v)$

$= 2(u+v)\alpha d(u+v) + (u+v)k(\alpha)(u+v) - 2u\alpha d(u) - uk(\alpha)u - 2v\alpha d(v) - vk(\alpha)v$

$= 2u\alpha d(u) + 2v\alpha d(u) + 2u\alpha d(v) + 2v\alpha d(v) + uk(\alpha)u + uk(\alpha)v + vk(\alpha)u + vk(\alpha)v - 2u\alpha d(u) - uk(\alpha)u - 2v\alpha d(v) - vk(\alpha)v$

$= 2u\alpha d(v) + 2v\alpha d(u) + uk(\alpha)v + vk(\alpha)u$

(ii): Replace v by $u\alpha v + v\alpha u$ we get

L.S. = $d(u\alpha(u\alpha v + v\alpha u) + (u\alpha v + v\alpha u)\alpha u)$

$= d(u\alpha u\alpha v + u\alpha v\alpha u + u\alpha v\alpha u + v\alpha u\alpha u)$

$= d(u\alpha v\alpha u + u\alpha v\alpha u) + d((u\alpha u)\alpha v + v\alpha(u\alpha u))$

$= d(2u\alpha v\alpha u) + 2u\alpha u\alpha d(v) + 2v\alpha d(u\alpha u) + u\alpha uk(\alpha)v + vk(\alpha)u\alpha u$

$= 2d(u\alpha v\alpha u) + 2u\alpha u\alpha d(v) + 2v\alpha(2u\alpha d(u) + uk(\alpha)u + u\alpha uk(\alpha)v + vk(\alpha)u\alpha u$

$= 2d(u\alpha v\alpha u) + 2u\alpha u\alpha d(v) + 4v\alpha u\alpha d(u) + 2v\alpha uk(\alpha)u + u\alpha uk(\alpha)v + vk(\alpha)u\alpha u$

And R. S. = $2u\alpha d(u\alpha v + v\alpha u) + 2(u\alpha v + v\alpha u)\alpha d(u) + uk(\alpha)(u\alpha v + v\alpha u) + (u\alpha v + v\alpha u)k(\alpha)u$

$= 2u\alpha(2u\alpha d(v) + 2v\alpha d(u) + uk(\alpha)v + vk(\alpha)u + 2u\alpha v\alpha d(u) + 2v\alpha u\alpha d(u) + uk(\alpha)u\alpha v + uk(\alpha)v\alpha u + u\alpha vk(\alpha)u + v\alpha uk(\alpha)u$

$= 4u\alpha u\alpha d(v) + 4u\alpha v\alpha d(u) + 2u\alpha uk(\alpha)v + 2u\alpha vk(\alpha)u + 2u\alpha v\alpha d(u) + 2v\alpha u\alpha d(u) + uk(\alpha)u\alpha v + uk(\alpha)v\alpha u + u\alpha vk(\alpha)u + v\alpha uk(\alpha)u$

$= 4u\alpha u\alpha d(v) + 6u\alpha v\alpha d(u) + 2u\alpha uk(\alpha)v + 3u\alpha vk(\alpha)u + 2v\alpha u\alpha d(u) + uk(\alpha)u\alpha v + uk(\alpha)v\alpha u + v\alpha uk(\alpha)u$

Computing R. S. and L. S. we get

$2d(u\alpha v\alpha u) = 2u\alpha u\alpha d(v) + 6u\alpha v\alpha d(u) + u\alpha uk(\alpha)v + 3u\alpha vk(\alpha)u - 2v\alpha u\alpha d(u) + uk(\alpha)v\alpha u - v\alpha uk(\alpha)u - vk(\alpha)u\alpha u$

Using the hypothesis $u\alpha v\beta w = u\beta v\alpha w$, for every $u, v, w \in U$ and $\alpha, \beta \in \Gamma$ we have

$2d(u\alpha v\alpha u) = 2u\alpha u\alpha d(v) + 6u\alpha v\alpha d(u) + 4u\alpha vk(\alpha)u + 2u\alpha uk(\alpha)v - 2v\alpha u\alpha d(u) - 2v\alpha uk(\alpha)u$

Since M is 2-torsion free, hence

$d(u\alpha v\alpha u) = u\alpha u\alpha d(v) + (3u\alpha v - v\alpha u)\alpha d(u) + u\alpha(2vk(\alpha)u + uk(\alpha)v - v\alpha uk(\alpha)u$

(iii) : Putting u+w for u in the above , we have

L.S. = $d((u+w)\alpha v\alpha(u+w))$

$= d(u\alpha v\alpha u + w\alpha v\alpha u + u\alpha v\alpha w + w\alpha v\alpha w)$

$= d(u\alpha v\alpha w + w\alpha v\alpha u) + d(u\alpha v\alpha u) + d(w\alpha v\alpha w)$

$= d(u\alpha v\alpha w + w\alpha v\alpha u) + u\alpha u\alpha d(v) + (3u\alpha v - v\alpha u)\alpha d(u) + u\alpha(2vk(\alpha)u + uk(\alpha)v - v\alpha uk(\alpha)u + w\alpha w\alpha d(v) + (3w\alpha v - v\alpha w)\alpha d(w) + w\alpha(2vk(\alpha)w + wk(\alpha)v) - v\alpha wk(\alpha)w$

$= d(u\alpha v\alpha w + w\alpha v\alpha u) + u\alpha u\alpha d(v) + 3u\alpha v\alpha d(u) - v\alpha u\alpha d(u) + 2u\alpha vk(\alpha)u + u\alpha uk(\alpha)v - u\alpha uk(\alpha)u + w\alpha w\alpha d(v) + 3w\alpha v\alpha d(w) - v\alpha w\alpha d(w) + 2w\alpha vk(\alpha)w + w\alpha wk(\alpha)v - v\alpha wk(\alpha)w$

R. S. $= (u+w)\alpha(u+w)\alpha d(v) + (3(u+w)\alpha v - v\alpha(u+w))\alpha d(u+w) + (u+w)\alpha(2vk(\alpha)(u+w) + (u+w)k(\alpha)v) - v\alpha(u+w)k(\alpha)(u+w)$

$= u\alpha u\alpha d(v) + w\alpha u\alpha d(v) + u\alpha w\alpha d(v) + w\alpha w\alpha d(v) + 3u\alpha v\alpha d(u) + 3w\alpha v\alpha d(u) + 3u\alpha v\alpha d(w) + 3w\alpha v\alpha d(w) - v\alpha u\alpha d(u) - v\alpha w\alpha d(u) - v\alpha u\alpha d(w) - v\alpha w\alpha d(w) + 2u\alpha vk(\alpha)u + 2w\alpha vk(\alpha)u + 2u\alpha vk(\alpha)w + 2w\alpha vk(\alpha)w + u\alpha uk(\alpha)v + w\alpha uk(\alpha)v + u\alpha wk(\alpha)v + w\alpha wk(\alpha)v - v\alpha uk(\alpha)u - v\alpha wk(\alpha)u - v\alpha uk(\alpha)w - v\alpha wk(\alpha)w$

Computing both sides we get

$d(u\alpha v\alpha w + w\alpha v\alpha u) = (w\alpha u + u\alpha w)\alpha d(v) + (3w\alpha v - v\alpha w)\alpha d(u) + (3u\alpha v - v\alpha u)\alpha d(w) + (2w\alpha v - v\alpha w)k(\alpha)u + (2u\alpha v - v\alpha u)k(\alpha)w + (u\alpha w + w\alpha u)k(\alpha)v$

(iv) Consider $A = d(u\alpha v\alpha u\alpha v + u\alpha v\alpha v\alpha u)$

$= d(uv(uv) + (uv)vu)$

$= (u\alpha v\alpha u + u\alpha u\alpha v)\alpha d(v) + (3u\alpha v\alpha v - v\alpha u\alpha v)\alpha d(u) + (3u\alpha v - v\alpha u)\alpha d(u\alpha v) + (2u\alpha v\alpha v - v\alpha u\alpha v)k(\alpha)u + (2u\alpha v - v\alpha u)k(\alpha)u\alpha v + (u\alpha u\alpha v + u\alpha v\alpha u)k(\alpha)v$

$= (u\alpha u\alpha v + u\alpha v\alpha u)\alpha d(v) + (3u\alpha v - v\alpha u)\alpha d(u\alpha v) + (3u\alpha v\alpha v - v\alpha u\alpha v)\alpha d(u) + u\alpha(2vk(\alpha)u\alpha v + u\alpha vk(\alpha)v) + u\alpha v\alpha(2vk(\alpha)u + uk(\alpha)v) - v\alpha(uk(\alpha)u\alpha v + u\alpha vk(\alpha)u)$

Again $A = d((u\alpha v)\alpha(u\alpha v)) + d(u\alpha(v\alpha v)\alpha u)$

$= 2u\alpha v\alpha d(u\alpha v) + u\alpha vk(\alpha)u\alpha v + u\alpha u\alpha d(v\alpha v) + (3u\alpha v\alpha v - v\alpha v\alpha u)d(u) + u\alpha(2v\alpha vk(\alpha)u + uk(\alpha)v\alpha v) - v\alpha vuk(\alpha)u$

= 2uαvαd(uαv) + uαvk(α)uαv + 2uαuvαd(v) + uαuαvk(α)v + (3uαvαv – vαvαu)αd(u) + uα(2vαvk(α)u + uk(α)vαv) – vαvαuk(α)u

Equating these two expressions for A, we have

(3uαv – 2uαv – vαu)αd(uαv) = (uαv – 2uαv)k(α)uαv + (2uαuαv – uαvαu – uαuαv)αd(v) + (uαuαv – uαuαv)k(α)v + (3uαvαv – vαvαu – 3uαvαv + vαuαv)αd(u) + (2uαvαv- vαvαu – 2uαvαv + vαuαv)k(α)u + (uαuk(α)v – uαvk(α)u + vαuk(α)u)αv

= (uαuαv – uαvαu)αd(v) + (vαuαv – vαvαu)αd(u) + (vαuαv – vαvαu)k(α)u + (uk(α)uαv – uk(α)vαu)αv + (vαu – uαv)k(α)uαsv

That implies

(uαv – vαu)αd(uαv) = uα(uαv – vαu)αd(v) + vα(uαv – vαu)αd(u) + vα(uαv – vαu)k(α)u + uk(α)(uαv – vαu)αv – (uαv – vαu)k(α)uαv.

Replacing u+v for v (which keeps uαv – vαu unaltered), we have

(uαv – vαu)αd(uα(u+v)) = uα(uαv – vαu)αd(u+v) + (u+v)α(uαv – vαu)αd(u) + (u+v)α(uαv – vαu)k(α)u + uk(α)(uαv – vαu)α(u+v) – (uαv – vαu)k(α)uα(u+v)

Then (uαv – vαu)α2uαd(u) + (uαv – vαu)αuk(α)u + uα(uαv – vαu)αd(v) + vα(uαv – vαu)αd(u) + vα(uαv –vαu)k(α)u + uk(α)(uαv –vαu)αv + (uαv –vαu)k(α)uαv = uα(uαv – vαu)αd(u) + uα(uαv – vαu)αd(v) +(u+v)α(uαv –vαu)αd(u) + uα(uαv – vαu)k(α)u + vα(uαv – vαu)k(α)u + uk(α)(uαv – vαu)αu +uk(α)(uαv – vαu)αv – (uαv – vαu)k(α)uαu + (uαv – vαu)k(α)uαv

That implies (uαv – vαu)αuαd(v) + (uαv – vαu)αvαd(u) = uα(uαv – vαu)αd(v) + vα(uαv – vαu)αd(u) + uα(uαv – vαu)k(α)v + vα(uαv – vαu)k(α)u – (uαv – vαu)αuk(α)v – (uαv – vαu)αvk(α)u

Therefore we have,

(uαv – vαu)αuαd(v) + (uαv – vαu)αvαd(u) = (uαv – vαu)αd(uαv) – (uαv – vαu)αvk(α)u [using (*) and (4)

Thus (uαv – vαu)α(d(uαv) – uαd(v) – vαd(u) – vk(α)u) = 0.

(vi) Put $v\alpha u$ for v in (i), we have ,

$d(u\alpha v\alpha u + v\alpha u\alpha u) = 2u\alpha d(v\alpha u) + 2v\alpha u\alpha d(u) + uk(\alpha)v\alpha u + v\alpha uk(\alpha)u$ .... (6)

Again put $u\alpha v$ for v in (i) we have

$d(u\alpha u\alpha v + u\alpha v\alpha u) = 2u\alpha d(u\alpha v) + 2u\alpha v\alpha d(u) + uk(\alpha)u\alpha v + u\alpha vk(\alpha)u$........... (7)

Subtracting (6) from (7) we obtain

$d(u\alpha u\alpha v + v\alpha v\alpha u - u\alpha v\alpha u - v\alpha u\alpha u) = 2u\alpha d(u\alpha v) + 2u\alpha v\alpha d(u) + uk(\alpha)u\alpha v + u\alpha vk(\alpha)u - 2u\alpha d(v\alpha u) - 2v\alpha u\alpha d(u) - uk(\alpha)v\alpha u - v\alpha uk(\alpha)u$

Then $d(u\alpha u\alpha v - v\alpha u\alpha u) = 2u\alpha(d(u\alpha v - v\alpha u) + 2(u\alpha v - v\alpha u)\alpha d(u) + uk(\alpha)u\alpha v - v\alpha uk(\alpha)u$ .... (8) [using (*)]

Also replacing $u\alpha u$ for u in (i) we get

$d(u\alpha u\alpha v + v\alpha u\alpha u) = 2u\alpha u\alpha d(v) + 2v\alpha d(u\alpha u) + u\alpha uk(\alpha)v + vk(\alpha)u\alpha u$

$= 2u\alpha u\alpha d(v) + 4v\alpha u\alpha d(u) + 3v\alpha uk(\alpha)u + u\alpha uk(\alpha)v$ ....(9)

Adding (8) and (9) we have $d(u\alpha u\alpha v - v\alpha u\alpha u + u\alpha u\alpha v + v\alpha u\alpha u) = 2u\alpha d(u\alpha v - v\alpha u) + 2(u\alpha v - v\alpha u)\alpha d(u) + uk(\alpha)u\alpha v - v\alpha uk(\alpha)u + 2u\alpha u\alpha d(v) + 4v\alpha u\alpha d(u) + 3v\alpha uk(\alpha)u + u\alpha uk(\alpha)v$

That implies

$2d(u\alpha u\alpha v) = 2u\alpha d(u\alpha v - v\alpha u) + 2(u\alpha v + v\alpha u)\alpha d(u) + 2u\alpha u\alpha d(v) + 2uk(\alpha)u\alpha v + 2v\alpha uk(\alpha)u$

Since M is 2-torsion free, we have

$d(v\alpha u\alpha u) = u\alpha u\alpha d(v) + (3v\alpha u - u\alpha v)\alpha d(u) - u\alpha d(u\alpha v - v\alpha u) + 2vk(\alpha)u\alpha u$ ....(11)

(viii) : From (i) we have

$d(u\alpha v + v\alpha u) = 2u\alpha d(v) + 2v\alpha d(u) + uk(\alpha)v + vk(\alpha)u$, and then

$d(u\alpha v) = - d(v\alpha u) + 2u\alpha d(v) + 2v\alpha d(u) + uk(\alpha)v + vk(\alpha)u$

Substituting this into (v) we have

$0 = (u\alpha v - v\alpha u)\alpha(- d(v\alpha u) + 2u\alpha d(v) + 2v\alpha d(u) + uk(\alpha)v + vk(\alpha)u - u\alpha d(v) - v\alpha d(u) - vk(\alpha)u)$

$= (u\alpha v - v\alpha u)\alpha(- d(v\alpha u) + u\alpha d(v) + v\alpha d(u) + uk(\alpha)v)$

$= (u\alpha v - v\alpha u)\alpha(d(v\alpha u - u\alpha d(v) - v\alpha d(u) - uk(\alpha)v)$

Subtracting this from (v) we have

$(u\alpha v - v\alpha u)\alpha(d(u\alpha v) - u\alpha d(v) - v\alpha d(u) - vk(\alpha)u - d(v\alpha u) + u\alpha d(v) + v\alpha d(u) + uk(\alpha)v) = 0$

That implies

$(u\alpha v - v\alpha u)\alpha(d(u\alpha v - v\alpha u) + uk(\alpha)v - vk(\alpha)u) = 0$ ….(12).

(ix) : From the definition of d we have

$d(u\alpha v - v\alpha u)\alpha(u\alpha v - v\alpha u) = 2(u\alpha v - v\alpha u)\alpha d(u\alpha v - v\alpha u) + (u\alpha v - v\alpha u)k(\alpha)(u\alpha v - v\alpha u)$

$= 2(u\alpha v - v\alpha u)\alpha(vk(\alpha)u - uk(\alpha)v) + (u\alpha v - v\alpha u)k(\alpha)(u\alpha v - v\alpha u)$

$= 2u\alpha v\alpha vk(\alpha)u - 2u\alpha v\alpha uk(\alpha)v - 2v\alpha u\alpha vk(\alpha)u + 2v\alpha u\alpha uk(\alpha)v + u\alpha vk(\alpha)u\alpha v - u\alpha vk(\alpha)v\alpha u - v\alpha uk(\alpha)u\alpha v + v\alpha uk(\alpha)v\alpha u$

Again , $d((u\alpha v - v\alpha u)\alpha(u\alpha v - v\alpha u))$

$= d(u\alpha v\alpha u\alpha v - u\alpha v\alpha v\alpha u - v\alpha u\alpha u\alpha v + v\alpha u\alpha v\alpha u)$

$= d(u\alpha(v\alpha u\alpha v) + (v\alpha u\alpha v)\alpha u) - d(u\alpha(v\alpha v)\alpha u - d(v\alpha(u\alpha u)\alpha v)$

$= 2u\alpha d(v\alpha u\alpha v) + 2(v\alpha u\alpha v)\alpha d(u) + uk(\alpha)(v\alpha u\alpha v) + (v\alpha u\alpha v)k(\alpha)u - (v\alpha vd(u\alpha u) + (3v\alpha u\alpha u - u\alpha u\alpha v)\alpha d(v) + v\alpha(2u\alpha uk(\alpha)v + vk(\alpha)u\alpha u) - u\alpha u\alpha vk(\alpha)v) - (u\alpha u\alpha d(v\alpha v) + (3u\alpha v\alpha v - v\alpha v\alpha u)\alpha d(u) + u\alpha(2v\alpha vk(\alpha)u + uk(\alpha)v\alpha v - v\alpha v\alpha uk(\alpha)u)$

$= 2u\alpha(v\alpha v\alpha d(u) + (3v\alpha u - u\alpha v)\alpha d(v) + v\alpha(2uk(\alpha)v + vk(\alpha)u) - u\alpha vk(\alpha)v) + 2v\alpha u\alpha v\alpha d(u) + uk(\alpha)v\alpha u\alpha v + v\alpha u\alpha vk(\alpha)u - (v\alpha v\alpha(2u\alpha d(u) + uk(\alpha)u) + (3v\alpha u\alpha u - u\alpha u\alpha v)\alpha d(v) + v\alpha(2u\alpha uk(\alpha)v + vk(\alpha)u\alpha u) - u\alpha u\alpha vk(\alpha)v) - (u\alpha u\alpha(2v\alpha d(v) + vk(\alpha)v) + (3u\alpha v\alpha v - v\alpha v\alpha u)\alpha d(u) + u\alpha(2v\alpha vk(\alpha)u + uk(\alpha)v\alpha v) - v\alpha v\alpha uk(\alpha)u)$

$= 2u\alpha v\alpha v\alpha d(u) + 6u\alpha v\alpha u\alpha d(v) - 2u\alpha u\alpha v\alpha d(v) + 4u\alpha v\alpha uk(\alpha)v + 2u\alpha v\alpha vk(\alpha)u - 2u\alpha u\alpha vk(\alpha)v + 2v\alpha u\alpha v\alpha d(u) + uk(\alpha)v\alpha u\alpha v + v\alpha u\alpha vk(\alpha)u - 2v\alpha v\alpha u\alpha d(u) - v\alpha v\alpha uk(\alpha)u - 3v\alpha u\alpha u\alpha d(v) + u\alpha u\alpha v\alpha d(v) - 2v\alpha u\alpha uk(\alpha)v - v\alpha vk(\alpha)u\alpha u + u\alpha u\alpha vk(\alpha)v - 2u\alpha u\alpha v\alpha d(v) - u\alpha u\alpha vk(\alpha)v - 3u\alpha v\alpha v\alpha d(u) + v\alpha v\alpha u\alpha d(u) - 2u\alpha v\alpha vk(\alpha)u - u\alpha uk(\alpha)v\alpha v + v\alpha v\alpha uk(\alpha)u$

Comparing these two expressions of d((uαv – vαu)α(uαv – vαu)) ,
we have,

2uαvαvk(α)u – 2uαvαuk(α)v – 2vαuαvk(α)u + 2vαuαuk(α)v + uαvk(α)uαv – uαvk(α)vαu – vαuk(α)uαv + vαuk(α)vαu = 2uαvαvαd(u) + 6uαvαuαd(v) – 2uαuαvαd(v) + 4uαvαuk(α)v + 2uαvαvk(α)u – 2uαuαvk(α)v + 2vαuαvαd(u) + uk(α)vαuαv + vαuαvk(α)u – 2vαvαuαd(u) – vαvαuk(α)u – 3vαuαuαd(v) + uαuαvαd(v) – 2vαuαuk(α)v – vαvk(α)uαu + uαuαvk(α)v – 2uαuαvαd(v) – uαuαvk(α)v – 3uαvαvαd(u) + vαvαuαd(u) – 2uαvαvk(α)u – uαuk(α)vαv + vαvαuk(α)u

Using (*) we get

3(uαuαv – 2uαvαu + vαuαu)αd(v) + (vαvαu – 2vαuαv + uαvαv)αd(u) + uk(α)(vαvαu – 6vαuαv + 3uαvαv) + vk(α)(3uαuαv - 2uαvαu + vαuαu) = 0. …………..(13)

From (4) we have

(uαv – vαu)αuαd(u) = uα(uαv – vαu)αd(u) + uα(uαv – vαu)k(α)u – (uαv – vαu)αuk(α)u

That implies

(uαuαv – 2uαvαu + vαuαu)αd(u) + uk(α)(uαv – vαu)αu – (uαv – vαu)αuk(α)u = 0

Replace u+v for u (which keeps (uαv – vαu) unaltered).

Then we have ,

0 = (u+v)α(u+v)αv – 2(u+v)αvα(u+v) + vα(u+v)α(u+v)αd(u+v) + (u+v)k(α)(uαv – vαu)(u+v) – (uαv – vαu)(u+v)k(α)(u+v)

= uαuαv + uαvαv +vαuαv + vαvαv – 2uαvαu – 2uαvαv – 2vαvαu – 2vαvαv + vαuαu+ vαuαv + vαvαu + vαvαv)α(d(u) + d(v)) + uk(α)(uαv – vαu)αu +vk(α)(uαv – vαu)αu + uk(α)(uαv – vαu)αv + vk(α)(uαv – vαu)αv – (uαv – vαu)αuk(α)u – (uαv – vαu)αvk(α)u – uαv – vαu)αvk(α)v

= [(uαuαv – 2uαvαu + vαuαu)αd(v) + uk(α)(uαv – vαu)αu – (uαv – vαu)αuk(α)u] – [(vαvαu – 2vαuαv + uαvαv)αd(v) + vk(α)(vαu – uαv)αv – (vαu – uαv)vk(α)v] + (- uαvαv + 2vαuαv – vαvαu)αd(u) + (uαuαv –

$2u\alpha v\alpha u + v\alpha u\alpha u)\alpha d(v) + uk(\alpha)(u\alpha v - v\alpha u)\alpha v + vk(\alpha)(u\alpha v - v\alpha u)\alpha u - (u\alpha v\alpha uk(\alpha)v - v\alpha u\alpha uk(\alpha)v) - (u\alpha v\alpha vk(\alpha)u - v\alpha u\alpha vk(\alpha)u)$

But in the third bracket of first two terms vanish. That implies

$0 = (u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\alpha\, d(v) - (u\alpha v\alpha v - 2v\alpha u\alpha v + v\alpha v\alpha u)\alpha d(u) + uk(\alpha)(u\alpha v\alpha v - v\alpha u\alpha v) + vk(\alpha)(u\alpha v\alpha u - v\alpha u\alpha u) - (u\alpha v\alpha uk(\alpha)v - v\alpha u\alpha uk(\alpha)v) - (u\alpha v\alpha vk(\alpha)u - v\alpha u\alpha vk(\alpha)u)$

$= (u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\alpha d(v) - (u\alpha v\alpha v - 2v\alpha u\alpha v + v\alpha v\alpha u)\alpha d(u) + uk(\alpha)(u\alpha v\alpha v - v\alpha u\alpha v - v\alpha u\alpha v - v\alpha v\alpha u) + vk(\alpha)(u\alpha v\alpha u - v\alpha u\alpha u + u\alpha u\alpha v + u\alpha v\alpha u)$

That implies

$(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\alpha d(v) - (u\alpha v\alpha v - 2v\alpha u\alpha v + v\alpha v\alpha u)\alpha d(u) + uk(\alpha)(u\alpha v\alpha v - 2v\alpha u\alpha v - v\alpha v\alpha u) + vk(\alpha)(2u\alpha v\alpha u - v\alpha u\alpha u + u\alpha u\alpha v) = 0$……..................(14)

Adding (13) and (14) we have ,

$4(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\alpha d(v) + 4uk(\alpha)(u\alpha v\alpha u - 2v\alpha u\alpha v) + 4vk(\alpha)u\alpha u\alpha v = 0$,

Then ,

$(u\alpha u\alpha v - 2u\alpha v\alpha u + v\alpha u\alpha u)\alpha d(v) + 4uk(\alpha)(u\alpha v\alpha u - 2v\alpha u\alpha v) + 4vk(\alpha)u\alpha u\alpha v = 0$…………………..(15)

Hence from (14) we have,

$(2v\alpha u\alpha v - v\alpha v\alpha u - u\alpha v\alpha v)\alpha d(u) - uk(\alpha)v\alpha v\alpha u + vk(\alpha)(2u\alpha v\alpha u - v\alpha u\alpha u) = 0$

That implies

$(v\alpha v\alpha u - 2v\alpha u\alpha v + u\alpha v\alpha v)\alpha d(u) + uk(\alpha)v\alpha v\alpha u + vk(\alpha)(v\alpha u\alpha u - 2u\alpha v\alpha u) = 0$

The proof is complete.

11.3 Theorem : Let M be a 2-torsion free completely prime Γ-ring and U a Lie ideal of M such that $u\alpha u \in U$ for all $u \in U$ and $\alpha \in \Gamma$. If d is a nonzero additive mapping such that $d(u\alpha u) = 2u\alpha d(u) + uk(\alpha)u$ for all $u \in$

U and $\alpha \in \Gamma$, then $d(u\alpha v) = u\alpha d(v) + uk(\alpha)v + v\alpha d(u)$ for all $u, v \in U$ and $\alpha \in \Gamma$.

Proof : From (viii) we have

$(u\alpha v - v\alpha u)\alpha(d(u\alpha v - v\alpha u) + uk(\alpha)v - vk(\alpha)u) = 0$

Since M is completely prime, we get $u\alpha v - v\alpha u = 0$

or $d(u\alpha v - v\alpha u) + uk(\alpha)v - vk(\alpha)u = 0$

If $u\alpha v - v\alpha u = 0$, then $u\alpha v = v\alpha u$ for every $u, v \in M$; $\alpha \in \Gamma$.

Therefore M is commutative.

And if $d(u\alpha v - v\alpha u) + uk(\alpha)v - vk(\alpha)u = 0$

Then $d(u\alpha v - v\alpha u) = vk(\alpha)u - uk(\alpha)v$

That implies $d(u\alpha v) = d(v\alpha u) + vk(\alpha)u - uk(\alpha)v$

Replacing v by $u\alpha v$ we have

$d(u\alpha u\alpha v) = d(u\alpha v\alpha u) + u\alpha vk(\alpha)u - uk(\alpha)u\alpha v$

Using (2) , (6) and (*) we get

$u\alpha u\alpha d(v) + (u\alpha v + v\alpha u)\alpha d(u) + u\alpha d(u\alpha v - v\alpha u) + uk(\alpha)u\alpha v + vk(\alpha)u\alpha u = u\alpha u d(v) + 3u\alpha v - v\alpha u)\alpha d(u) + u\alpha(2vk(\alpha)u + uk(\alpha)v) - v\alpha uk(\alpha)u + u\alpha vk(\alpha)u - uk(\alpha)u\alpha v$

That implies $0 = (3u\alpha v - v\alpha u - u\alpha v - v\alpha u)\alpha d(u) + 2uk(\alpha)v\alpha u + uk(\alpha)u\alpha v - vk(\alpha)u\alpha u - uk(\alpha)u\alpha v + u\alpha vk(\alpha)u - u\alpha d(u\alpha v - v\alpha u) - uk(\alpha)u\alpha v - vk(\alpha)u\alpha u$

$= 2(u\alpha v - v\alpha u)\alpha d(u) - u\alpha(d(u\alpha v - v\alpha u) + uk(\alpha)v - vk(\alpha)u + 2(u\alpha v - v\alpha u)k(\alpha)u$

$= 2(u\alpha v - v\alpha u)\alpha d(u) + 2(u\alpha v - v\alpha u)k(\alpha)u$

Since M is 2-torsion free

$(u\alpha v - v\alpha u)\alpha d(u) + (u\alpha v - v\alpha u)k(\alpha)u = 0$

Putting $u\alpha v$ for v again, we get

$0 = (u\alpha u\alpha v - u\alpha v\alpha u)\alpha d(u) + (u\alpha u\alpha v - u\alpha v\alpha u)k(\alpha)u$

$= u\alpha(u\alpha v - v\alpha u)\alpha d(u) + u\alpha(u\alpha v - v\alpha u)k(\alpha)u$

$= (u\alpha v - v\alpha u)\alpha u\alpha d(u) + (u\alpha v - v\alpha u)\alpha k(\alpha)u$ [using (iv)]

$= (u\alpha v - v\alpha u)\alpha(uk(\alpha)u + u\alpha d(u))$

Since M is completely prime , hence $u\alpha v - v\alpha u = 0$ or $uk(\alpha)u + u\alpha d(u) = 0$

If $uk(\alpha)u + u\alpha d(u) = 0$, then $2u\alpha d(u) + uk(\alpha)u - u\alpha d(u) = 0$

That implies $d(u\alpha u) - u\alpha d(u) = 0$ and hence $d(u\alpha u) = u\alpha d(u)$, which is a contradiction to the definition of d (since $d \neq 0$). Hence $u\alpha v - v\alpha u = 0$ implies

$u\alpha v = v\alpha u$. Therefore, U is commutative.

Then from (i) we have,

$d(u\alpha v + v\alpha u) = 2u\alpha d(v) + 2v\alpha d(u) + 2uk(\alpha)v$

That implies $2d(u\alpha v) = 2u\alpha d(v) + 2v\alpha d(u) + 2uk(\alpha)v$ and hence

$d(u\alpha v) = u\alpha d(v) + v\alpha d(u) + uk(\alpha)v$, for all $u, v \in U$ and $\alpha \in \Gamma$.

Hence we have the following Corollary,

11.4 Corollary : Let M be a 2-torsion free completely prime Γ-ring satisfying the condition (*) and d is a Jordan left k- derivation on M. Then d is a left k- derivation on M.

Bibliography

[1] Mansoor Ahmad , Lie and Jordan ideals in Prime rings with derivations, American Mathematical Society, vol. 55, Number 3, march 1976 (proceedings).

[2] ………….. , On a Theorem of Posner, American Mathematical Society, vol. 66, Number 1, September 1977 (proceedings).

[3] Argac and Yenigul, Lie ideals and symmetric bi-derivation on prime and semiprime rings, Pure and Applied Math. Sci. 44 (1-2), 17 – 21 (1996).

[4] Asma Ali, V. De Fillippis and Faiza Shujat, Results Concerning Symmetric Generalized Bi-derivations of Prime and Semiprime Rings, 66, 4(2014), 410 – 417.

[5] Mohammad Ashraf, On symmetric bi-derivations in Rings, Rend. Istit. Mat. Univ. Trieste, vol. xxxi, 25 – 36 (1999).

[6] R. Awtar, Lie ideals and Jordan derivations of prime rings, Proc. Amer Math. Soc. Vol. 90(1) (1984, 9 – 14).

[7] W. E. Barnes , On the Γ- rings of Nobusawa, Pacific J. Math ,18(1966) , 411-422.

[8] H. E. Bell and W. S. Martindale, Centralizing mapping of semiprime rings, Canad. Math. Bull. 30 (1987) , 92 – 101.

[9] Jeffrey Bergen, I. N. Herstein and Jeanne Wald Kerr, Lie ideals and Derivations of Prime Rings, Journal of Algebra 71, 256 – 267 (1981).

[10] M. Bresar, On a generalization of the notion of centralizing mappings, Proc. Amer. Math.Soc. 114 (1992), 641 – 649.

[11] . ………., Centralizing mappings and derivations in prime rings, J. Algebra 156 (1993), 385 – 394.

[12] ……..., Commuting traces of bi-additive mappings and Lie mappings, Trans.Amer. Math. Soc. 335 (1993), 525 – 546.

[13] Y. Ceven and M. A . Ozturk , on Jordan generalized derivations in Gamma rings , Itacezzepe J. Math . and staf , 33 (2004) , 11 - 14 .

[14] Y. Ceven, Jordan left derivations on completely prime gamma rings, C. U. Fen Edebiyat Fakultesi , Fen Bilimlari Dergisi (2002)Cilt 23Sayi 2.

[15] S. Chakraborty and A. C. Paul , On Jordan k- derivations of 2- torsion free prime Γ_N- rings, Punjab Univ. J. Math. vol. 40 (2008) , 97-101.

[16] …….. , On Jordan k- derivations of a 2- torsion free Prime Γ_N -rings, Punjab Univ. J. Math . vol. 40(2008), 97 - 101.

[17] ……………….., on Jordan generalized k- derivations of semiprime Γ_N - rings , Bull. Iranian Math . Soe. vol. 36, No. 1 (2010), 41 - 53.

[18] ………………. , Jordan generalized k- derivations of completely semiprime Γ_N - rings , Bull . Allahabad Math. Soc. , V . 24 , Part 1, 2009, 21 - 30.

[19] …………………… , On Jordan generalized k- derivations of 2 - torsion free prime Γ_N - rings , International Mathematical Forum , vol. 2 , No. 57 (2007) ,2823 - 2829.

[20] ……………………, Jordan k- derivations of completely prime Γ_N - rings ,Southeast Asian Bulletin of Math. , vol . 35 , (2011) , 29 - 34.

[21] …………………. , Jordan k- derivations of certain Nobusawa Γ - rings , GANIT: Journal of Bangladesh Mathematical Soc. , vol . 31 , 920110 , 53 - 64.

[22] …………, k-Derivations and k-Homomorphisms of Gamma rings, Lap Lambert Academic Publishing GmbH &

Co. KG. Heinrich – bcking – str. 6 – 8, 66121, Saarbrcken, Germany.

[23] W. Cortes and C. Haetinger, On Lie ideals and left Jordan σ-centralizers of 2- torsion free rings, Math. J. Okayama Univ. Vol. 51 (2009), 111 – 119.

[24] M. N. daif and H. E. Bell. Remarks on derivations on semiprime rings., Internat. J. Math & Math. Sci. 15(1992), 205 – 206.

[25] K..K. Dey and A. C. Paul, On Commutativity of σ-Prime Γ- Rings, Kyungpook Math. J. 00(0000), 000-000

[26] B. felzenszwalb, Derivations in prime rings,American Mathematical Society, vol. 84, no. 1, January, 1982.

[27] ………., A Commutativity Theorem for Rings with Derivations, Pacific Journal of Mathematics, vol. 102, no. 1, 1982.

[28] A. K. Halder and A. C. Paul , Jordan left derivations on Lie ideals of prime gamma rings , Punjab Uni. J. Math Vol. 44(2012), 23 – 29.

[29] ……………….., Jordan left derivations of Two Torsion Free ΓM – Modules, Journal of Physical Sciences, vol. 13, 2009, 13 – 19 .

[30] I. N. Herstein, Topics in Ring Theory, The University of Chicago Press, Chicago, 111. London , 1969.

[31] …………, A note on derivations, Canada Math. Bull. Vol. 21(3), 1978(369 – 370).

[32] …………, A note on derivations II, Canada Math. Bull. Vol. 22(4), 1979.

[33] M. Hongan. N. U. Rehman and R. M. Al-Omary , Lie ideals and Jordan triple derivations in rings, Rend. Sem. Mat. Uni. Padova, Vol. 120 (2011), 147 – 156.

[34] Motoshi Hongan and Andrzej Trzepizur, A Note on Semiprime Rings with Derivation, Inteernat. J. Math. & Math. Sci. vol. 20 no. 2 (1997), 413 – 415.

[35] M. F. Hoque and A. C. Paul, On Centralizers of Semiprime Gamma Rings , International Mathematical Forum. Vol. 6 (2011), No. 13, 627 - 638.

[36] ………………………, Centralizers on Semiprime Gamma Rings, Italian J. of Pure and Applied Mathematics . Vol. 30 (2013), 289 – 302.

[37] ………………………., An Equation Related to Centralizers in Semiprime Gamma Rings, Annals of Pure and Applied Mathematics . Vol. 1 (2012), 84 - 90.

[38] ………………………, An Equation Related to θ-Centralizers in Semiprime Gamma Rings, International J. Math Combin.Vol.4, (2013), 17 -26.

[39] ………………………, The θ-Centralizers of Semiprime Gamma Rings, Research Journal of Applied Sciences, Engineering and Technology 6 (22) , (2013), 4129 - 4137.

[40] H. Kandamar , The k- derivations of a gamma ring, Turkish J. Math, 24 (2000) , 221-231.

[41] S. Kyuno, On prime gamma ring. Pacific J. Math, 75 (1978) ,185 - 190. [33]

[42] P. H. Lee and T. k. Lee, Lie ideals of prime rings with derivations, Bulletin of the Institute of Mathematics Academia Sinica, vol. 11, no. 1, march 1983.

[43] L. Luh, On the theory of simple Gamma rings , Michigan Math. J. 16 (1969) , 65 – 75.

[44] G. Maksa, A remark on symmetric bi-additive functions having no-negative diagonalization, Glasnik Mat. 15(1980), 279 – 280.

[45] ………., On the trace of symmetric bi-derivation, C. R. Math. Rep. Acad. Sci. Canada 9(1987), 303 – 307.

[46] J. Mayne, Centralizing mappings of prime rings, Canada Math. Bull. 27 (1984), 122 – 126.

[47] N. M. Muthana, Left centralizer traces, generalized bi-derivations left bi-multiplies and generalized Jordan bi-derivations, The Aligarh Bull. of Maths. 26 (2), 33 - 45, (2007).

[48] N. Nobusawa, On the generalization of ring theory, Osaka J. Math. 1(1964) , 81-89.

[49] Mehmet Ali Ozturk, Mehmet Sapanci, Muharrem Soyturk And Kyuno Ho Kim, " Symmetric bi-derivation on Prime Gamma rings " Scientiae Mathematicae Vol. 3, No. 2 (2000) , 273 – 281.

[50] A. C. Paul and Md . Sabur Uddin , Simple Gamma rings with involutions , IOSR Journal of Mathematics vol. 4, Issue 3 (Nov - Dec 2012) PP 40 - 48.

[51] …………………, Lie and Jordan structure in simple gamma rings, J. Physical Sciences, 14(2010), 77 – 86.

[52] Edward C. Posner, Derivations in Prime rings, Proc. Amer. Math. Soc. 8(1957), 1093 – 1100.

[53] M. M. Rahman and A. C. Paul, "Jodan generalized derivations on Lie ideals of prime Γ- rings". South Asian Journal of Mathematics, 2013, vol 3(3): 148 - 153.

[54] …………………, Jordan Derivations on Lie ideals of Prime Γ-rings. Mathematical Theory and Modeling (2013), vol. 3(3); 128 -135.

[55] I. S. Rakhimov , K. K. Dey and A. C. Paul, On Commutativity of completely prime Gamma Rings, Malaysia Journal of Mathematical Sciences 7(2), 283 – 295 (2013).

[56] Nadeem Ur Rehman and Abu Zaid Ansari,On Lie ideals with symmetric bi-additive maps in rings, Palestine Journal of Mathematics , Vol. 2(1) (2013), 14 – 21.

[57] N. U. Rehman and M. Hongan. Generalized Jordan derivations on Lie ideals associate with Hoehschild 2-coeyeles of rings , Rend. Cire, Mat. Palermo, Vol. 60(2011), 437 – 444.

[58] M. Sapanci and A. Nakajima , Jordan derivations on completely prime Γ- rings ,Math. Japanica 46(1997), 47-51.

[59] ………………., k-Derivations and k- homomorphisms of Gamma rings , Lambert Academic Publishing CmbH Co, KG Heinrich - Beking - Str . 6 - 8 , 66121, saarbrchen, Germany, 2012.

[60] J. Vukman, Commuting and centralizing mappings in prime rings, Proc. Amer. Math. Soc. 109 (1990), 47 – 52.

[61] ……………, Symmetric bi-derivations in prime and semiprime rings, Aequationes Math. 38(1989), 245 – 254.

[62] …………, Two results concerning Symmetric bi-derivations on prime rings, Aequationes Math. 40 (1990), 181 – 189.

[63] ………, Derivations in semiprime rings, Bull. Austral. Math. Soc. 53(1995), 353 -359.

[64]………., Centralizers on semiprime rings, Comment Math Univ. Carolinae 42, 2(2001), 237 – 245.

[65] B. Zalar , On centralizers of semiprime rings, Comment Math Univ. Carolinae 32, (1991), 609 – 614.