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On Fuzzy Compactness

Talukder, Md. Abdul Mottalib

University of Rajshahi

Ph. D.
Thesis

On Fuzzy Compactness

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A THESIS

SUBMITTED TO THE

UNIVERSITY OF RAJSHAHI

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

BY

MD. ABDUL MOTTALIB TALUKDER

In the

Department of Mathematics

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DEDICATED

TO MY

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CERTIFICATE

This is certified that the thesis entitled "**On Fuzzy Compactness**" submitted by **Md. Abdul Mottalib Talukder** in fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics, Faculty of Science, University of Rajshahi, Rajshahi-6205, Bangladesh has been completed under my supervision. I believe that this research work is an original one and it has not been submitted elsewhere for any degree.

I wish him every success and consequentially a bright future in life.

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STATEMENT OF ORIGINALITY

I declare that the contents in my Ph. D. thesis entitled "**On Fuzzy Compactness**" is original and accurate to the best of my knowledge. I also declare that the materials contained in my research work have not been previously published or written by any person for any degree or diploma.

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CONTENTS

ABSTRACT

The fundamental concept of a fuzzy set and fuzzy set operations was first introduced by L. A. Zadeh [175] in 1965 and it provides a natural foundation for treating mathematically the fuzzy phenomena, which exists pervasively in our real world and for building new branches of fuzzy mathematics. This also provides a natural frame work for generalizing various branches of mathematics such as fuzzy topology, fuzzy groups, fuzzy rings, fuzzy vector spaces, fuzzy supra topology, fuzzy infra topology, fuzzy bitopology etc. C. L. Chang [19] in 1968 first introduced the concept of fuzzy topological spaces by using fuzzy sets. C. K. Wong [160, 161, 162, 162], R. Lowen [107, 108, 109, 110,111], B. Hutton [70, 71, 72], T. E. Gantner et al. [54], P. P. Ming and L. Y. Ming [121, 122], etc., discussed various aspects of fuzzy topology by using fuzzy sets. A. J. Klein [91] defines α -level sets and α -level topology. Fuzzy compactness occupies a very important place in fuzzy topological spaces and so does some of its forms. Fuzzy compactness first discussed by C. L. Chang [19], T. E. Gantner et al. [54] introduced α -compactness, A. D. Concilio and G. Gerla [27] discussed almost compact spaces and M. N. Mukherjee and A. Bhattacharyya [130] discussed almost α -compact spaces.

The purpose of this thesis is to contribute about different types of fuzzy compactness and establish theorems, corollaries and examples in fuzzy topological spaces by using the definitions of C. L. Chang [19], T. E. Gantner et al. [54], A. D. Concilio and G. Gerla [27] and M. N. Mukherjee and A. Bhattacharyya [130]. We study several properties of these definitions along with the different theorems from existing there. Moreover to suggest new definitions of fuzzy δ -compact spaces, δ -compact fuzzy sets, δ - α -compact spaces, partially α -compact and partially δ - α -compact fuzzy sets,

Q-compact and δ - Q-compact fuzzy sets, $Q\alpha$ -compact and δ - $Q\alpha$ -compact fuzzy sets, almost partially α -compact and almost partially δ - α -compact fuzzy sets, almost $Q\alpha$ -compact and almost δ - $Q\alpha$ -compact fuzzy sets and also to study their several properties in fuzzy topological spaces have been done in the work.

Chapter one incorporates some fundamental definitions and results of fuzzy sets, fuzzy set operations, fuzzy mapping, fuzzy topology, fuzzy separation axioms, good extension property and fuzzy productivity. These results are ready bibliographies for the study in the next chapters. Results are stated without proof and can be found in the thesis referred to.

Our works start from chapter two. Chapter two deals with fuzzy compact spaces due to C. L Chang [19] which is global property. In this chapter, we have discussed some theorems, corollaries and examples in fuzzy topological spaces, fuzzy subspaces, mappings in fuzzy topological spaces, fuzzy T_1-spaces, fuzzy Hausdorff spaces, fuzzy regular spaces and good extension property about fuzzy compact spaces. Also we have defined δ -open fuzzy sets, δ -cover, fuzzy δ -compact spaces and investigated difference between fuzzy compact and fuzzy δ -compact spaces.

We aim to study α -compact spaces in the sense of T. E. Gantner et al. [54] in chapter three which is global property and we have introduced α -level continuous mapping. In this chapter, we have established some theorems, corollaries and examples in fuzzy topological spaces, fuzzy subspaces, mappings in fuzzy topological spaces, fuzzy T_1-spaces, fuzzy Hausdorff spaces, fuzzy regular spaces, α -level topological spaces, cofinite topological spaces, good extension property and fuzzy product spaces and give some examples about α -compact spaces. Also we have constructed δ - α -shading,

δ-α-compact spaces and identified difference between α-compact and δ-α-compact spaces.

We have discussed compact fuzzy sets due to C. L. Chang [19] in chapter four which is local property. In this chapter, we have investigated some theorems, corollaries and examples of compact fuzzy sets in fuzzy topological spaces, fuzzy subspaces, fuzzy mappings, fuzzy T_1-spaces, fuzzy Hausdorff spaces, fuzzy regular spaces, good extension property and fuzzy productivity about compact fuzzy sets. Also we have introduced δ-compact fuzzy sets and found difference between compact and δ-compact fuzzy sets.

In chapter five, we have defined partial α-shading, partial α-subshading, open partial α-shading, partially α-compact fuzzy sets. We have discussed some theorems, corollaries and examples of partially α-compact fuzzy sets in fuzzy topological spaces, fuzzy subspaces, fuzzy mappings, α-level continuous mapping, fuzzy T_1-spaces, fuzzy Hausdorff spaces, fuzzy regular spaces, α-level topological spaces, good extension property and fuzzy productivity about partially α-compact fuzzy sets. Also we have introduced partial δ-α-shading, partial δ-α-subshading and partially δ-α-compact fuzzy sets and indicated the difference between partially α-compact and partially δ-α-compact fuzzy sets.

In chapter six, we have constructed Q-cover, Q-subcover, open Q-cover, Q-compact fuzzy sets, $Q\alpha$-cover, $Q\alpha$-subcover, open $Q\alpha$-cover, $Q\alpha$-compact fuzzy sets, δ-Q-cover, δ-Q-subcover, δ-Q-compact fuzzy sets, δ-$Q\alpha$-compact fuzzy sets. We have also studied some theorems, corollaries and examples in fuzzy topological spaces, fuzzy subspaces, fuzzy T_1-spaces, fuzzy Hausdorff spaces, fuzzy regular spaces,

α-level topological spaces, good extension property and fuzzy productivity about Q-compact, $Q\alpha$-compact, δ-Q-compact, δ-$Q\alpha$-compact fuzzy sets. Furthermore, we have found difference between Q-compact and $Q\alpha$-compact fuzzy sets, Q-compact and δ-Q-compact fuzzy sets, $Q\alpha$-compact and δ-$Q\alpha$-compact fuzzy sets. Moreover, we have compared compact fuzzy stes (Chang's sense [19]) with Q-compact and $Q\alpha$-compact fuzzy sets, δ-compact fuzzy stes (Chang's sense [19]) with δ-Q-compact and δ-$Q\alpha$-compact fuzzy sets.

In chapter seven, we have studied almost compact fuzzy sets due to A. D. Concilio and G. Gerla [27] which is local property. We have established some theorems, corollary and give some examples in fuzzy topological spaces, fuzzy subspaces, fuzzy mappings, fuzzy T_1-spaces, fuzzy regular spaces, good extension property and fuzzy productivity about almost compact fuzzy sets. Also we have introduced proximate δ-cover, proximate δ-subcover, almost δ-compact fuzzy sets and found different characterizations between almost compact and almost δ-compact fuzzy sets.

We have dealth with almost α-compact spaces due to M. N. Mukherjee and A. Bhattacharyya [130] in chapter eight which is global property. In this chapter, we have established some theorems, corollary and give some examples in fuzzy topological spaces, fuzzy subspaces, fuzzy mappings, fuzzy T_1-spaces, fuzzy regular spaces, α-level topological spaces, α-level continuous mapping and good extension property about almost α-compact spaces. Also we have introduced proximate δ-α-shading, proximate δ-α-subshading, almost δ-α-compact spaces and found difference between almost α-compact and almost δ-α-compact spaces.

In chapter nine, we have introduced proximate partial α-shading, proximate partial α-subshading, almost partially α-compact fuzzy sets. We have also established some theorems, corollary and give some examples in fuzzy topological spaces, fuzzy subspaces, fuzzy mappings, fuzzy T_1-spaces, fuzzy regular spaces, α-level topological spaces, α-level continuous mapping, good extension property and fuzzy productivity about almost partially α-compact fuzzy sets. In addition to that, we have defined proximate partial δ-α-shading, proximate partial δ-α-subshading, almost partially δ-α-compact fuzzy sets and investigated different characterizations between almost partially α-compact and almost partially δ-α-compact fuzzy sets.

In chapter ten, we have defined proximate $Q\alpha$-cover, proximate $Q\alpha$-subcover, almost $Q\alpha$-compact fuzzy sets. We have also studied some theorems, corollary and give some examples in fuzzy topological spaces, fuzzy subspaces, fuzzy T_1-spaces, fuzzy regular spaces, α-level topological spaces, good extension property and fuzzy productivity about almost $Q\alpha$-compact fuzzy sets. Moreover, we have introduced proximate δ-$Q\alpha$-cover, proximate δ-$Q\alpha$-subcover, almost δ-$Q\alpha$-compact fuzzy sets and found different characterizations between almost $Q\alpha$-compact and almost δ-$Q\alpha$-compact fuzzy sets.

Chapter One

Preliminaries

Introduction 1.1: In this chapter incorporates concepts and results of fuzzy sets, fuzzy mappings, fuzzy topological spaces, subspace of a fuzzy topological space, fuzzy product topological space and its characterizations which are to be used as references for understanding the next chapters. Most of the results are quoted from the various research articles. Through the sequel, we make use of the following notations.

X	: Non-empty set
J	: Index set
J_n	: Finite subset of J
R	: Set of real numbers
$+$	: Sum
$\cup$	: Union
$\cap$	: Intersection
$\subset$	: Strictly subset or proper subset
$\subseteq$	: Subset
$\in$	: Belongs to
$\notin$	: Not belongs to
$\Rightarrow$	: Implies that
$I = [0, 1]$	: Closed unit interval
$I_1 = [0, 1)$	: Right open unit interval
$I_0 = (0, 1]$	: Left open unit interval
$A, B, C, \ldots$	: Ordinary sets or Classical sets

$u\ ,\ v\ ,\ \lambda\ ,\ \mu\ ,\ldots$	: Fuzzy sets
(X, T)	: General topological space
(X, t)	: Fuzzy topological space
(A, t_A)	: Subspace of (X, t)
$\prod_{i \in J} X_i$	: Usual product of X_i
$(X \times X, t \times t)$	: Product fuzzy topological space
$\alpha(u) = \{\ x \in X\ :\ u(x) > \alpha\ \}$	: Subset of X
$t_\alpha = \{\ \alpha(u) :\ u \in t\ \}$	: General topology on X
$\omega(T) = \{\ u \in I^X :\ u^{-1}(a, 1] \in T\ ,\ a \in I_1\ \}$	: Fuzzy topology on X

This thesis deals with various fuzzy compactness in fuzzy topological spaces. To present our work in a systematic way, we consider in this chapter, various concepts and results on fuzzy sets and fuzzy topological spaces found in various research papers. For this we begin with.

Definition 1.2[175]: Let X be a non-empty set and I is the closed unit interval $[0, 1]$. A fuzzy set in X is a function $u :\ X \to I$ which assigns to every element $x \in X$. $u(x)$ denotes a degree or the grade of membership of x. The set of all fuzzy sets in X is denoted by I^X. A member of I^X may also be called a fuzzy subset of X.

Definition 1.3[121]: A fuzzy set is empty iff its grade of membership is identically zero. It is denoted by 0_X.

Definition 1.4[121]: A fuzzy set is whole iff its grade of membership is identically one in X. It is denoted by 1_X.

Definition 1.5[175]: Let X be a non-empty set and $A \subseteq X$. Then the characteristic function $1_A(x): X \to \{0, 1\}$ defined by $1_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$

Hence we say that A is fuzzy set in X and this fuzzy set is denoted by 1_A. Thus we can consider any subset of a set X as a fuzzy set whose range is $\{0, 1\}$.

Definition 1.6[19]: Let u and v be two fuzzy sets in X. Then we define

(i) $u = v$ iff $u(x) = v(x)$ for all $x \in X$

(ii) $u \subseteq v$ iff $u(x) \leq v(x)$ for all $x \in X$

(iii) $\lambda = u \cup v$ iff $\lambda(x) = (u \cup v)(x) = \max[u(x), v(x)]$ for all $x \in X$

(iv) $\mu = u \cap v$ iff $\mu(x) = (u \cap v)(x) = \min[u(x), v(x)]$ for all $x \in X$

(v) $\gamma = u^c$ iff $\gamma(x) = 1 - u(x)$ for all $x \in X$ and we say that u^c is complement of u.

Remark: Two fuzzy sets u and v are disjoint iff $u \cap v = 0$.

Definition 1.7[19]: In general, if $\{ u_i : i \in J \}$ is family of fuzzy sets in X, then union $\bigcup u_i$ and intersection $\bigcap u_i$ are defined by

$\bigcup u_i(x) = \sup\{ u_i(x) : i \in J \text{ and } x \in X \}$

$\bigcap u_i(x) = \inf\{ u_i(x) : i \in J \text{ and } x \in X \}$, where J is an index set.

De-Morgan's laws 1.8[175]: De-Morgan's Laws valid for fuzzy sets in X i.e. if u and v are any fuzzy sets in X, then

(i) $1 - (u \cup v) = (1 - u) \cap (1 - v)$

(ii) $1 - (u \cap v) = (1 - u) \cup (1 - v)$

For any fuzzy set in u in X , $u \cap (1-u)$ need not be zero and $u \cup (1-u)$ need not be one.

Distributive laws 1.9[175]: Distributive laws remain valid for fuzzy sets in X i.e. if u , v and w are fuzzy sets in X , then

(i) $u \cup (v \cap w) = (u \cup v) \cap (u \cup w)$

(ii) $u \cap (v \cup w) = (u \cap v) \cup (u \cap w)$.

Definition 1.10[121]: Let λ be a fuzzy set in X , then the set $\{ x \in X : \lambda(x) > 0 \}$ is called the support of λ and is denoted by λ_0 or supp λ .

Definition 1.11[121]: A fuzzy set in X is called a fuzzy point iff it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is r ($0 < r < 1$), we denote this fuzzy point by x_r , where the point x is called its support.

Definition 1.12[121]: A fuzzy set λ in X is called quasi-coincident (in short q-coincident) with a fuzzy set μ in X , denoted by $\lambda q \mu$ iff $\lambda(x) + \mu(x) > 1$ for some $x \in X$.

Definition 1.13[19]: Let $f : X \to Y$ be a mapping and u be a fuzzy set in X . Then the image of u , written $f(u)$, is a fuzzy set in Y whose membership function is given by

$$f(u)(y) = \begin{cases} \sup\{ u(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases} .$$

Definition 1.14[19]: Let $f: X \to Y$ be a mapping and v be a fuzzy set in Y. Then the inverse of v, written $f^{-1}(v)$, is a fuzzy set in X whose membership function is given by $f^{-1}(v)(x) = v(f(x))$.

Definition 1.15[131]: Let $f: X \to Y$ be a mapping. Then f is said to be one-one (one-to-one) iff $f(a) = f(b) \Rightarrow a = b$.

Definition 1.16[131]: Let $f: X \to Y$ be a mapping. Then f is said to be onto (surjective) iff $f(X) = Y$.

Definition 1.17[131]: Let $f: X \to Y$ be a mapping. Then f is said to be bijective iff it is both one-one and onto.

Theorem 1.18[168]: Let $f: X \to Y$ be a mapping and u_1, u_2 be fuzzy sets in X. If $u_1 \subseteq u_2$, then $f(u_1) \subseteq f(u_2)$.

Theorem 1.19[168]: Let $f: X \to Y$ be a mapping and v_1, v_2 be fuzzy sets in Y. If $v_1 \subseteq v_2$, then $f^{-1}(v_1) \subseteq f^{-1}(v_2)$.

Theorem 1.20[168]: Let $f: X \to Y$ be one-to-one mapping and u be a fuzzy set in X, then $f^{-1}(f(u)) = u$.

Theorem 1.21[168]: Let $f: X \to Y$ be onto mapping and v be a fuzzy set in Y, then $f(f^{-1}(v)) = v$.

Theorems (1.20) and (1.21) will be used again and again in our next works.

Theorem 1.22[159]: Let $f : X \to Y$ be a mapping, u_i , $i \in J$ be fuzzy sets in X and v_i , $i \in J$ be fuzzy sets in Y . Then

(i) $f\left(\bigcap_{i\in J} u_i\right) \subseteq \bigcap_{i\in J} f(u_i)$

(ii) $f^{-1}\left(\bigcap_{i\in J} v_i\right) = \bigcap_{i\in J} f^{-1}(v_i)$

(iii) $f\left(\bigcup_{i\in J} u_i\right) = \bigcup_{i\in J} f(u_i)$

(iv) $f^{-1}\left(\bigcup_{i\in J} v_i\right) = \bigcup_{i\in J} f^{-1}(v_i)$.

Definition 1.23[106]: Let X be a non-empty set and T be a family of subsets of X . Then T is said to be topology on X if

(i) ϕ , $X \in T$

(ii) if $A_i \in T$ for each $i \in J$, then $\bigcup_{i\in J} A_i \in T$

(iii) if A , $B \in T \Rightarrow A \cap B \in T$

The pair (X, T) is called topological space, any member $U \in T$ is called open set in the topology T and its complement i.e. U^c is called closed set in the topology T .

Definition 1.24[106]: Let U denote the class of all open sets of real numbers **R**. Then U is a topology on **R**; it is called the usual topology on **R**.

Definition 1.25[106]: Let X be a non-empty set and T denote the class of all subsets of X whose complements are finite together with the empty set ϕ. This class T is also a topology on X. It is called the cofinite topology on X.

Definition 1.26[106]: A subset A of a topological space (X, T) is compact iff every open cover of A has a finite subcover.

Definition 1.27[19]: Let X be a non-empty set and $t \subseteq I^X$ i.e. t is a collection of fuzzy sets in X. Then t is called a fuzzy topology on X if

(i) $0, 1 \in t$

(ii) if $u_i \in t$ for each $i \in J$, then $\bigcup_{i \in J} u_i \in t$

(iii) if u, $v \in t$, then $u \cap v \in t$

The pair (X, t) is called a fuzzy topological space and in short, fts. Every member of t is called a t-open fuzzy set. A fuzzy set is t-closed iff its complements is t-open. In the sequel, when no confusion is likely to arise, we shall call a t-open (t-closed) fuzzy set simply an open (closed) fuzzy set.

Definition 1.28[19]: A fuzzy topology t_1 is said to be coarser than a fuzzy topology t_2 if and only if $t_1 \subset t_2$.

Definition 1.29[121]: Let λ be a fuzzy set in an fts (X, t). Then the interior of λ is denoted by λ^0 or $\text{int}\,\lambda$ and defined by $\lambda^0 = \bigcup\{\mu : \mu \subseteq \lambda \text{ and } \mu \in t\}$.

Remark [1]: The interior of a fuzzy set λ is the largest open fuzzy set contained in λ and trivially, a fuzzy set λ is fuzzy open if and only if $\lambda = \lambda^0$.

Definition 1.30[121]: Let λ be a fuzzy set in an fts (X, t). Then the closure of λ is denoted by $\bar{\lambda}$ or cl λ and defined by $\bar{\lambda} = \bigcap\{ \mu : \lambda \subseteq \mu \text{ and } \mu \in t^c \}$.

Remark [1]: The closure of a fuzzy set λ is the smallest closed fuzzy set containing λ and trivially, a fuzzy set λ is a fuzzy closed if and only if $\lambda = \bar{\lambda}$.

Theorem 1.31[1]: Let (X, t) be a fuzzy topological space and u, v be two fuzzy sets in X. Then

(i) $\bar{0} = 0$, $\bar{1} = 1$

(ii) $(u^0)^0 = u^0$, $(\bar{\bar{u}}) = \bar{u}$

(iii) $u^0 \subseteq u \subseteq \bar{u}$

(iv) $\overline{u \cap v} \subseteq \bar{u} \cap \bar{v}$

(v) If $u \subseteq v$, then $u^0 \subseteq v^0$

(vi) If $u \subseteq v$, then $\bar{u} \subseteq \bar{v}$.

Theorem 1.32[27]: Let (X, t) be an fts and u be an open fuzzy set in t. Then $u \subseteq (\bar{u})^0$.

Definition 1.33[121]: Let (X, t) be an fts and $A \subseteq X$. Then the collection $t_A = \{ u \,|\, A : u \in t \}$ is fuzzy topology on A, called the subspace fuzzy topology on A and the pair (A, t_A) is referred to as a fuzzy subspace of (X, t).

Definition 1.34[19]: Let (X, t) and (Y, s) be two fuzzy topological spaces. A mapping $f : (X, t) \to (Y, s)$ is called a fuzzy continuous iff the inverse of each s-open fuzzy set is t-open or equivalently for each s-closed fuzzy set is t-closed.

Definition 1.35[161]: Let (X, t) and (Y, s) be two fuzzy topological spaces. Let $f : (X, t) \to (Y, s)$ be a mapping from an fts (X, t) to another fts (Y, s). Then f is called

(i) a fuzzy open mapping iff $f(u) \in s$ for each $u \in t$.

(ii) a fuzzy closed mapping iff $f(v)$ is a closed fuzzy set of Y, for each closed fuzzy set v of X.

Definition 1.36[116]: Let f be a mapping from an fts (X, t) into an fts (Y, s). Then f is fuzzy closed iff $\overline{f(u)} \subseteq f(\overline{u})$ for each fuzzy set u in X.

Theorem 1.37[122]: Let $f : (X, t) \to (Y, s)$ be a fuzzy continuous mapping. Then

(i) $f(\overline{u}) \subseteq \overline{f(u)}$, for any fuzzy set u in X.

(ii) $\overline{f^{-1}(v)} \subseteq f^{-1}(\overline{v})$, for any fuzzy set v in Y.

Definition 1.38[49]: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively and f is a mapping from (X, t) to (Y, s), then we say that f is a mapping from (A, t_A) to (B, s_B) if $f(A) \subseteq B$.

Definition 1.39[49]: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively. Then a mapping $f : (A, t_A) \to (B, s_B)$ is relatively fuzzy continuous iff for each $v \in s_B$, then $f^{-1}(v) \mid A \in t_A$.

Definition 1.40[49]: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively. Then a mapping $f : (A, t_A) \to (B, s_B)$ is relatively fuzzy open iff for each $v \in t_A$, the image $f(v) \in s_B$.

Definition 1.41[3]: Let (X, T) be a topological space. A function $f : X \to \mathbf{R}$ (with usual topology) is called lower semi-continuous (l. s. c.) if for each $a \in \mathbf{R}$, the set $f^{-1}(a, \infty) \in T$. For a topology T on a set X, let $\omega(T)$ be the set of all l. s. c. functions from (X, T) to I (with usual topology); thus $\omega(T) = \{ u \in I^X : u^{-1}(a, 1] \in T ,\ a \in I_1 \}$. It can be shown that $\omega(T)$ is a fuzzy topology on X.

Let P be a property of topological spaces and FP be its fuzzy topology analogue. Then FP is called a 'good extension' of P "iff the statement (X, T) has P iff $(X, \omega(T))$ has FP" holds good for every topological space (X, T). Thus characteristic functions are l . s. c.

Definition 1.42[106]: Let $\{ X_i : i \in J \}$ be any family of sets and let X denote the Cartesian product of these sets i.e. $X = \prod_{i \in J} X_i$. Note that X contains all points $p = < a_i : i \in J >$ where $a_i \in X_i$. Recall that, for each $j_0 \in J$, we define the projection π_{j_0} from the product set X to the coordinate space X_{j_0} i.e. $\pi_{j_0} : X \to X_{j_0}$ by $\pi_{j_0}(< a_i : i \in J >) = a_{j_0}$. These projections are used to define the product topology.

Definition 1.43[9]: Let $\lambda \in I^X$ and $\mu \in I^Y$. Then $(\lambda \times \mu)$ is a fuzzy set in $X \times Y$ for which $(\lambda \times \mu)\ (x, y) = \min\{\ \lambda(x)\ ,\ \mu(y)\ \}$, for every $(x, y) \in X \times Y$.

Definition 1.44[161]: Given a family $\{(X_i, t_i) : i \in J\}$ of fts's, we define their product $\prod_{i \in J}(X_i, t_i)$ to be the fts (X, t), where $X = \prod_{i \in J} X_i$ is the usual product set and t is the coarsest fuzzy topology on X for which the projections $\pi_i :\ X \to X_i$ are fuzzy continuous for each $i \in J$. The fuzzy topology t is called the product fuzzy topology on X and (X, t) is a product fts.

Definition 1.45[150]: An fts (X, t) is said to be fuzzy T_1-space iff for every x , $y \in X$, $x \neq y$, there exist u , $v \in t$ such that $u(x) = 1$, $u(y) = 0$ and $v(x) = 0$, $v(y) = 1$.

Definition 1.46[85]: An fts (X, t) is said to be fuzzy T_1-space iff for all x , $y \in X$, $x \neq y$, there exist u , $v \in t$ such that $u(x) > 0$, $u(y) = 0$ and $v(x) = 0$, $v(y) > 0$.

Definition 1.47[54]: An fts (X, t) is said to be fuzzy Hausdorff space iff for all x , $y \in X$, $x \neq y$, there exist u , $v \in t$ such that $u(x) = 1$, $v(y) = 1$ and $u \cap v = 0$.

Definition 1.48[85]: An fts (X, t) is said to be fuzzy Hausdorff iff for all x , $y \in X$, $x \neq y$, there exist u , $v \in t$ such that $u(x) > 0$, $v(y) > 0$ and $u \cap v = 0$.

Definition 1.49[93]: An fts (X, t) is said to be fuzzy Hausdorff iff for every pair of distinct fuzzy points x_r, y_s in X, there exist u, $v \in t$ such that $x_r \in u$, $y_s \in v$ and $u \cap v = 0$.

Definition 1.50[116]: An fts (X, t) is said to be fuzzy Hausdorff iff for all x, $y \in X$, $x \neq y$, there exist u, $v \in t$ such that $u(x) = 1$, $v(y) = 1$ and $u \subseteq 1 - v$.

Definition 1.51[116]: An fts (X, t) is said to be fuzzy regular iff for each $x \in X$ and $u \in t^c$ with $u(x) = 0$, there exist v, $w \in t$ such that $v(x) = 1$, $u \subseteq w$ and $v \subseteq 1 - w$.

Definition 1.52[27]: An fts (X, t) is said to be fuzzy regular iff each open fuzzy set u of X is a union of open fuzzy sets u_i of X such that $\overline{u_i} \subseteq u$ for each i.

Chapter Two

Fuzzy Compact Spaces

Fuzzy compact spaces was first introduced by Chang [19] in fuzzy topological spaces and mentioned some properties which are global property. In this chapter, we have discussed various other properties of this concept and established some theorems, corollaries and examples. Also we have defined fuzzy δ-compact spaces and found different characterizations between fuzzy compact and fuzzy δ-compact spaces.

Definition 2.1[19]: Let (X, t) be an fts and λ be a fuzzy set in X. Let $M = \{ u_i : i \in J \}$ be a family of fuzzy sets. Then $M = \{u_i\}$ is called a cover of λ iff $\lambda \subseteq \bigcup \{ u_i : i \in J \}$. If each u_i is open, then $M = \{u_i\}$ is called an open cover of λ. Furthermore, if a finite subfamily of M is also cover λ i.e. there exist u_{i_1}, u_{i_2}, ……, $u_{i_n} \in M$ such that $\lambda \subseteq u_{i_1} \cup u_{i_2} \cup \ldots\ldots \cup u_{i_n}$, then M is said to be reducible to a finite cover or contains a finite subcover or has a finite subcover.

Definition 2.2[19]: An fts (X, t) is compact iff each open cover has a finite subcover.

Theorem 2.3: Let (X, t) be a compact fts, $A \subset X$ with 1_A is closed. Then 1_A is also compact.

Proof: Let $M = \{ u_i : \ i \in J \}$ be an open cover of 1_A i.e. $1_A \subseteq \bigcup_{i \in J} u_i$. Then $1_X = \left(\bigcup_{i \in J} u_i \right) \cup 1_{A^c}$ that is $M^* = \{u_i\} \cup \ \{1_{A^c}\}$ is an open cover of 1_X. But 1_{A^c} is open, since 1_A is closed. So M^* is an open cover of 1_X. As (X, t) is compact; hence M^* has a finite subcover i.e. there exist $u_{i_k} \in M \ \ (k = 1 , \ 2 \ , \ ...... \ , n \ )$ such that $1_X = u_{i_1} \cup u_{i_2} \cup ...... \cup u_{i_n} \cup 1_{A^c}$. But 1_A and 1_{A^c} are disjoint; hence $1_A \subseteq \ u_{i_1} \cup u_{i_2} \cup \ ...... \ \cup u_{i_n}$; $u_{i_k} \in M \ \ (k = 1 , \ 2 \ , \ ...... \ , n \ )$. We have just shown that any open cover $M = \{u_i\}$ of 1_A contains a finite subcover i.e. 1_A is compact.

Definition 2.4[10]: A family M of fuzzy sets has the finite intersection property iff the intersection of the members of each finite subfamily of M is non-empty.

Theorem 2.5: An fts (X, t) is compact iff each family of closed fuzzy sets which has the finite intersection property has a non-empty intersection.

Proof: cf.[19].

Theorem 2.6: For an fts (X, t), the following statements are equivalent :

(i) (X, t) is compact.

(ii) For each $\{ 1_{A_i} : \ i \in J \ \}$ of closed subsets of (X, t) ; $\bigcap_{i \in J} 1_{A_i} = 0_X$ implies $\{ 1_{A_i} : \ i \in J \ \}$ contains a finite subfamily $\{ 1_{A_{i_1}} , \ 1_{A_{i_2}} , \ .... \ , \ 1_{A_{i_n}} \ \}$ with $1_{A_{i_1}} \cap 1_{A_{i_2}} \cap ...... \cap 1_{A_{i_n}} = 0_X$.

Proof: (i) $\Rightarrow$ (ii) : Suppose $\bigcap_{i \in J} 1_{A_i} = 0_X$. Then by De-Morgan's law, $1_X = (0_X)^c = \left(\bigcap_{i \in J} 1_{A_i} \right)^c = \bigcup_{i \in J} 1_{A_i^c}$. So $\{1_{A_i^c}\}$ is an open cover of (X, t), since each 1_{A_i} is closed. As (X, t) is compact, then there exist $1_{A_{i_1}^c}, 1_{A_{i_2}^c}, \ldots\ldots, 1_{A_{i_n}^c} \in \{1_{A_i^c}\}$ such that $1_X = 1_{A_{i_1}^c} \cup 1_{A_{i_2}^c} \cup \ldots\ldots \cup 1_{A_{i_n}^c}$. Thus by De-Morgan's law, $0_X = (1_X)^c = (1_{A_{i_1}^c} \cup 1_{A_{i_2}^c} \cup \ldots\ldots \cup 1_{A_{i_n}^c})^c = 1_{A_{i_1}} \cap 1_{A_{i_2}} \cap \ldots\ldots \cap 1_{A_{i_n}}$ and we have shown that (i) $\Rightarrow$(ii).

(ii) $\Rightarrow$ (i) : Let $\{ u_i : i \in J \}$ be an open cover of (X, t) i.e. $1_X = \bigcup_{i \in J} u_i$. By De-Morgan's law, we have $0_X = (1_X)^c = \left(\bigcup_{i \in J} u_i \right)^c = \bigcap_{i \in J} u_i^c$. Since each u_i is open, then $\{ u_i^c : i \in J \}$ is a family of closed fuzzy sets and so by above has an empty intersection. Hence by hypothesis, there exist $u_{i_1}^c, u_{i_2}^c, \ldots, u_{i_n}^c \in \{u_i^c\}$ such that $u_{i_1}^c \cap u_{i_2}^c \cap \ldots\ldots \cap u_{i_n}^c = 0_X$. Thus by De-Morgan's law, we get $1_X = (0_X)^c = (u_{i_1}^c \cap u_{i_2}^c \cap \ldots\ldots \cap u_{i_n}^c)^c = u_{i_1} \cup u_{i_2} \cup \ldots\ldots \cup u_{i_n}$. Accordingly, (X, t) is compact and so (ii) $\Rightarrow$ (i).

Theorem 2.7: Let (X, t) be an fts and $A \subset X$. Then 1_A is compact in (X, t) iff 1_A is compact in (A, t_A).

Proof: Suppose 1_A is compact in (X, t). Let $\{ u_i : i \in J \}$ be an open cover of 1_A in (A, t_A). Then there exist $v_i \in t$ such that $u_i = v_i \mid A \subseteq v_i$. Hence $1_A \subseteq \bigcup_{i \in J} u_i \subseteq \bigcup_{i \in J} v_i$ and therefore $\{ v_i : i \in J \}$ is an open cover of 1_A. Since 1_A is compact, so $\{ v_i : i \in J \}$ contains a finite subcover, say $\{ v_{i_k} : k \in J_n \}$ such that $1_A \subseteq v_{i_1} \cup v_{i_2} \cup \ldots\ldots \cup v_{i_n}$. But, then $1_A \subseteq (v_{i_1} \cup v_{i_2} \cup \ldots\ldots \cup v_{i_n}) \mid A = (v_{i_1} \mid A) \cup (v_{i_2} \mid A) \cup \ldots\ldots \cup (v_{i_n} \mid A)$

$= u_{i_1} \cup u_{i_2} \cup ...... \cup u_{i_n}$. Thus $\{ u_i : i \in J \}$ contains a finite subcover $\{ u_{i_1}, u_{i_2}, ..., u_{i_n} \}$ and 1_A is compact in (A, t_A).

Conversely, suppose 1_A is compact in (A, t_A). Let $\{ v_i : i \in J \}$ be an open cover of 1_A in (X, t). Set $u_i = v_i \mid A$, then $1_A \subseteq \bigcup_{i \in J} v_i$ implies that $1_A \subseteq \left(\bigcup_{i \in J} v_i \right) \mid A = \bigcup_{i \in J} (v_i \mid A) = \bigcup_{i \in J} u_i$. But $u_i \in t_A$, so $\{ u_i : i \in J \}$ is an open cover of 1_A in (A, t_A). As 1_A is compact in (A, t_A), thus $\{ u_i : i \in J \}$ contains a finite subcover, say $\{ u_{i_1}, u_{i_2}, ..., u_{i_n} \}$. Accordingly, $1_A \subseteq u_{i_1} \cup u_{i_2} \cup ...... \cup u_{i_n} = (v_{i_1} \mid A) \cup (v_{i_2} \mid A) \cup ...... \cup (v_{i_n} \mid A) = (v_{i_1} \cup v_{i_2} \cup ...... \cup v_{i_n}) \mid A \subseteq v_{i_1} \cup v_{i_2} \cup ...... \cup v_{i_n}$. Thus $\{ v_i : i \in J \}$ contains a finite subcover $\{ v_{i_1}, v_{i_2}, ..., v_{i_n} \}$ and therefore 1_A is compact in (X, t).

Corollary 2.8: Let (Y, t^*) be a subspace of (X, t) and A be a subset of (Y, t^*) such that $A \subset Y \subset X$. Then 1_A is compact in (X, t) iff 1_A is compact in (Y, t^*).

Proof: Let t_A and t_A^* be the subspaces of fuzzy topologies on A. Then by preceding theorem (2.7), 1_A is compact in (X, t) or (Y, t^*) iff 1_A is compact in (A, t_A) or (A, t_A^*); but $t_A = t_A^*$.

Theorem 2.9: Let (X, t_1) and (X, t_2) be two fts's and (X, t_1) be compact. If t_2 is coarser than t_1, then (X, t_2) is also compact.

The proof is easy.

Theorem 2.10: Let (X, t) be an fts and $\{1_{Y_s}\} \subseteq 1_X$, where $\{1_{Y_s}\}$ be a finite family. If each 1_{Y_s} is compact, then $\bigcup 1_{Y_s}$ is a compact subspace of (X, t).

Proof: Let $\{ u_i : i \in J \}$ be an open cover of $\bigcup 1_{Y_s}$. Then $\{ u_i : i \in J \}$ is an open cover of 1_{Y_s} for each $s \in J$. Since 1_{Y_s} is compact, then $\{ u_i : i \in J \}$ contains a finite subcover, say $\{ u_{i_k} : k \in J_n \}$ which is a cover of 1_{Y_s} . The union of these families is a finite subcover of $\bigcup 1_{Y_s}$. Thus $\bigcup 1_{Y_s}$ is compact.

Theorem 2.11: Let (X, t) and (Y, s) be two fts's and $f : (X, t) \to (Y, s)$ be bijective, fuzzy open and fuzzy continuous. Then (X, t) is compact iff (Y, s) is compact.

The necessary part of this theorem has already been proof by Chang [19].

Suppose (Y, s) is compact. Let $M = \{ u_i : i \in J \}$ be an open cover of (X, t) with $\bigcup_{i \in J} u_i = 1_X$. Since f is fuzzy open, so $f(u_i) \in s$ and hence $\{ f(u_i) : i \in J \}$ is an open cover of (Y, s). As (Y, s) is compact, then for each $y \in Y$, we have $\bigcup_{i \in J} f(u_i)(y) = 1_Y$.

Hence there exist $f(u_{i_k}) \in \{ f(u_i) : i \in J \}$ $(k \in J_n)$ such that $\bigcup_{k \in J_n} f(u_{i_k})(y) = 1_Y$.

Again, let v be any fuzzy set in X . Since f is bijective, then $f^{-1}(f(v)) = v$. Hence

$1_X = f^{-1}(1_Y) = f^{-1}\left(\bigcup_{k \in J_n} f(u_{i_k}) \right) = \bigcup_{k \in J_n} \left(f^{-1}(f(u_{i_k})) \right) = \bigcup_{k \in J_n} u_{i_k}$. Thus (X, t) is compact.

Theorem 2.12: Let (X, t) be an fts and (A, t_A) be a subspace of (X, t) with (X, t) is fuzzy compact. Let $f : (X, t) \to (A, t_A)$ be fuzzy continuous and onto, then (A, t_A) is fuzzy compact.

Proof: Let $M = \{ u_i : i \in J \}$ be an open cover of (A, t_A) with $\bigcup_{i \in J} u_i = 1_A$. Put $u_i = v_i \mid A$, where $v_i \in t$. Since f is fuzzy continuous, then $f^{-1}(u_i) \in t$ implies that $f^{-1}(v_i \mid A) \in t$. As (X, t) is fuzzy compact, then we have for each $x \in X$,

$\bigcup_{i \in J} f^{-1}(v_i \mid A)(x) = 1_X$. Thus we see that $\{ f^{-1}(v_i \mid A) : i \in J \}$ is an open cover of (X, t).

Hence there exist $f^{-1}(v_{i_1} \mid A)$, $f^{-1}(v_{i_2} \mid A)$, ……, $f^{-1}(v_{i_n} \mid A) \in \{ f^{-1}(v_i \mid A)\}$ such that $\bigcup_{k=1}^{n} f^{-1}(v_{i_k} \mid A)(x) = 1_X$ for every $x \in X$. Again, let u be any fuzzy set in A. Since f is onto, then we have $f(f^{-1}(u)) = u$. Hence $1_A = f(1_X) = f\left(\bigcup_{k=1}^{n} f^{-1}(v_{i_k} \mid A)\right)$ $= \bigcup_{k=1}^{n} f\left(f^{-1}(v_{i_k} \mid A)\right) = \bigcup_{k=1}^{n}(v_{i_k} \mid A) = \bigcup_{k=1}^{n} u_{i_k}$. Therefore (A, t_A) is fuzzy compact.

Theorem 2.13: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively with (A, t_A) is compact. Let $f : (A, t_A) \to (B, s_B)$ be relatively fuzzy continuous and surjective mapping. Then (B, s_B) is compact.

Proof: Assume that $f(A) = B$, as f is surjective. Let $\{ v_i : v_i \in s_B \}$ for each $i \in J$ be an open cover of (B, s_B) i.e. $\bigcup_{i \in J} v_i = 1_B$. As f is relatively fuzzy continuous, then $f^{-1}(v_i) \mid A \in t_A$ and hence $\{ f^{-1}(v_i) \mid A : i \in J \}$ is an open cover of (A, t_A). Since (A, t_A) is compact, so $\{ f^{-1}(v_i) \mid A : i \in J \}$ has a finite subcover i.e. there exist $f^{-1}(v_{i_k}) \mid A \in \{ f^{-1}(v_i) \mid A \}$ $(k = 1, 2, \ldots\ldots, n)$ such that $1_A = \bigcup_{k=1}^{n}(f^{-1}(v_{i_k}) \mid A)$.

Again, let v be any fuzzy set in B. As f is surjective, so we have $f\left(f^{-1}(v)\right) = v$. Therefore $1_B = f(1_A) = f\left(\bigcup_{k=1}^{n}(f^{-1}(v_{i_k}) \mid A)\right) = \bigcup_{k=1}^{n} f\,(f^{-1}(v_{i_k}) \mid A) = \bigcup_{k=1}^{n}(v_{i_k} \mid f(A))$ $= \bigcup_{k=1}^{n}(v_{i_k} \mid B) = \bigcup_{k=1}^{n} v_{i_k}$, as $v_i \mid B \subseteq v_i$. Thus (B, s_B) is compact.

Theorem 2.14: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively. Let $f:\ (A, t_A) \to (B, s_B)$ be relatively fuzzy open and bijective mapping with (B, s_B) is compact. Then (A, t_A) is also compact.

Proof: We have $f(A)=B$, as f is bijective. Let $\{ u_i :\ u_i \in t_A \}$ be an open cover of (A, t_A) for every $i \in J$ i.e $\bigcup_{i \in J} u_i = 1_A$. As $u_i \in t_A$, then there exists $v_i \in t$ such that $u_i = v_i \mid A$ and so $\bigcup_{i \in J} (v_i \mid A) = 1_A$. As f is relatively fuzzy open, then $f(u_i) \in s_B$ and hence $\{ f(u_i) :\ i \in J \}$ is an open cover of (B, s_B) implies that $\{ f(v_i \mid A) :\ i \in J \}$ $= \{ f(v_i) \mid f(A) :\ i \in J \}$ $= \{ f(v_i) | B :\ i \in J \}$ is an open cover of (B, s_B). Since (B, s_B) is compact, then $\{ f(v_i) | B :\ i \in J \}$ has a finite subcover, say $\{ f(v_{i_k}) | B :\ k \in J_n \}$ such that $\bigcup_{k \in J_n} (f(v_{i_k}) \mid B) = 1_B$. Let v be any fuzzy set in A. As f is bijective, then we have $f^{-1}(f(v)) = v$. Hence $1_A = f^{-1}(1_B) = f^{-1}\left(\bigcup_{k \in J_n} (f(v_{i_k}) \mid B) \right)$ $= \bigcup_{k \in J_n} (v_{i_k} \mid f^{-1}(B)) = \bigcup_{k \in J_n} (v_{i_k} \mid A) = \bigcup_{k \in J_n} u_{i_k}$. Thus $\{ u_{i_k} :\ k \in J_n \}$ is a finite subcover of $\{ u_i :\ u_i \in t_A \}$. Hence (A, t_A) is compact.

Theorem 2.15: Let (X, t) be a fuzzy T_1-space (as def. 1.45), $A \subset X$ and 1_A be a compact subset in (X, t). Suppose $x \in A^c$, then there exist u, $v \in t$ such that $u(x) = 1$ and $A \subseteq v^{-1}(0, 1]$.

Proof: Let $y \in A$. Since $x \notin A$ ($x \in A^c$), then clearly $x \neq y$. As (X, t) is fuzzy T_1-space, then there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $u_y(y) = 0$ and $v_y(x) = 0$, $v_y(y) = 1$. Hence $1_A \subseteq \bigcup \{ v_y :\ y \in A \}$ i.e. $\{ v_y :\ y \in A \}$ is an open cover of 1_A. Since 1_A is compact, then it has a finite subcover, say v_{y_1}, v_{y_2}, ..., $v_{y_n} \in \{v_y\}$ such that

$1_A \subseteq v_{y_1} \cup v_{y_2} \cup \ldots\ldots. \cup v_{y_n}$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ldots.. \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, $u \in t$. Furthermore, $A \subseteq v^{-1}(0,1]$ and $u(x) = 1$, since each $u_{y_k}(x) = 1$ individually.

Theorem 2.16: Let (X, t) be a fuzzy T_1-space (as def. 1.45) and 1_A, 1_B be disjoint compact subsets in (X, t) (A, $B \subset X$). Then there exist u, $v \in t$ such that $A \subseteq u^{-1}(0,1]$ and $B \subseteq v^{-1}(0,1]$.

Proof: Let $y \in A$. Then $y \notin B$, as 1_A and 1_B are disjoint. Since 1_B is compact, then by theorem (2.15), there exist u_y, $v_y \in t$ such that $u_y(y) = 1$ and $B \subseteq v_y^{-1}(0,1]$. Since $u_y(y) = 1$, then $\{ u_y : y \in A \}$ is an open cover of 1_A. As 1_A is compact, so it has a finite subcoer, say u_{y_1}, u_{y_2}, ……, $u_{y_n} \in \{u_y\}$ such that $1_A \subseteq u_{y_1} \cup u_{y_2} \cup \ldots\ldots. \cup u_{y_n}$. Furthermore, $1_B \subseteq v_{y_1} \cap v_{y_2} \cap \ldots\ldots. \cap v_{y_n}$, as $B \subseteq v_{y_k}^{-1}(0,1]$ for each k. Again, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots.. \cap v_{y_n}$. Thus we see that $A \subseteq u^{-1}(0,1]$ and $B \subseteq v^{-1}(0,1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, $v \in t$.

Theorem 2.17: Let (X, t) be a fuzzy T_1-space (as def. 1.45) and $A \subset X$. If 1_A is compact in (X, t), then 1_A is closed.

Proof: Let $x \in A^c$. We have to show that, there exist $u \in t$ such that $u(x) = 1$ and $u \subseteq A^p$, where A^p is the characteristic function of A^c. Indeed, for each $y \in A$, there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $u_y(y) = 0$ and $v_y(x) = 0$, $v_y(y) = 1$. Hence we see that $1_A \subseteq \bigcup \{ v_y : y \in A \}$ i.e. $\{ v_y : y \in A \}$ is an open cover of 1_A. Since 1_A is compact in (X, t), so 1_A has a finite subcover, say $\{ v_{y_k} : \ y \in A \}$ ($k \in J_n$) such that $1_A \subseteq v_{y_1} \cup v_{y_2} \cup \ \ldots\ldots. \cup v_{y_n}$. Now, let $u = \ u_{y_1} \cap \ u_{y_2} \cap \ \ldots\ldots \ \cap u_{y_n}$. Thus we see that $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k. For, each $z \in A$, there exists a k such that $\bigcup \{v_{y_k}\}(z) = 1$ ($k = 1, \ 2 , \ \ldots\ldots , n$) and so $u(z) = 0$. Hence $u \subseteq A^p$. Therefore, 1_{A^c} is open in (X, t). Thus 1_A is closed in (X, t).

Theorem 2.18: Let (X, t) be a fuzzy T_1-space (as def. 1.46), $A \subset X$ and 1_A be a compact subset in (X, t). Suppose $x \in A^c$, then there exist u , $v \in t$ such that $u(x) > 0$ and $A \subseteq v^{-1}(0, 1]$.

Such fuzzy T_1-space have no compact subset. So the above theorem (2.18) is vacuously true for there space.

Theorem 2.19: Let (X, t) be a fuzzy Hausdorff space (as def. 1.47), $A \subset X$ and 1_A be a compact subset in (X, t). Suppose $x \in A^c$, then there exist u , $v \in t$ such that $u(x) = 1$, $A \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$.

Proof: Let $y \in A$. Since $x \notin A$ ($x \in A^c$), then clearly $x \neq y$. As (X, t) is fuzzy Hausdorff, then there exist u_y , $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \cap v_y = 0$. Hence $1_A \subseteq \bigcup \{ v_y : \ y \in A \}$ i.e. $\{ v_y : \ y \in A \}$ is an open cover of 1_A. Since 1_A is compact, then there exist v_{y_1} , v_{y_2} ,......, $v_{y_n} \in \{v_y\}$ such that $1_A \subseteq v_{y_1} \cup v_{y_2} \cup \ \ldots\ldots. \cup v_{y_n}$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ \ldots\ldots \cup v_{y_n}$ and $u = \ u_{y_1} \cap \ u_{y_2} \cap \ \ldots\ldots \ \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $u \in t$. Furthermore, $A \subseteq v^{-1}(0, 1]$ and $u(x) = 1$, since each $u_{y_k}(x) = 1$ individually.

Finally, we claim that $u \cap v = 0$. We observe that $u_{y_k} \cap v_{y_k} = 0$ implies that $u \cap v_{y_k} = 0$, by distributive law, we have $u \cap v = u \cap (\, v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}\,)$ $= (\, u \cap v_{y_1}\,) \cup (\, u \cap v_{y_2}\,) \cup \ldots\ldots \cup (\, u \cap v_{y_n}\,) = 0$.

Corollary 2.20: Let (X, t) be a fuzzy Hausdorff space (as def. 1.47), $A \subset X$ and 1_A be a compact subset in (X, t). Let $x \notin A$, then there exist $u \in t$ such that $u(x) = 1$ and $u^{-1}(0,1] \subseteq A^c$.

Proof: By theorem (2.19), there exist u, $v \in t$ such that $u(x) = 1$, $A \subseteq v^{-1}(0,1]$ and $u \cap v = 0$. Hence $u^{-1}(0,1] \cap v^{-1}(0,1] = \phi$. If not, there exists $x \in u^{-1}(0,1] \cap v^{-1}(0,1]$ $\Rightarrow$ $x \in u^{-1}(0,1]$ and $x \in v^{-1}(0,1]$ $\Rightarrow$ $u(x) > 0$ and $v(x) > 0$ $\Rightarrow$ $u \cap v \neq 0$. Hence $u^{-1}(0,1] \cap A = \phi$ and consequently $u^{-1}(0,1] \subseteq A^c$.

Theorem 2.21: Let (X, t) be a fuzzy Hausdorff space (as def. 1.47) and 1_A, 1_B be disjoint compact subsets in (X, t) (A, $B \subset X$). Then there exist u, $v \in t$ such that $A \subseteq u^{-1}(0,1]$, $B \subseteq v^{-1}(0,1]$ and $u \cap v = 0$.

Proof: Let $y \in A$. Then $y \notin B$, as 1_A and 1_B are disjoint. Since 1_B is compact, then by theorem (2.19), there exist u_y, $v_y \in t$ such that $u_y(y) = 1$, $B \subseteq v_y^{-1}(0,1]$ and $u_y \cap v_y = 0$. Since $u_y(y) = 1$, then $\{\, u_y :\ y \in A\,\}$ is an open cover of 1_A. As 1_A is compact, then there exist u_{y_1}, u_{y_2}, ……, $u_{y_n} \in \{u_y\}$ such that $1_A \subseteq u_{y_1} \cup u_{y_2} \cup \ldots\ldots\ldots \cup u_{y_n}$. Furthermore, $1_B \subseteq v_{y_1} \cap v_{y_2} \cap \ldots\ldots\ldots \cap v_{y_n}$, as $B \subseteq v_{y_k}^{-1}(0,1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Thus we see that $A \subseteq u^{-1}(0,1]$ and $B \subseteq v^{-1}(0,1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, $v \in t$.

Finally, we have to show that $u \cap v = 0$. First, we observe that $u_{y_k} \cap v_{y_k} = 0$ for each k, implies that $u_{y_k} \cap v = 0$, by distributive law, we see that $u \cap v = (u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}) \cap v = (u_{y_1} \cap v) \cup (u_{y_2} \cap v) \cup \ldots\ldots \cup (u_{y_n} \cap v) = 0$.

Theorem 2.22: Let (X, t) be a fuzzy Hausdorff space (as def. 1.47), $A \subset X$ and 1_A be a compact subset in (X, t). Then 1_A is closed.

Proof: Let $x \in A^c$. We have to show that, there exists $u \in t$ such that $u(x) = 1$ and $u \subseteq A^p$, where A^p is the characteristic function of A^c. Now, let $y \in A$, then there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \cap v_y = 0$. Thus we see that $1_A \subseteq \bigcup \{ v_y : y \in A \}$ i.e. $\{ v_y : y \in A \}$ is an open cover of 1_A. Since 1_A is compact, so it has a finite subcover, say v_{y_1}, v_{y_2}, ……, $v_{y_n} \in \{v_y\}$ such that $1_A \subseteq v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$. Again, let $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$ and $v = v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$. Hence we observe that $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k and $u \cap (v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}) = 0$. For each $z \in A$, it is clear that $\bigcup \{v_{y_k}\}(z) = 1$ ($k = 1, 2, \ldots\ldots, n$). Thus $u(z) = 0$ and hence $u \subseteq A^p$. Therefore, 1_{A^c} is open and so 1_A is closed.

Theorem 2.23: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48), $A \subset X$ and 1_A be a compact subset in (X, t). Suppose $x \in A^c$, then there exist u, $v \in t$ such that $u(x) > 0$, $A \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$.

Such fuzzy Hausdorff space have no compact subset. So the above theorem (2.23) is vacuously true for there space.

Theorem 2.24: Let (X, t) be a fuzzy Hausdorff space (as def. 1.49), $A \subset X$ and 1_A be a compact subset in (X, t). Suppose $x \in A^c$, then there exist u, $v \in t$ such that $x_r \in u$, $A \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$.

Such fuzzy Hausdorff space have no compact subset. So the above theorem (2.24) is vacuously true for there space.

Theorem 2.25: Let (X, t) be a fuzzy Hausdorff space (as def. 1.50), $A \subset X$ and 1_A be a compact subset in (X, t). Suppose $x \in A^c$, then there exist u, $v \in t$ such that $u(x) = 1$, $A \subseteq v^{-1}(0, 1]$ and $u \subseteq 1 - v$.

Proof: Let $y \in A$. Since $x \notin A$ ($x \in A^c$), then clearly $x \neq y$. As (X, t) is fuzzy Hausdorff, then there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \subseteq 1 - v_y$. Hence $1_A \subseteq \bigcup\{ v_y : y \in A \}$ i.e. $\{ v_y : y \in A \}$ is an open cover of 1_A. Since 1_A is compact, then there exist v_{y_1}, $v_{y_2}, \ldots\ldots, v_{y_n} \in \{v_y\}$ such that $1_A \subseteq v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ldots.. \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, $u \in t$. Furthermore, $A \subseteq v^{-1}(0, 1]$ and $u(x) = 1$, since each $u_{y_k}(x) = 1$ for each k.

Finally, we have to show that $u \subseteq 1 - v$. As $u_y \subseteq 1 - v_y$ implies $u \subseteq 1 - v_y$. Since $u_{y_k}(x) \leq 1 - v_{y_k}(x)$ for all $x \in X$ and for each k, then $u \subseteq 1 - v$. If not, there exists $x \in X$ such that $u_y(x) > 1 - v_y(x)$. We have $u_y(x) \leq u_{y_k}(x)$ for each k. Then for some k, $u_{y_k}(x) > 1 - v_{y_k}(x)$. But this is a contradiction, as $u_{y_k}(x) \leq 1 - v_{y_k}(x)$ for each k. Hence $u \subseteq 1 - v$.

Theorem 2.26: Let (X, t) be a fuzzy Hausdorff space (as def. 1.50) and 1_A, 1_B be disjoint compact subsets in (X, t) (A, $B \subset X$). Then there exist u, $v \in t$ such that $A \subseteq u^{-1}(0,1]$, $B \subseteq v^{-1}(0,1]$ and $u \subseteq 1-v$.

Proof: Let $y \in A$. Then $y \notin B$, as 1_A and 1_B are disjoint. Since 1_B is compact, then by theorem (2.25), there exist u_y, $v_y \in t$ such that $u_y(y)=1$, $B \subseteq v_y^{-1}(0,1]$ and $u_y \subseteq 1-v_y$. Since $u_y(y)=1$, then $\{ u_y : y \in A \}$ is an open cover of 1_A. As 1_A is compact, then there exist u_{y_1}, u_{y_2}, ……, $u_{y_n} \in \{u_y\}$ such that $1_A \subseteq u_{y_1} \cup u_{y_2} \cup$ ……. $\cup\, u_{y_n}$. Furthermore, $1_B \subseteq v_{y_1} \cap v_{y_2} \cap$ ……. $\cap\, v_{y_n}$, as $B \subseteq v_{y_k}^{-1}(0,1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup$ …… $\cup\, u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap$ ….. $\cap\, v_{y_n}$. Thus we see that $A \subseteq u^{-1}(0,1]$ and $B \subseteq v^{-1}(0,1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, $v \in t$.

Finally, we have to show that $u \subseteq 1-v$. First, we observe that $u_{y_k} \subseteq 1-v_{y_k}$ for each k, implies that $u_{y_k} \subseteq 1-v$ for each k and it is clear that $u \subseteq 1-v$.

Theorem 2.27: Let (X, t) be a fuzzy Hausdorff space (as def. 1.50), $A \subset X$ and 1_A be a compact subset in (X, t). Then 1_A is closed.

Proof: Let $x \in A^c$. We have to show that, there exists $u \in t$ such that $u(x)=1$ and $u \subseteq A^p$, where A^p is the characteristic function of A^c. Now, let $y \in A$, then there exist u_y, $v_y \in t$ such that $u_y(x)=1$, $v_y(y)=1$ and $u_y \subseteq 1-v_y$. Thus we see that $1_A \subseteq \bigcup \{ v_y : y \in A \}$ i.e. $\{ v_y : y \in A \}$ is an open cover of 1_A. Since 1_A is compact, so it has a finite subcover, say v_{y_1}, v_{y_2}, ……, $v_{y_n} \in \{v_y\}$ such that $1_A \subseteq v_{y_1} \cup v_{y_2} \cup$ ……$\cup\, v_{y_n}$. Again, let $u = u_{y_1} \cap u_{y_2} \cap$ …… $\cap\, u_{y_n}$ and $v = v_{y_1} \cup v_{y_2} \cup$ ….. $\cup\, v_{y_n}$.

Hence we observe that $u(x)=1$, as $u_{y_k}(x)=1$ for each k and $u_y \subseteq 1-v_y$ implies that $u \subseteq 1-v_y$. As $u_{y_k}(x) \le 1-v_{y_k}(x)$ for all $x \in X$ and for each k, then $u \subseteq 1-v$. If not, there exists $x \in X$ such that $u_y(x) > 1-v_y(x)$. We have $u_y(x) \le u_{y_k}(x)$ for each k. Then for some k, $u_{y_k}(x) > 1-v_{y_k}(x)$. But this is a contradiction as $u_{y_k}(x) \le 1-v_{y_k}(x)$ for each k. Hence $u \subseteq 1-v$. For each $z \in A$, it is clear that $\bigcup\{v_{y_k}\}(z)=1$ ($k=1,\ 2,\ \ldots\ldots,\ n$). Thus $u(z)=0$ and hence $u \subseteq A^p$. Therefore, 1_{A^c} is open and so 1_A is closed.

Theorem 2.28: Let (X, t) be a fuzzy regular space (as def. 1.51), $A \subset X$ and 1_A be a compact subset in (X, t). Suppose $x \in A$ and $u \in t^c$ with $u(x)=0$. Then there exist v, $w \in t$ such that $v(x)=1$, $u \subseteq w$, $A \subseteq v^{-1}(0,1]$ and $v \subseteq 1-w$.

Proof: Suppose $x \in A$ and $u \in t^c$ we have $u(x)=0$. Since (X, t) is fuzzy regular, then there exist v_x, $w_x \in t$ such that $v_x(x)=1$, $u_x \subseteq w_x$ and $v_x \subseteq 1-w_x$. Hence $1_A \subseteq \bigcup\{\ v_x :\ x \in A\ \}$ i.e. $\{\ v_x :\ x \in A\ \}$ is an open cover of 1_A. Since 1_A is compact, so it has a finite subcover, say v_{x_1}, $v_{x_2}, \ldots\ldots, v_{x_n} \in \{v_x\}$ such that $1_A \subseteq v_{x_1} \cup v_{x_2} \cup \ldots\ldots \cup v_{x_n}$. Now, let $v = v_{x_n} \cup v_{x_2} \cup \ldots\ldots \cup v_{x_n}$ and $w = w_{x_1} \cap w_{x_2} \cap \ldots\ldots \cap w_{x_n}$. Thus we see that v and w are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, $w \in t$. Furthermore, $A \subseteq v^{-1}(0,1]$, $v(x)=1$ and $u \subseteq w$, as $u \subseteq w_k$ individually.

Finally, we have to show that $v \subseteq 1-w$. As $v_{x_k} \subseteq 1-w_{x_k}$ for each k implies that $v_{x_k} \subseteq 1-w$ for each k and it is clear that $v \subseteq 1-w$.

Theorem 2.29: A topological space (X, T) is compact iff $(X, \omega(T))$ is fuzzy compact.

Proof: Suppose (X, T) is compact. Let $\{ u_i : i \in J \}$ be an open cover of $(X, \omega(T))$ i.e $1_X = \bigcup_{i \in J} u_i$. Then $u_i^{-1}(a, 1] \in T$ for $a \in I_1$ and $\{ u_i^{-1}(a, 1] : u_i^{-1}(a, 1] \in T \}$ is an open cover of (X, T). Since (X, T) is compact, so it has a finite subcover, say $\{ u_{i_k}^{-1}(a, 1] : k \in J_n \}$ such that $X = u_{i_1}^{-1}(a, 1] \cup u_{i_2}^{-1}(a, 1] \cup ......\cup u_{i_n}^{-1}(a, 1]$. Now, we can write $1_X = u_{i_1} \cup u_{i_2} \cup ......\cup u_{i_n}$ and it is seen that $\{ u_{i_k} : k \in J_n \}$ is a finite subcover of $\{ u_i : i \in J \}$. Thus $(X, \omega(T))$ is fuzzy compact.

Conversely, suppose that $(X, \omega(T))$ is fuzzy compact. Let $\{ V_j : j \in J \}$ be an open cover of (X, T) i.e. $X = \bigcup_{j \in J} V_j$. Since 1_{V_j} are l. s. c. then $1_{V_j} \in \omega(T)$ and $\{ 1_{V_j} : 1_{V_j} \in \omega(T) \}$ is an open cover of $(X, \omega(T))$. Since $(X, \omega(T))$ is fuzzy compact, so it has a finite subcover, say $\{ 1_{V_{jk}} : k \in J_n \}$ such that $1_X = 1_{V_{j1}} \cup 1_{V_{j2}} \cup ......\cup 1_{V_{jn}}$. Now, we can write $X = V_{j_1} \cup V_{j_2} \cup ......\cup V_{j_n}$ and it is seen that $\{ V_{j_k} : k \in J_n \}$ is a finite subcover of $\{ V_j : j \in J \}$. Thus (X, T) is compact.

Theorem 2.30: If $\{(X_i, t_i) : i \in J \}$ is a family of fuzzy compact fuzzy topological spaces, then the product space $\left(\prod_{i \in J} X_i , \prod_{i \in J} t_i \right)$ is also fuzzy compact.

Proof: cf.[108].

Definition 2.31: Let (X, t) be an fts and $0 < \delta \leq 1$. A fuzzy set $u \in t$ is said to be δ-open in X iff $u(x) \geq \delta$ for all $x \in u_0$. If $\delta = 0$, then u is open. A fuzzy set is said to be δ-closed iff its complement is δ-open.

Example 2.32: Let $X = \{a, b\}$, $I = [0, 1]$ and $0 < \delta \leq 1$. Again, let u, $v \in I^X$ defined by $u(a) = 0.4$, $u(b) = 0.3$ and $v(a) = 0.7$, $v(b) = 0.5$. Consider $t = \{0, u, v, 1\}$, then (X, t) is an fts. Take $\delta = 0.4$. Then u is not δ-open in X, as $u(b) < \delta$ for $b \in u_0$. But v is δ-open in X, as $v(a)$, $v(b) > \delta$ for a, $b \in v_0$.

Definition 2.33: Let $M = \{u_i : i \in J\}$ be a family of δ-open fuzzy sets in an fts (X, t) and λ be a fuzzy set in X. Then M is said to be δ-cover of λ iff $\lambda \subseteq \bigcup_{i \in J} \{u_i : u_i \in M\}$. A subfamily of a δ-cover of λ which is also a δ-cover of λ is said to be δ-subcover.

Example 2.34: Let $X = \{a, b\}$, $I = [0, 1]$ and $0 < \delta \leq 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 1$, $u_1(b) = 0.4$; $u_2(a) = 0.5$, $u_2(b) = 1$ and $u_3(a) = 0.5$, $u_3(b) = 0.4$. Now, take $t = \{0, u_1, u_2, u_3, 1\}$, then we see that (X, t) is an fts. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.6$, $\lambda(b) = 0.7$. Take $\delta = 0.4$. Clearly u_1, u_2 and u_3 are δ-open fuzzy sets in (X, t). Now, we observe that $\lambda \subset u_1 \cup u_2$. So $\{u_1, u_2\}$ is a δ-cover of λ in (X, t).

Definition 2.35: Let (X, t) be an fts and $0 < \delta \leq 1$. An fts (X, t) is δ-compact iff every δ-cover of X has a finite δ-subcover.

Theorem 2.36: Any fuzzy δ-compact space is fuzzy compact. The converse is not necessarily true in general.

The proof is straightforward.

Now, for the converse, we consider the following example.

Let $X = [0, 1]$, $I = [0, 1]$ and $0 < \delta \leq 1$. Let u_1, u_2, $u_3 \in I^X$ defined by

$$u_1(x) = \begin{cases} 1 & \text{for } 0 \leq x < 0.4 \\ 1 & \text{for x} = 0.4 \\ 0.6 & \text{for } 0.4 < x \leq 1 \end{cases}, \quad u_2(x) = \begin{cases} 0.5 & \text{for } 0 \leq x < 0.4 \\ 1 & \text{for x} = 0.4 \\ 1 & \text{for } 0.4 < x \leq 1 \end{cases} \text{ and}$$

$$u_3(x) = \begin{cases} 0.5 & \text{for } 0 \leq x < 0.4 \\ 1 & \text{for x} = 0.4 \\ 0.6 & \text{for } 0.4 < x \leq 1 \end{cases}.$$

Now, take $t = \{0, u_1, u_2, u_3, 1\}$, then we see that (X, t) is an fts. Clearly (X, t) is fuzzy compact. Take $\delta = 0.8$. Then there is no finite δ-open fuzzy sets u_k for $k = 1, 2, 3$ in (X, t). Thus (X, t) is not δ-compact.

Chapter Three

α -Compact Spaces

α -compact spaces have been introduced first by Gantner et al. [54] in fuzzy topological spaces and discussed some characterizations of this concept. We aim to study various other properties of this concept and established some theorems, corollaries and examples. Also we have defined δ - α -compact spaces and found different properties between α -compact and δ - α -compact spaces.

Definition 3.1[54]: Let (X, t) be an fts and $\alpha \in I$. A collection M of fuzzy sets is called an α -shading, $0 \leq \alpha < 1$ (res. α^* -shading, $0 < \alpha \leq 1$) of X if for each $x \in X$ there exists a $u \in M$ such that $u(x) > \alpha$ (resp. $u(x) \geq \alpha$). A subcollection of an α -shading (res. α^* -shading) of X which is also an α -shading (resp. α^* -shading) is called an α -subshading (res. α^* -subshading) of X .

Definition 3.2[54]: An fts (X, t) is said to be α -compact, $0 \leq \alpha < 1$ (res. α^* -compact, $0 < \alpha \leq 1$) iff each α -shading (res. α^* -shading) of X by open fuzzy sets has a finite α -subshading (res. α^* -subshading), where $\alpha \in I$.

Theorem 3.3: Let (X, t) be an fts and $A \subset X$. Then 1_A is α -compact (resp. α^* -compact) in (X, t) iff 1_A is α -compact (resp. α^* -compact) in (A, t_A).

Proof: Suppose 1_A is α -compact in (X, t). Let $M = \{ u_i : i \in J \}$ be an open α -shading of 1_A in (A, t_A). Then there exist $v_i \in t$ such that $u_i = v_i \mid A \subseteq v_i$. Hence $\{ v_i : i \in J \}$ is an open α -shading of 1_A in (X, t). Since 1_A is α -compact in (X, t), then $\{ v_i : i \in J \}$ has a finite α -subshading, say $\{ v_{i_k} : k \in J_n \}$ such that $v_{i_k}(x) > \alpha$ for each $x \in A$. For, if $x \in A$, then there exists $v_{i_{k0}}$ such that $v_{i_{k0}}(x) > \alpha$ implies that $(v_{i_{k0}} | A)(x) > \alpha$ and consequently $u_{i_{k0}}(x) > \alpha$, as $A \subset X$. Hence $u_{i_{k0}} \in M$ and so $\{ u_{i_k} : k \in J_n \}$ is a finite α -subshading of M. Therefore, 1_A is α -compact in (A, t_A).

Conversely, suppose 1_A is α -compact in (A, t_A). Let $H = \{ v_i : i \in J \}$ be an open α -shading of 1_A in (X, t). Put $u_i = v_i | A$. To show this, let $x \in X$. If $x \in A$, then there exists $v_{i_0} \in H$ such that $u_{i_0} = v_{i_0} | A$. But $u_{i_0} \in t_A$, so $u_{i_0}(x) > \alpha$ for each $x \in A$. Therefore, $\{ u_i : i \in J \}$ be an open α -shading of 1_A in (A, t_A). Since 1_A is α -compact in (A, t_A), then $\{ u_i : i \in J \}$ has a finite α -subshading, say $\{ u_{i_k} : k \in J_n \}$ such that $u_{i_k}(x) > \alpha$ for each $x \in A$. For, if $x \in A$, then there exists $u_{i_{k0}}$ such that $u_{i_{k0}}(x) > \alpha \Rightarrow (v_{i_{k0}} | A)(x) > \alpha \Rightarrow v_{i_{k0}}(x) > \alpha$, as $A \subset X$. Thus $\{ v_{i_k} : k \in J_n \}$ is a finite α -subshading of H. Hence 1_A is α -compact in (X, t).

Similar proof for α^*-compactness can be given.

Corollary 3.4: Let (Y, t^*) be a fuzzy subspace of (X, t) and $A \subset Y \subset X$. Then 1_A is α -compact (resp. α^*-compact) in (X, t) iff 1_A is α -compact (resp. α^*-compact) in (Y, t^*).

Proof: Let t_A and t_A^* be the subspace fuzzy topologies on A. Then by preceding theorem (3.3), 1_A is α -compact in (X, t) or (Y, t^*) iff 1_A is α -compact in (A, t_A) or (A, t_A^*). But $t_A = t_A^*$.

Similar work for α^*-compactness can be given.

Theorem 3.5: Let (X,t) be an fts and $A \subset X$. If (X,t) is α-compact (resp. α^*-compact) and 1_A is closed, then (A, t_A) is α-compact (resp. α^*-compact) subspace of (X,t).

Proof: Let $M = \{ u_i : i \in J \}$ be an open α-shading of (A, t_A). Then there exist $v_i \in t$ such that $u_i = v_i \mid A$. Let $H = \{ v_i \in t : v_i \mid A \in M \}$. Then $H \cup \{ 1_{X-A} \}$ is a family and is an open α-shading of (X,t). To prove this, let $x \in X$. If $x \in A$, then there exists $u_{i_0} \in M$ such that $u_{i_0}(x) > \alpha$. Let $v_i^/ \in t$ such that $v_i^/ \mid A = u_{i_0}$. Thus $v_i^/ \in H$ and $v_i^/(x) > \alpha$. If $x \in X - A$, then $(1_{X-A})(x) > \alpha$. Since (X,t) is α-compact, so $H \cup \{1_{X-A}\}$ has a finite α-subshading, say $\{ v_{i_k}, 1_{X-A} \}$ ($k \in J_n$). Also 1_A and 1_{X-A} are disjoint, so we can exclude 1_{X-A} from this α-shading. Hence $\{ v_{i_k} \mid A \}$ ($k \in J_n$) is a finite α-subshading of M. For if $x \in A$ and $\{ v_{i_k}, 1_{X-A} \}$ ($k \in J_n$) is an open α-shading of (X,t), then there exists v_{i_0} such that $v_{i_0}(x) > \alpha$. Therefore $(v_{i_0} \mid A)(x) > \alpha$ and $v_{i_0} \mid A \in M$. Hence (A, t_A) is α-compact.

The proof is similar for α^*-compactness can be given.

Note: This theorem have been proved in Gantner et. al. [54] in a different form.

Theorem 3.6: Let (X,t) and (Y,s) be two fuzzy topological spaces with (X,t) is α-compact (resp. α^*-compact). Let $f: (X,t) \to (Y,s)$ be fuzzy continuous and surjective mapping. Then (Y,s) is α-compact (resp. α^*-compact).

Proof: Let $\{ u_i : u_i \in s \}$ be an open α-shading of (Y,s) for every $i \in J$. Since f is fuzzy continuous, then $f^{-1}(u_i) \in t$. We see that, for each $x \in X$, $f^{-1}(u_i)(x) > \alpha$ and so $\{ f^{-1}(u_i)\}$ is an open α-shading of (X,t), $i \in J$. Since (X,t) is α-compact, then $\{ f^{-1}(u_i)\}$ has a finite α-subshading, say $\{ f^{-1}(u_{i_k}) : k \in J_n \}$. Now, if $y \in Y$, then

$y = f(x)$ for some $x \in X$. Then there exists $u_{i_k} \in \{u_i\}$ such that $f^{-1}(u_{i_k})(x) > \alpha$ which implies that $u_{i_k}(f(x)) > \alpha$ or $u_{i_k}(y) > \alpha$. Thus $\{u_i\}$ has a finite α-subshading $\{ u_{i_k} : k \in J_n \}$. Hence (Y, s) is α-compact.

Similar proof for α^*-compactness can be given.

Note: This theorem was proved in Gantner et. al. [54] in a different form.

Theorem 3.7: Let (X, t) and (Y, s) be two fuzzy topological spaces with (Y, s) is α-compact (resp. α^*-compact). Let $f : (X, t) \to (Y, s)$ be fuzzy open and bijective mapping. Then (X, t) is α-compact (resp. α^*-compact).

Proof: Let $M = \{ u_i : i \in J \}$ be an open α-shading of (X, t). Since f is fuzzy open, then $f(u_i) \in s$ and hence $f(M) = \{ f(u_i) : i \in J \}$ is also an open α-shading of (Y, s). For, if $y \in Y$, then $f^{-1}(y) \in f^{-1}(Y)$. So there exists $u_{i_0} \in M$ such $u_{i_0}(f^{-1}(y)) > \alpha$ which implies that $f(u_{i_0})(y) > \alpha$. As (Y, s) is α-compact, then $f(M)$ has a finite α-subshading, say $\{ f(u_{i_k}) : k \in J_n \}$ such that $f(u_{i_k})(y) > \alpha$ for each $y \in Y$. For, if $x \in f^{-1}(Y)$, then $x = f^{-1}(y)$ for $y \in Y$. Therefore, there exists $u_{i_k} \in M$ such that $f(u_{i_k})(y) > \alpha$ which implies that $u_{i_k}(f^{-1}(y)) > \alpha$ or $u_{i_k}(x) > \alpha$. Thus M has a finite α-subshading $\{ u_{i_k} : k \in J_n \}$. Hence Then (X, t) is α-compact.

Similar work for α^*-compactness can be done.

Theorem 3.8: Let (X, t) be an fts and (A, t_A) be a subspace of an fts (X, t). Let $f : (X, t) \to (A, t_A)$ be fuzzy continuous and onto mapping with (X, t) is α-compact (resp. α^*-compact). Then (A, t_A) is α-compact (resp. α^*-compact).

Proof: Let $M = \{ u_i : i \in J \}$ be an open α-shading of (A, t_A). Put $u_i = v_i \mid A$, where $v_i \in t$. Since f is fuzzy continuous, then $f^{-1}(u_i) \in t$ and so $f^{-1}(v_i \mid A) \in t$. Thus we have for every $x \in X$, $f^{-1}(v_i \mid A)(x) > \alpha$ and hence $f^{-1}(M) = \{ f^{-1}(u_i) : u_i \in M \}$ i.e $f^{-1}(M) = \{ f^{-1}(v_i \mid A) : i \in J \}$ is an open α-shading of (X, t). As (X, t) is α-compact, then $f^{-1}(M)$ has a finite α-subshading, say $\{ f^{-1}(v_{i_1} \mid A)$, $f^{-1}(v_{i_2} \mid A)$, ... , $f^{-1}(v_{i_n} \mid A) \}$. Now, if $y \in A$, then $y = f(x)$ for some $x \in X$. Then there exists k such that $f^{-1}(v_{i_k} \mid A)(x) > \alpha$ which implies that $(v_{i_k} \mid A)\left(f(x)\right) > \alpha$ or $u_{i_k}(y) > \alpha$. Hence (A, t_A) is α-compact.

Similar work for α^*-compactness can be given.

Theorem 3.9: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively and $f : (A, t_A) \to (B, s_B)$ be relatively fuzzy continuous and onto mapping with (A, t_A) is α-compact (resp. α^*-compact). Then (B, s_B) is α-compact (resp. α^*-compact).

Proof: We have $f(A)=B$, as f is onto. Let $\{ v_i : v_i \in s_B \}$ be an open α-shading of (B, s_B) for every $i \in J$ i.e $v_i(y) > \alpha$ for each $y \in B$. Since $v_i \in s_B$, then there exists $u_i \in s$ such that $v_i = u_i \mid B$ and so $(u_i \mid B)(y) > \alpha$ for each $y \in B$. As f is relatively fuzzy continuous, then $f^{-1}(v_i) | A \in t_A$. Thus we observe that, for each $x \in A$, $(f^{-1}(v_i)|A)(x) > \alpha$ and hence $\{ f^{-1}(v_i)|A : i \in J \}$ is an open α-shading of (A, t_A) implies that $\{ f^{-1}(u_i \mid B)|A : i \in J \} = \{ f^{-1}(u_i) | (f^{-1}(B) \cap A) : i \in J \}$ $= \{ f^{-1}(u_i)|A : i \in J \}$ is an open α-shading of (A, t_A). Since (A, t_A) is α-compact, then $\{ f^{-1}(u_i)|A : i \in J \}$ has a finite α-subshading, say $\{ f^{-1}(u_{i_k})|A \}$ $(k \in J_n)$ such that $(f^{-1}(u_{i_k})|A)(x) > \alpha$ for each $x \in A$. Now, if $y \in B$, then $y = f(x)$ for some $x \in A$. Then there exists k we have $(f^{-1}(u_{i_k})|A)(x) > \alpha$ implies that $(u_{i_k} | f(A))\,(f(x)) > \alpha$ implies that $(u_{i_k} | B)(y) > \alpha$, as f is onto or $v_{i_k}(y) > \alpha$. Hence it is clear that $\{ v_{i_k} : k \in J_n \}$ is a finite α-subshading of $\{ v_i : v_i \in s_B \}$. Thus (B, s_B) is α -compact.

The proof is similar for α^*-compactness can be given.

Theorem 3.10: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively. Let $f : (A, t_A) \to (B, s_B)$ be relatively fuzzy open and bijective mapping with (B, s_B) is α -compact (resp. α^*-compact). Then (A, t_A) is also α -compact (resp. α^*-compact).

Proof: We have $f(A) = B$, as f is bijective. Let $\{ u_i : u_i \in t_A \}$ be an open α-shading of (A, t_A) for every $i \in J$ i.e $u_i(x) > \alpha$ for each $x \in A$. Since $u_i \in t_A$, then there exists $v_i \in t$ such that $u_i = v_i | A$ and so $(v_i | A)(x) > \alpha$ for each $x \in A$. As f is relatively fuzzy open, then $f(u_i) \in s_B$. Thus we observe that, for each $y \in B$, $f(u_i)(y) > \alpha$ and hence $\{ f(u_i) : i \in J \}$ is an open α-shading of (B, s_B) implies that $\{ f(v_i | A) : i \in J \} = \{ f(v_i) | f(A) : i \in J \} = \{ f(v_i)|B : i \in J \}$ is an open α -shading of (B, s_B). Since (B, s_B) is α -compact, then $\{ f(v_i)|B : i \in J \}$ has a finite α -subshading, say $\{ f(v_{i_k})|B : k \in J_n \}$ such that $\left(f(v_{i_k}) | B\right)(y) > \alpha$ for each $y \in B$. Now, if $x \in f^{-1}(B)$, then $x = f^{-1}(y)$ for $y \in B$. Then there exists k, we have $\left(f(v_{i_k}) | B\right)(y) > \alpha$ implies that $(v_{i_k} | f^{-1}(B))\,(f^{-1}(y)) > \alpha$ implies that $(v_{i_k} | A)(x) > \alpha$ or $u_{i_k}(x) > \alpha$. Hence it is clear that $\{ u_{i_k} : k \in J_n \}$ is a finite α-subshading of $\{ u_i : u_i \in t_A \}$. Thus (A, t_A) is α -compact.

Similar work for α^*-compactness can be done.

Theorem 3.11: Let (X, t) be an fts. If every family of closed fuzzy sets in (X, t) which has empty intersection has a finite subfamily with empty intersection, then (X, t) is α -compact (resp. α^*-compact). The converse is not true in general.

Proof: Let $M = \{ u_i : i \in J \}$ be an open α -shading of (X, t). From the first condition of the theorem, we have $\bigcap_{i \in J} u_i^c = 0_X$. Thus $\bigcup_{i \in J} u_i = 1_X$. Again, by the second condition of the theorem, we get $\bigcap_{k \in J_n} u_{i_k}^c = 0_X$ implies that $\bigcup_{k \in J_n} u_{i_k} = 1_X$ and hence $u_{i_k}(x) > \alpha$ for each $x \in X$. It is clear that $\{ u_{i_k} : k \in J_n \}$ is a finite α -subshading of M. Therefore (X, t) is α -compact.

Now, for the converse, consider the following example.

Let $X = \{a, b\}$, $I = [0, 1]$ and $0 \le \alpha < 1$. Let u, $v \in I^X$ defined by $u(a) = 0.3$, $u(b) = 0.4$ and $v(a) = 0.6$, $v(b) = 0.7$. Put $t = \{ 0 , u , v , 1\}$, then we see that (X, t) is an fts. Take $\alpha = 0.5$. Then (X, t) is an α -compact. Now, closed fuzzy sets are $u^c(a) = 0.7$, $u^c(b) = 0.6$ and $v^c(a) = 0.4$, $v^c(b) = 0.3$. We observe that $u^c \cap v^c \neq 0$. Thus the converse of the theorem is not necessarily true in general.

The work is similar for α^*-compactness can be given.

Definition 3.12[91]: Let (X, t) be an fts and $0 \le \alpha < 1$, then the family $t_\alpha = \{ \alpha(u) : u \in t \}$ of all subsets of X of the form $\alpha(u) = \{ x \in X : u(x) > \alpha \}$ is called α -level sets, forms a topology on X and is called the α -level topology on X and the pair (X, t_α) is called α -level topological space.

Theorem 3.13: Let $0 \le \alpha < 1$. An fts (X, t) is α -compact iff (X, t_α) is compact topological space.

Proof: For proof cf.[12].

Theorem 3.14[106]: If T is a cofinite topology on X , then (X, T) is compact.

Theorem 3.15: Let (X, t) be an fts and if t_α becomes a cofinite topology on X , then (X, t) is α -compact.

Proof: Let $M = \{ u_i : \ i \in J \}$ be an open α-shading of (X, t). Then $t_\alpha = \{ \alpha(u_i) : \ u_i \in t \}$, where $\alpha(u_i) = \{ x \in X : \ u_i(x) > \alpha \}$ and by the theorem t_α is a cofinite topology on X . We see that $H = \{ \alpha(u_i) : \ i \in J \}$ is an open cover of (X, t_α) . For let, $x \in X$, then there exists a $u_{i_0} \in M$ such that $u_{i_0}(x) > \alpha$. Therefore, $x \in \alpha(u_{i_0})$ and $\alpha(u_{i_0}) \in H$. As (X, t_α) is cofinite, hence compact which implies that H has a finite subcover, say $\{ \alpha(u_{i_k}) \}$ ($k \in J_n$), where $u_{i_k} \in t$ and $\alpha(u_{i_k}) \in t_\alpha$. Then the family $\{u_{i_k}\}$ ($k \in J_n$) forms a finite α -subshading of M and hence (X, t) is α -compact.

Definition 3.16: A mapping $f : (X, t_\alpha) \to (X, t)$ is said to be α -level continuous iff $\alpha(f^{-1}(u)) \in t_\alpha$ for every $u \in t$.

Example 3.17: Let $X = \{a, b, c\}$, $I = [0, 1]$ and $0 \le \alpha < 1$. Let u_1 , u_2 , u_3 , $u_4 \in I^X$ defined by $u_1(a) = 0.4$, $u_1(b) = 0.2$, $u_1(c) = 0.6$; $u_2(a) = 0.2$, $u_2(b) = 0.4$, $u_2(c) = 0.6$; $u_3(a) = 0.4$, $u_3(b) = 0.4$, $u_3(c) = 0.6$ and $u_4(a) = 0.2$, $u_4(b) = 0.2$, $u_4(c) = 0.6$. Now, put $t = \{ 0 , u_1 , u_2 , u_3 , u_4 , 1 \}$, then we see that (X, t) is an fts. Now, we have $t_\alpha = \{ \alpha(\mu) : \ \mu \in t \}$ and $\alpha(\mu) = \{ x \in X : \ \mu(x) > \alpha \}$. Put $\alpha = 0.3$. Then we have $0.3(0)=\phi$, $0.3(1)=X$, $0.3(u_1)=\{a, c\}$, $0.3(u_2)=\{b, c\}$, $0.3(u_3)=X$, $0.3(u_4)=\{c\}$. Therefore, $t_{0.3}=\{\phi, X, \{a, c\}, \{b, c\}, \{c\}\}$ is a topology on X. Let $f:\ (X, t_\alpha)\to(X, t)$ defined by $f(a)=b$, $f(b)=a$, $f(c)=c$. Now, $f^{-1}(0)(X)=0(f(X))=0$, $f^{-1}(1)(X)=1$ for all $x\in X$; $f^{-1}(u_1)(a)=u_1(f(a))=u_1(b)=0.2$, $f^{-1}(u_1)(b)=u_1(f(b))=u_1(a)=0.4$, $f^{-1}(u_1)(c)=u_1(f(c))=u_1(c)=0.6$; $f^{-1}(u_2)(a)=u_2(f(a))=u_2(b)=0.4$, $f^{-1}(u_2)(b)=u_2(f(b))=u_2(a)=0.2$, $f^{-1}(u_2)(c)=u_2(f(c))=u_2(c)=0.6$; $f^{-1}(u_3)(a)=u_3(f(a))=u_3(b)=0.4$, $f^{-1}(u_3)(b)=u_3(f(b))=u_3(a)=0.4$, $f^{-1}(u_3)(c)=u_3(f(c))=u_3(c)=0.6$; $f^{-1}(u_4)(a)=u_4(f(a))=u_4(b)=0.2$, $f^{-1}(u_4)(b)=u_4(f(b))=u_4(a)=0.2$, $f^{-1}(u_4)(c)=u_4(f(c))=u_4(c)=0.6$. Then we observe that $0.3(f^{-1}(0))=\phi$, $0.3(f^{-1}(1))=X$, $0.3(f^{-1}(u_1))=\{b, c\}$, $0.3(f^{-1}(u_2))=\{a, c\}$, $0.3(f^{-1}(u_3))=X$, $0.3(f^{-1}(u_4))=\{c\}$. Therefore ϕ, X, $\{b, c\}$, $\{a, c\}$, $\{c\}\in t_{0.3}$ i.e. $0.3(f^{-1}(0))$, $0.3(f^{-1}(1))$, $0.3(f^{-1}(u_1))$, $0.3(f^{-1}(u_2))$, $0.3(f^{-1}(u_3))$, $0.3(f^{-1}(u_4))\in t_{0.3}$. Hence f is α -level continuous.

Theorem 3.18: Let $f:\ (X, t_\alpha)\to(X, t)$ be α-level continuous and bijective mapping with (X, t_α) is compact. Then (X, t) is α -compact.

Proof: Let $M=\{u_i:\ i\in J\}$ be an open α-shading of (X, t). As f is α-level continuous, then $\alpha(f^{-1}(u_i))\in t_\alpha$ and hence $\{\alpha(f^{-1}(u_i)):\ i\in J\}$ is an open cover of (X, t_α). Since (X, t_α) is compact, then $\{\alpha(f^{-1}(u_i)):\ i\in J\}$ has a finite subcover, say $\{\alpha(f^{-1}(u_{i_k}))\}$ ($k\in J_n$). Now, we have $f(x)=y$ for $y\in X$, as f is bijective. But $\{\alpha(f^{-1}(u_{i_k}))\}$ is finite subcover of $\{\alpha(f^{-1}(u_i)):\ i\in J\}$, there exist some k such that

$u_{i_k}(f(x)) > \alpha$ implies that $u_{i_k}(y) > \alpha$ for each $y \in X$. Thus { u_{i_k} : $k \in J_n$ } is a finite α -subshading of M . Therefore (X, t) is α -compact.

Theorem 3.19: Let (X, t) be a fuzzy T_1-space (as def. 1.45), $A \subset X$ and 1_A be an α -compact (resp. α^*-compact) subset in (X, t). Let $x \in A^c$, then there exist u , $v \in t$ such that $u(x) = 1$ and $A \subseteq v^{-1}(0, 1]$.

Proof: Let $y \in A$. Since $x \notin A$ ($x \in A^c$), then clearly $x \neq y$. As (X, t) is fuzzy T_1-space, then there exist u_y , $v_y \in t$ such that $u_y(x) = 1$, $u_y(y) = 0$ and $v_y(x) = 0$, $v_y(y) = 1$. Let us take $\alpha \in I_1$ such that $v_y(y) > \alpha > 0$. Thus we see that { v_y : $y \in A$ } is an open α -shading of 1_A. Since 1_A is α -compact in (X, t), so it has a finite α -subshading, say { v_{y_k} : $y \in A$ } ($k \in J_n$) such that $v_{y_k}(y) > \alpha$ for each $y \in A$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ldots.. \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $u \in t$. Moreover, $A \subseteq v^{-1}(0, 1]$ and $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k .

Similar proof for α^*-compactness can be given.

Theorem 3.20: Let (X, t) be a fuzzy T_1-space (as def. 1.45) and 1_A , 1_B be disjoint α -compact (resp. α^*-compact) subsets in (X, t) (A , $B \subset X$). Then there exist u , $v \in t$ such that $A \subseteq u^{-1}(0, 1]$ and $B \subseteq v^{-1}(0, 1]$.

Proof: Let $y \in A$. Then $y \notin B$, as 1_A and 1_B are disjoint . Since 1_B is α -compact, then by theorem (3.19), there exist u_y , $v_y \in t$ such that $u_y(y) = 1$ and $B \subseteq v_y^{-1}(0, 1]$. Let us take $\alpha \in I_1$ such that $u_y(y) > \alpha > 0$. As $u_y(y) = 1$, then we see that { u_y : $y \in A$ } is an open α-shading of 1_A. Since 1_A is α-compact in (X, t), so it has a finite α-subshading, say $\{ u_{y_k} : y \in A \}$ $(k \in J_n)$ such that $u_{y_k}(y) > \alpha$ for each $y \in A$. Furthermore, since 1_B is α-compact, so 1_B has a finite α-subshading, say $\{ v_{y_k} : x \in B \}$ $(k \in J_n)$ such that $v_{y_k}(x) > \alpha$ for each $x \in B$, as $B \subseteq v_{y_k}^{-1}(0, 1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots.. \cap v_{y_n}$. Thus we see that $A \subseteq u^{-1}(0, 1]$ and $B \subseteq v^{-1}(0, 1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u , $v \in t$.

Similar work for α^*-compactness can be given.

Theorem 3.21: Let (X, t) be a fuzzy T_1-space (as def. 1.45) and $A \subset X$. If 1_A is α-compact (resp. α^*-compact) subset in (X, t), then 1_A is closed.

Proof: Let $x \in A^c$. We have to show that, there exist $u \in t$ such that $u(x) = 1$ and $u \subseteq A^p$, where A^p is the characteristic function of A^c. Indeed, for each $y \in A$, there exist u_y , $v_y \in t$ such that $u_y(x) = 1$, $u_y(y) = 0$ and $v_y(x) = 0$, $v_y(y) = 1$. Let us take $\alpha \in I_1$ such that $v_y(y) > \alpha > 0$. Thus we see that $\{ v_y : y \in A \}$ is an α-shading of 1_A. Since 1_A is α-compact in (X, t), so it has a finite α-subshading, say $\{ v_{y_k} : y \in A \}$ $(k \in J_n)$ such that $v_{y_k}(y) > \alpha$ for each $y \in A$. Now, let $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Thus we see that $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k. For, each $z \in A$, there exists a k such that $v_{y_k}(z) > \alpha \geq 0$ and so $u(z) = 0$. Hence $u \subseteq A^p$. Therefore, 1_{A^c} is open in (X, t). Thus 1_A is closed in (X, t).

The proof is similar for α^*-compactness can be given.

Theorem 3.22: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and $A \subset X$. If 1_A is α-compact (resp. α^*-compact) subset in (X, t) and $x \in A^c$, then there exist u, $v \in t$ such that $u(x) > 0$ and $A \subseteq v^{-1}(0,1]$.The converse of the theorem is not necessarily true in general.

The proof is similar as that of theorem (3.19).

Now, for the converse, consider the following example.

Let $X = \{a, b\}$, $I =$ $[0,1]$ and $0 \le \alpha < 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) =$ 0.2, $u_1(b) = 0$; $u_2(a) = 0$, $u_2(b) = 0.3$ and $u_3(a) = 0.2$, $u_3(b) = 0.3$. Now, put $t = \{ 0 , u_1 , u_2 , u_3 , 1 \}$, then we see that (X, t) is a fuzzy T_1-space. Again, let $1_A \in I^X$ defined by $1_A(a) = 0$, $1_A(b) = 1$. Hence we observe that $A = \{b\}$ and $a \in A^c$. Now u_1, $u_2 \in t$ where $u_1(a) > 0$ and $u_2^{-1}(0,1] = \{b\}$. Hence $A \subseteq u_2^{-1}(0,1]$. Take $\alpha = 0.8$. Then we see that 1_A is not α-compact in (X , t), as $u_k(b) < \alpha$ for $b \in A$ and $k = 1, 2,$ 3. Thus the converse of the theorem is not true in general.

Similar work for α^*-compactness can be given.

Theorem 3.23: Let (X , t) be a fuzzy T_1-space (as def. 1.46) and $A, B \subset X$. If 1_A and 1_B are disjoint α-compact (resp. α^*-compact) subsets in (X, t), then there exist u, $v \in t$ such that $A \subseteq u^{-1}(0,1]$ and $B \subseteq v^{-1}(0,1]$. The converse of the theorem is not true in general.

The proof is similar as that of theorem (3.20).

Now, for the converse, consider the fuzzy T_1-space (X, t) in the example of the theorem (3.22). Let 1_A, $1_B \in I^X$ defined by $1_A(a) = 1$, $1_A(b) = 0$ and $1_B(a) = 0$, $1_B(b) = 1$. Hence we observe that $A = \{a\}$ and $B = \{b\}$. Now u_1, $u_2 \in t$ where $u_1^{-1}(0,1]$ $= \{a\}$ and $u_2^{-1}(0,1] = \{b\}$. Hence we observe that $A \subseteq u_1^{-1}(0,1]$ and $B \subseteq u_2^{-1}(0,1]$, where 1_A and 1_B

are disjoint. Take $\alpha = 0.8$. Then we see that 1_A and 1_B are not α-compact in (X, t), as $u_k(a) < \alpha$ for $a \in A$ and $u_k(b) < \alpha$ for $b \in B$, where $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

Similar proof for α^*-compactness can be given.

The following example will show that the α-compact subsets in fuzzy T_1-space (as def. 1.46) need not be closed.

Example 3.24: Consider the fuzzy T_1-space (X, t) in the example of the theorem (3.22). Again, let $1_A \in I^X$ defined by $1_A(a) = 0$, $1_A(b) = 1$. Take $\alpha = 0.2$. Then clearly 1_A is α-compact in (X, t). But 1_A is not closed, as its complements 1_{A^c} is not open in (X, t).

Theorem 3.25: Let (X, t) be a fuzzy Hausdorff space (as def. 1.47), $A \subset X$ and 1_A be an α-compact (resp. α^*-compact) subset in (X, t). Let $x \in A^c$, then there exist u, $v \in t$ such that $u(x) = 1$, $A \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$.

Proof: Let $y \in A$. Since $x \notin A$ ($x \in A^c$), then clearly $x \neq y$. As (X, t) is fuzzy Hausdorff, then there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \cap v_y = 0$. Let us take $\alpha \in I_1$ such that $v_y(y) > \alpha > 0$. Thus we see that $\{ v_y : y \in A \}$ is an open α-shading of 1_A. Since 1_A is α-compact in (X, t), so it has a finite α-subshading, say $\{ v_{y_k} : y \in A \}$ ($k \in J_n$) such that $v_{y_k}(y) > \alpha$ for each $y \in A$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ldots.. \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $u \in t$. Moreover, $A \subseteq v^{-1}(0,1]$ and $u(x)=1$, as $u_{y_k}(x)=1$ for each k.

Finally, we claim that $u \cap v = 0$. As $u_{y_k} \cap v_{y_k} = 0$ implies that $u \cap v_{y_k} = 0$, by distributive law, we see that $u \cap v =$ $u \cap (\ v_{y_1} \cup v_{y_2} \cup \ldots.. \cup v_{y_n}\ ) = 0$.

Similar work for α^*-compactness can be given.

Corollary 3.26: Let (X,t) be a fuzzy Hausdroff space (as def. 1.47), $A \subset X$ and 1_A be an α-compact (resp. α^*-compact) subset in (X,t). Let $x \notin A$, then there exists $u \in t$ such that $u(x)=1$ and $u^{-1}(0,1] \subseteq A^c$.

Proof: By theorem (3.25), there exist u , $v \in t$ such that $u(x)=1$, $A \subseteq v^{-1}(0,1]$ and $u \cap v = 0$. Hence $u^{-1}(0,1] \cap v^{-1}(0,1] = \phi$. If not, there exists $x \in u^{-1}(0,1] \cap v^{-1}(0,1]$ $\Rightarrow$ $x \in u^{-1}(0,1]$ and $x \in v^{-1}(0,1]$ $\Rightarrow$ $u(x) > 0$ and $v(x) > 0$ $\Rightarrow$ $u \cap v \neq 0$. Hence $u^{-1}(0,1] \cap A = \phi$ and consequently $u^{-1}(0,1] \subseteq A^c$.

Similar proof for α^*-compactness can be given.

Theorem 3.27: Let (X,t) be a fuzzy Hausdorff space (as def. 1.47) and 1_A , 1_B be disjoint α-compact (resp. α^*-compact) subsets in (X,t) (A , $B \subset X$). Then there exist u , $v \in t$ such that $A \subseteq u^{-1}(0,1]$, $B \subseteq v^{-1}(0,1]$ and $u \cap v = 0$.

Proof: Let $y \in A$. Then $y \notin B$, as 1_A and 1_B are disjoint. Since 1_B is α-compact , then by theorem (3.25), there exist u_y , $v_y \in t$ such that $u_y(y)=1$, $B \subseteq v_y^{-1}(0,1]$ and $u_y \cap v_y = 0$. Let us take $\alpha \in I_1$ such that $u_y(y) > \alpha > 0$. As $u_y(y)=1$, then we see that $\{\ u_y :\ y \in A\ \}$ is an open α-shading of 1_A. Since 1_A is α-compact in (X,t), so it has a finite α-subshading, say $\{\ u_{y_k} :\ y \in A\ \}$ ($k \in J_n$) such that $u_{y_k}(y) > \alpha$ for each $y \in A$.

Furthermore, since 1_B is α -compact, so 1_B has a finite α -subshading, say $\{ v_{y_k} : x \in B \}$ ($k \in J_n$) such that $v_{y_k}(x) > \alpha$ for each $x \in B$, as $B \subseteq v_{y_k}^{-1}(0,1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Thus we see that $A \subseteq u^{-1}(0,1]$ and $B \subseteq v^{-1}(0,1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u , $v \in t$.

Lastly, we have to show that $u \cap v = 0$. First, we observe that $u_{y_k} \cap v_{y_k} = 0$ for each k implies that $u_{y_k} \cap v = 0$, by distributive law , we see that $u \cap v =$ ($u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$) $\cap v = 0$.

Similar proof for α^*-compactness can be given.

Theorem 3.28: Let (X, t) be a fuzzy Hausdorff space (as def. 1.47), $A \subset X$. If 1_A is α -compact (resp. α^*-compact) subset in (X, t), then 1_A is closed.

Proof: cf. [54].

Theorem 3.29: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and $A \subset X$. If 1_A is α -compact (resp. α^*-compact) subset in (X, t) and $x \in A^c$, then there exist u , $v \in t$ such that $u(x) > 0$, $A \subseteq v^{-1}(0,1]$ and $u \cap v = 0$ .The converse of the theorem is not necessarily true in general.

The proof is similar as that of theorem (3.25).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (3.22), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Again, let $1_A \in I^X$ defined by $1_A(a) = 0$, $1_A(b) = 1$. Hence we observe that $A = \{b\}$ and $a \in A^c$. Now u_1 , $u_2 \in t$ where $u_1(a) > 0$ and $u_2^{-1}(0,1] = \{b\}$. Hence $A \subseteq u_2^{-1}(0,1]$ and $u_1 \cap u_2 = 0$.

Take $\alpha = 0.8$. Then we see that 1_A is not α -compact in (X, t), as $u_k(b) < \alpha$ for $b \in A$ and $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

Similar work for α^*-compactness can be given.

Corollary 3.30: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and $A \subset X$. If 1_A is α -compact (resp. α^*-compact) subset in (X, t) and $x \notin A$, then there exists $u \in t$ such that $u(x) > 0$ and $u^{-1}(0, 1] \subseteq A^c$. The converse is not true in general.

The proof is similar as that of corollary (3.26).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (3.22), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Let $1_A \in I^X$ defined by $1_A(a) = 0$, $1_A(b) = 1$. Hence we observe that $A = \{b\}$ and $a \notin A$. Now $u_1 \in t$ where $u_1(a) > 0$ and then $u_1^{-1}(0, 1] = \{a\}$. Hence we have $u_1^{-1}(0, 1] \subseteq A^c$. Take $\alpha = 0.8$. Thus we see that 1_A is not α -compact in (X, t) i.e. $u_k(b) < \alpha$ for $b \in A$, where $k = 1, 2, 3$. Thus the converse of the corollary is not true in general.

Similar proof for α^*-compactness can be given.

Theorem 3.31: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and $A, B \subset X$. If 1_A and 1_B are disjoint α -compact (resp. α^*-compact) subsets in (X, t), then there exist $u, v \in t$ such that $A \subseteq u^{-1}(0, 1]$, $B \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$. The converse of the theorem is not true in general.

The proof is similar as that of theorem (3.27).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (3.22), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Let $1_A, 1_B \in I^X$ defined by $1_A(a) = 1$, $1_A(b) = 0$ and $1_B(a) = 0$, $1_B(b) = 1$. Hence we observe that $A = \{a\}$ and

$B = \{b\}$. Now u_1, $u_2 \in t$ where $u_1^{-1}(0,1] = \{a\}$ and $u_2^{-1}(0,1] = \{b\}$. Hence we observe that $A \subseteq u_1^{-1}(0,1]$, $B \subseteq u_2^{-1}(0,1]$ and $u_1 \cap u_2 = 0$, where 1_A and 1_B are disjoint. Take $\alpha = 0.8$. Then we see that 1_A and 1_B are not α-compact in (X, t), as $u_k(a) < \alpha$ for $a \in A$ and $u_k(b) < \alpha$ for $b \in B$, where $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

Similar work for α^*-compactness can be given.

The following example will show that the α-compact subsets in fuzzy Hausdorff space (as def. 1.48) need not be closed.

Example 3.32: Consider the fuzzy topology t in the example of the theorem (3.22), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Again, let $1_A \in I^X$ defined by $1_A(a) = 1$, $1_A(b) = 0$. Take $\alpha = 0.1$. Then clearly 1_A is α-compact in (X, t). But 1_A is not closed, as its complement 1_{A^c} is not open in (X, t).

Theorem 3.33: Let (X, t) be a fuzzy Hausdorff space (as def. 1.49), $A \subset X$ and 1_A be an α-compact (resp. α^*-compact) subset in (X, t). Suppose x_r be a fuzzy point in 1_{A^c}, then there exist u, $v \in t$ such that $x_r \in u$, $A \subseteq v^{-1}(0,1]$ and $u \cap v = 0$.

Proof: Let y_s ($s > \alpha$) be fuzzy point in 1_A, then clearly $x \neq y$ i.e. x_r and y_s are distinct . As (X, t) is fuzzy Hausdorff, then there exist u_{y_s}, $v_{y_s} \in t$ such that $x_r \in u_{y_s}$, $y_s \in v_{y_s}$ and $u_{y_s} \cap v_{y_s} = 0$ and this is true for any value of s. Hence this is also true for $s > \alpha$. Let us take $\alpha \in I_1$ such that $v_{y_s}(y) > \alpha > 0$. Thus we see that $\{ v_{y_s} : y_s \in 1_A \}$ is an open α-shading of 1_A. Since 1_A is α-compact in (X, t) , so it has a finite α-subshading, say $\{ v_{y_{s_k}} : y_s \in 1_A \}$ ($k \in J_n$) such that $v_{y_{s_k}}(y) > \alpha$. Let $v = v_{y_{s_1}} \cup v_{y_{s_2}}$

$\cup \ldots\ldots \cup v_{y_{s_n}}$ and $u = u_{y_{s_1}} \cap u_{y_{s_2}} \cap \ldots\ldots \cap u_{y_{s_n}}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $u \in t$. Moreover, $A \subseteq v^{-1}(0,1]$ and $x_r \in u$, since $x_r \in u_{y_{s_k}}$ for each k.

Finally, we claim that $u \cap v = 0$. As $u_{y_{s_k}} \cap v_{y_{s_k}} = 0$ for each k implies that $u \cap v_{y_{s_k}} = 0$, by distributive law, we therefore observe that $u \cap v =$ $u \cap (v_{y_{s_1}} \cup v_{y_{s_2}} \cup \ldots\ldots \cup v_{y_{s_n}}) = 0$.

The proof is similar for α^*-compactness can be done.

Corollary 3.34: Let (X,t) be a fuzzy Hausdroff space (as def. 1.49), $A \subset X$ and 1_A be an α-compact (resp. α^*-compact) subset in (X,t). Let $x_r \notin 1_A$, then there exists $u \in t$ such that $x_r \in u$ and $u^{-1}(0,1] \subseteq A^c$.

Proof: By theorem (3.33), there exist u , $v \in t$ such that $x_r \in u$, $A \subseteq v^{-1}(0,1]$ and $u \cap v = 0$. Hence $u^{-1}(0,1] \cap v^{-1}(0,1] = \phi$. If not, there exists $x \in u^{-1}(0,1] \cap v^{-1}(0,1]$ $\Rightarrow$ $x \in u^{-1}(0,1]$ and $x \in v^{-1}(0,1]$ $\Rightarrow$ $u(x) > 0$ and $v(x) > 0$ $\Rightarrow$ $u \cap v \neq 0$. Hence $u^{-1}(0,1] \cap A = \phi$ and consequently $u^{-1}(0,1] \subseteq A^c$.

Similar proof for α^*-compactness can be given.

Theorem 3.35: Let (X,t) be a fuzzy Hausdorff space (as def. 1.49) and 1_A , 1_B be disjoint α-compact (resp. α^*-compact) subsets in (X,t) (A , $B \subset X$). Then there exist u , $v \in t$ such that $A \subseteq u^{-1}(0,1]$, $B \subseteq v^{-1}(0,1]$ and $u \cap v = 0$.

Proof: Let $y_s \in 1_A$ ($s > \alpha$), then clearly $y_s \notin 1_B$, as 1_A and 1_B are disjoint . Since 1_B is α-compact, then by theorem (3.33), there exist u_{y_s} , $v_{y_s} \in t$ such that $y_s \in u_{y_s}$, $B \subseteq v_{y_s}^{-1}(0,1]$ and $u_{y_s} \cap v_{y_s} = 0$ and this is true for any value of s. Hence this is also true for $s > \alpha$. Let us take $\alpha \in I_1$ such that $u_{y_s}(y) > \alpha > 0$. Since $y_s \in u_{y_s}$, then $\{ u_{y_s} : y_s \in 1_A \}$ is an open α -shading of 1_A. Since 1_A is α -compact in (X, t), so it has a finite α -subshading, say $\{ u_{y_{s_k}} : y_s \in 1_A \}$ ($k \in J_n$) such that $u_{y_{s_k}}(y) > \alpha$. Furthermore, since 1_B is α -compact, so 1_B has a finite α -subshading, say $\{ v_{y_{s_k}} : x_r \in 1_B \}$ ($k \in J_n$) such that $v_{y_{s_k}}(x) > \alpha$, as $B \subseteq v_{y_{s_k}}^{-1}(0, 1]$ for each k. Now, let $u = u_{y_{s_1}} \cup u_{y_{s_2}} \cup \ldots\ldots \cup u_{y_{s_n}}$ and $v = v_{y_{s_1}} \cap v_{y_{s_2}} \cap \ldots\ldots \cap v_{y_{s_n}}$. Thus we see that $A \subseteq u^{-1}(0, 1]$ and $B \subseteq v^{-1}(0, 1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u , $v \in t$.

Finally, we have to show that $u \cap v = 0$. First , we observe that $u_{y_{s_k}} \cap v_{y_{s_k}} = 0$ for each k implies that $u_{y_{s_k}} \cap v = 0$, by distributive law, we see that $u \cap v = (u_{y_{s_1}} \cup u_{y_{s_2}} \cup \ldots\ldots \cup u_{y_{s_n}}) \cap v = 0$.

Similar work for α^*-compactness can be given.

The following example will show that the α -compact (resp. α^* -compact) subsets in fuzzy Hausdorff space (as def. 1.49) need not be closed.

Example 3.36: Let $X = \{a, b\}$, $I = [0, 1]$ and $0 \le \alpha < 1$. Again, let u_1 , u_2 , $u_3 \in I^X$ with $u_1(a) = 0.6$, $u_1(b) = 0$; $u_2(a) = 0$, $u_2(b) = 0.8$ and $u_3(a) = 0.6$, $u_3(b) = 0.8$. Put $t = \{ 0 , u_1 , u_2 , u_3 , 1 \}$, then (X, t) is an fts. Now, let $a_{0.4}$ and $b_{0.7}$ be fuzzy points in X. Therefore (X, t) is also a fuzzy Hausdorff space (as def. 1.49). Again, let $1_A \in I^X$ defined by $1_A(a) = 1$, $1_A(b) = 0$. Take $\alpha = 0.5$. Then clearly 1_A is α -compact in (X, t). But 1_A is not closed, as its complement 1_{A^c} is not open in (X, t).

Theorem 3.37: Let (X,t) be a fuzzy Hausdorff space (as def. 1.50), $A \subset X$ and 1_A be an α-compact (resp. α^*-compact) subset in (X,t). Suppose $x \in A^c$, then there exist u, $v \in t$ such that $u(x)=1$, $A \subseteq v^{-1}(0, 1]$ and $u \subseteq 1-v$.

Proof: Let $y \in A$. Since $x \notin A$ ($x \in A^c$), then clearly $x \neq y$. As (X,t) is fuzzy Hausdorff, then there exist u_y, $v_y \in t$ such that $u_y(x)=1$, $v_y(y)=1$ and $u_y \subseteq 1-v_y$. Let us take $\alpha \in I_1$ such that $v_y(y) > \alpha > 0$. Thus we see that $\{ v_y : y \in A \}$ is an open α-shading of 1_A. Since 1_A is α-compact in (X,t), so it has a finite α-subshading, say $\{ v_{y_k} : y \in A \}$ ($k \in J_n$) such that $v_{y_k}(y) > \alpha$ for each $y \in A$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ldots.. \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, $u \in t$. Moreover, $A \subseteq v^{-1}(0,1]$ and $u(x)=1$, as $u_{y_k}(x)=1$ for each k.

Finally, we claim that $u \subseteq 1-v$. As $u_y \subseteq 1-v_y$, so $u \subseteq 1-v_y$. Since $u_{y_k}(x) \leq 1-v_{y_k}(x)$ for all $x \in X$ and for each k, then $u \subseteq 1-v$. If not, there exists $x \in X$ such that $u_y(x) > 1-v_y(x)$. We have $u_y(x) \leq u_{y_k}(x)$ for each k. Then for some k, $u_{y_k}(x) > 1-v_{y_k}(x)$. But this is a contradiction, as $u_{y_k}(x) \leq 1-v_{y_k}(x)$ for each k. Hence $u \subseteq 1-v$.

Similar proof of α^*-compactness can be given.

Theorem 3.38: Let (X,t) be a fuzzy Hausdorff space (as def. 1.50) and 1_A, 1_B be disjoint α-compact (resp. α^*-compact) subsets in (X,t) (A, $B \subset X$). Then there exist u, $v \in t$ such that $A \subseteq u^{-1}(0,1]$, $B \subseteq v^{-1}(0,1]$ and $u \subseteq 1-v$.

Proof: Let $y \in A$. Then $y \notin B$, as 1_A and 1_B are disjoint. Since 1_B is α-compact, then by theorem (3.37), there exist u_y, $v_y \in t$ such that $u_y(y)=1$, $B \subseteq v_y^{-1}(0,1]$ and

$u_y \subseteq 1 - v_y$. Let us take $\alpha \in I_1$ such that $u_y(y) > \alpha > 0$. As $u_y(y) = 1$, then we observe that $\{ u_y :\ y \in A \}$ is an open α -shading of 1_A. Since 1_A is α -compact in (X, t), so it has a finite α -subshading, say $\{ u_{y_k} :\ y \in A \}$ ($k \in J_n$) such that $u_{y_k}(y) > \alpha$ for each $y \in A$. Furthermore, since 1_B is α -compact, so 1_B has a finite α -subshading, say $\{ v_{y_k} :\ x \in B \}$ ($k \in J_n$) such that $v_{y_k}(x) > \alpha$ for each $x \in B$, as $B \subseteq v_{y_k}^{-1}(0, 1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Thus we see that $A \subseteq u^{-1}(0, 1]$ and $B \subseteq v^{-1}(0, 1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u , $v \in t$.

Finally, we have to show that $u \subseteq 1 - v$. First we observe that $u_{y_k} \subseteq 1 - v_{y_k}$ for each k implies that $u_{y_k} \subseteq 1 - v$ for each k and it is clearly shows that $u \subseteq 1 - v$.

Similar proof for α^*-compactness can be done.

Theorem 3.39: Let (X, t) be a fuzzy Hausdorff space (as def. 1.50) and $A \subset X$. If 1_A is α -compact (resp. α^*-compact) subset in (X, t), then 1_A is closed.

Proof: Let $x \in A^c$. We have to show that, there exist $u \in t$ such that $u(x) = 1$ and $u \subseteq A^p$, where A^p is the characteristic function of A^c. Suppose, for each $y \in A$, there exist u_y , $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \subseteq 1 - v_y$. Let us take $\alpha \in I_1$ such that $v_y(y) > \alpha > 0$. Thus we see that $\{ v_y :\ y \in A \}$ is an open α -shading of 1_A. Since 1_A is α -compact in (X, t), so it has a finite α -subshading, say $\{ v_{y_k} :\ y \in A \}$ ($k \in J_n$) such that $v_{y_k}(y) > \alpha$ for each $y \in A$. Now, let $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$ and $v = v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$. Thus we see that $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k and $u_y \subseteq 1 - v_y$ implies that $u \subseteq 1 - v_y$. But $u_{y_k}(x) \leq 1 - v_{y_k}(x)$ for all $x \in X$ and for each k, then $u \subseteq 1-v$. If not, there exists $x \in X$ such that $u_y(x) > 1-v_y(x)$. We have $u_y(x) \le u_{y_k}(x)$ for each k. Then for some k, $u_{y_k}(x) > 1-v_{y_k}(x)$. But this is a contradiction, as $u_{y_k}(x) \le 1-v_{y_k}(x)$ for each k. Hence $u \subseteq 1-v$. For, each $z \in A$, there exists k such that $v_{y_k}(z) > \alpha \ge 0$ and so $u(z) = 0$. Hence $u \subseteq A^p$. Therefore, 1_{A^c} is open in (X, t). Thus 1_A is closed in (X, t).

The proof is similar for α^*-compactness can be done.

Theorem 3.40: Let (X, t) be a fuzzy regular space (as def. 1.51), $A \subset X$ and 1_A be an α-compact (resp. α^*-compact) subset in (X, t). If for each $x \in A$, there exists $u \in t^c$ with $u(x) = 0$, we have v, $w \in t$ such that $v(x) = 1$, $u \subseteq w$, $A \subseteq v^{-1}(0,1]$ and $v \subseteq 1-w$.

Proof: Suppose $x \in A$ and $u \in t^c$ we have $u(x) = 0$. As (X, t) is fuzzy regular, then there exist v_x, $w_x \in t$ such that $v_x(x) = 1$, $u_x \subseteq w_x$ and $v_x \subseteq 1-w_x$. Let us take $\alpha \in I_1$ such that $v_x(x) > \alpha > 0$. Thus we observe that $\{ v_x : x \in A \}$ is an open α-shading of 1_A. Since 1_A is α-compact in (X, t), then it has a finite α-subshading, say $\{ v_{x_k} : x \in A \}$ ($k \in J_n$) such that $v_{x_k}(x) > \alpha$ for each $x \in A$. Let $v = v_{x_1} \cup v_{x_2} \cup \ldots.. \cup v_{x_n}$ and $w = w_{x_1} \cap w_{x_2} \cap \ldots.. \cap w_{x_n}$. Thus we see that v and w are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, $w \in t$. Furthermore, $A \subseteq v^{-1}(0,1]$, $v(x) = 1$, and $u \subseteq w$, as $u \subseteq w_{x_k}$ for each k.

Finally, we have to show that $v \subseteq 1-w$. As $v_{x_k} \subseteq 1-w_{x_k}$ for each k implies that $v_{x_k} \subseteq 1-w$ for each k and hence it is clear that $v \subseteq 1-w$.

Similar proof for α^*-compactness can be given.

Theorem 3.41: A topological space (X, T) is compact iff $(X, \omega(T))$ is α -compact (resp. α^*-compact).

Proof: Suppose (X, T) is compact. Let $M = \{ u_i : i \in J \}$ be an open α -shading of $(X, \omega(T))$. Then $u_i^{-1}(a, 1] \in T$ and $\{ u_i^{-1}(a, 1] : u_i^{-1}(a, 1] \in T \}$ is an open cover of (X, T). As (X, T) is compact, so it has a finite subcover i.e. there exist $u_{i_k}^{-1}(a, 1] \in T$ ($k \in J_n$) such that $X = u_{i_1}^{-1}(a, 1] \cup u_{i_2}^{-1}(a, 1] \cup ..... \cup u_{i_n}^{-1}(a, 1]$. Now, we observe that there exist $u_{i_k} \in \{u_i\}$ ($k \in J_n$) such that $u_{i_k}(x) > \alpha$ for each $x \in X$ and it is shows that $\{u_{i_k}\}$ ($k \in J_n$) is a finite α -subshading of M . Therefore, $(X, \omega(T))$ is α -compact.

Conversely, suppose that $(X, \omega(T))$ is α -compact. Let $\{ V_j : i \in J \}$ be open cover of (X, T) i.e. $X = \bigcup_{j \in J}\{ V_j : V_j \in T \}$. Since 1_{V_j} is l. s. c, then $1_{V_j} \in \omega(T)$ and $\{ 1_{V_j} : 1_{V_j} \in \omega(T) \}$ is an open α -shading of $(X, \omega(T))$. As $(X, \omega(T))$ is α -compact, so it has a finite α -subshading, say $\{ 1_{V_{j_k}} : 1_{V_{j_k}} \in \omega(T) \}$ ($k \in J_n$) such that $1_{V_{j_k}}(x) > \alpha$ for each $x \in X$. Therefore, we can write $X = V_{j_1} \cup V_{j_2} \cup .....$ $\cup V_{j_n}$ and it is clear that $\{V_{j_k}\}$ ($k \in J_n$) is a finite subcover of (X, T). Hence (X, T) is compact.

Similar work for α^*-compactness can be given.

Theorem 3.42: Let (X, t) and (Y, s) be two fuzzy topological spaces. Then the product space $(X \times Y, t \times s)$ is α -compact iff (X, t) and (Y, s) are α -compact.

Proof: First suppose that $(X \times Y, \sigma)$, where $\sigma = \{ g_i \times h_i : g_i \in t \text{ and } h_i \in s \}$ is α -compact. Now we can define a fuzzy projection mappings $\pi_x : (X \times Y, \delta) \to (X, t)$ such that $\pi_x(x, y) = x$ for all $(x, y) \in X \times Y$ and $\pi_y : (X \times Y, \delta) \to (Y, s)$ such that

$\pi_y(x, y) = y$ for all $(x, y) \in X \times Y$ which we know are continuous. Hence (X, t) and (Y, s) are continuous images of $(X \times Y, \sigma)$ which are therefore α-compact when $(X \times Y, \sigma)$ is given to be α-compact.

Conversely, let (X, t) and (Y, s) be α-compact. Let $\sigma = \{ g_i \times h_i : g_i \in t \text{ and } h_i \in s \}$, where g_i and h_i are open fuzzy sets in t and s respectively. Therefore $\{ g_i : i \in J \}$ is an α-shading of (X, t) and $\{ h_i : i \in J \}$ is an α-shading of (Y, s). That is $g_i(x) > \alpha$ for all $x \in X$, $h_i(y) > \alpha$ for all $y \in Y$. We see that $(g_i \times h_i)(x, y) = \min\{ g_i(x), h_i(y) \} > \alpha$. As (X, t) and (Y, s) are α-compact, there exist $g_{i_k} \in t$ such that $g_{i_k}(x) > \alpha$ for each $x \in X$ and $h_{i_k} \in s$ such that $h_{i_k}(y) > \alpha$ for each $y \in Y$ respectively. Hence we have $\sigma = \{ g_i \times h_i : g_i \in t \text{ and } h_i \in s \}$ has a finite α-subshading, say $\{ g_{i_k} \times h_{i_k} : k \in J_n \}$ such that $(g_{i_k} \times h_{i_k})(x, y) > \alpha$ for each $(x, y) \in X \times Y$. Thus $(X \times Y, \sigma)$ is α-compact.

Definition 3.43: Let (X, t) be an fts and $0 < \delta \leq 1$, $\alpha \in I$. A family M of δ-open fuzzy sets is called a δ-α-shading, $0 \leq \alpha < 1$ (resp. δ-α^*-shading, $0 < \alpha \leq 1$) of X if for each $x \in X$ there exists a $u \in M$ with $u(x) > \alpha$ (resp. $u(x) \geq \alpha$). A subfamily of a δ-α-shading (resp. δ-α^*-shading) of X which is also a δ-α-shading (resp. δ-α^*-shading) of X is called a δ-α-subshading (resp. δ-α^*-subshading) of X.

Example 3.44: Let $X = \{a, b\}$, $I = [0, 1]$ and $0 < \delta \leq 1$, $0 \leq \alpha < 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 1$, $u_1(b) = 0.6$; $u_2(a) = 0.7$, $u_2(b) = 1$ and $u_3(a) = 0.7$, $u_3(b) = 0.6$. Now, take $t = \{ 0, u_1, u_2, u_3, 1 \}$, then we see that (X, t) is an fts. Take $\delta = 0.6$. Clearly u_1, u_2 and u_3 are δ-open fuzzy sets in (X, t). Again, take $\alpha = 0.8$.

Hence we observe that $u_1(a) > \alpha$, $u_2(b) > \alpha$ for a , $b \in X$. So $\{ u_1 ,\ u_2 \}$ is a δ - α -shading of X .

Similarly, we can give of δ - α^* -shading of X .

Definition 3.45: Let (X, t) be an fts and $0 < \delta \le 1$, $\alpha \in I$. Then (X, t) is said to be δ - α -compact, $0 \le \alpha < 1$ (resp. δ - α^* -compact, $0 < \alpha \le 1$) iff every δ - α -shading (resp. δ - α^* -shading) of X has a finite δ - α -subshading (resp. δ - α^* -subshading).

Theorem 3.46: Every δ - α -compact (resp. δ - α^* -compact) spaces is α -compact (resp. α^* -compact). But the converse is not true.

The proof is straightforward.

Now, for the converse, consider the following example.

Let $X = [0, 1]$, $I = [0, 1]$ and $0 < \delta \le 1$, $0 \le \alpha < 1$. Let u_1 , u_2 , $u_3 \in I^X$ defined by

$$u_1(x) = \begin{cases} 1 & \text{for } 0 \le x < 0.7 \\ 1 & \text{for x} = 0.7 \\ 0.4 & \text{for } 0.7 < x \le 1 \end{cases} \quad , \quad u_2(x) = \begin{cases} 0.6 & \text{for } 0 \le x < 0.7 \\ 1 & \text{for x} = 0.7 \\ 1 & \text{for } 0.7 < x \le 1 \end{cases} \quad \text{and}$$

$$u_3(x) = \begin{cases} 0.6 & \text{for } 0 \le x < 0.7 \\ 1 & \text{for x} = 0.7 \\ 0.4 & \text{for } 0.7 < x \le 1 \end{cases}$$

. Now, take $t = \{ 0 ,\ u_1 ,\ u_2 ,\ u_3 ,\ 1 \}$, then we see that (X, t) is an fts. Take $\alpha = 0.8$. Clearly (X, t) is α -compact. Again take $\delta = 0.9$. Then there is no finite δ -open fuzzy sets u_k for $k = 1 ,\ 2 ,\ 3$ in (X, t). Thus (X, t) is not δ - α -compact.

Similarly, we can prove for δ - α^* -compact spaces.

Chapter Four

Compact Fuzzy Sets

Compact fuzzy sets due to Chang [19] is local property. In this chapter, we have discussed various properties of this concept and established some theorems, corollaries and examples. Also we have defined δ-compact fuzzy sets and found different properties between compact and δ-compact fuzzy sets.

Definition 4.1[19]: A fuzzy set λ in X is said to be compact iff every open cover of λ has a finite subcover i.e. there exist u_{i_1}, u_{i_2}, ……, $u_{i_n} \in \{u_i\}$ such that $\lambda \subseteq u_{i_1} \cup u_{i_2} \cup \ldots\ldots \cup u_{i_n}$ or equivalently, a fuzzy set λ in X is said to be compact iff every open cover of λ has a finite subcover. If $\mu \subseteq \lambda$ and $\mu \in I^X$, then μ is also compact. Thus we can say that, any other subsets of a compact fuzzy set is also compact. If $\lambda(x) = 1$ for all $x \in X$, then this definition coincides an fts (X, t) with that of Chang [19].

Theorem 4.2: Let (X, t) be an fts, $A \subset X$ and λ be a fuzzy set in X with $\lambda_0 \subseteq A$. Then λ is compact in (X, t) iff λ is compact in (A, t_A).

Proof: Suppose λ is compact in (X, t). Let $\{ u_i : i \in J \}$ be an open cover of λ in (A, t_A). Then there exist $v_i \in t$ such that $u_i = v_i \mid A \subseteq v_i$. Hence $\lambda \subseteq \bigcup_{i \in J} u_i \subseteq \bigcup_{i \in J} v_i$ and consequently $\{ v_i : i \in J \}$ is an open cover of λ in (X, t). As λ is compact in (X, t), then $\{ v_i : i \in J \}$ contains a finite subcover i.e. there exist v_{i_1}, v_{i_2}, ……, $v_{i_n} \in \{v_i\}$ such that $\lambda \subseteq v_{i_1} \cup v_{i_2} \cup \ldots\ldots \cup v_{i_n}$. But, then $\lambda \subseteq (v_{i_1} \cup v_{i_2} \cup \ldots\ldots \cup v_{i_n}) \mid A$

$= (v_{i_1} \mid A) \cup (v_{i_2} \mid A) \cup \ldots\ldots \cup (v_{i_n} \mid A) = u_{i_1} \cup u_{i_2} \cup \ldots\ldots\ldots \cup u_{i_n}$, as $\lambda_0 \subseteq A$. Thus $\{ u_i : i \in J \}$ contains a finite subcover $\{ u_{i_1}, u_{i_2}, \ldots\ldots, u_{i_n} \}$ and hence λ is compact in (A, t_A).

Conversely, suppose λ is compact in (A, t_A). Let $\{ v_i : i \in J \}$ be an open cover of λ in (X, t). Set $u_i = v_i \mid A$, then $\lambda \subseteq \bigcup_{i \in J} v_i$ implies that $\lambda \subseteq (\bigcup_{i \in J} v_i) \mid A \subseteq \bigcup_{i \in J} (v_i \mid A) \subseteq \bigcup_{i \in J} u_i$. But $u_i \in t_A$, so $\{ u_i : i \in J \}$ is an open cover of λ in (A, t_A). As λ is compact in (A, t_A), then $\{ u_i : i \in J \}$ contains a finite subcover, say $\{ u_{i_k} : k \in J_n \}$. Accordingly, $\lambda \subseteq u_{i_1} \cup u_{i_2} \cup \ldots\ldots\ldots \cup u_{i_n} \subseteq (v_{i_1} \mid A) \cup (v_{i_2} \mid A) \cup \ldots\ldots \cup (v_{i_n} \mid A) \subseteq (v_{i_1} \cup v_{i_2} \cup \ldots\ldots \cup v_{i_n}) \mid A \subseteq v_{i_1} \cup v_{i_2} \cup \ldots\ldots \cup v_{i_n}$, as $\lambda_0 \subseteq A$. Thus $\{ v_i : i \in J \}$ contains a finite subcover $\{ v_{i_k} : k \in J_n \}$ and therefore λ is compact in (X, t).

Note: This theorem is different form of H. K. Abdulla and N. R. Kareem [1].

Corollary 4.3: Let (Y, t^*) be a fuzzy subspace of (X, t) and $A \subset Y \subset X$. Let $\lambda \in I^X$ and $\lambda_0 \subseteq A$. Then λ is compact in (X, t) if and only if λ is compact in (Y, t^*).

Proof: Let t_A and t_A^* be the subspace fuzzy topologies on A. Then by theorem (4.2), λ is compact in (X, t) or (Y, t^*) if and only if λ is compact in (A, t_A) or (A, t_A^*). But $t_A = t_A^*$.

Theorem 4.4: Let (X, t) and (Y, s) be two fuzzy topological spaces and $f : (X, t) \to (Y, s)$ be fuzzy continuous and onto mapping. If λ is compact fuzzy set in (X, t), then $f(\lambda)$ is also compact fuzzy set in (Y, s).

Proof: cf.[107].

Theorem 4.5: Let (X, t) and (Y, s) be two fuzzy topological spaces and $f : (X, t) \to (Y, s)$ be fuzzy open and bijective mapping. If λ is compact fuzzy set in (Y, s), then $f^{-1}(\lambda)$ is also compact in (X, t).

Proof: Let $\{ u_i : i \in J \}$ be an open cover of $f^{-1}(\lambda)$ in (X, t) i.e $f^{-1}(\lambda) \subseteq \bigcup_{i \in J} u_i$. As f is fuzzy open, then $f(u_i) \in s$ and hence $\{ f(u_i) : i \in J \}$ is an open cover of λ in (Y, s). Since λ is compact fuzzy set in (Y, s), then λ has a finite subcover i.e. there exist $f(u_{i_1})$, $f(u_{i_2})$, ……, $f(u_{i_n}) \in \{ f(u_i)\}$ such that $\lambda \subseteq f(u_{i_1}) \cup f(u_{i_2}) \cup \ldots\ldots \cup f(u_{i_n})$. Again, let u be any fuzzy set in X. Since f is bijective, then we have $f^{-1}(f(u) = u$. Hence $f^{-1}(\lambda) \subseteq f^{-1}(\, f(u_{i_1}) \cup f(u_{i_2}) \cup \ldots\ldots \cup f(u_{i_n})) \subseteq u_{i_1} \cup u_{i_2} \cup \ldots\ldots \cup u_{i_n}$. Therefore $f^{-1}(\lambda)$ is compact in (X, t).

Theorem 4.6: Let (X, t) be an fts, (A, t_A) be subspace of (X, t) and $f : (X, t) \to (A, t_A)$ be fuzzy continuous and onto mapping. If λ is compact fuzzy set in (X, t), then $f(\lambda)$ is also compact fuzzy set in (A, t_A).

Proof: Let $\{ u_i : i \in J \}$ be an open cover of $f(\lambda)$ in (A, t_A) i.e. $f(\lambda) \subseteq \bigcup_{i \in J} u_i$. Put $u_i = v_i \mid A$, where $v_i \in t$. Since f is fuzzy continuous, then $f^{-1}(u_i) \in t$ implies that $f^{-1}(v_i \mid A) \in t$ and consequently $\{ f^{-1}(u_i) : i \in J \}$ i.e $\{ f^{-1}(v_i \mid A) : i \in J \}$ is an open cover of λ in (X, t). As λ is compact fuzzy set in (X, t), then λ has a finite subcover i.e. there exist $f^{-1}(v_{i_k} \mid A) \in \{ f^{-1}(v_i \mid A)\}$ ($k = 1, 2, \ldots\ldots, n$) such that $\lambda \subseteq \bigcup_{k=1}^{n} f^{-1}(v_{i_k} \mid A)$. Again, let u be any fuzzy set in A. Since f is onto, then we have

$f(f^{-1}(u)) = u$. Hence $f(\lambda) \subseteq f\left(\bigcup_{k=1}^{n} f^{-1}(v_{i_k} \mid A)\right) = \bigcup_{k=1}^{n} f\ (\ f^{-1}(v_{i_k} \mid A)\ ) = \bigcup_{k=1}^{n} (v_{i_k} \mid A)$

$= \bigcup_{k=1}^{n} u_{i_k}$. Therefore $f(\lambda)$ is compact in (A, t_A).

Theorem 4.7: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively. Let λ be a compact fuzzy set in (A, t_A) and $f:\ (A, t_A) \to (B, s_B)$ be relatively fuzzy continuous and onto mapping. Then $f(\lambda)$ is also compact in (B, s_B).

Proof: Assume that $f(A)=B$, as f is onto. Let λ be compact in (A, t_A) and $M = \{\ v_i : i \in J\ \}$ be an open cover of $f(\lambda)$ in (B, s_B) i.e. $f(\lambda) \subseteq \bigcup_{i \in J} v_i$. Since $v_i \in s_B$, then there exist $u_i \in s$ such that $v_i = u_i \mid B$. Hence $f(\lambda) \subseteq \bigcup_{i \in J}(\ u_i \mid B\ )$. As f is relatively fuzzy continuous, then $f^{-1}(v_i) \mid A \in t_A$ and hence $\{\ f^{-1}(v_i) \mid A :\ i \in J\ \}$ is an open cover of λ in (A, t_A) i.e. $\{\ f^{-1}\ (\ u_i \mid B\ ) \mid A :\ i \in J\ \} = \{\ f^{-1}(u_i) \mid (\ f^{-1}(B) \cap A\ ) :$ $i \in J\ \} = \{\ f^{-1}(u_i) \mid A :\ i \in J\ \}$ is an open cover of λ in (A, t_A). Since λ is compact in (A, t_A), then there exist $f^{-1}(u_{i_k}) \mid A \in \{\ f^{-1}(u_i) \mid A\ \}$ $(\ k \in J_n\ )$ such that $\lambda \subseteq \bigcup_{k \in J_n}(\ f^{-1}(u_{i_k}) \mid A\ )$. Again, let v be any fuzzy set in B. Since f is onto, then we have $f(f^{-1}(v)) = v$. Therefore $f(\lambda) \subseteq f(\bigcup_{k \in J_n}(f^{-1}(u_{i_k}) \mid A))$ implies that $f(\lambda) \subseteq$ $\bigcup_{k \in J_n} f(f^{-1}(u_{i_k}) \mid A)$ implies that $f(\lambda) \subseteq \bigcup_{k \in J_n}(\ u_{i_k} \mid f(A)\ )$ implies that $f(\lambda) \subseteq \bigcup_{k \in J_n}(\ u_{i_k} \mid B\ )$ implies that $f(\lambda) \subseteq \bigcup_{k \in J_n} v_{i_k}$. Thus $f(\lambda)$ is compact in (B, s_B).

Theorem 4.8: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively. Let λ be a compact fuzzy set in (B, s_B) and $f : (A, t_A) \to (B, s_B)$ be relatively fuzzy open and bijective mapping. Then $f^{-1}(\lambda)$ is compact in (A, t_A).

Proof: We have $f(A)=B$, as f is bijective. Let $\{ u_i : u_i \in t_A \}$ be an open cover of $f^{-1}(\lambda)$ in (A, t_A) for every $i \in J$ i.e $f^{-1}(\lambda) \subseteq \bigcup_{i \in J} u_i$. Since $u_i \in t_A$, then there exists $v_i \in t$ such that $u_i = v_i \mid A$ and so $f^{-1}(\lambda) \subseteq \bigcup_{i \in J} (v_i \mid A)$. As f is relatively fuzzy open, then $f(u_i) \in s_B$ and hence $\{ f(u_i) : i \in J \}$ is an open cover of λ in (B, s_B) implies that $\{ f(v_i \mid A) : i \in J \} = \{ f(v_i) \mid f(A) : i \in J \} = \{ f(v_i) | B : i \in J \}$ is an open cover of λ in (B, s_B). Since λ is compact in (B, s_B), then $\{ f(v_i) | B : i \in J \}$ has a finite subcover, say $\{ f(v_{i_k}) | B : k \in J_n \}$ such that $\lambda \subseteq \bigcup_{k=1}^{n} (f(v_{i_k}) \mid B)$. Again, let u be any fuzzy set in X. Since f is bijective, then we have $f^{-1}(f(u) = u$. Hence $f^{-1}(\lambda) \subseteq f^{-1}\left(\bigcup_{k=1}^{n} (f(v_{i_k}) \mid B)\right) = \bigcup_{k=1}^{n} f^{-1}(f(v_{i_k}) \mid B) = \bigcup_{k=1}^{n} (v_{i_k} \mid f^{-1}(B)) = \bigcup_{k=1}^{n} (v_{i_k} \mid A)$ $= \bigcup_{k=1}^{n} u_{i_k}$. Therefore $\{ u_{i_k} : k \in J_n \}$ is a finite subcover of $\{ u_i : u_i \in t_A \}$. Thus $f^{-1}(\lambda)$ is compact in (A, t_A).

Theorem 4.9: Let (X, t) be an fts and λ be a fuzzy set in X. If every family of closed fuzzy sets in (X, t) which has empty intersection has a finite subfamily with empty intersection, then λ is compact. The converse is not true in general.

Proof: Let $\{ u_i : i \in J \}$ be an open cover of λ in (X, t) i.e. $\lambda \subseteq \bigcup_{i \in J} u_i$. By the first condition of the theorem, we have $\bigcap_{i \in J} u_i^c = 0_X$. Hence we can write $\bigcup_{i \in J} u_i = 1_X$. Again, by the second condition of the theorem, we can write $\bigcap_{k \in J_n} u_{i_k}^c = 0_X$ implies that $\bigcup_{k \in J_n} u_{i_k} = 1_X$ and hence $\lambda \subseteq \bigcup_{k \in J_n} u_{i_k}$. Thus we see that $\{ u_{i_k} : k \in J_n \}$ is a finite subcover of $\{ u_i : i \in J \}$. Therefore λ is compact.

Now, for the converse, we consider the following example.

Let $X = \{a, b\}$ and $I = [0, 1]$. Let $u_1,\ u_2 \in I^X$ defined by $u_1(a) = 0.3$, $u_1(b) = 0.6$ and $u_2(a) = 0.4$, $u_2(b) = 0.8$. Now, take $t = \{ 0, u_1, u_2, 1\}$, then we see that (X, t) is an fts. Let $\lambda \in I^X$ defined by $\lambda(a) = 0.2$, $\lambda(b) = 0.7$. Clearly λ is compact in (X, t). Now, closed fuzzy sets are $u_1^c(a) = 0.7$, $u_1^c(b) = 0.4$ and $u_2^c(a) = 0.6$, $u_2^c(b) = 0.2$. We observe that $u_1^c \cap u_2^c \neq 0$. Thus the converse of the theorem is not necessarily true in general.

Theorem 4.10: Let λ and μ be compact fuzzy sets in an fts (X, t). Then $\lambda \cup \mu$ is also compact.

Proof: Let $M = \{ u_i : i \in J \}$ be any open cover $\lambda \cup \mu$. Then M is an open cover of both λ and μ respectively. Since λ is compact in (X, t), then λ has a finite subcover i.e. there exist $u_{i_k} \in M$ $(k \in J_n)$ such that $\lambda \subseteq \bigcup_{k \in J_n}^{n} u_{i_k}$. Again μ is compact in (X, t), then μ has a finite subcover i.e. there exist $u_{i_r} \in M$ $(r \in J_n)$ such that $\lambda \subseteq \bigcup_{r \in J_n}^{n} u_{i_r}$. Therefore $\{v_{i_k},\ w_{i_r}\}$ is a finite subcover of M. Hence $\lambda \cup \mu$ is compact in (X, t).

Theorem 4.11: Let λ and μ be compact fuzzy sets ($\lambda \cap \mu \neq 0$) in an fts (X, t). Then $\lambda \cap \mu$ is also compact.

Proof: Since $\lambda \cap \mu \subseteq \lambda$, $\lambda \cap \mu \subseteq \mu$ and λ , μ are compact in (X, t), then $\lambda \cap \mu$ is also compact.

The following example will show that the compact fuzzy sets in an fts need not be closed.

Example 4.12: Let $X = \{a, b\}$ and $I = [0, 1]$. Let u_1 , u_2 , u_3 , $u_4 \in I^X$ defined by $u_1(a) = 0.4$, $u_1(b) = 0.7$; $u_2(a) = 0.5$, $u_2(b) = 0.3$; $u_3(a) = 0.5$, $u_3(b) = 07$; $u_4(a) = 0.4$, $u_4(b) = 0.3$. Now, take $t = \{ 0 , u_1 , u_2 , u_3 , u_4 , 1\}$, then we see that (X, t) is an fts. Let $\lambda \in I^X$ defined by $\lambda(a) = 0.5$, $\lambda(b) = 0.4$. Clearly λ is compact. But λ is not closed, as its complement λ^c is not open in (X, t).

The following example will show that the closure of compact fuzzy sets in an fts need not be compact.

Example 4.13: Let $X = \{a, b\}$ and $I = [0, 1]$. Let u_1 , u_2 , u_3 , $u_4 \in I^X$ defined by $u_1(a) = 0.1$, $u_1(b) = 0.3$; $u_2(a) = 0.4$, $u_2(b) = 0.5$; $u_3(a) = 0.6$, $u_3(b) = 0.7$; $u_4(a) = 0.8$, $u_4(b) = 0.9$. Now, take $t = \{ 0 , u_1 , u_2 , u_3 , u_4 , 1 \}$, then we see that (X, t) is an fts. Let $\lambda \in I^X$ defined by $\lambda(a) = 0.2$, $\lambda(b) = 0.7$. Clearly λ is compact. Now, closed fuzzy sets are $0^c(a) = 1$, $0^c(b) = 1$; $u_1^c(a) = 0.9$, $u_1^c(b) = 0.7$; $u_2^c(a) = 0.6$, $u_2^c(b) = 0.5$; $u_3^c(a) = 0.4$, $u_3^c(b) = 0.3$; $u_4^c(a) = 0.2$, $u_4^c(b) = 0.1$. So we have $\bar{\lambda} = \bigcap\{0^c, u_1^c\} = u_1^c$ i.e. $\bar{\lambda}(a) = 0.9$, $\bar{\lambda}(b) = 0.7$. Hence we observe that, there is no finite subcover of $\bar{\lambda}$ in (X, t). Thus $\bar{\lambda}$ is not compact.

Theorem 4.14: Let (X, t) be a fuzzy T_1-space (as def. 1.45) and λ be a compact fuzzy set in X with $\lambda_0 \subset X$. Let $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u, $v \in t$ such that $u(x) = 1$ and $\lambda_0 \subseteq v^{-1}(0, 1]$.

Proof: Let $y \in \lambda_0$. Then clearly $x \neq y$. As (X, t) is fuzzy T_1-space, then there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $u_y(y) = 0$ and $v_y(x) = 0$, $v_y(y) = 1$. Hence we see that $\lambda \subseteq \bigcup \{ v_y : y \in \lambda_0 \}$ i.e. $\{ v_y : y \in \lambda_0 \}$ is an open cover of λ in (X, t). Since λ is compact, then $\{ v_y : y \in \lambda_0 \}$ has a finite subcover i.e. there exist v_{y_1}, v_{y_2}, ……, $v_{y_n} \in \{v_y\}$ such that $\lambda \subseteq v_{y_1} \cup v_{y_2} \cup \ldots\ldots. \cup v_{y_n}$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ldots.. \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Then we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, $u \in t$. Furthermore, $\lambda_0 \subseteq v^{-1}(0, 1]$ and $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k.

Theorem 4.15: Let (X, t) be a fuzzy T_1-space (as def. 1.45) and λ and μ be disjoint compact fuzzy sets in X with λ_0, $\mu_0 \subset X$. Then there exist u, $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$.

Proof: Let $y \in \lambda_0$. Then $y \notin \mu_0$, as λ and μ are disjoint. Since μ is compact in (X, t), then by theorem (4.14), there exist u_y, $v_y \in t$ such that $u_y(y) = 1$ and $\mu_0 \subseteq v_y^{-1}(0, 1]$. As $u_y(y) = 1$, then $\{ u_y : y \in \lambda_0 \}$ is an open cover of λ in (X, t). Since λ is compact, then $\{ u_y : y \in \lambda_0 \}$ has a finite subcover i.e. there exist u_{y_1}, u_{y_2}, ……, $u_{y_n} \in \{u_y\}$ such that $\lambda \subseteq u_{y_1} \cup u_{y_2} \cup \ldots\ldots. \cup u_{y_n}$. Furthermore, $\mu \subseteq v_{y_1} \cap v_{y_2} \cap \ldots\ldots. \cap v_{y_n}$, as $\mu_0 \subseteq v_{y_k}^{-1}(0, 1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots.. \cap v_{y_n}$. Thus we see that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $u\ ,\ v \in t$.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (4.14) and (4.15) are not at all true.

The following example will show that the compact fuzzy sets in fuzzy T_1-space (as def. 1.45) need not be closed.

Example 4.16: Let $X = \{a, b\}$ and $I = [0, 1]$. Let $u\ ,\ v \in I^X$ defined by $u(a) = 1$, $u(b) = 0$ and $v(a) = 0$, $v(b) = 1$. Now, put $t = \{\ 0\ ,\ u\ ,\ v\ ,\ 1\}$, then we see that (X, t) is a fuzzy T_1-space. Let $\lambda \in I^X$ defined by $\lambda(a) = 0.3$, $\lambda(b) = 0.7$. Clearly λ is compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 4.17: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is compact in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist $u\ ,\ v \in t$ such that $u(x) > 0$ and $\lambda_0 \subseteq v^{-1}(0, 1]$. The converse of the theorem is not necessarily true in general.

The proof is similar as that of theorem (4.14).

Now, for the converse, we consider the following example.

Let $X = \{a, b\}$ and $I = [0, 1]$. Let $u_1\ ,\ u_2\ ,\ u_3 \in I^X$ defined by $u_1(a) = 0.2$, $u_1(b) = 0$; $u_2(a) = 0$, $u_2(b) = 0.3$ and $u_3(a) = 0.2$, $u_3(b) = 0.3$. Now, take $t = \{\ 0\ ,\ u_1\ ,\ u_2\ ,\ u_3\ ,\ 1\}$, then we see that (X, t) is a fuzzy T_1-space. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.6$. Hence we observe that $\lambda_0 = \{b\}$ and $a \notin \lambda_0$. Now $u_1\ ,\ u_2 \in t$ where

$u_1(a) > 0$ and $u_2^{-1}(0,1] = \{b\}$. Hence $\lambda_0 \subseteq u_2^{-1}(0,1]$. But λ is not compact, as there is no finite subcover of λ in (X,t). Thus the converse of the theorem is not true in general.

Theorem 4.18: Let (X,t) be a fuzzy T_1-space (as def. 1.46) and λ , μ be fuzzy sets in X with λ_0 , $\mu_0 \subset X$. If λ and μ are disjoint compact fuzzy sets in (X,t), then there exist u , $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0,1]$ and $\mu_0 \subseteq v^{-1}(0,1]$. The converse of the theorem is not true in general.

The proof is similar as that of theorem (4.15).

Now, for the converse, consider the fuzzy T_1-space (X,t) in the example of the theorem (4.17). Let λ , $\mu \in I^X$ defined by $\lambda(a) = 0.8$, $\lambda(b) = 0$ and $\mu(a) = 0$, $\mu(b) = 0.6$. Hence we observe that $\lambda_0 = \{a\}$ and $\mu_0 = \{b\}$. Now u_1 , $u_2 \in t$ where $u_1^{-1}(0,1] = \{a\}$ and $u_2^{-1}(0,1] = \{b\}$. Thus we see that $\lambda_0 \subseteq u_1^{-1}(0,1]$ and $\mu_0 \subseteq u_2^{-1}(0,1]$, where λ and μ are disjoint. But λ and μ are not compact, as there is no finite suvcover of λ and μ in (X,t) respectively. Thus the converse of the theorem is not true in general.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (4.17) and (4.18) are not at all true.

The following example will show that the compact fuzzy sets in fuzzy T_1-space (as def. 1.46) need not be closed.

Example 4.19: Consider the fuzzy T_1-space (X,t) in the example of the theorem (4.17). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.1$, $\lambda(b) = 0.2$. Clearly λ is compact in (X,t). But λ is not closed, as its complement λ^c is not open in (X,t).

Theorem 4.20: Let (X, t) be a fuzzy Hausdorff space (as def. 1.47) and λ be a compact fuzzy set in X with $\lambda_0 \subset X$. Suppose $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u , $v \in t$ such that $u(x) = 1$, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$.

Proof: Let $y \in \lambda_0$. Then clearly $x \neq y$. As (X, t) is fuzzy Hausdorff, then there exist u_y , $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \cap v_y = 0$. Hence $\lambda \subseteq \bigcup\{ v_y : y \in \lambda_0 \}$ i.e. $\{ v_y : y \in \lambda_0 \}$ is an open cover of λ. Since λ is compact in (X, t), then there exist v_{y_1} , v_{y_2} ,……, $v_{y_n} \in \{v_y\}$ such that $\lambda \subseteq v_{y_1} \cup v_{y_2} \cup$ …… $\cup v_{y_n}$. Now, let $v = v_{y_1} \cup v_{y_2} \cup$ …… $\cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap$ …… $\cap u_{y_n}$. Then we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $u \in t$. Furthermore, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k.

Finally, we have to show that $u \cap v = 0$. As $u_{y_k} \cap v_{y_k} = 0$ implies that $u \cap v_{y_k} = 0$, by distributive law, we see that $u \cap v = u \cap (v_{y_1} \cup v_{y_2} \cup$ …... $\cup v_{y_n}) = 0$.

Corollary 4.21: Let (X, t) be a fuzzy Hausdorff space (as def. 1.47) and λ be a compact fuzzy set in X with $\lambda_0 \subset X$. Let $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exists $u \in t$ such that $u(x) = 1$ and $u^{-1}(0,1] \subseteq \lambda_0^c$.

Proof: By theorem (4.20), there exists u , $v \in t$ such that $u(x) = 1$, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$. Hence $u^{-1}(0,1] \cap v^{-1}(0,1] = \phi$. If not, there exists $x \in u^{-1}(0,1] \cap v^{-1}(0,1]$ $\Rightarrow$ $x \in u^{-1}(0,1]$ and $x \in v^{-1}(0,1]$ $\Rightarrow$ $u(x) > 0$ and $v(x) > 0$ $\Rightarrow$ $u \cap v \neq 0$. Hence $u^{-1}(0,1] \cap \lambda_0 = \phi$ and consequently $u^{-1}(0,1] \subseteq \lambda_0^c$.

Theorem 4.22: Let (X, t) be a fuzzy Hausdorff space (as def. 1.47) and λ , μ be disjoint compact fuzzy sets in X with λ_0 , $\mu_0 \subset X$. Then there exist u , $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0,1]$, $\mu_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$.

Proof: Let $y \in \lambda_0$. Then $y \notin \mu_0$, as λ and μ are disjoint. Since μ is compact in (X, t), then by theorem (4.20), there exist u_y , $v_y \in t$ such that $u_y(y) = 1$, $\mu_0 \subseteq v_y^{-1}(0,1]$ and $u_y \cap v_y = 0$. As $u_y(y) = 1$, then $\{ u_y : y \in \lambda_0 \}$ is an open cover of λ. Since λ is compact in (X, t), then there exist u_{y_1} , u_{y_2} ,……, $u_{y_n} \in \{u_y\}$ such that $\lambda \subseteq u_{y_1} \cup u_{y_2} \cup$ ……. $\cup\, u_{y_n}$. Furthermore, $\mu \subseteq v_{y_1} \cap v_{y_2} \cap$ ……. $\cap\, v_{y_n}$, as $\mu_0 \subseteq v_{y_k}^{-1}(0,1]$ for each k . Now, let $u = u_{y_1} \cup u_{y_2} \cup$ …… $\cup\, u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap$ ….. $\cap\, v_{y_n}$. Thus we see that $\lambda_0 \subseteq u^{-1}(0,1]$ and $\mu_0 \subseteq v^{-1}(0,1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u , $v \in t$.

Lastly, we have to show that $u \cap v = 0$. First , we observe that $u_{y_k} \cap v_{y_k} = 0$ implies that $u_{y_k} \cap v = 0$, by distributive law, we see that $u \cap v = (u_{y_1} \cup u_{y_2} \cup$ …… $\cup\, u_{y_n}) \cap v = 0$.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (4.20), (4.22) and corollary (4.21) are not at all true.

Note: The compact fuzzy sets in fuzzy Hausdorff space (as def. 1.47) need not be closed.

Consider the fuzzy topology t in the example (4.16), then (X, t) is also a fuzzy Hausdorff space (as def. 1.47) and will serve the purpose that the compact fuzzy sets in fuzzy Hausdroff space need not be closed.

Theorem 4.23: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is compact in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u , $v \in t$ such that $u(x) > 0$, $\lambda_0 \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$. The converse of the theorem is not necessarily true in general.

The proof is similar as that of theorem (4.20).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (4.17), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.6$. Hence we observe that $\lambda_0 = \{b\}$ and $a \notin \lambda_0$. Now u_1 , $u_2 \in t$ where $u_1(a) > 0$ and $u_2^{-1}(0, 1] = \{b\}$. Hence $\lambda_0 \subseteq u_2^{-1}(0, 1]$ and $u_1 \cap u_2 = 0$. But λ is not compact, as there is no finite subcover of λ in (X, t). Thus the converse of the theorem is not true in general.

Corollary 4.24: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is compact in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exists $u \in t$ such that $u(x) > 0$ and $u^{-1}(0, 1] \subseteq \lambda_0^c$.

The proof is similar as that of corollary (4.21).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (4.17), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.6$. Hence we observe that $\lambda_0 = \{b\}$ and $a \notin \lambda_0$. Now $u_1 \in t$ where $u_1(a) > 0$ and then $u_1^{-1}(0, 1] = \{a\}$. Hence we have $u_1^{-1}(0, 1] \subseteq \lambda_0^c$. But λ is not compact, as there is no finite subcover of λ in (X, t). Thus the converse is not true in general.

Theorem 4.25: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and λ, μ be fuzzy sets in X with λ_0, $\mu_0 \subset X$. If λ and μ are disjoint compact fuzzy sets in (X, t), then there exist u, $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0,1]$, $\mu_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$. The converse of the theorem is not true in general.

The proof is similar as that of theorem (4.22).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (4.17), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Again, let λ, $\mu \in I^X$ defined by $\lambda(a) = 0.8$, $\lambda(b) = 0$ and $\mu(a) = 0$, $\mu(b) = 0.6$. Hence we observe that $\lambda_0 = \{a\}$ and $\mu_0 = \{b\}$. Now u_1, $u_2 \in t$ where $u_1^{-1}(0,1] = \{a\}$ and $u_2^{-1}(0,1] = \{b\}$. Thus we see that $\lambda_0 \subseteq u_1^{-1}(0,1]$, $\mu_0 \subseteq u_2^{-1}(0,1]$ and $u_1 \cap u_2 = 0$, where λ and μ are disjoint. But λ and μ are not compact, as there is no finite subcover of λ and μ in (X, t) respectively. Thus the converse of the theorem is not true in general.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (4.23), (4.25) and corollary (4.24) are not at all true.

The following example will show that the compact fuzzy sets in fuzzy Hausdorff space (as def. 1.48) need not be closed.

Example 4.26: Consider the fuzzy topology t in the example of the theorem (4.17), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.2$, $\lambda(b) = 0.1$. Clearly λ is compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 4.27: Let λ be a compact fuzzy set in a fuzzy Hausdorff space (X, t) (as def. 1.50) with $\lambda_0 \subset X$. Suppose $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u , $v \in t$ such that $u(x) = 1$, $\lambda_0 \subseteq v^{-1}(0, 1]$ and $u \subseteq 1 - v$.

Proof: Let $y \in \lambda_0$. Then clearly $x \neq y$. As (X, t) is fuzzy Hausdorff, then there exist u_y , $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \subseteq 1 - v_y$. Hence $\lambda \subseteq \bigcup\{ v_y : y \in \lambda_0 \}$ i.e. $\{ v_y : y \in \lambda_0 \}$ is an open cover of λ. Since λ is compact in (X, t), then there exist v_{y_1} , $v_{y_2}, \ldots\ldots, v_{y_n} \in \{v_y\}$ such that $\lambda \subseteq v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ldots.. \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Then we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $u \in t$. Furthermore, $\lambda_0 \subseteq v^{-1}(0, 1]$ and $u(x) = 1$, as $u_{y_k}(x) = 1$ individually.

Lastly, we have to show that $u \subseteq 1 - v$. As $u_y \subseteq 1 - v_y$ implies that $u \subseteq 1 - v_y$. Since $u_{y_k}(x) \leq 1 - v_{y_k}(x)$ for all $x \in X$ and for each k, then $u \subseteq 1 - v$. If not, then there exist $x \in X$ such that $u_y(x) > 1 - v_y(x)$. We have $u_y(x) \leq u_{y_k}(x)$ for each k. Then for some k, $u_{y_k}(x) > 1 - v_{y_k}(x)$. But this is a contradiction, as $u_{y_k}(x) \leq 1 - v_{y_k}(x)$ for each k. Hence $u \subseteq 1 - v$.

Theorem 4.28: Let λ and μ be disjoint compact fuzzy sets in a fuzzy Hausdorff space (X, t) (as def. 1.50) with λ_0 , $\mu_0 \subset X$. Then there exist u , $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0, 1]$, $\mu_0 \subseteq v^{-1}(0, 1]$ and $u \subseteq 1 - v$.

Proof: Let $y \in \lambda_0$. Then $y \notin \mu_0$, as λ and μ are disjoint . Since μ is compact in (X, t), then by theorem (4.27), there exist u_y , $v_y \in t$ such that $u_y(y) = 1$, $\mu_0 \subseteq v_y^{-1}(0, 1]$

and $u_y \subseteq 1 - v_y$. As $u_y(y) = 1$, then $\{ u_y : y \in \lambda_0 \}$ is an open cover of λ. Since λ is compact in (X, t), then there exist $u_{y_1}, u_{y_2}, \ldots\ldots, u_{y_n} \in \{u_y\}$ such that $\lambda \subseteq u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$. Furthermore, $\mu \subseteq v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$, as $\mu_0 \subseteq v_{y_k}^{-1}(0,1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots.. \cap v_{y_n}$. Thus we see that $\lambda_0 \subseteq u^{-1}(0,1]$ and $\mu_0 \subseteq v^{-1}(0,1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, $v \in t$.

Finally, we have to show hat $u \subseteq 1 - v$. First, we observe that $u_{y_k} \subseteq 1 - v_{y_k}$ for each k implies that $u_{y_k} \subseteq 1 - v$ for each k and it is clear that $u \subseteq 1 - v$.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (4.27) and (4.28) are not at all true.

Note: The compact fuzzy sets in fuzzy Hausdorff space (as def. 1.50) need not be closed

Consider the fuzzy topology t in the example (4.16), then (X, t) is also a fuzzy Hausdorff space (as def. 1.50) and will serve the purpose that the compact fuzzy sets in fuzzy Hausdorff space need not be closed.

Theorem 4.29: Let λ be a compact fuzzy set in a fuzzy regular space (X, t) (as def. 1.51) with $\lambda_0 \subset X$. If for each $x \in \lambda_0$ and $u \in t^c$ with $u(x) = 0$, there exist v, $w \in t$ we have $v(x) = 1$, $u \subseteq w$, $\lambda_0 \subseteq v^{-1}(0,1]$ and $v \subseteq 1 - w$.

Proof: Let (X, t) be a fuzzy regular space and λ be a compact fuzzy set in (X, t). Now, if each $x \in \lambda_0$, there exists $u \in t^c$ with $u(x) = 0$, by fuzzy regularity of (X, t), we have v_x, $w_x \in t$ such that $v_x(x) = 1$, $u_x \subseteq w_x$ and $v_x \subseteq 1 - w_x$. Hence $\lambda \subseteq \bigcup\{ v_x : x \in \lambda_0 \}$

i.e. $\{ v_x : x \in \lambda_0 \}$ is an open cover of λ. Since λ is compact in (X, t), then $\{ v_x : x \in \lambda_0 \}$ has a finite subcover i.e. there exist v_{x_1}, v_{x_2}, ……, $v_{x_n} \in \{v_x\}$ such that $\lambda \subseteq v_{x_1} \cup v_{x_2} \cup$ ……. $\cup\, v_{x_n}$. Now, let $v = v_{x_n} \cup v_{x_2} \cup \ldots\ldots \cup v_{x_n}$ and $w = w_{x_1} \cap w_{x_2} \cap \ldots\ldots \cap w_{x_n}$. Then we see that v and w are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, $w \in t$. Furthermore, $\lambda_0 \subseteq v^{-1}(0,1]$, $v(x) = 1$ and $u \subseteq w$, as $u \subseteq w_{x_k}$ individually.

Lastly, we have to show that $v \subseteq 1 - w$. As $v_{x_k} \subseteq 1 - w_{x_k}$ implies that $v_{x_k} \subseteq 1 - w$ for each k and hence it is clear that $v \subseteq 1 - w$.

Theorem 4.30: Let (X, T) be a topological space and $(X, \omega(T))$ be an fts. If λ is any compact fuzzy set in $(X, \omega(T))$, then λ_0 is compact in (X, T). The converse is not true in general.

Proof: Suppose λ be any compact fuzzy set in $(X, \omega(T))$. Let $\{ V_i : i \in J \}$ be an open cover of λ_0 in (X, T) i.e. $\lambda_0 \subseteq \bigcup_{i \in J} V_i$. As 1_{V_i} is l.s.c., then $1_{V_i} \in \omega(T)$ and $\{ 1_{V_i} : 1_{V_i} \in \omega(T) \}$ is an open cover of λ in $(X, \omega(T))$. Since λ is compact in $(X, \omega(T))$, then λ has a finite subcover i.e. there exist $1_{V_{i_1}}$, $1_{V_{i_2}}$, ..., $1_{V_{i_n}} \in \{1_{V_i}\}$ such that $\lambda \subseteq 1_{V_{i_1}} \cup 1_{V_{i_2}} \cup \ldots\ldots \cup 1_{V_{i_n}}$. Hence, we can write $\lambda_0 \subseteq V_{i_1} \cup V_{i_2} \cup \ldots\ldots \cup V_{i_n}$ and therefore λ_0 is compact in (X, T).

Now, for the converse, we give the following example.

Let $X = \{a, b, c\}$ and $T = \{ \phi, \{b\}, \{c\}, \{b, c\}, X \}$, then (X, T) is a topological space. Let u_1, u_2, $u_3 \in I^X$ with $u_1(a) = 0$, $u_1(b) = 0.6$, $u_1(c) = 0$; $u_2(a) = 0$, $u_2(b) = 0$, $u_2(c) = 0.8$ and $u_3(a) = 0$, $u_3(b) = 0.6$, $u_3(c) = 0.8$. Then

$\omega(T) = \{ 0 , u_1 , u_2 , u_3 , 1\}$ and $(X, \omega(T))$ is an fts. Again, let $\lambda \in I^X$ with $\lambda(a) = 0$, $\lambda(b) = 0.7$, $\lambda(c) = 0.9$. Hence $\lambda_0 = \{b, c\}$. Then clearly λ_0 is compact in (X, T). But λ is not compact in $(X, \omega(T))$, as there do not exist $u_k \in \{ \omega(T)\}$ ($k = 1, 2, 3$) such that $\lambda \subseteq u_1 \cup u_2 \cup u_3$. Thus the converse of the theorem is not true in general.

Theorem 4.31: If λ and μ are compact fuzzy sets in an fts (X, t), then $(\lambda \times \mu)$ is also compact in $(X \times X, t \times t)$.

Proof: Suppose λ and μ are compact fuzzy sets in an fts (X, t). Let $\{ u_i : i \in J \}$ and $\{ v_i : i \in J \}$ be open cover of λ and μ respectively, where u_i , $v_i \in t$. Hence it can be easily shown that, $\min(\lambda(x) , \mu(y)) \subseteq \bigcup_{i \in J} \min (u_i(x) , v_i(y))$ for every $(x, y) \in X \times X$. Then $\{ u_i \times v_i : i \in J \}$ is an open cover of $(\lambda \times \mu)$ in $(X \times X, t \times t)$ i.e. $(\lambda \times \mu) \subseteq \bigcup_{i \in J} (u_i \times v_i)$. Since λ and μ are compact, then $\{ u_i : i \in J \}$ and $\{ v_i : i \in J \}$ have finite subcovers, say $\{ u_{i_k} : k \in J_n \}$ and $\{ v_{i_k} : k \in J_n \}$ such that $\lambda \subseteq \bigcup_{k \in J_n} u_{i_k}$ and $\mu \subseteq \bigcup_{k \in J_n} v_{i_k}$ respectively. Thus we can write $(\lambda \times \mu) \subseteq \bigcup_{k \in J_n} (u_{i_k} \times v_{i_k})$. Therefore $\{ u_{i_k} \times v_{i_k} : k \in J_n \}$ is a finite subcover of $\{ u_i \times v_i : i \in J \}$. Thus $(\lambda \times \mu)$ is compact in $(X \times X, t \times t)$.

Definition 4.32: Let (X, t) be an fts, $0 < \delta \leq 1$ and λ be a fuzzy set in X . Then λ is said to be δ-compact iff every δ-cover of λ has a finite δ-subcover. If $\mu \subseteq \lambda$ and

$\mu \in I^X$, then μ is also δ-compact. Thus we can say that, any other subsets of a δ-compact fuzzy set in an fts is also δ-compact.

Theorem 4.33: Any δ-compact fuzzy set in an fts is compact. The converse is not true in general.

The proof of the theorem is straightforward.

Now, for the converse, consider the following example.

Let $X = \{a, b\}$, $I = [0, 1]$ and $0 < \delta \leq 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 1$, $u_1(b) = 0.4$; $u_2(a) = 0.7$, $u_2(b) = 1$ and $u_3(a) = 0.7$, $u_3(b) = 0.4$. Now, take $t = \{0$, u_1, u_2, u_3, $1\}$, then we see that (X, t) is an fts. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.9$, $\lambda(b) = 0.8$. Clearly λ is compact in (X, t). Take $\delta = 0.6$. Then we observe that there is no finite δ-subcover of λ in (X, t). Hence λ is not δ-compact in (X, t). Thus the converse of theorem is not necessarily true.

Chapter Five

Partially α-Compact Fuzzy Sets

In this chapter, we have introduced partially α-compact fuzzy sets. Furthermore, we have established some theorems, corollaries and examples of partially α-compact fuzzy sets. Also we have defined partially δ-α-compact fuzzy sets and found different properties between partially α-compact and partially δ-α-compact fuzzy sets.

Definition 5.1: Let (X, t) be an fts and $\alpha \in I$. A family M of fuzzy sets is called a partial α-shading, $0 \le \alpha < 1$ (resp. partial α^*-shading, $0 < \alpha \le 1$), in short, $p\alpha$-shading (resp. $p\alpha^*$-shading) of a fuzzy set λ in X if for each $x \in \lambda_0$, ($\lambda_0 \neq X$) there exists a $u \in M$ with $u(x) > \alpha$ (resp. $u(x) \ge \alpha$). If each u is open, then M is called an open $p\alpha$-shading (resp. open $p\alpha^*$-shading) of λ in (X, t).

A subfamily of a $p\alpha$-shading (resp. $p\alpha^*$-shading) of λ which is also a $p\alpha$-shading (resp. $p\alpha^*$-shading) of λ is called a $p\alpha$-subshading (resp. $p\alpha^*$-subshading) of λ.

If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then $p\alpha$-shading (resp. $p\alpha^*$-shading) and α-shading (resp. α^*-shading) will be same.

Example 5.2: Let $X = \{a, b, c\}$, $I = [0, 1]$ and $0 \le \alpha < 1$. Let u_1, $u_2 \in I^X$ defined by $u_1(a) = 0.7$, $u_1(b) = 0.4$, $u_1(c) = 0.2$ and $u_2(a) = 0.3$, $u_2(b) = 0.9$, $u_2(c) = 0.1$. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.8$, $\lambda(b) = 0.4$, $\lambda(c) = 0$. Now, take $\alpha = 0.6$. Hence we observe that $u_1(a) > \alpha$, $u_2(b) > \alpha$ where a, $b \in \lambda_0$. Therefore $\{ u_1, u_2 \}$ is a $p\alpha$-shading of λ.

Again, if we take $\alpha = 0.7$, then $\{ u_1 , u_2 \}$ is a $p\alpha^*$ -shading of λ.

Example 5.3: Let $X = \{a, b, c\}$, $I = [0,1]$ and $0 \le \alpha < 1$. Let $u_1 , u_2 , u_3 \in I^X$ defined by $u_1(a) = 1$, $u_1(b) = 1$, $u_1(c) = 0$; $u_2(a) = 0$, $u_2(b) = 0.2$, $u_2(c) = 1$ and $u_3(a) = 0$, $u_3(b) = 0.2$, $u_3(c) = 0$. Put $t = \{ 0 , u_1 , u_2 , u_3 , 1 \}$, then (X, t) is an fts. Again, let $\lambda \in I^X$ with $\lambda(a) = 0$, $\lambda(b) = 0.4$, $\lambda(c) = 0.6$. Now, take $\alpha = 0.7$. Hence we observe that $u_1(b) > \alpha$, $u_2(c) > \alpha$ where $b , c \in \lambda_0$. Therefore $\{ u_1 , u_2 \}$ is an open $p\alpha$ -shading of λ in (X, t).

Again, if we take $\alpha = 1$, then $\{ u_1 , u_2 \}$ is an open $p\alpha^*$ -shading of λ in (X, t).

Definition 5.4: Let (X, t) be an fts and $\alpha \in I$. A fuzzy set λ in X is said to be partially α -compact, $0 \le \alpha < 1$ (resp. partially α^*-compact, $0 < \alpha \le 1$), in short, $p\alpha$ -compact (resp. $p\alpha^*$ -compact) iff every open $p\alpha$ -shading (resp. $p\alpha^*$ -shading) of λ has a finite $p\alpha$ -subshading (resp. $p\alpha^*$ -subshading).

Theorem 5.5: Let (X, t) be an fts, $A \subset X$ and λ be a fuzzy set in X with $\lambda_0 \subseteq A$. Then λ is $p\alpha$ -compact (resp. $p\alpha^*$ -compact) in (X, t) iff λ is $p\alpha$ -compact (resp. $p\alpha^*$ -compact) in (A, t_A).

Proof: Suppose λ is $p\alpha$ -compact in (X, t). Let $M = \{ u_i : i \in J \}$ be an open $p\alpha$ -shading of λ in (A, t_A). Then there exist $v_i \in t$ such that $u_i = v_i \mid A \subseteq v_i$. Hence $\{ v_i : i \in J \}$ is an open $p\alpha$ -shading of λ in (X, t). As λ is $p\alpha$ -compact in (X, t), then $\{ v_i : i \in J \}$ has a finite $p\alpha$ -subshading, say $\{ v_{i_k} : k \in J_n \}$ such that $v_{i_k}(x) > \alpha$ for all $x \in \lambda_0$. For, if $x \in \lambda_0$, then there exists $v_{i_{k_0}}$ such that $v_{i_{k_0}}(x) > \alpha$ implies that

$(v_{i_{k_0}} \mid A)(x) > \alpha$ and consequently $u_{i_{k_0}}(x) > \alpha$, as $\lambda_0 \subseteq A$. Thus $u_{i_{k_0}} \in M$ and hence $\{ u_{i_k} : k \in J_n \}$ is a finite $p\alpha$ -subshading of M. Therefore λ is $p\alpha$ -compact in (A, t_A).

Conversely, suppose λ is $p\alpha$ -compact in (A, t_A). Let $\{ v_i : i \in J \}$ be an open $p\alpha$ -shading of λ in (X, t). Put $u_i = v_i \mid A$. To show this, let $x \in X$. If $x \in A$, then there exists $v_{i_0} \in \{ v_i : i \in J \}$ such that $u_{i_0} = v_{i_0} \mid A$. But $u_{i_0} \in t_A$, so $u_{i_0}(x) > \alpha$ for all $x \in \lambda_0$. Therefore, $\{ u_i : i \in J \}$ is an open $p\alpha$ -shading of λ in (A, t_A). Since λ is $p\alpha$ -compact in (A, t_A), then $\{ u_i : i \in J \}$ has a finite $p\alpha$ -subshading, say $\{ u_{i_k} : k \in J_n \}$ such that $u_{i_k}(x) > \alpha$ for all $x \in \lambda_0$. For, if $x \in \lambda_0$, then there exists $u_{i_{k_0}}$ such that $u_{i_{k_0}}(x) > \alpha \Rightarrow (v_{i_{k_0}} \mid A)(x) > \alpha \Rightarrow v_{i_{k_0}}(x) > \alpha$, as $\lambda_0 \subseteq A$. Thus $\{ v_{i_k} : k \in J_n \}$ is a finite $p\alpha$ -subshading of $\{ v_i : i \in J \}$. Hence λ is $p\alpha$ -compact in (X, t).

The proof is similar for $p\alpha^*$ -compactness can be given.

Corollary 5.6: Let (Y, t^*) be a fuzzy subspace of (X, t) and $A \subset Y \subset X$. Let $\lambda \in I^X$ with $\lambda_0 \subseteq A$. Then λ is $p\alpha$ -compact (resp. $p\alpha^*$ -compact) in (X, t) iff λ is $p\alpha$ -compact (resp. $p\alpha^*$ -compact) in (A, t_A).

Proof: Let t_A and t_A^* be the subspace fuzzy topologies on A. Then by preceding theorem (5.5), λ is $p\alpha$ -compact in (X, t) or (Y, t^*) if and only if λ is $p\alpha$ -compact in (A, t_A) or (A, t_A^*). But $t_A = t_A^*$.

Similar work for $p\alpha^*$ -compactness can be done.

Theorem 5.7: Let $f: X \to Y$ be any mapping and $\lambda \in I^X$. Then $f(\lambda_0) = (f(\lambda))_0$.

Proof: Let $y \in f(\lambda_0)$, then there exists an $x \in \lambda_0$ such that $y = f(x)$. Now, $\lambda(x) > 0$ and therefore $\sup\{ \lambda(x) : \ x \in f^{-1}(y) \} > 0$ which implies that $f(\lambda)(y) > 0$. Hence $y \in \left(f(\lambda)\right)_0$. Therefore $f(\lambda_0) \subseteq \left(f(\lambda)\right)_0$.

Again, let $y \in \left(f(\lambda)\right)_0$, then $f(\lambda)(y) > 0$ which implies that $\sup\{ \lambda(x) : \ f(x) = y ,\ f^{-1}(y) \neq \phi \} > 0$. Then there exists an $x_0 \in X$, $y = f(x_0)$ and $x \in \lambda_0$. Therefore $f(x_0) \in f(\lambda_0)$ implies that $y \in f(\lambda_0)$. Therefore $\left(f(\lambda)\right)_0 \subseteq f(\lambda_0)$. Hence $f(\lambda_0) = \left(f(\lambda)\right)_0$.

Theorem 5.8: Let (X, t) and (Y, s) be two fuzzy topological spaces and $f : (X, t) \rightarrow (Y, s)$ be fuzzy continuous and onto mapping. If λ is $p\alpha$ -compact (resp. $p\alpha^*$ -compact) in (X, t), then $f(\lambda)$ is $p\alpha$ -compact (resp. $p\alpha^*$ -compact) in (Y, s).

Proof: Let $M = \{ u_i : \ i \in J \}$ be an open $p\alpha$-shading of $f(\lambda)$ in (Y, s). Since f is fuzzy continuous, then $f^{-1}(u_i) \in t$ and hence $f^{-1}(M) = \{ f^{-1}(u_i) : \ u_i \in M \}$ is an open $p\alpha$-shading of λ in (X, t). For, if $x \in \lambda_0$, then $f(x) \in \left(f(\lambda)\right)_0$. So there exists $u_{i_0} \in M$ such that $u_{i_0}\left(f(x)\right) > \alpha$ which implies that $f^{-1}(u_{i_0})(x) > \alpha$. As λ is $p\alpha$ -compact in (X, t), then $f^{-1}(M)$ has a finite $p\alpha$ -subshading, say $\{ f^{-1}(u_{i_1}),\ f^{-1}(u_{i_2}),\ \ldots\ldots,\ f^{-1}(u_{i_n}) \}$. Now, if $y \in \left(f(\lambda)\right)_0$, then $y = f(x)$ for some $x \in \lambda_0$. Then there exists k such that $f^{-1}(u_{i_k})(x) > \alpha$ which implies that $u_{i_k}\left(f(x)\right) > \alpha$ or $u_{i_k}(y) > \alpha$. Hence $f(\lambda)$ is $p\alpha$ -compact in (Y, s).

Similar work for $p\alpha^*$ -compactness can be given.

Theorem 5.9: Let (X, t) and (Y, s) be two fuzzy topological spaces and $f : (X, t) \rightarrow (Y, s)$ be fuzzy open and bijective mapping. If λ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (Y, s), then $f^{-1}(\lambda)$ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (X, t).

Proof: Let $M = \{ u_i : i \in J \}$ be an open $p\alpha$-shading of $f^{-1}(\lambda)$ in (X, t). As f is fuzzy open, then $f(u_i) \in s$ and so $f(M) = \{ f(u_i) : u_i \in M \}$ is an open $p\alpha$-shading of λ in (Y, s). For, if $y \in \lambda_0$, then $f^{-1}(y) \in (f^{-1}(\lambda))_0$. So there exists $u_{i_0} \in M$ such that $u_{i_0}(f^{-1}(y)) > \alpha$ which implies that $f(u_{i_0})(y) > \alpha$. Since λ is $p\alpha$-compact in (Y, s), then $f(M)$ has a finite $p\alpha$-subshading, say $\{ f(u_{i_1}), f(u_{i_2}), \ldots\ldots, f(u_{i_n}) \}$. For, if $x \in (f^{-1}(\lambda))_0$, then $x = f^{-1}(y)$ for some $y \in \lambda_0$. Therefore, there exists k such that $f(u_{i_k})(y) > \alpha$ which implies that $u_{i_k}(f^{-1}(y)) > \alpha$ or $u_{i_k}(x) > \alpha$. Hence $f^{-1}(\lambda)$ is $p\alpha$-compact in (X, t).

The work is similar for $p\alpha^*$-compactness can be given.

Theorem 5.10: Let (X, t) be an fts, $A \subset X$ and λ be a fuzzy set in X with $\lambda_0 \subseteq A$. Let (A, t_A) be a fuzzy subspace of (X, t) and $f : (X, t) \rightarrow (A, t_A)$ be fuzzy continuous and onto mapping. If λ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (X, t), then $f(\lambda)$ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (A, t_A).

Proof: Let $M = \{ u_i : i \in J \}$ be an open $p\alpha$-shading of $f(\lambda)$ in (A, t_A). Put $u_i = v_i \mid A$, where $v_i \in t$. Since f is fuzzy continuous, then $f^{-1}(u_i) \in t$ implies that $f^{-1}(v_i \mid A) \in t$ and hence $f^{-1}(M) = \{ f^{-1}(u_i) : u_i \in M \}$ i.e $f^{-1}(M) = \{ f^{-1}(v_i \mid A) : i \in J \}$ is an open $p\alpha$-shading of λ in (X, t). For, if $x \in \lambda_0$, then $f(x) \in (f(\lambda))_0$. So there exists $u_{i_0} \in M$ such that $u_{i_0}(f(x)) > \alpha$ which implies that $f^{-1}(u_{i_0})(x) > \alpha$ i.e.

$f^{-1}(v_{i_0} \mid A)(x) > \alpha$. As λ is $p\alpha$-compact in (X, t), then $f^{-1}(M)$ has a finite $p\alpha$-subshading, say { $f^{-1}(v_{i_1} \mid A)$, $f^{-1}(v_{i_2} \mid A)$, …… , $f^{-1}(v_{i_n} \mid A)$ }. Now, if $y \in (f(\lambda))_0$, then $y = f(x)$ for some $x \in \lambda_0$. Then there exists k such that $f^{-1}(v_{i_k} \mid A)(x) > \alpha$ which implies that $(v_{i_k} \mid A)\,(f(x)) > \alpha$ or $u_{i_k}(y) > \alpha$. Hence $f(\lambda)$ is $p\alpha$-compact in (A, t_A).

Similar work for $p\alpha^*$-compactness can be given.

Theorem 5.11: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fuzzy topological spaces (X, t) and (Y, s) respectively and $f : (A, t_A) \to (B, s_B)$ be relatively fuzzy continuous and onto mapping. If λ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (A, t_A), then $f(\lambda)$ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (B, s_B).

Proof: We have $f(A)=B$, as f is onto. Let { $v_i : v_i \in s_B$ } be an open $p\alpha$-shading of $f(\lambda)$ in (B, s_B) for every $i \in J$ i.e $v_i(y) > \alpha$ for every $y \in (f(\lambda))_0$. Since $v_i \in s_B$, then there exists $u_i \in s$ such that $v_i = u_i \mid B$ and so ($u_i \mid B$) $(y) > \alpha$ for every $y \in (f(\lambda))_0$. As f is relatively fuzzy continuous, then $f^{-1}(v_i) | A \in t_A$. Thus we observe that, for each $x \in \lambda_0$, $(f^{-1}(v_i)|A)(x) > \alpha$ and hence { $f^{-1}(v_i)|A : i \in J$ } is an open $p\alpha$-shading of λ in (A, t_A) implies that { $(f^{-1}(u_i \mid B)) \mid A : i \in J$ } = { $f^{-1}(u_i) \mid (f^{-1}(B) \cap A) : i \in J$ } = { $f^{-1}(u_i) | A : i \in J$ } is an open $p\alpha$-shading of λ in (A, t_A). Since λ is $p\alpha$-compact in (A, t_A), then { $f^{-1}(u_i)|A : i \in J$ } has a finite $p\alpha$-subshading, say { $f^{-1}(u_{i_k})|A$ } ($k \in J_n$) such that $(f^{-1}(u_{i_k}) | A)(x) > \alpha$ for each $x \in \lambda_0$. Now, if $y \in (f(\lambda))_0$, then $y = f(x)$ for some $x \in \lambda_0$. Then there exists k we have $(f^{-1}(u_{i_k})|A)(x) > \alpha$ implies that $(u_{i_k} \mid f(A))\,(f(x)) > \alpha$ implies that $(u_{i_k} |B)(y) > \alpha$, as f is onto or $v_{i_k}(y) > \alpha$.

Hence it is clear that $\{ v_{i_k} : k \in J_n \}$ is a finite $p\alpha$-subshading of $\{ v_i : v_i \in s_B \}$. Thus $f(\lambda)$ is $p\alpha$-compact in (B, s_B).

The work is similar for $p\alpha^*$-compactness can be given.

Theorem 5.12: Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively. Let $f : (A, t_A) \to (B, s_B)$ be relatively fuzzy open and bijective mapping. If λ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (B, s_B), then $f^{-1}(\lambda)$ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (A, t_A).

Proof: We have $f(A)=B$, as f is bijective. Let $\{ u_i : u_i \in t_A \}$ be an open $p\alpha$-shading of $f^{-1}(\lambda)$ in (A, t_A) for every $i \in J$ i.e $u_i(x) > \alpha$ for every $x \in (f^{-1}(\lambda))_0$. Since $u_i \in t_A$, then there exists $v_i \in t$ such that $u_i = v_i \mid A$ and so $(v_i \mid A)(x) > \alpha$ for every $x \in (f^{-1}(\lambda))_0$. As f is relatively fuzzy open, then $f(u_i) \in s_B$. Thus we observe that, for each $y \in \lambda_0$, $f(u_i)(y) > \alpha$ and hence $\{ f(u_i) : i \in J \}$ is an open $p\alpha$-shading of λ in (B, s_B) implies that $\{ f(v_i \mid A) : i \in J \} = \{ f(v_i) \mid f(A) : i \in J \} = \{ f(v_i) \mid B : i \in J \}$ is an open $p\alpha$-shading of λ in (B, s_B). Since λ is $p\alpha$-compact in (B, s_B), then $\{ f(v_i) \mid B : i \in J \}$ has a finite $p\alpha$-subshading, say $\{ f(v_{i_k}) \mid B : k \in J_n \}$ such that $(f(v_{i_k}) \mid B)(y) > \alpha$ for each $y \in \lambda_0$. Now, if $x \in (f^{-1}(\lambda))_0$, then $x = f^{-1}(y)$ for each $y \in \lambda_0$. Then there exists k we have $(f(v_{i_k}) \mid B)(y) > \alpha$ implies that $(v_{i_k} \mid f^{-1}(B)) (f^{-1}(y)) > \alpha$ implies that $(v_{i_k} \mid A)(x) > \alpha$ or $u_{i_k}(x) > \alpha$. Hence it is clear that $\{ u_{i_k} : k \in J_n \}$ is a finite $p\alpha$-subshading of $\{ u_i : u_i \in t_A \}$. Thus $f^{-1}(\lambda)$ is $p\alpha$-compact in (A, t_A).

Similar work for $p\alpha^*$-compactness can be done.

Theorem 5.13: Let (X,t) be an fts and λ be a fuzzy set in X with $\lambda_0 \subset X$. If every family of closed fuzzy sets in (X,t) which has empty intersection has a finite subfamily with empty intersection, then λ is $p\alpha$-compact (resp. $p\alpha^*$-compact). The converse is not true.

Proof: Let $M = \{ u_i : i \in J \}$ be an open $p\alpha$-shading of λ in (X,t) i.e. $u_i(x) > \alpha$ for all $x \in \lambda_0$. First condition from the given theorem, we have $\bigcap_{i \in J} u_i^c = 0_X$. Hence we can write $\bigcup_{i \in J} u_i = 1_X$. Again, by the second condition of the theorem, we get $\bigcap_{k \in J_n} u_{i_k}^c = 0_X$ implies that $\bigcup_{k \in J_n} u_{i_k} = 1_X$ and hence $u_{i_k}(x) > \alpha$ for all $x \in \lambda_0$. Hence it is clear that $\{ u_{i_k} : k \in J_n \}$ is a finite $p\alpha$-subshading of M. Therefore λ is $p\alpha$-compact.

Now, for the converse, consider the following example.

Let $X = \{a, b, c\}$, $I = [0,1]$ and $0 \le \alpha < 1$. Let u_1, u_2, u_3, $u_4 \in I^X$ defined by $u_1(a) = 0.4$, $u_1(b) = 0.3$, $u_1(c) = 0.2$; $u_2(a) = 0.8$, $u_2(b) = 0.4$, $u_2(c) = 0.1$; $u_3(a) = 0.8$, $u_3(b) = 0.4$, $u_3(c) = 0.2$ and $u_4(a) = 0.4$, $u_4(b) = 0.3$, $u_4(c) = 0.1$. Now, put $t = \{ 0, u_1, u_2, u_3, u_4, 1 \}$, then we see that (X,t) is an fts. Let $\lambda \in I^X$ defined by $\lambda(a) = 0.2$, $\lambda(b) = 0.5$, $\lambda(c) = 0$. Take $\alpha = 0.2$. Then clearly λ is $p\alpha$-compact in (X,t). Now, closed fuzzy sets are $u_1^c(a) = 0.6$, $u_1^c(b) = 0.7$, $u_1^c(c) = 0.8$; $u_2^c(a) = 0.2$, $u_2^c(b) = 0.6$, $u_2^c(c) = 0.9$; $u_3^c(a) = 0.2$, $u_3^c(b) = 0.6$, $u_3^c(c) = 0.8$ and $u_4^c(a) = 0.6$, $u_4^c(b) = 0.7$, $u_4^c(c) = 0.9$. Thus we see that $u_1^c \cap u_2^c \cap u_3^c \cap u_4^c \neq 0$. Therefore the converse of the theorem is not necessarily true.

The work is similar for $p\alpha^*$-compactness can be given.

Note: The $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy sets in an fts need not be closed.

Consider the example in the theorem (5.13), then we have λ is $p\alpha$-compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Again, take $\alpha = 0.4$. Then λ is $p\alpha^*$-compact in (X, t) and λ is not closed.

Theorem 5.14: Let λ be a $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy set in fuzzy T_1-space (X, t) (as def. 1.45) with $\lambda_0 \subset X$. Let $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u , $v \in t$ such that $u(x) = 1$ and $\lambda_0 \subseteq v^{-1}(0, 1]$.

Proof: Suppose $y \in \lambda_0$. Then clearly $x \neq y$. As (X, t) is fuzzy T_1-space, there exist u_y , $v_y \in t$ such that $u_y(x) = 1$, $u_y(y) = 0$ and $v_y(x) = 0$, $v_y(y) = 1$. Let us take $0 \leq \alpha < 1$. Then $v_y(y) > \alpha > 0$, as $v_y(y) = 1$. Hence we see that $\{ v_y : y \in \lambda_0 \}$ is an open $p\alpha$-shading of λ in (X, t). Since λ is $p\alpha$-compact, then $\{ v_y : y \in \lambda_0 \}$ has a finite $p\alpha$-subshading, say $\{ v_{y_k} : y \in \lambda_0 \}$ ($k \in J_n$) such that $v_{y_k}(y) > \alpha$ for each $y \in \lambda_0$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ ..... \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ ...... \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $u \in t$. Moreover, $\lambda_0 \subseteq v^{-1}(0, 1]$ and $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k.

Similar proof for $p\alpha^*$-compact can be done.

Theorem 5.15: Let λ and μ be disjoint $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy sets in fuzzy T_1-space (X, t) (as def. 1.45) with λ_0 , $\mu_0 \subset X$. Then there exist u , $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$.

Proof: Suppose $y \in \lambda_0$. Then $y \notin \mu_0$, as λ and μ are disjoint. Since μ is $p\alpha$-compact in (X, t), then by theorem (5.14), there exist u_y , $v_y \in t$ such that $u_y(y) = 1$ and $\mu_0 \subseteq v_y^{-1}(0, 1]$. Let us take $0 \le \alpha < 1$ with $u_y(y) > \alpha > 0$, as $u_y(y) = 1$. Thus we see that $\{ u_y : y \in \lambda_0 \}$ is an open $p\alpha$-shading of λ in (X, t). Since λ is $p\alpha$-compact, then $\{ u_y : y \in \lambda_0 \}$ has a finite $p\alpha$-subshading, say $\{ u_{y_k} : y \in \lambda_0 \}$ ($k \in J_n$) such that $u_{y_k}(y) > \alpha$ for each $y \in \lambda_0$. Furthermore, μ is $p\alpha$-compact , so $\{ v_y : x \in \mu_0 \}$ has a finite $p\alpha$-subshading, say $\{ v_{y_k} : x \in \mu_0 \}$ ($k \in J_n$) such that $v_{y_k}(x) > \alpha$ for each $x \in \mu_0$, as $\mu_0 \subseteq v_{y_k}^{-1}(0, 1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Hence we see that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$. Thus u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u , $v \in t$.

Similar proof for $p\alpha^*$-compact can be given.

Remark: If $\lambda(x) \ne 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above theorems (5.14) and (5.15) are not at all true.

The following example will show that the $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy sets in fuzzy T_1-space (as def. 1.45) need not be closed.

Example 5.16: Let $X = \{a, b\}$, $I = [0, 1]$ and $0 \le \alpha < 1$. Let u_1 , $u_2 \in I^X$ defined by $u_1(a) = 1$, $u_1(b) = 0$ and $u_2(a) = 0$, $u_2(b) = 1$. Put $t = \{ 0 , u_1 , u_2 , 1 \}$, then we have (X, t) is a fuzzy T_1-space. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.2$, $\lambda(b) = 0$. Now, take $\alpha = 0.4$. Then λ is $p\alpha$-compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Again, if we take $\alpha = 1$, then this example is also applicable for $p\alpha^*$-compactness.

Theorem 5.17: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u, $v \in t$ such that $u(x) > 0$ and $\lambda_0 \subseteq v^{-1}(0,1]$. The converse is not true in general.

The proof is similar as that of theorem (5.14).

Now, for the converse, we give the following example.

Let $X = \{a, b\}$, $I = [0,1]$ and $0 \le \alpha < 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 0.2$, $u_1(b) = 0$; $u_2(a) = 0$, $u_2(b) = 0.3$ and $u_3(a) = 0.2$, $u_3(b) = 0.3$. Now, put $t = \{ 0, u_1, u_2, u_3, 1\}$, then we see that (X, t) is a fuzzy T_1-space. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.3$. Hence we observe that $\lambda_0 = \{b\}$ and $a \notin \lambda_0$. Here u_1, $u_2 \in t$ where $u_1(a) > 0$ and $u_2^{-1}(0,1] = \{b\}$. Therefore $\lambda_0 \subseteq u_2^{-1}(0,1]$. Now, take $\alpha = 0.4$. But we see that λ is not $p\alpha$-compact in (X, t), as $u_k(b) < \alpha$ where $b \in \lambda_0$, for $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

This example is also valid for $p\alpha^*$-compactness.

Theorem 5.18: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and λ, μ be fuzzy sets in X with λ_0, $\mu_0 \subset X$. If λ and μ are disjoint $p\alpha$-compacts (resp. $p\alpha^*$-compacts) in (X, t), then there exist u, $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0,1]$ and $\mu_0 \subseteq v^{-1}(0,1]$. The converse is not true in general.

Similar proof as theorem (5.15).

Now, for the converse, consider the fuzzy T_1-space (X, t) in the example of the theorem (5.17). Let λ, $\mu \in I^X$ with $\lambda(a) = 0.3$, $\lambda(b) = 0$ and $\mu(a) = 0$, $\mu(b) = 0.1$. Thus we see that $\lambda_0 = \{a\}$ and $\mu_0 = \{b\}$. Now u_1, $u_2 \in t$ where $u_1^{-1}(0,1] = \{a\}$ and

$u_2^{-1}(0,1]=\{b\}$. Hence we observe that $\lambda_0 \subseteq u_1^{-1}(0,1]$ and $\mu_0 \subseteq u_2^{-1}(0,1]$, where λ and μ are disjoint. Take $\alpha = 0.4$. Hence we observe that λ and μ are not $p\alpha$-compacts in (X, t), as $u_k(a) < \alpha$ where $a \in \lambda_0$ and $u_k(b) < \alpha$ where $b \in \mu_0$, for $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

This example is also applicable for $p\alpha^*$-compactness.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above theorems (5.17) and (5.18) are not at all true.

The following example will show that the $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy sets in fuzzy T_1-space (as def. 1.46) need not be closed.

Example 5.19: Consider the fuzzy T_1-space (X, t) in the example of the theorem (5.17). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.8$. Then $\lambda_0 = \{b\}$. Take $\alpha = 0.2$. Clearly λ is $p\alpha$-compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Again, if we take $\alpha = 0.3$, then this example is also applicable for $p\alpha^*$-compactness.

Theorem 5.20: Let λ be a $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy set in a fuzzy Hausdorff space (X, t) (as def. 1.47) with $\lambda_0 \subset X$. Let $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u, $v \in t$ such that $u(x) = 1$, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$.

Proof: Let $y \in \lambda_0$. Then clearly $x \neq y$. Since (X, t) is fuzzy Hausdorff space, there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \cap v_y = 0$. Let us take $0 \leq \alpha < 1$ such that $v_y(y) > \alpha > 0$, as $v_y(y) = 1$. Hence we see that $\{ v_y : y \in \lambda_0 \}$ is an open $p\alpha$-shading of λ in (X, t). As λ is $p\alpha$-compact in (X, t), then $\{ v_y : y \in \lambda_0 \}$ has a finite $p\alpha$-subshading, say $\{ v_{y_k} : y \in \lambda_0 \}$ ($k \in J_n$) such that $v_{y_k}(y) > \alpha$ for each $y \in \lambda_0$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ldots.. \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $u \in t$. Moreover, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k.

Finally, we have to show that $u \cap v = 0$. As $u_{y_k} \cap v_{y_k} = 0$ implies that $u \cap v_{y_k} = 0$, by distributive law, we see that $u \cap v = u \cap (v_{y_1} \cup v_{y_2} \cup \ldots... \cup v_{y_n}) = 0$.

Similar work for $p\alpha^*$ -compactness can be given.

Corollary 5.21: Let λ be a $p\alpha$ -compact (resp. $p\alpha^*$ -compact) fuzzy set in a fuzzy Hausdorff space (X, t) (as def. 1.47) with $\lambda_0 \subset X$. Let $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exists $u \in t$ such that $u(x) = 1$ and $u^{-1}(0,1] \subseteq \lambda_0^c$.

Proof: By theorem (5.20), there exist u , $v \in t$ such that $u(x) = 1$, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$. Hence $u^{-1}(0,1] \cap v^{-1}(0,1] = \phi$. If not, there exists $x \in u^{-1}(0,1] \cap v^{-1}(0,1]$ $\Rightarrow x \in u^{-1}(0,1]$ and $x \in v^{-1}(0,1]$ $\Rightarrow u(x) > 0$ and $v(x) > 0$ $\Rightarrow u \cap v \neq 0$. Hence $u^{-1}(0,1] \cap \lambda_0 = \phi$ and consequently $u^{-1}(0,1] \subseteq \lambda_0^c$.

Similar work for $p\alpha^*$ -compactness can be given.

Theorem 5.22: Let λ and μ be disjoint $p\alpha$ -compact (resp. $p\alpha^*$ -compact) fuzzy sets in a fuzzy Hausdorff space (X, t) (as def. 1.47) with λ_0 , $\mu_0 \subset X$. Then there exist u , $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0,1]$, $\mu_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$.

Proof: Suppose $y \in \lambda_0$. Then $y \notin \mu_0$, as λ and μ are disjoint. Since μ is $p\alpha$-compact fuzzy set in (X, t), then by theorem (5.20), there exist u_y , $v_y \in t$ such that $u_y(y) = 1$, $\mu_0 \subseteq v_y^{-1}(0,1]$ and $u_y \cap v_y = 0$. Let us take $0 \le \alpha < 1$ such that $u_y(y) > \alpha > 0$, as $u_y(y) = 1$. Then we see that { u_y : $y \in \lambda_0$ } is an open $p\alpha$-shading of λ in (X, t). Since λ is $p\alpha$-compact in (X, t), then { u_y : $y \in \lambda_0$ } has a finite $p\alpha$-subshading, say { u_{y_k} : $y \in \lambda_0$ } ($k \in J_n$) such that $u_{y_k}(y) > \alpha$ for each $y \in \lambda_0$. Furthermore, μ is $p\alpha$-compact, then { v_y : $x \in \mu_0$ } has a finite $p\alpha$-subshading, say { v_{y_k} : $x \in \mu_0$ } ($k \in J_n$) such that $v_{y_k}(x) > \alpha$ for each $x \in \mu_0$, as $\mu_0 \subseteq v_{y_k}^{-1}(0,1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Hence we see that $\lambda_0 \subseteq u^{-1}(0,1]$ and $\mu_0 \subseteq v^{-1}(0,1]$. Thus u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u , $v \in t$.

Finally, we have to show that $u \cap v = 0$. We observe that $u_{y_k} \cap v_{y_k} = 0$ for each k implies that $u_{y_k} \cap v = 0$ for each k, by distributive law, we see that $u \cap v =$ ($u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$) $\cap v = 0$.

Similar proof of $p\alpha^*$-compactness can be given.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (5.20), (5.22) and corollary (5.21) are not at all true.

Note: The $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy sets in fuzzy Hausdorff space (as def. 1.47) need not be closed.

Consider the fuzzy topology t in the example (5.16), then (X, t) is fuzzy Hausdorff space (as def. 1.47) and also will serve the purpose.

Theorem 5.23: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u, $v \in t$ such that $u(x) > 0$, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$. The converse of the theorem is not necessarily true in general.

The proof is similar as that of theorem (5.20).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (5.17), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.3$. Hence we observe that $\lambda_0 = \{b\}$ and $a \notin \lambda_0$. Here, u_1, $u_2 \in t$ where $u_1(a) > 0$ and $u_2^{-1}(0,1] = \{b\}$. Therefore $\lambda_0 \subseteq u_2^{-1}(0,1]$ and $u_1 \cap u_2 = 0$. Now, take $\alpha = 0.4$. But we see that λ is not $p\alpha$-compact in (X, t), as $u_k(b) < \alpha$ where $b \in \lambda_0$, for $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

Similar work for $p\alpha^*$-compactness can be done.

Corollary 5.24: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exists $u \in t$ such that $u(x) > 0$ and $u^{-1}(0,1] \subseteq \lambda_0^c$.

The proof is similar as that of corollary (5.21).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (5.17), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.3$. Hence we observe that $\lambda_0 = \{b\}$ and $a \notin \lambda_0$ and. Now, $u_1 \in t$ where $u_1(a) > 0$ and then $u_1^{-1}(0,1] = \{a\}$. Hence we have $u_1^{-1}(0,1] \subseteq \lambda_0^c$. Now, take

$\alpha = 0.4$. Thus we see that λ is not $p\alpha$-compact, as $u_k(b) < \alpha$ where $b \in \lambda_0$, for $k = 1$, 2, 3. Thus the converse is not true in general.

The work is similar for $p\alpha^*$-compactness can be given.

Theorem 5.25: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and λ, μ be fuzzy sets in X with λ_0, $\mu_0 \subset X$. If λ and μ are disjoint $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy sets in (X, t), then there exist u, $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0,1]$, $\mu_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$. The converse of the theorem is not true in general.

The proof is similar as that of theorem (5.22).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (5.17), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Again, let λ, $\mu \in I^X$ with $\lambda(a) = 0.3$, $\lambda(b) = 0$ and $\mu(a) = 0$, $\mu(b) = 0.1$. Thus we see that $\lambda_0 = \{a\}$ and $\mu_0 = \{b\}$. Now u_1, $u_2 \in t$ where $u_1^{-1}(0,1] = \{a\}$ and $u_2^{-1}(0,1] = \{b\}$. Hence we observe that $\lambda_0 \subseteq u_1^{-1}(0,1]$ and $\mu_0 \subseteq u_2^{-1}(0,1]$ and $u_1 \cap u_2 = 0$, where λ and μ are disjoint. Take $\alpha = 0.4$. Hence we observe that λ and μ are not $p\alpha$-compacts in (X, t), as $u_k(a) < \alpha$ where $a \in \lambda_0$ and $u_k(b) < \alpha$ where $b \in \mu_0$, for $k = 1$, 2, 3. Thus the converse of the theorem is not true in general.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (5.23), (5.25) and corollary (5.24) are not at all true.

The following example will show that the $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy sets in fuzzy Hausdorff space (as def. 1.48) need not be closed.

Example 5.26: Consider the fuzzy topology t in the example of the theorem (5.17), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Again, Let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.3$. Hence we observe that $\lambda_0 = \{b\}$. Now, take $\alpha = 0.2$. Clearly λ is $p\alpha$-compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t). Again, if we take $\alpha = 0.3$, then this example is also applicable for $p\alpha^*$-compactness.

Theorem 5.27: Let (X, t) be a fuzzy Hausdorff space (as def. 1.49) and λ be a fuzzy set in X with $\lambda(x) = 0$ for at least one $x \in X$. If λ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (X, t), then there exist u , $v \in t$ such that $x_r \in u$, $\lambda_0 \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$, where x_r is a fuzzy point in X. The converse is not true in general.

Proof: Suppose (X, t) is a fuzzy Hausdorff space and λ is a $p\alpha$-compact fuzzy set in X. Let x_r , y_s be two fuzzy points in X with y_s ($s > \alpha$) in λ. Now, we see that $x \neq y$, as $\lambda(x) = 0$. As (X, t) is fuzzy Hausdorff, then there exist u_{y_s} , $v_{y_s} \in t$ such that $x_r \in u_{y_s}$, $y_s \in v_{y_s}$ and $u_{y_s} \cap v_{y_s} = 0$ and this is true for any value of s. Hence this is also true for $s > \alpha$. Let us take $\alpha \in I_1$ such that $v_{y_s}(y) > \alpha > 0$. Thus we see that $\{ v_{y_s} : y_s \in \lambda \}$ is an open $p\alpha$-shading of λ. Since λ is $p\alpha$-compact in (X, t), so $\{ v_{y_s} : y_s \in \lambda \}$ has a finite $p\alpha$-subshading, say $\{ v_{y_{s_k}} : y_s \in \lambda \}$ ($k \in J_n$) such that $v_{y_{s_k}}(y) > \alpha$. Let $v = v_{y_{s_1}} \cup v_{y_{s_2}} \cup \ldots\ldots \cup v_{y_{s_n}}$ and $u = u_{y_{s_1}} \cap u_{y_{s_2}} \cap \ldots\ldots \cap u_{y_{s_n}}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $u \in t$. Moreover, $\lambda_0 \subseteq v^{-1}(0, 1]$ and $x_r \in u$, since $x_r \in u_{y_{s_k}}$ for each k.

Finally, we claim that $u \cap v = 0$. As $u_{y_{s_k}} \cap v_{y_{s_k}} = 0$ for each k implies that $u \cap v_{y_{s_k}} = 0$, by distributive law, we therefore observe that $u \cap v =$ $u \cap (v_{y_{s_1}} \cup v_{y_{s_2}} \cup \ldots\ldots \cup v_{y_{s_n}}) = 0$.

Now, for the converse, consider the fuzzy topology t in the example of the theorem (5.17). Let $a_{0.1}$ and $b_{0.2}$ be fuzzy points in X. Then (X, t) is fuzzy Hausdorff space (as def. 1.49). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.6$. Hence we observe that $\lambda_0 = \{b\}$. Now u_1, $u_2 \in t$ where $a_{0.1} \in u_1$ and $u_2^{-1}(0, 1] = \{b\}$. Hence $\lambda_0 \subseteq u_2^{-1}(0, 1]$ and $u_1 \cap u_2 = 0$. Take $\alpha = 0.8$. Then we see that λ is not $p\alpha$-compact in (X , t), as $u_k(b) < \alpha$ where $b \in \lambda_0$, for $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

Similar work for $p\alpha^*$-compactness can be given.

Corollary 5.28: Let (X, t) be a fuzzy Hausdroff space (as def. 1.49) and λ be a fuzzy set in X with $\lambda(x) = 0$ for at least one $x \in X$. If λ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in (X, t), then there exist $u \in t$ such that $x_r \in u$ and $u^{-1}(0, 1] \subseteq \lambda_0^c$, where x_r is a fuzzy point in X. The converse is not true in general.

Proof: By theorem (5.27), there exists u , $v \in t$ such that $x_r \in u$, $\lambda_0 \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$. Hence $u^{-1}(0, 1] \cap v^{-1}(0, 1] = \phi$. If not, there exists $x \in u^{-1}(0, 1] \cap v^{-1}(0, 1]$ $\Rightarrow$ $x \in u^{-1}(0, 1]$ and $x \in v^{-1}(0, 1]$ $\Rightarrow$ $u(x) > 0$ and $v(x) > 0$ $\Rightarrow$ $u \cap v \neq 0$. Hence $u^{-1}(0, 1] \cap \lambda_0 = \phi$ and consequently $u^{-1}(0, 1] \subseteq \lambda_0^c$.

Now, for the converse, consider fuzzy Hausdorff space (as def. 1.49) in the example of the theorem (5.27). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.6$. Hence we observe that $\lambda_0 = \{b\}$. Now, $u_1 \in t$ where $a_{0.1} \in u_1$ and $u_1^{-1}(0,1] = \{a\}$. Hence $u_1^{-1}(0,1] \subseteq \lambda_0^c$. Take $\alpha = 0.8$. Then we see that λ is not $p\alpha$-compact in (X, t) i.e. $u_k(b) < \alpha$ where $b \in \lambda_0$, for $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

Similar work for $p\alpha^*$-compactness can be given.

Theorem 5.29: Let (X, t) be a fuzzy Hausdorff space (as def. 1.49) and λ, μ be disjoint fuzzy sets in X with λ_0, $\mu_0 \subset X$. If λ and μ are $p\alpha$-compacts (resp. $p\alpha^*$-compacts) in (X, t), then there exist u, $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0,1]$, $\mu_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$. The converse is not true in general.

Proof: Let y_s ($s > \alpha$) be a fuzzy point in λ. Then y_s is not a fuzzy point in μ, as λ and μ are disjoint. Since μ is $p\alpha$-compact, then by theorem (5.27), there exist u_{y_s}, $v_{y_s} \in t$ such that $y_s \in u_{y_s}$, $\mu_0 \subseteq v_{y_s}^{-1}(0,1]$ and $u_{y_s} \cap v_{y_s} = 0$ and this is true for any value of s. Hence this is also true for $s > \alpha$. Let us take $\alpha \in I_1$ such that $u_{y_s}(y) > \alpha > 0$. Since $y_s \in u_{y_s}$, then $\{ u_{y_s} : y_s \in \lambda \}$ is an open $p\alpha$-shading of λ. Since λ is $p\alpha$-compact in (X, t), so $\{ u_{y_s} : y_s \in \lambda \}$ has a finite $p\alpha$-subshading, say $\{ u_{y_{s_k}} : y_s \in \lambda \}$ ($k \in J_n$) such that $u_{y_{s_k}}(y) > \alpha$. Furthermore, μ is $p\alpha$-compact, so $\{ v_{y_{s_k}} : x_r \in \mu \}$ has a finite $p\alpha$-subshading, say $\{ v_{y_{s_k}} : x_r \in \mu \}$ ($k \in J_n$) such that $v_{y_{s_k}}(x) > \alpha$, as $\mu_0 \subseteq v_{y_{s_k}}^{-1}(0,1]$ for each k. Now, let $u = u_{y_{s_1}} \cup u_{y_{s_2}} \cup \ldots\ldots \cup u_{y_{s_n}}$ and $v = v_{y_{s_1}} \cap v_{y_{s_2}} \cap \ldots\ldots \cap v_{y_{s_n}}$. Thus we see that $\lambda_0 \subseteq u^{-1}(0,1]$ and $\mu_0 \subseteq v^{-1}(0,1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u, $v \in t$.

Finally, we have to show that $u \cap v = 0$. First we observe that $u_{y_{s_k}} \cap v_{y_{s_k}} = 0$ for each k implies that $u_{y_{s_k}} \cap v = 0$, by distributive law, we see that $u \cap v = (u_{y_{s_1}} \cup u_{y_{s_2}} \cup \ldots\ldots \cup u_{y_{s_n}}) \cap v = 0$.

Now, for the converse, consider fuzzy Hausdorff space (as def. 1.49) in the example of the theorem (5.27). Again, let λ, $\mu \in I^X$ with $\lambda(a) = 0.3$, $\lambda(b) = 0$ and $\mu(a) = 0$, $\mu(b) = 0.1$. Thus we see that $\lambda_0 = \{a\}$ and $\mu_0 = \{b\}$. Now u_1, $u_2 \in t$ where $u_1^{-1}(0,1] = \{a\}$ and $u_2^{-1}(0,1] = \{b\}$. Hence we observe that $\lambda_0 \subseteq u_1^{-1}(0,1]$ and $\mu_0 \subseteq u_1^{-1}(0,1]$ and $u_1 \cap u_2 = 0$, where λ and μ are disjoint. Take $\alpha = 0.4$. Hence we observe that λ and μ are not $p\alpha$-compacts in (X, t), as $u_k(a) < \alpha$ where $a \in \lambda_0$ and $u_k(b) < \alpha$ where $b \in \mu_0$, for $k = 1$, 2, 3. Thus the converse of the theorem is not true in general.

Similar work for $p\alpha^*$-compactness can be given.

The following example will show that the $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy sets in fuzzy Hausdorff space (as def. 1.49) need not be closed.

Example 5.30: Let $X = \{a, b\}$, $I = [0,1]$ and $0 \le \alpha < 1$. Again, let u_1, u_2, $u_3 \in I^X$ with $u_1(a) = 0.6$, $u_1(b) = 0$; $u_2(a) = 0$, $u_2(b) = 0.8$ and $u_3(a) = 0.6$, $u_3(b) = 0.8$. Put $t = \{ 0, u_1, u_2, u_3, 1\}$, then (X, t) is an fts. Now, let $a_{0.4}$ and $b_{0.7}$ be fuzzy points in X. Therefore (X, t) is also a fuzzy Hausdorff space (as def. 1.49). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.9$. Take $\alpha = 0.5$. Then clearly λ is $p\alpha$-compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Similar work for $p\alpha^*$-compactness can be given.

Theorem 5.31: Let λ be a $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy set in a fuzzy Hausdorff space (X, t) (as def. 1.50) with $\lambda_0 \subset X$. Let $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u , $v \in t$ such that $u(x) = 1$, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u \subseteq 1 - v$.

Proof: Suppose $y \in \lambda_0$. Then clearly $x \neq y$. Since (X, t) is fuzzy Hausdorff space, there exist u_y , $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \subseteq 1 - v_y$. Let us take $0 \leq \alpha < 1$ such that $v_y(y) > \alpha > 0$, as $v_y(y) = 1$. Thus we see that $\{ v_y : \; y \in \lambda_0 \}$ is an open $p\alpha$-shading of λ in (X, t). Since λ is $p\alpha$-compact in (X, t), then $\{ v_y : \; y \in \lambda_0 \}$ has a finite $p\alpha$-subshading, say $\{ v_{y_k} : \; y \in \lambda_0 \}$ ($k \in J_n$) such that $v_{y_k}(y) > \alpha$ for each $y \in \lambda_0$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \; ..... \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \; ...... \cap u_{y_n}$. Thus we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $u \in t$. Moreover, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k.

Finally, we have to show that $u \subseteq 1 - v$. Since $u_y \subseteq 1 - v_y$ implies that $u \subseteq 1 - v_y$. As $u_{y_k}(x) \leq 1 - v_{y_k}(x)$ for all $x \in X$ and for each k, then $u \subseteq 1 - v$. If not, then there exist $x \in X$ such that $u_y(x) > 1 - v_y(x)$. We have $u_y(x) \; \leq \; u_{y_k}(x)$ for each k. Then for some k , $u_{y_k}(x) > 1 - v_{y_k}(x)$. But this is a contradiction, as $u_{y_k}(x) \; \leq 1 - v_{y_k}(x)$ for each k. Hence $u \subseteq 1 - v$.

Similar proof for $p\alpha^*$-compactness can be given.

Theorem 5.32: Let λ and μ be disjoint $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy sets in fuzzy Hausdorff space (X, t) (as def. 1.50) with λ_0 , $\mu_0 \subset X$. Then there exist u , $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0,1]$, $\mu_0 \subseteq v^{-1}(0,1]$ and $u \subseteq 1 - v$.

Proof: Suppose $y \in \lambda_0$. Then $y \notin \mu_0$, as λ and μ are disjoint. Since μ is $p\alpha$-compact in (X, t), then by theorem (5.31), there exist u_y, $v_y \in t$ such that $u_y(y) = 1$, $\mu_0 \subseteq v_y^{-1}(0, 1]$ and $u_y \subseteq 1 - v_y$. Let us assume that $0 \leq \alpha < 1$ such that $u_y(y) > \alpha > 0$, as $u_y(y) = 1$. Then we see that $\{ u_y :\ y \in \lambda_0 \}$ is an open $p\alpha$-shading of λ in (X, t). Since λ is $p\alpha$-compact in (X, t), then $\{ u_y :\ y \in \lambda_0 \}$ has a finite $p\alpha$-subshading, say $\{ u_{y_k} :\ y \in \lambda_0 \}$ ($k \in J_n$) such that $u_{y_k}(y) > \alpha$ for each $y \in \lambda_0$. Furthermore, μ is $p\alpha$-compact , then $\{ v_y :\ x \in \mu_0 \}$ has a finite $p\alpha$-subshading, say $\{ v_{y_k} :\ x \in \mu_0 \}$ ($k \in J_n$) such that $v_{y_k}(x) > \alpha$ for each $x \in \mu_0$, as $\mu_0 \subseteq v_{y_k}^{-1}(0, 1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Hence we see that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$. Thus u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u , $v \in t$.

Finally, we have to show that $u \subseteq 1 - v$. Since $u_{y_k} \subseteq 1 - v_{y_k}$ for each k implies that $u_{y_k} \subseteq 1 - v$ for each k and it is clear that $u \subseteq 1 - v$.

Similar proof of $p\alpha^*$ -compactness can be given.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (5.31) and (5.32) are not at all true.

Note: The $p\alpha$ -compact (resp. $p\alpha^*$ -compact) fuzzy sets in fuzzy Hausdorff space (as def. 1.50) need not be closed .

Consider the fuzzy topology t in the example (5.16), then (X, t) is fuzzy Hausdorff space (as def. 1.50) and also will serve the purpose.

Theorem 5.33: Let λ be a $p\alpha$-compact (resp. $p\alpha^*$-compact) fuzzy set in a fuzzy regular space (X, t) (as def. 1.51) with $\lambda_0 \subset X$. If for each $x \in \lambda_0$, there exist $u \in t^c$ with $u(x) = 0$, we have v, $w \in t$ such that $v(x) = 1$, $u \subseteq w$, $\lambda_0 \subseteq v^{-1}(0, 1]$ and $v \subseteq 1 - w$.

Proof: Let (X, t) be a fuzzy regular space and λ be a $p\alpha$-compact fuzzy set in (X, t). Then for each $x \in \lambda_0$, there exists $u \in t^c$ with $u(x) = 0$. As (X, t) is fuzzy regular, we have v_x, $w_x \in t$ such that $v_x(x) = 1$, $u_x \subseteq w_x$ and $v_x \subseteq 1 - w_x$. Let us take $0 \le \alpha < 1$, then $v_x(x) > \alpha > 0$, as $v_x(x) = 1$. Hence we see that $\{ v_x : x \in \lambda_0 \}$ is an open $p\alpha$-shading of λ in (X, t). Since λ is $p\alpha$-compact in (X, t), then $\{ v_x : x \in \lambda_0 \}$ has a finite $p\alpha$-subshading, say $\{ v_{x_k} : x \in \lambda_0 \}$ ($k \in J_n$) such that $v_{x_k}(x) > \alpha$ for each $x \in \lambda_0$. Now, let $v = v_{x_1} \cup v_{x_2} \cup \ldots.. \cup v_{x_n}$ and $w = w_{x_1} \cap w_{x_2} \cap \ldots.. \cap w_{x_n}$. Thus v and w are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, $w \in t$. Moreover, $\lambda_0 \subseteq v^{-1}(0, 1]$, $v(x) = 1$ and $u \subseteq w$, as $u \subseteq w_{x_k}$ for each k.

Finally, we have to show that $v \subseteq 1 - w$. First we observe that $v_{x_k} \subseteq 1 - w_{x_k}$ for each k implies that $v_{x_k} \subseteq 1 - w$ for each k and hence it is clear that $v \subseteq 1 - w$.

Similar proof for $p\alpha^*$-compactness can be given.

Theorem 5.34: Let (X, t) be an fts and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ_0 is compact in (X, t_α), then λ is $p\alpha$-compact in (X, t). The converse is not true in general.

Proof: Suppose λ_0 is compact in (X, t_α). Let $M = \{ v_i : i \in J \}$ is an open $p\alpha$-shading of λ in (X, t). Then the family $W = \{ \alpha(v_i) : i \in J \}$ is an open cover of λ_0 in (X, t_α).

For, let $x \in \lambda_0$. Then there exists a $v_{i_0} \in M$ such that $v_{i_0}(x) > \alpha$. Therefore $x \in \alpha(v_{i_0})$ and thus $\alpha(v_{i_0}) \in W$. Since λ_0 is compact in (X, t_α), so W has a finite subcover, say $\{ \alpha(v_{i_k}) : k \in J_n \}$. Then the family $\{ v_{i_k} : k \in J_n \}$ forms a finite $p\alpha$-subshading of M and hence λ is $p\alpha$-compact in (X, t).

Now, for the converse, we consider the following example.

Let $X = \{a, b, c\}$, $I = [0,1]$ and $0 \le \alpha < 1$. Let u_1, u_2, u_3, $u_4 \in I^X$ defined by $u_1(a) = 0.3$, $u_1(b) = 0.9$, $u_1(c) = 0.1$; $u_2(a) = 0.5$, $u_2(b) = 0.4$, $u_2(c) = 0.6$; $u_3(a) = 0.5$, $u_3(b) = 0.9$, $u_3(c) = 0.6$ and $u_4(a) = 0.3$, $u_4(b) = 0.4$, $u_4(c) = 0.1$. Now, put $t = \{ 0, u_1, u_2, u_3, u_4, 1 \}$, then we see that (X, t) is an fts . Let $\lambda \in I^X$ defined by $\lambda(a) = 0.8$, $\lambda(b) = 0.7$, $\lambda(c) = 0$. Then $\lambda_0 = \{a, b\}$. Take $\alpha = 0.3$.Then clearly λ is $p\alpha$-compact in (X, t). Now, we have $t_{0.3} = \{ \phi, \{b\}, X \}$. It is clear that λ_0 is not compact in $(X, t_{0.3})$.

Theorem 5.35: Let λ be a fuzzy set in X with $\lambda_0 \subset X$ and $f : (X, t_\alpha) \to (X, t)$ be α-level continuous and bijective mapping. If λ_0 is compact in (X, t_α), then $f(\lambda)$ is $p\alpha$-compact in (X, t).

Proof: Let $M = \{ u_i : i \in J \}$ be an open $p\alpha$-shading of $f(\lambda)$ in (X, t). As f is α-level continuous, then $\alpha(f^{-1}(u_i)) \in t_\alpha$ and hence $\{ \alpha(f^{-1}(u_i)) : i \in J \}$ is an open cover of λ_0 in (X, t_α). Since λ_0 is compact in (X, t_α), then $\{ \alpha(f^{-1}(u_i)) : i \in J \}$ has a finite subcover, say $\{ \alpha(f^{-1}(u_{i_k})) \} (k \in J_n)$. Now, if $y \in f(\lambda)_0$, then $y = f(x)$ for $x \in \lambda_0$, as f is bijective. But $\{ \alpha(f^{-1}(u_{i_k})) \}$ is finite subcover of $\{ \alpha(f^{-1}(u_i)) : i \in J \}$, there exist some k such that $u_{i_k}(f(x)) > \alpha$ implies that $u_{i_k}(y) > \alpha$ for each $y \in f(\lambda)_0$.

Thus $\{ u_{i_k} : k \in J_n \}$ is a finite $p\alpha$-subshading of M. Therefore $f(\lambda)$ is $p\alpha$-compact in (X, t).

Theorem 5.36: Let (X, T) be a topological space, $(X, \omega(T))$ be an fts and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is $p\alpha$-compact (resp. $p\alpha^*$-compact) in $(X, \omega(T))$, then λ_0 is compact in (X, T). The converse is not true in general.

Proof: Suppose λ is $p\alpha$-compact fuzzy set in $(X, \omega(T))$. Let $W = \{ V_i : i \in J \}$ be an open cover of λ_0 in (X, T). Then, since for each V_i, there exists a $u_i \in \omega(T)$ such that $V_i = u_i^{-1}(0,1]$, we have $W = \{ u_i^{-1}(0,1] : i \in J \}$. Then the family $G = \{ u_i : i \in J \}$ is an open $p\alpha$-shading of λ in $(X, \omega(T))$. Since W is an open cover of λ_0, then there exists a $V_{i_0} \in W$ such that $x \in V_{i_0}$. But $V_{i_0} = u_{i_0}^{-1}(0,1]$ for some $u_{i_0} \in \omega(T)$. Therefore $x \in u_{i_0}^{-1}(0,1]$ which implies that $u_{i_0}(x) > \alpha$. By $p\alpha$-compactness of λ, G has a finite $p\alpha$-subshading, say $\{ u_{i_k} : k \in J_n \}$. Then $\{ u_{i_k}^{-1}(0,1] : k \in J_n \}$ forms a finite subcover of W and hence λ_0 is compact in (X, T).

Now, for the converse, we consider the following example.

Let $X = \{a, b, c\}$, $I = [0,1]$, $0 \le \alpha < 1$ and $T = \{ \{b\}, \{c\}, \{b, c\}, \phi, X \}$. Then (X, T) is a topological space. Let u_1, u_2, $u_3 \in I^X$ with $u_1(a) = 0$, $u_1(b) = 0.6$, $u_1(c) = 0$; $u_2(a) = 0$, $u_2(b) = 0$, $u_2(c) = 0.8$ and $u_3(a) = 0$, $u_3(b) = 0.6$, $u_3(c) = 0.8$. Then $\omega(T) = \{ u_1, u_2, u_3, 0, 1 \}$ and $(X, \omega(T))$ is an fts. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.4$, $\lambda(c) = 0.3$. Then $\lambda_0 = \{b, c\}$. Then clearly λ_0 is compact in (X, T). Now, take $\alpha = 0.9$. Then λ is not $p\alpha$-compact in $(X, \omega(T))$, as there do not exist $u_k \in \{\omega(T)\}$ ($k = 1, 2, 3$) such that $u_k(b) > \alpha$ for $b \in \lambda_0$. Thus the converse of the theorem is not necessarily true in general.

The work is similar for $p\alpha^*$ -compactness can be given.

Theorem 5.37: Let λ , $\mu \in I^X$. Then $\lambda_0 \times \mu_0 = (\lambda \times \mu)_0$.

Proof: Let $(x,y) \in \lambda_0 \times \mu_0$. Then $x \in \lambda_0$ and $y \in \mu_0$. So $\lambda(x) > 0$ and $\mu(y) > 0$. Therefore $(\lambda \times \mu)(x,y) > 0$ implies that $(x,y) \in (\lambda \times \mu)_0$. Hence $\lambda_0 \times \mu_0 \subseteq (\lambda \times \mu)_0$. Again, let $(x,y) \in (\lambda \times \mu)_0$. Then $(\lambda \times \mu)(x,y) > 0$. Thus $\lambda(x) > 0$ and $\mu(y) > 0$ implies that $x \in \lambda_0$ and $y \in \mu_0$. Therefore $(x,y) \in \lambda_0 \times \mu_0$. Hence $(\lambda \times \mu)_0 \subseteq \lambda_0 \times \mu_0$. Therefore $\lambda_0 \times \mu_0 = (\lambda \times \mu)_0$.

Theorem 5.38: Let λ and μ be $p\alpha$ -compact (resp. $p\alpha^*$ -compact) fuzzy sets in an fts (X, t). Then $(\lambda \times \mu)$ is also $p\alpha$ -compact (resp. $p\alpha^*$ -compact) in $(X \times X, t \times t)$.

Proof: Suppose $\{ u_i : i \in J \}$ is an open $p\alpha$ -shading of λ in (X, t) i.e. $u_i(x) > \alpha$ for each $x \in \lambda_0$ and $\{ v_i : i \in J \}$ is an open $p\alpha$ -shading of μ in (X, t) i.e. $v_i(y) > \alpha$ for each $y \in \mu_0$. Now, let $M = \{ u_i \times v_i : u_i , v_i \in t \}$ be an open $p\alpha$ -shading of $(\lambda \times \mu)$ in $(X \times X, t \times t)$. Thus we see that $(u_i \times v_i)(x, y) = \min(u_i(x), v_i(y)) > \alpha$, for each $(x, y) \in (\lambda \times \mu)_0$. As λ and μ are $p\alpha$ -compact in (X, t), then $\{ u_i : i \in J \}$ and $\{ v_i : i \in J \}$ have finite $p\alpha$ -subshading, say $\{ u_{i_k} : k \in J_n \}$ and $\{ v_{i_k} : k \in J_n \}$ such that $u_{i_k}(x) > \alpha$ and $v_{i_k}(y) > \alpha$ for each $x \in \lambda_0$ and $y \in \mu_0$ respectively. Hence we have M has a finite $p\alpha$ -subshading, say $\{ u_{i_k} \times v_{i_k} : k \in J_n \}$ such that $(u_{i_k} \times v_{i_k})(x, y) = \min(u_{i_k}(x), v_{i_k}(y)) > \alpha$ for each $(x, y) \in (\lambda \times \mu)_0$. Therefore $(\lambda \times \mu)$ is $p\alpha$ -compact in $(X \times X, t \times t)$

Similar proof for $p\alpha^*$ -compactness can be given.

Definition 5.39: Let (X, t) be an fts and $0 < \delta \le 1$, $\alpha \in I$. A family M of δ-open fuzzy sets is called a partial δ-α-shading, $0 \le \alpha < 1$ (resp. partial δ-α^*-shading, $0 < \alpha \le 1$), in short, $p\delta\alpha$-shading (resp. $p\delta\alpha^*$-shading) of a fuzzy set λ in X if for each $x \in \lambda_0$, ($\lambda_0 \ne X$) there exists a $u \in M$ with $u(x) > \alpha$ (resp. $u(x) \ge \alpha$). A subfamily of a $p\delta\alpha$-shading (resp. $p\delta\alpha^*$-shading) of λ which is also a $p\delta\alpha$-shading (resp. $p\delta\alpha^*$-shading) of λ is called a $p\delta\alpha$-subshading (resp. $p\delta\alpha^*$-subshading) of λ.

If $\lambda(x) \ne 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then $p\delta\alpha$-shading (resp. $p\delta\alpha^*$-shading) and δ-α-shading (resp. δ-α^*-shading) will be same.

Example 5.40: Let $X = \{a, b, c\}$, $I = [0,1]$ and $0 < \delta \le 1$, $0 \le \alpha < 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 1$, $u_1(b) = 1$, $u_1(c) = 0.3$; $u_2(a) = 0.4$, $u_2(b) = 0.2$, $u_2(c) = 1$ and $u_3(a) = 0.4$, $u_3(b) = 0.2$, $u_3(c) = 0.3$. Put $t = \{0, u_1, u_2, u_3, 1\}$, then (X, t) is an fts. Again, let $\lambda \in I^X$ with $\lambda(a) = 0$, $\lambda(b) = 0.4$, $\lambda(c) = 0.6$. Then $\lambda_0 = \{b, c\}$. Now, take $\delta = 0.2$ and $\alpha = 0.7$. Hence we observe that u_1, u_2, u_3 are δ-open fuzzy sets and $u_1(b) > \alpha$, $u_2(c) > \alpha$ for b, $c \in \lambda_0$. Therefore $\{u_1, u_2\}$ is an $p\delta\alpha$-shading of λ in (X, t).

Again, if we take $\alpha = 1$, then $\{u_1, u_2\}$ is an $p\delta\alpha^*$-shading of λ in (X, t).

Definition 5.41: Let (X, t) be an fts and $0 < \delta \le 1$, $\alpha \in I$. A fuzzy set λ in X is said to be partially δ-α-compact, $0 \le \alpha < 1$ (resp. partially δ-α^*-compact, $0 < \alpha \le 1$), in short, $p\delta\alpha$-compact (resp. $p\delta\alpha^*$-compact) iff every $p\delta\alpha$-shading (resp. $p\delta\alpha^*$-shading) of λ has a finite $p\delta\alpha$-subshading (resp. $p\delta\alpha^*$-subshading).

Theorem 5.42: Every $p\delta\alpha$-compact (resp. $p\delta\alpha^*$-compact) fuzzy set in an fts is $p\alpha$-compact (resp. $p\alpha^*$-compact). But the converse is not true.

The proof is straightforward.

Now, for the converse, we consider the following example.

Let $X = \{a, b, c\}$, $I = [0, 1]$ and $0 < \delta \le 1$, $0 \le \alpha < 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 0.2$, $u_1(b) = 1$, $u_1(c) = 1$; $u_2(a) = 1$, $u_2(b) = 0.4$, $u_2(c) = 0.7$ and $u_3(a) = 0.2$, $u_3(b) = 0.4$, $u_3(c) = 0.7$. Put $t = \{0, u_1, u_2, u_3, 1\}$, then (X, t) is an fts. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.9$, $\lambda(b) = 0.4$, $\lambda(c) = 0$. Then $\lambda_0 = \{a, b\}$. Now, take $\alpha = 0.7$. Clearly λ is $p\alpha$-compact in (X, t). Again take $\delta = 0.5$. Hence we observe that there is no finite δ-open fuzzy sets in (X, t) such that $u_k(a) > \alpha$ for $k = 1, 2, 3$ and $a \in \lambda_0$. Thus λ is not $p\delta\alpha$-compact in (X, t).

Similarly we can prove for $p\delta\alpha^*$-compact fuzzy sets.

Chapter Six

Q-Compact Fuzzy Sets

In this chapter, we have introduced Q-compact and $Q\alpha$-compact fuzzy sets. Furthermore, we have established some theorems, corollaries and examples of Q-compact fuzzy sets and discussed different characterizations of Q-compact and $Q\alpha$-compact fuzzy sets. Also we have defined δ-Q-compact and δ-$Q\alpha$-compact fuzzy sets and found different properties between Q-compact and δ-Q-compact fuzzy sets, $Q\alpha$-compact and δ-$Q\alpha$-compact fuzzy sets.

Definition 6.1: Let (X, t) be an fts and λ be a fuzzy set in X. Let $M = \{ u_i : i \in J \}$ be a family of fuzzy sets. Then $M = \{u_i\}$ is called a Q-cover of λ iff $\lambda(x) + u_i(x) \geq 1$ for each $x \in X$ and for some u_i. If each u_i is open, then $M = \{u_i\}$ is called an open Q-cover of λ. A subfamily of Q-cover of a fuzzy set λ in X which is also a Q-cover of λ is called Q-subcover of λ.

Example 6.2: Let $X = \{a, b\}$ and $I = [0, 1]$. Let u_1, $u_2 \in I^X$ defined by $u_1(a) = 0.4$, $u_1(b) = 0.1$ and $u_2(a) = 0.3$, $u_2(b) = 0.2$. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.6$, $\lambda(b) = 0.8$. Hence we observe that $\lambda(a) + u_1(a) \geq 1$, $\lambda(b) + u_2(b) \geq 1$. Therefore $\{ u_1, u_2 \}$ is a Q-cover of λ.

Example 6.3: Let $X = \{a, b\}$ and $I = [0, 1]$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 1$, $u_1(b) = 0.3$; $u_2(a) = 0.4$, $u_2(b) = 1$ and $u_3(a) = 0.4$, $u_3(b) = 0.3$.

Put $t = \{ 0 , u_1 , u_2 , u_3 , 1 \}$, then (X, t) is an fts. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.2$, $\lambda(b) = 0.6$. Hence we observe that $\lambda(a) + u_1(a) > 1$, $\lambda(b) + u_2(b) > 1$. Therefore $\{ u_1 , u_2 \}$ is an open Q-cover of λ in (X, t).

Definition 6.4: A fuzzy set λ in X is said to be Q-compact iff every open Q-cover of λ has a finite Q-subcover i.e. there exist $u_{i_1} , u_{i_2} , \ldots , u_{i_n} \in \{u_i\}$ such that $\lambda(x) + u_{i_k}(x) \geq 1$ for each $x \in X$. If $\lambda \subseteq \mu$ and $\mu \in I^X$, then μ is also Q-compact i.e. every super sets of Q-compact fuzzy set is also Q-compact.

Theorem 6.5: Let (X, t) be an fts, $A \subset X$ and λ be a fuzzy set in A. Then λ is Q-compact in (X, t) iff λ is Q-compact in (A, t_A).

Proof: Suppose λ is Q-compact in (X, t). Let $\{ u_i : i \in J \}$ be an open Q-cover of λ in (A, t_A). Then there exist $v_i \in t$ such that $u_i = v_i \mid A \subseteq v_i$. Hence $\lambda(x) + u_i(x) \geq 1$ for each $x \in A$ and consequently $\lambda(x) + v_i(x) \geq 1$ for each $x \in A$. Therefore $\{ v_i : i \in J \}$ is an open Q-cover of λ in (X, t). As λ is Q-compact in (X, t), then λ has finite Q-subcover i.e. there exist $v_{i_k} \in \{v_i\}$ $(k \in J_n)$ such that $\lambda(x) + v_{i_k}(x) \geq 1$ for each $x \in A$. But, then $\lambda(x) + (v_{i_k} \mid A)(x) \geq 1$ for each $x \in A$ and therefore $\lambda(x) + u_{i_k}(x) \geq 1$ for each $x \in A$. Thus $\{u_i\}$ contains a finite Q-subcover $\{ u_{i_1} , u_{i_2} , \ldots , u_{i_n} \}$ and hence λ is Q-compact in (A, t_A).

Conversely, suppose λ is Q-compact in (A, t_A). Let $\{ v_i : i \in J \}$ be an open Q-cover of λ in (X, t). Set $u_i = v_i \mid A$, then $\lambda(x) + v_i(x) \geq 1$ for each $x \in A$ and hence $\lambda(x) + (v_i \mid A)(x) \geq 1$ for each $x \in A$ implies that $\lambda(x) + u_i(x) \geq 1$ for each $x \in A$. But $u_i \in t_A$, so $\{ u_i : i \in J \}$ is an open Q-cover of λ in (A, t_A). As λ is Q-compact in

(A, t_A), then there exist $u_{i_k} \in \{u_i\}$ ($k \in J_n$) such that $\lambda(x) + u_{i_k}(x) \geq 1$ for each $x \in A$. Thus we have $\lambda(x) + (v_{i_k} \mid A)(x) \geq 1$ for each $x \in A$ and consequently $\lambda(x) + v_{i_k}(x) \geq 1$ for each $x \in A$. Thus $\{v_i\}$ contains a finite Q-subcover { v_{i_1} , v_{i_2} , ….., v_{i_n} } and therefore λ is Q-compact in (X, t).

Corollary 6.6: Let (Y, t^*) be a fuzzy subspace of (X, t) and $A \subset Y \subset X$. Let λ be a fuzzy set in A. Then λ is Q-compact in (X, t) if and only if λ is Q-compact in (Y, t^*).

Proof: Let t_A and t_A^* be the subspace fuzzy topologies on A. Then by preceding theorem (6.5), λ is Q-compact in (X, t) or (Y, t^*) if and only if λ is Q-compact in (A, t_A) or (A, t_A^*). But $t_A = t_A^*$.

Theorem 6.7: Let (X, t) be an fts and λ be a Q-compact fuzzy set in X. If $\mu \subseteq \lambda$ and $\mu \in t^c$, then μ is also Q-compact in (X, t).

Proof: Let { u_i : $i \in J$ } be an open Q-cover of μ. Then $\{u_i\} \cup \mu^c$ is an open Q-cover of λ. As $\mu(x) + u_i(x) \geq 1$ for each $x \in X$, then $\lambda(x) + \max(\ u_i(x),\ \mu^c(x)\ ) \geq 1$ for each $x \in X$. Hence $\mu(x) + u_i(x) \leq \lambda(x) + u_i(x) \geq 1$ for each $x \in X$. Since λ is Q-compact in (X, t), then each open Q-cover of λ has a finite Q-subcover i.e. there exist a finite subset $J_n \subset J$ such that { u_{i_k} : $k \in J_n$ } $\cup$ μ^c is an open Q-cover of λ. Then { u_{i_k} : $k \in J_n$ } is a finite subfamily of { u_i : $i \in J$ } and is an open Q-cover of μ i.e. { u_{i_k} : $k \in J_n$ } is a finite Q-subcover of μ. Hence μ is Q-compact.

Theorem 6.8: Let (X, t) be an fts and λ and μ be Q-compact fuzzy sets in X. Then $\lambda \cap \mu$ is also Q-compact in (X, t).

Proof: Let $M = \{ u_i : i \in J \}$ be an open Q-cover of $\lambda \cap \mu$. Then M is open Q-cover of both λ and μ. Since λ is Q-compact in (X, t), then each open Q-cover of λ has a finite Q-subcover i.e. there exist, say $u_{i_k} \in M$ $(k \in J_n)$ such that $\lambda(x) + u_{i_k}(x) \geq 1$ for each $x \in X$. Again, μ is Q-compact in (X, t), then each open Q-cover of μ has a finite Q-subcover i.e. there exist, say $u_{i_r} \in M$ $(r \in J_n)$ such that $\mu(x) + u_{i_r}(x) \geq 1$ for each $x \in X$. Therefore $\{ u_{i_k} , u_{i_r} \}$ is a finite Q-subcover of M. Hence $\lambda \cap \mu$ is Q-compact in (X, t).

Theorem 6.9: Let λ and μ be Q-compact fuzzy sets in an fts (X, t). Then $\lambda \cup \mu$ is also Q-compact in (X, t).

Proof: Since $\lambda \subseteq \lambda \cup \mu$, $\mu \subseteq \lambda \cup \mu$ and λ, μ are Q-compacts in (X, t), then $\lambda \cup \mu$ is also Q-compact in (X, t).

Theorem 6.10: Let (X, t) be an fts and λ be a fuzzy set in X. If every family of closed fuzzy sets in (X, t) which has empty intersection has a finite subfamily with empty intersection, then λ is Q-compact. The converse is not true in general.

Proof: Let $\{ u_i : i \in J \}$ be an open Q-cover of λ i.e. $\lambda(x) + u_i(x) \geq 1$ for each $x \in X$. By the first condition of the theorem, we have $\bigcap_{i \in J} u_i^c = 0_X$. Hence we can write $\bigcup_{i \in J} u_i = 1_X$. Again, by the second condition, we have $\bigcap_{k \in J_n} u_{i_k}^c = 0_X$ implies that $\bigcup_{k \in J_n} u_{i_k} = 1_X$ and consequently $\lambda(x) + u_{i_k}(x) \geq 1$ for each $x \in X$. Hence it is clear that $\{ u_{i_k} : k \in J_n \}$ is a finite Q-subcover of $\{ u_i : i \in J \}$. Therefore λ is Q-compact.

Now, for the converse, we consider the following example.

Let $X = \{a, b\}$ and $I = [0, 1]$. Let u, $v \in I^X$ defined by $u(a) = 0.4$, $u(b) = 0.3$ and $v(a) = 0.6$, $v(b) = 0.8$. Take $t = \{ 0 , u , v , 1\}$, then (X, t) is an fts. Let $\lambda \in I^X$ with $\lambda(a) = 0.8$, $\lambda(b) = 0.9$. Clearly λ is Q-compact in (X, t). Now, closed fuzzy sets are $u^c(a) = 0.6$, $u^c(b) = 0.7$ and $v^c(a) = 0.4$, $v^c(b) = 0.2$. Hence We observe that $u^c \cap v^c \neq 0$. Therefore the converse of the theorem is not necessarily true.

The following example will show that the Q-compact fuzzy sets in an fts need not be closed.

Example 6.11: Let $X = \{a, b\}$ and $I = [0, 1]$. Let u_1, $u_2 \in I^X$ defined by $u_1(a) = 0.2$, $u_1(b) = 0.4$ and $u_2(a) = 0.5$, $u_2(b) = 0.6$. Now, put $t = \{ 0 , u_1 , u_2 , 1\}$, then we see that (X, t) is an fts. Let $\lambda \in I^X$ defined by $\lambda(a) = 0.9$, $\lambda(b) = 0.7$. Clearly λ is Q-compact in (X, t). But λ is not closed, as its complements λ^c is not open in (X, t).

The following example will show that the subsets of Q-compact fuzzy set in an fts need not be Q-compact.

Example 6.12: Let $X = \{a, b\}$ and $I = [0, 1]$. Let u_1, $u_2 \in I^X$ defined by $u_1(a) = 0.3$, $u_1(b) = 0.5$ and $u_2(a) = 0.6$, $u_2(b) = 0.7$. Now, put $t = \{ 0 , u_1 , u_2 , 1\}$, then we see that (X, t) is an fts. Let λ, $\mu \in I^X$ defined by $\lambda(a) = 0.8$, $\lambda(b) = 0.6$ and

$\mu(a) = 0.3$, $\mu(b) = 0.6$. Hence we see that $\mu \subseteq \lambda$. Clearly λ is Q-compact in (X, t). But $\mu(a) + u_k(a) < 1$ for $a \in X$ and $k = 1, 2$. Hence μ is not Q-compact in (X, t).

Theorem 6.13: Let λ be a Q-compact fuzzy set in fuzzy T_1-space (X, t) (as def. 1.45) with $\lambda_0 \subset X$. Let $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u, $v \in t$ such that $u(x) = 1$ and $\lambda_0 \subseteq v^{-1}(0, 1]$.

Proof: Let $y \in \lambda_0$. Then clearly $x \neq y$. As (X, t) is fuzzy T_1-space, then there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $u_y(y) = 0$ and $v_y(x) = 0$, $v_y(y) = 1$. Therefore $\lambda(x) + u_y(x) \geq 1$, $x \in X$ and $\lambda(y) + v_y(y) \geq 1$, $y \in \lambda_0$ i.e. $\{ u_y, v_y : y \in \lambda_0 \}$ is an open Q-cover of λ. Since λ is Q-compact fuzzy set in (X, t), then λ has a finite Q-subcover i.e. there exist u_{y_1}, u_{y_2}, ……. , $u_{y_n} \in \{u_y\}$ and v_{y_1}, v_{y_2}, ……., $v_{y_n} \in \{v_y\}$ such that $\lambda(x) + u_{y_k}(x) \geq 1$ for each $x \in X$ when $\lambda(x) = 0$ and some $u_{y_k} \in \{u_y\}$ and $\lambda(y) + v_{y_k}(y) \geq 1$ for each $y \in X$ when $\lambda(y) > 0$ and some $v_{y_k} \in \{v_y\}$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \; ….. \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \; …… \cap u_{y_n}$. Hence v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, $u \in t$. Furthermore, $\lambda_0 \subseteq v^{-1}(0, 1]$ and $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k.

Theorem 6.14: Let λ and μ be disjoint Q-compact fuzzy sets in fuzzy T_1-space (X, t) (as def. 1.45) with λ_0, $\mu_0 \subset X$. Then there exist u, $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$.

Proof: Let $y \in \lambda_0$. Then $y \notin \mu_0$, as λ and μ are disjoint. Since μ is Q-compact in (X, t), then by theorem (6.13), there exist u_y, $v_y \in t$ such that $u_y(y) = 1$ and

$\mu_0 \subseteq v_y^{-1}(0,1]$. As $u_y(y)=1$, then $\lambda(x)+v_y(x)\geq 1$, $x\in X$ and $\lambda(y)+u_y(y)\geq 1$, $y\in\lambda_0$ i.e. $\{v_y, u_y : y\in\lambda_0\}$ is an open Q-cover of λ. Since λ is Q-compact fuzzy set in (X,t), then λ has a finite Q-subcover i.e. there exist v_{y_1}, v_{y_2}, ……. , $v_{y_n}\in\{v_y\}$ and u_{y_1}, u_{y_2}, ……. , $u_{y_n}\in\{u_y\}$ such that $\lambda(x)+v_{y_k}(x)\geq 1$ for each $x\in X$ when $\lambda(x)=0$ and some $v_{y_k}\in\{v_y\}$ and $\lambda(y)+u_{y_k}(y)\geq 1$ for each $y\in X$ when $\lambda(y)>0$ and some $u_{y_k}\in\{u_y\}$. Again, since μ is Q-compact in (X,t), then we have $\mu(x)+v_{y_k}(x)\geq 1$ for each $x\in X$ when $\mu(x)>0$ and some $v_{y_k}\in\{v_y\}$ and $\mu(y)+u_{y_k}(y)\geq 1$ for each $y\in X$ when $\mu(y)=0$ and some $u_{y_k}\in\{u_y\}$ and also $\mu_0\subseteq v_{y_k}^{-1}(0,1]$ for each k. Now, let $u=u_{y_1}\cup u_{y_2}\cup \ldots\ldots \cup u_{y_n}$ and $v=v_{y_1}\cap v_{y_2}\cap \ldots\ldots \cap v_{y_n}$. Thus we see that $\lambda_0\subseteq u^{-1}(0,1]$ and $\mu_0\subseteq v^{-1}(0,1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u , $v\in t$.

Remark: If $\lambda(x)\neq 0$ for all $x\in X$ i.e. $\lambda_0=X$, then the above two theorems (6.13) and (6.14) are not at all true.

The following example will show that the Q-compact fuzzy sets in fuzzy T_1-space (as def. 1.45) need not be closed.

Example 6.15: Let $X=\{a,b\}$ and $I=[0,1]$. Let u_1, $u_2\in I^X$ defined by $u_1(a)=1$, $u_1(b)=0$ and $u_2(a)=0$, $u_2(b)=1$. Now, put $t=\{0, u_1, u_2, 1\}$, then we see that (X,t) is a fuzzy T_1-space. Let $\lambda\in I^X$ defined by $\lambda(a)=0.6$, $\lambda(b)=0.4$. Clearly λ is Q-compact in (X,t). But λ is not closed, as its complement λ^c is not open in (X,t).

Theorem 6.16: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is Q-compact in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u , $v \in t$ such that $u(x) > 0$ and $\lambda_0 \subseteq v^{-1}(0, 1]$. The converse is not true in general.

The proof is similar as theorem (6.13).

Now, for the converse, we give the following example.

Let $X = \{a, b\}$ and $I = [0, 1]$. Let u_1 , u_2 , $u_3 \in I^X$ defined by $u_1(a) = 0.2$, $u_1(b) = 0$; $u_2(a) = 0$, $u_2(b) = 0.3$ and $u_3(a) = 0.2$, $u_3(b) = 0.3$. Now, put $t = \{ 0 , u_1 , u_2 , u_3 , 1 \}$, then we see that (X, t) is a fuzzy T_1-space. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.3$. Hence we observe that $\lambda_0 = \{b\}$ and $a \notin \lambda_0$. Here u_1 , $u_2 \in t$ where $u_1(a) > 0$ and $u_2^{-1}(0, 1] = \{b\}$. Hence $\lambda_0 \subseteq u_2^{-1}(0, 1]$. But we see that λ is not Q-compact in (X, t), as $\lambda(a) + u_k(a) < 1$ for $a \in X$ and $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

Theorem 6.17: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and λ , μ be fuzzy sets in X with λ_0 , $\mu_0 \subset X$. If λ and μ are disjoint Q-compacts in (X, t), then there exist u , $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0, 1]$ and $\mu_0 \subseteq v^{-1}(0, 1]$. The converse is not true in general.

Similar proof as theorem (6.14).

Now, for the converse, consider the fuzzy T_1-space (X, t) in the example of the theorem (6.16). Let λ , $\mu \in I^X$ with $\lambda(a) = 0.3$, $\lambda(b) = 0$ and $\mu(a) = 0$, $\mu(b) = 0.1$. Thus we see that $\lambda_0 = \{a\}$ and $\mu_0 = \{b\}$. Now u_1 , $u_2 \in t$ where $u_1^{-1}(0, 1] = \{a\}$ and $u_2^{-1}(0, 1] = \{b\}$. Hence we observe that $\lambda_0 \subseteq u_1^{-1}(0, 1]$ and $\mu_0 \subseteq u_2^{-1}(0, 1]$, where λ and μ are disjoint. But λ and μ are not Q-compacts in (X, t), as $\lambda(b) + u_k(b) < 1$ for

$b \in X$ and $\mu(a) + u_k(a) < 1$ for $a \in X$ and $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (6.16) and (6.17) are not at all true.

The following example will show that the Q-compact fuzzy sets in fuzzy T_1-space (as def. 1.46) need not be closed.

Example 6.18: Consider the fuzzy T_1-space (X, t) in the example of the theorem (6.16). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.9$, $\lambda(b) = 0.8$. Clearly λ is Q-compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 6.19: Let λ be a Q-compact fuzzy set in fuzzy Hausdorff space (X, t) (as def. 1.47) with $\lambda_0 \subset X$. Suppose $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u, $v \in t$ such that $u(x) = 1$, $\lambda_0 \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$.

Proof: Let $y \in \lambda_0$. Then clearly $x \neq y$. As (X, t) is fuzzy Hausdorff, then there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$ and $u_y \cap v_y = 0$. Hence $\lambda(x) + u_y(x) \geq 1$, $x \in X$ and $\lambda(y) + v_y(y) \geq 1$, $y \in \lambda_0$ i.e. $\{ u_y, v_y : y \in \lambda_0 \}$ is an open Q-cover of λ. Since λ is Q-compact in (X, t), then there exist u_{y_1}, u_{y_2}, ……., $u_{y_n} \in \{u_y\}$ and v_{y_1}, v_{y_2}, ……., $v_{y_n} \in \{v_y\}$ such that $\lambda(x) + u_{y_k}(x) \geq 1$ for each $x \in X$ when $\lambda(x) = 0$ and some $u_{y_k} \in \{u_y\}$ and $\lambda(y) + v_{y_k}(y) \geq 1$ for each $y \in X$ when $\lambda(y) > 0$ and some $v_{y_k} \in \{v_y\}$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ldots.. \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ldots\ldots \cap u_{y_n}$. Then we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $u \in t$. Furthermore, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u(x)=1$, as $u_{y_k}(x)=1$ for each k.

Finally, we have to show that $u \cap v = 0$. As $u_{y_k} \cap v_{y_k} = 0$ implies $u \cap v_{y_k} = 0$, by distributive law, we see that $u \cap v = u \cap (v_{y_1} \cup v_{y_2} \cup \ldots\ldots \cup v_{y_n}) = 0$.

Corollary 6.20: Let λ be a Q-compact fuzzy set in fuzzy Hausdorff space (X, t) (as def. 1.47) with $\lambda_0 \subset X$. Let $x \notin \lambda_0$ ($\lambda(x)=0$), then there exists $u \in t$ such that $u(x)=1$ and $u^{-1}(0,1] \subseteq \lambda_0^c$.

Proof: By theorem (6.19), there exist u , $v \in t$ such that $u(x)=1$, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$. Hence $u^{-1}(0,1] \cap v^{-1}(0,1] = \phi$. If not, there exists $x \in u^{-1}(0,1] \cap v^{-1}(0,1]$ $\Rightarrow$ $x \in u^{-1}(0,1]$ and $x \in v^{-1}(0,1]$ $\Rightarrow$ $u(x) > 0$ and $v(x) > 0$ $\Rightarrow$ $u \cap v \neq 0$. Hence $u^{-1}(0,1] \cap \lambda_0 = \phi$ and consequently $u^{-1}(0,1] \subseteq \lambda_0^c$.

Theorem 6.21: Let λ and μ be disjoint Q-compact fuzzy sets in fuzzy Hausdorff space (X, t) (as def. 1.47) with λ_0, $\mu_0 \subset X$. Then there exist u , $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0,1]$, $\mu_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$.

Proof: Let $y \in \lambda_0$. Then $y \notin \mu_0$, as λ and μ are disjoint. Since μ is Q-compact in (X, t), then by theorem (6.19), there exist u_y , $v_y \in t$ such that $u_y(y)=1$, $\mu_0 \subseteq v_y^{-1}(0,1]$ and $u_y \cap v_y = 0$. As $u_y(y)=1$, then $\lambda(x) + v_y(x) \geq 1$, $x \in X$ and $\lambda(y) + u_y(y) \geq 1$, $y \in \lambda_0$ i.e. $\{ v_y , u_y : y \in \lambda_0 \}$ is an open Q-cover of λ. Since λ is Q-compact in (X, t), then there exist v_{y_1} , v_{y_2} , ..., $v_{y_n} \in \{v_y\}$ and u_{y_1} , u_{y_2} , ..., $u_{y_n} \in \{u_y\}$ such that $\lambda(x) + v_{y_k}(x) \geq 1$ for each $x \in X$ when $\lambda(x)=0$ and some

$v_{y_k} \in \{v_y\}$ and $\lambda(y) + u_{y_k}(y) \geq 1$ for each $y \in X$ when $\lambda(y) > 0$ and some $u_{y_k} \in \{u_y\}$. Again, since μ is Q-compact in (X, t), then we have $\mu(x) + v_{y_k}(x) \geq 1$ for each $x \in X$ when $\mu(x) > 0$ and some $v_{y_k} \in \{v_y\}$ and $\mu(y) + u_{y_k}(y) \geq 1$ for each $y \in X$ when $\mu(y) = 0$ and some $u_{y_k} \in \{u_y\}$ and also $\mu_0 \subseteq v_{y_k}^{-1}(0,1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Thus we see that $\lambda_0 \subseteq u^{-1}(0,1]$ and $\mu_0 \subseteq v^{-1}(0,1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u , $v \in t$.

Finally, we have to show that $u \cap v = 0$. As $u_{y_k} \cap v_{y_k} = 0$ implies $u_{y_k} \cap v = 0$, by distributive law, we see that $u \cap v = (u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}) \cap v = 0$.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (6.19), (6.21) and corollary (6.20) are not at all true.

Note: The Q-compact fuzzy sets in fuzzy Hausdorff space (as def. 1.47) need not be closed.

Consider the fuzzy topology t in the example (6.15), then (X, t) is also a fuzzy Hausdorff space (as def. 1.47) and will serve the purpose.

Theorem 6.22: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is Q-compact in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u , $v \in t$ such that $u(x) > 0$, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u \cap v = 0$. The converse of the theorem is not necessarily true in general.

The proof is similar as that of theorem (6.19).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (6.16) then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.3$. Hence we observe that $\lambda_0 = \{b\}$ and $a \notin \lambda_0$. Here u_1 , $u_2 \in t$ where $u_1(a) > 0$ and $u_2^{-1}(0, 1] = \{b\}$. Hence $\lambda_0 \subseteq u_2^{-1}(0, 1]$ and $u_1 \cap u_2 = 0$. But we see that λ is not Q-compact in (X, t), as $\lambda(a) + u_k(a) < 1$ for $a \in X$ and $k = 1$, 2 , 3. Thus the converse of the theorem is not true in general.

Corollary 6.23: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is Q-compact in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exists $u \in t$ such that $u(x) > 0$ and $u^{-1}(0, 1] \subseteq \lambda_0^c$.

The proof is similar as that of corollary (6.20).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (6.16), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.3$. Hence we observe that $\lambda_0 = \{b\}$ and $a \notin \lambda_0$. Now $u_1 \in t$ where $u_1(a) > 0$ and then $u_1^{-1}(0, 1] = \{a\}$. Hence we have $u_1^{-1}(0, 1] \subseteq \lambda_0^c$. But λ is not Q-compact, as $\lambda(a) + u_k(a) < 1$ for $a \in X$ and $k = 1$, 2 , 3. Thus the converse is not true in general.

Theorem 6.24: Let (X, t) be a fuzzy Hausdorff space (as def. 1.48) and λ , μ be fuzzy sets in X with λ_0 , $\mu_0 \subset X$. If λ and μ are disjoint Q-compacts in (X, t), then there exist u , $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0, 1]$, $\mu_0 \subseteq v^{-1}(0, 1]$ and $u \cap v = 0$. The converse of the theorem is not true in general.

The proof is similar as that of theorem (6.21).

Now, for the converse, consider the fuzzy topology t in the example of the theorem (6.16) then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Again, let λ, $\mu \in I^X$ with $\lambda(a) = 0.3$, $\lambda(b) = 0$ and $\mu(a) = 0$, $\mu(b) = 0.1$. Thus we see that $\lambda_0 = \{a\}$ and $\mu_0 = \{b\}$. Now u_1, $u_2 \in t$ where $u_1^{-1}(0,1] = \{a\}$ and $u_2^{-1}(0,1] = \{b\}$. Hence we observe that $\lambda_0 \subseteq u_1^{-1}(0,1]$, $\mu_0 \subseteq u_2^{-1}(0,1]$ and $u_1 \cap u_2 = 0$, where λ and μ are disjoint. But λ and μ are not Q-compacts in (X, t), as $\lambda(b) + u_k(b) < 1$ for $b \in X$ and $\mu(a) + u_k(a) < 1$ for $a \in X$ and $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (6.22), (6.24) and corollary (6.23) are not at all true.

The following example will show that the Q-compact fuzzy sets in fuzzy Hausdorff space (as def. 1.48) need not be closed.

Example 6.25: Consider the fuzzy topology t in the example of the theorem (6.16), then (X, t) is also a fuzzy Hausdorff space (as def. 1.48). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.9$, $\lambda(b) = 0.8$. Clearly λ is Q-compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 6.26: Let λ be a Q-compact fuzzy set in fuzzy Hausdorff space (X, t) (as def. 1.50) with $\lambda_0 \subset X$. Suppose $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u, $v \in t$ such that $u(x) = 1$, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u \subseteq 1 - v$.

Proof: Let $y \in \lambda_0$. Then clearly $x \neq y$. As (X, t) is fuzzy Hausdorff, then there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $v_y(y) = 1$and $u_y \subseteq 1 - v_y$. Hence $\lambda(x) + u_y(x) \geq 1$,

$x \in X$ and $\lambda(y) + v_y(y) \geq 1$, $y \in \lambda_0$ i.e. $\{u_y, v_y : y \in \lambda_0\}$ is an open Q-cover of λ. Since λ is Q-compact in (X, t), then there exist u_{y_1}, u_{y_2}, ……., $u_{y_n} \in \{u_y\}$ and v_{y_1}, v_{y_2}, ……., $v_{y_n} \in \{v_y\}$ such that $\lambda(x) + u_{y_k}(x) \geq 1$ for each $x \in X$ when $\lambda(x) = 0$ and some $u_{y_k} \in \{u_y\}$ and $\lambda(y) + v_{y_k}(y) \geq 1$ for each $y \in X$ when $\lambda(y) > 0$ and some $v_{y_k} \in \{v_y\}$. Now, let $v = v_{y_1} \cup v_{y_2} \cup \ldots \cup v_{y_n}$ and $u = u_{y_1} \cap u_{y_2} \cap \ldots \cap u_{y_n}$. Then we see that v and u are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v, $u \in t$. Furthermore, $\lambda_0 \subseteq v^{-1}(0,1]$ and $u(x) = 1$, as $u_{y_k}(x) = 1$ for each k.

Finally, we have to show that $u \subseteq 1 - v$. As $u_y \subseteq 1 - v_y$ implies that $u \subseteq 1 - v_y$. Since $u_{y_k}(x) \leq 1 - v_{y_k}(x)$ for all $x \in X$ and for each k, then $u \subseteq 1 - v$. If not, then there exist $x \in X$ such that $u_y(x) > 1 - v_y(x)$. We have $u_y(x) \leq u_{y_k}(x)$ for each k. Then for some k, $u_{y_k}(x) > 1 - v_{y_k}(x)$. But this is a contradiction, as $u_{y_k}(x) \leq 1 - v_{y_k}(x)$ for each k. Hence $u \subseteq 1 - v$.

Theorem 6.27: Let λ and μ are disjoint Q-compact fuzzy sets in fuzzy Hausdorff space (X, t) (as def. 1.50) with λ_0, $\mu_0 \subset X$. Then there exist u, $v \in t$ such that $\lambda_0 \subseteq u^{-1}(0,1]$, $\mu_0 \subseteq v^{-1}(0,1]$ and $u \subseteq 1 - v$.

Proof: Let $y \in \lambda_0$. Then $y \notin \mu_0$, as λ and μ are disjoint. As μ is Q-compact in (X, t), then by theorem (6.26), there exist u_y, $v_y \in t$ such that $u_y(y) = 1$, $\mu_0 \subseteq v_y^{-1}(0,1]$ and $u_y \subseteq 1 - v_y$. As $u_y(y) = 1$, then $\lambda(x) + v_y(x) \geq 1$, $x \in X$ and $\lambda(y) + u_y(y) \geq 1$, $y \in \lambda_0$ i.e. $\{v_y, u_y : y \in \lambda_0\}$ is an open Q-cover of λ. Since λ is Q-compact in (X, t), then there exist v_{y_1}, v_{y_2}, ……., $v_{y_n} \in \{v_y\}$ and u_{y_1}, u_{y_2}, ……. ,

$u_{y_n} \in \{u_y\}$ such that $\lambda(x) + v_{y_k}(x) \geq 1$ for each $x \in X$ when $\lambda(x) = 0$ and some $v_{y_k} \in \{v_y\}$ and $\lambda(y) + u_{y_k}(y) \geq 1$ for each $y \in X$ when $\lambda(y) > 0$ and some $u_{y_k} \in \{u_y\}$. Again, since μ is Q-compact in (X, t), then we have $\mu(x) + v_{y_k}(x) \geq 1$ for each $x \in X$ when $\mu(x) > 0$ and some $v_{y_k} \in \{v_y\}$ and $\mu(y) + u_{y_k}(y) \geq 1$ for each $y \in X$ when $\mu(y) = 0$ and some $u_{y_k} \in \{u_y\}$ and also $\mu_0 \subseteq v_{y_k}^{-1}(0,1]$ for each k. Now, let $u = u_{y_1} \cup u_{y_2} \cup \ldots\ldots \cup u_{y_n}$ and $v = v_{y_1} \cap v_{y_2} \cap \ldots\ldots \cap v_{y_n}$. Thus we see that $\lambda_0 \subseteq u^{-1}(0,1]$ and $\mu_0 \subseteq v^{-1}(0,1]$. Hence u and v are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. u , $v \in t$.

Finally, we have to show hat $u \subseteq 1 - v$. First we observe that $u_{y_k} \subseteq 1 - v_{y_k}$ for each k implies that $u_{y_k} \subseteq 1 - v$ for each k and it is clear that $u \subseteq 1 - v$.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (6.26) and (6.27) are not at all true.

Note: The Q-compact fuzzy sets in fuzzy Hausdorff space (as def. 1.50) need not be closed.

Consider the fuzzy topology t in the example (6.15), then (X, t) is also a fuzzy Hausdorff space (as def. 1.50) and will serve the purpose.

Theorem 6.28: Let λ be a Q-compact fuzzy set in fuzzy regular space (X, t) (as def. 1.51) with $\lambda_0 \subset X$. If for each $x \in \lambda_0$, there exist $u \in t^c$ with $u(x) = 0$, we have v , $w \in t$ such that $v(x) = 1$, $u \subseteq w$, $\lambda_0 \subseteq v^{-1}(0,1]$ and $v \subseteq 1 - w$.

Proof: Let (X, t) be a fuzzy regular space and λ be a Q-compact fuzzy set in X. Then for each $x \in \lambda_0$, there exists $u \in t^c$ with $u(x) = 0$. As (X, t) is fuzzy regular, we have

v_x , $w_x \in t$ such that $v_x(x) = 1$, $u_x \subseteq w_x$ and $v_x \subseteq 1 - w_x$. Hence $\lambda(x) + v_x(x) \geq 1$ for each $x \in X$ i.e. $\{ v_x : x \in \lambda_0 \}$ is an open Q-cover of λ. As λ is Q-compact fuzzy set in (X, t), so λ has a finite subcover i.e. there exist $v_{x_k} \in \{v_x\}$ ($k = 1, 2, \ldots\ldots, n$) such that $\lambda(x) + v_{x_k}(x) \geq 1$ for each $x \in X$. Now, let $v = v_{x_1} \cup v_{x_2} \cup \ldots\ldots \cup v_{x_n}$ and $w = w_{x_1} \cap w_{x_2} \cap \ldots\ldots \cap w_{x_n}$. Thus v and w are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. v , $w \in t$. Furthermore, $\lambda_0 \subseteq v^{-1}(0,1]$, $v(x) = 1$ and $u \subseteq w$, as $u \subseteq w_{x_k}$ for each k.

Finally, we have to show that $v \subseteq 1 - w$. As $v_{x_k} \subseteq 1 - w_{x_k}$ for each k implies that $v_{x_k} \subseteq 1 - w$ for each k and hence it is clear that $v \subseteq 1 - w$.

The following example will show that the "good extension" property does not hold for Q-compact fuzzy sets.

Example 6.29: Let $X = \{a, b, c\}$ and $T = \{ \phi , \{a\} , \{b\} , \{a, b\} , X \}$. Then (X, T) is a topological space. Again, let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 1$, $u_1(b) = 0$, $u_1(c) = 0$; $u_2(a) = 0$, $u_2(b) = 0.7$, $u_2(c) = 0$; and $u_3(a) = 1$, $u_3(b) = 0.7$, $u_3(c) = 0$. Then $\omega(T) = \{ 0 , u_1 , u_2 , u_3 , 1\}$ and $(X, \omega(T))$ is an fts. Now, let $\lambda \in I^X$ with $\lambda(a) = 0.7$, $\lambda(b) = 0.4$, $\lambda(c) = 0$. Then $\lambda_0 = \{a, b\}$. Clearly λ_0 is compact in (X, T). But λ is not Q-compact in $(X, \omega(T))$, as there do not exist $u_k \in \omega(T)$ ($k = 1, 2, 3$) such that $\lambda(c) + u_k(c) \geq 1$. Again, let $\mu \in I^X$ with $\mu(a) = 0$, $\mu(b) = 0.5$, $\mu(c) = 1$. Then clearly μ is Q-compact in $(X, \omega(T))$, but $\mu_0 = \{b, c\}$ is not compact in (X, T). It is, therefore, observe that the "good extension property" does not hold good for Q-compact fuzzy sets.

Theorem 6.30: Let λ and μ be Q-compact fuzzy sets in an fts (X,t). Then $(\lambda\times\mu)$ is also Q-compact in $(X\times X, t\times t)$.

Proof: Let $M=\{\ a_i:\ a_i\in t\times t$ and $i\in J\ \}$ be a Q-cover of $(\lambda\times\mu)$ in $(X\times X, t\times t)$. Then $(\lambda\times\mu)\,(x,y)+a_i\,(x,y)\geq 1$ for each $(x,y)\in X\times X$. Now, we can write $a_i=\ u_i\times v_i$, where u_i, $v_i\in t$. Thus we have $(\lambda\times\mu)\,(x,y)+(u_i\times v_i)\,(x,y)\geq 1$ for each $(x,y)\in X\times X$. Hence it is clear that $\lambda(x)+u_i(x)\geq 1$ for each $x\in X$ and $\mu(y)+v_i(y)\geq 1$ for each $y\in X$. Therefore, $\{\ u_i:\ i\in J\ \}$ and $\{\ v_i:\ i\in J\ \}$ are open Q-cover of λ and μ respectively. Since λ and μ are Q-compacts, then $\{\ u_i:\ i\in J\ \}$ and $\{\ v_i:\ i\in J\ \}$ have finite Q-subcovers, say $\{\ u_{i_k}:\ k\in J_n\ \}$ and $\{\ v_{i_k}:\ k\in J_n\ \}$ such that $\lambda(x)+u_{i_k}(x)\geq 1$ for each $x\in X$ and $\mu(y)+v_{i_k}(y)\geq 1$ for each $y\in X$ respectively. Thus we can write $(\lambda\times\mu)\,(x,y)+(u_{i_k}\times v_{i_k})\,(x,y)\geq 1$ for each $(x,y)\in X\times X$. Hence $(\lambda\times\mu)$ is Q-compact in $(X\times X, t\times t)$.

Compact fuzzy sets in Chang's sense [19] and Q-compact fuzzy sets are independent. The following example will serve the purpose.

Example 6.31: Let $X=\{a,b\}$ and $I=[0,1]$. Let u_1, u_2, u_3, $u_4\in I^X$ defined by $u_1(a)=0.4$, $u_1(b)=0.6$; $u_2(a)=0.3$, $u_2(b)=0.7$; $u_3(a)=0.4$, $u_3(b)=0.7$; $u_4(a)=0.3$, $u_4(b)=0.6$. Now, take $t=\{\ 0\ ,\ u_1,\ u_2,\ u_3,\ u_4,\ 1\}$, then we see that (X,t) is an fts. Let $\lambda\in I^X$ defined by $\lambda(a)=0.4$, $\lambda(b)=0.5$. Clearly λ is compact in (X,t) in the sense of Chang. Now, we observe that $\lambda(a)+u_k(a)<1$ for $a\in X$ and $k=1,\ 2,\ 3,\ 4$. Hence λ is not Q-compact in (X,t).

Again, let $\mu \in I^X$ defined by $\mu(a) = 0.9$, $\mu(b) = 0.8$. Clearly μ is Q-compact in (X, t). But μ is not compact in (X, t) in the sense of Chang, as there do not exist u_k such that $\mu \subseteq \bigcup_{k=1}^{4} u_k$.

Definition 6.32: Let $M = \{ u_i : i \in J \}$ be a family of δ-open fuzzy sets in an fts (X, t) and λ be a fuzzy set in X. Then M is said to be δ-Q-cover of λ iff $\lambda(x) + u_i(x) \geq 1$ for each $x \in X$ and for some u_i. A subfamily of δ-Q-cover of a fuzzy set λ in X which is also a δ-Q-cover of λ is called δ-Q-subcover of λ.

Example 6.33: Let $X = \{a, b\}$, $I = [0, 1]$ and $0 < \delta \leq 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 1$, $u_1(b) = 0.4$; $u_2(a) = 0.5$, $u_2(b) = 1$ and $u_3(a) = 0.5$, $u_3(b) = 0.4$. Now, take $t = \{ 0, u_1, u_2, u_3, 1\}$, then we see that (X, t) is an fts. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.1$, $\lambda(b) = 0.2$. Take $\delta = 0.4$. Clearly u_1, u_2 and u_3 are δ-open fuzzy sets in (X, t). Now, we observe that $\lambda(a) + u_1(a) > 1$, $\lambda(b) + u_2(b) > 1$ for a, $b \in X$. So $\{ u_1, u_2 \}$ is a δ-Q-cover of λ in (X, t).

Definition 6.34: Let (X, t) be an fts, $0 < \delta \leq 1$ and λ be a fuzzy set in X. Then λ is said to be δ-Q-compact iff every δ-Q-cover of λ has a finite δ-Q-subcover. If $\lambda \subseteq \mu$ and $\mu \in I^X$, then μ is also δ-Q-compact. Thus we can say that any other supersets of δ-Q-compact fuzzy sets in an fts is also δ-Q-compact.

Theorem 6.35: Any δ-Q-compact fuzzy set in an fts is Q-compact. The converse is not true in general.

The proof of the theorem is straightforward.

Now, for the converse, consider the following example.

Let $X = \{a, b\}$, $I = [0, 1]$ and $0 < \delta \leq 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 1$, $u_1(b) = 0.4$; $u_2(a) = 0.7$, $u_2(b) = 1$ and $u_3(a) = 0.7$, $u_3(b) = 0.4$. Now, take $t = \{ 0 , u_1 , u_2 , u_3 , 1\}$, then we see that (X, t) is an fts. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.2$, $\lambda(b) = 0.3$. Clearly λ is Q-compact in (X, t). Take $\delta = 0.8$. Hence we observe that there is no finite δ-open fuzzy set in (X, t). Hence λ is not δ-Q-compact in (X, t). Thus the converse of theorem is not necessarily true.

δ-compact fuzzy sets (Chang's sense [19]) and δ-Q-compact fuzzy sets are independent. For this, we give the following example.

Example 6.36: Let $X = \{a, b\}$, $I = [0, 1]$ and $0 < \delta \leq 1$. Let u_1, u_2, u_3, $u_4 \in I^X$ defined by $u_1(a) = 0.5$, $u_1(b) = 0.2$; $u_2(a) = 0.3$, $u_2(b) = 0.4$; $u_3(a) = 0.5$, $u_3(b) = 0.4$ and $u_4(a) = 0.3$, $u_4(b) = 0.2$. Now, take $t = \{ 0 , u_1 , u_2 , u_3 , u_4 , 1\}$, then we see that (X, t) is an fts. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.3$, $\lambda(b) = 0.4$. Take $\delta = 0.2$. It is clear that λ is δ-compact (Chang's sense) in (X, t). But λ is not δ-Q-compact in (X, t), as $\lambda(a) + u_k(a) < 1$ for $a \in X$ and $k = 1, 2, 3, 4$. Again, let $\mu \in I^X$ with $\mu(a) = 0.6$, $\mu(b) = 0.8$. Clearly μ is δ-Q-compact in (X, t). But μ is not δ-compact (Chang's sense) in (X, t), as there do not exist δ-open fuzzy sets u_k such that $\mu \subseteq \bigcup_{k=1}^{4} u_k$.

Definition 6.37: Let (X, t) be an fts, λ be a fuzzy set in X and $0 < \alpha \leq 1$. Let $M = \{ u_i : i \in J \}$ be a family of fuzzy sets. Then M is said to be $Q\alpha$-cover of λ iff

$\lambda(x) + u_i(x) \geq \alpha$ for each $x \in X$ and for some u_i. If each u_i is open, then M is said to be an open $Q\alpha$ -cover of λ. A subfamily of a $Q\alpha$ -cover of λ which is also a $Q\alpha$ -cover of λ is said to be a $Q\alpha$ -subcover of λ.

Example 6.38: Let $X = \{a, b\}$, $I = [0,1]$ and $0 < \alpha \leq 1$. Let u_1, $u_2 \in I^X$ defined by $u_1(a) = 0.3$, $u_1(b) = 0.2$ and $u_2(a) = 0.1$, $u_2(b) = 0.4$. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.4$, $\lambda(b) = 0.3$. Take $\alpha = 0.7$. Hence we observe that $\lambda(a) + u_1(a) \geq \alpha$, $\lambda(b) + u_2(b) \geq \alpha$. Therefore $\{ u_1, u_2 \}$ is a $Q\alpha$ -cover of λ.

Example 6.39: Let $X = \{a, b\}$, $I = [0,1]$ and $0 < \alpha \leq 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 0.2$, $u_1(b) = 1$; $u_2(a) = 1$, $u_2(b) = 0.3$ and $u_3(a) = 0.2$, $u_3(b) = 0.3$. Put $t = \{ 0, u_1, u_2, u_3, 1\}$, then (X, t) is an fts. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.6$, $\lambda(b) = 0.5$. Take $\alpha = 0.9$. Hence we observe that $\lambda(a) + u_2(a) > \alpha$, $\lambda(b) + u_1(b) > \alpha$. Therefore $\{ u_1, u_2 \}$ is an open $Q\alpha$ -cover of λ in (X, t).

Definition 6.40: A fuzzy set λ is said to be $Q\alpha$ -compact iff every open $Q\alpha$ -cover of λ has a finite $Q\alpha$ -subcover.

Theorem 6.41: Every Q-compact fuzzy set in an fts is $Q\alpha$ -compact. But the converse is not true in general.

The proof of the theorem is straightforward.

Now, for the converse, consider the following example.

Let $X = \{a, b\}$, $I = [0,1]$ and $0 < \alpha \leq 1$. Let u_1, $u_2 \in I^X$ defined by $u_1(a) = 0.4$, $u_1(b) = 0.3$ and $u_2(a) = 0.6$, $u_2(b) = 0.5$. Put $t = \{ 0, u_1, u_2, 1\}$, then (X, t) is an fts.

Again, let $\lambda \in I^X$ with $\lambda(a) = 0.2$, $\lambda(b) = 0.4$. Take $\alpha = 0.8$. Clearly λ is $Q\alpha$ -compact in (X, t). But λ is not Q-compact, as $\lambda(a) + u_k(a) < 1$ for $a \in X$ and $k = 1, 2$.

Note: If we consider $\alpha = 0.9$, then example (6.31) will show that the compact fuzzy sets in Chang's sense [19] and $Q\alpha$ -compact fuzzy sets are independent.

Let (X, t) be an fts, $0 < \alpha \le 1$, (X, t_α) be a α -level topological space and λ be a fuzzy set in X .Then $Q\alpha$ -compactness of λ in (X, t) and compactness of λ_0 in (X, t_α) are independent. For this, we give the following examples.

Example 6.42: Let $X = \{a, b\}$, $I = [0, 1]$ and $0 < \alpha \le 1$. Let u_1 , $u_2 \in I^X$ defined by $u_1(a) = 0.4$, $u_1(b) = 0.3$ and $u_2(a) = 0.6$, $u_2(b) = 0.8$. Put $t = \{ 0 , u_1 , u_2 , 1 \}$, then (X, t) is an fts. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.2$, $\lambda(b) = 0$. Take $\alpha = 0.8$. Clearly λ is $Q\alpha$ -compact in (X, t). Now, we have $\lambda_0 = \{a\}$ and $t_{0.8} = \{ \phi , X \}$. Hence $(X, t_{0.8})$ is a 0.8-level topological space. Thus we see that λ_0 is not compact in $(X, t_{0.8})$, as there is no finite suvcover of λ_0 in $(X, t_{0.8})$.

Again, let $\mu \in I^X$ with $\mu(a) = 0$, $\mu(b) = 0.2$. So we have $\mu_0 = \{b\}$. Take $\alpha = 0.7$. Then we get $t_{0.7} = \{ \phi , \{b\} , X \}$. Hence $(X, t_{0.7})$ is a 0.7-level topological space. Clearly μ_0 is compact in $(X, t_{0.7})$. But μ is not $Q\alpha$ -compact in (X, t), as $\mu(a) + u_k(a) < \alpha$ for $a \in X$ and $k = 1, 2$.

Definition 6.43: Let (X, t) be an fts, λ be a fuzzy set in X and $0 < \delta \le 1$, $0 < \alpha \le 1$. Let $M = \{ u_i : i \in J \}$ be a family of δ -open fuzzy sets. Then M is said to be δ - $Q\alpha$ -cover of λ iff $\lambda(x) + u_i(x) \ge \alpha$ for each $x \in X$ and for some u_i. A subfamily of

δ-$Q\alpha$-cover of λ which is also a δ-$Q\alpha$-cover of λ is said to be δ-$Q\alpha$-subcover of λ.

Definition 6.44: A fuzzy set λ is said to be δ-$Q\alpha$-compact iff every δ-$Q\alpha$-cover of λ has a finite δ-$Q\alpha$-subcover.

Theorem 6.45: Every δ-$Q\alpha$-compact fuzzy set in an fts is $Q\alpha$-compact. But the converse is not true in general.

The proof of the theorem is straightforward.

Now, for the converse, we consider the following example.

Let $X = \{a, b\}$, $I = [0, 1]$ and $0 < \delta \le 1$, $0 < \alpha \le 1$. Let u_1, $u_2 \in I^X$ defined by $u_1(a) = 0.5$, $u_1(b) = 0.4$ and $u_2(a) = 0.7$, $u_2(b) = 0.6$. Put $t = \{ 0 , u_1, u_2 , 1\}$, then (X, t) is an fts. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.2$, $\lambda(b) = 0.3$. Take $\alpha = 0.9$. Clearly λ is $Q\alpha$-compact in (X, t). Again, take $\delta = 0.9$. But λ is not δ-$Q\alpha$-compact, as there is no finite δ-open fuzzy sets in (X, t).

Note: If we consider $\alpha = 0.9$, then example (6.36) will show that the δ-compact fuzzy sets in Chang's sense [19] and δ-$Q\alpha$-compact fuzzy sets are independent.

Chapter Seven

Almost Compact Fuzzy Sets

Almost compact fuzzy sets was first constructed by Concilio and Gerla [27] which is local property. In this chapter, we have discussed several characterizations of this concept and established some theorems, corollary and examples. Also we have defined almost δ-compact fuzzy sets and investigated different characterizations between almost compact and almost δ-compact fuzzy sets.

Definition 7.1[27]: Let λ be a fuzzy set in X. A family $\{ u_i : i \in J \}$ is a proximate cover of λ when $\{ \overline{u_i} : i \in J \}$ is a cover of λ i.e. $\lambda \subseteq \bigcup_{i \in J} \overline{u_i}$. A subfamily of $\{ u_i : i \in J \}$ which is also a proximate cover of λ is said to be proximate subcover of λ.

Definition 7.2[27]: A fuzzy set λ is said to be almost compact iff every open cover of λ has a finite subfamily whose closures is cover of λ or equivalently, every open cover of λ has a finite proximate subcover.

Every fuzzy subsets of an almost compact fuzzy set is also almost compact.

Theorem 7.3: Let (X, t) be an fts, $A \subset X$ and λ be a fuzzy set in X with $\lambda_0 \subseteq A$. Then λ is almost compact in (X, t) iff λ is almost compact in (A, t_A).

Proof: Suppose λ is almost compact in (X, t). Let $\{ u_i : i \in J \}$ be an open cover of λ in (A, t_A), then $\left\{ \left(\overline{u_i} \right)^0 : i \in J \right\}$ is also an open cover of λ in (A, t_A). Then there exist $v_i \in t$ such that $u_i = v_i \mid A \subseteq v_i$. Therefore $\{ v_i : i \in J \}$ is an open cover of λ in (X, t), so $\left\{\left(\overline{v_i}\right)^0 : i \in J\right\}$ is also an open cover of λ in (X, t). But from $\left(\overline{v_i}\right)^0 \subseteq \overline{v_i}$ and since λ is almost compact in (X, t), then $\left\{\left(\overline{v_i}\right)^0 : i \in J\right\}$ has a finite subfamily, say $\left\{\left(\overline{v_{i_k}}\right)^0 : k \in J_n\right\}$ such that $\lambda \subseteq \bigcup_{k \in J_n} \overline{v_{i_k}}$ i.e. $\lambda \subseteq \overline{v_{i_1}} \cup \overline{v_{i_2}} \cup \ldots\ldots \cup \overline{v_{i_n}}$. But $\overline{u_i} = \overline{v_i \mid A} \subseteq \overline{v_i} \mid A \subseteq \overline{v_i}$. Therefore, $\lambda \subseteq (\overline{v_{i_1}} \cup \overline{v_{i_2}} \cup \ldots\ldots \cup \overline{v_{i_n}}) \mid A = (\overline{v_{i_1}} \mid A) \cup (\overline{v_{i_2}} \mid A) \ldots\ldots \cup (\overline{v_{i_n}} \mid A)$ $= \overline{u_{i_1}} \cup \overline{u_{i_2}} \cup \ldots\ldots \cup \overline{u_{i_n}}$, as $\lambda_0 \subseteq A$ i.e. $\lambda \subseteq \overline{u_{i_1}} \cup \overline{u_{i_2}} \cup \ldots\ldots \cup \overline{u_{i_n}}$. Hence $\{ \overline{u_{i_k}} : k \in J_n \}$ is a finite proximate subcover of $\{ u_i : i \in J \}$. So λ is almost compact in (A, t_A).

Conversely, suppose λ is almost compact in (A, t_A). Let $\{ v_i : i \in J \}$ be an open cover of λ in (X, t), then $\left\{\left(\overline{v_i}\right)^0 : i \in J\right\}$ is also an open cover of λ in (X, t). Choose $u_i = v_i \mid A$, then we have $\lambda \subseteq \bigcup_{i \in J} v_i \Rightarrow \lambda \subseteq \left(\bigcup_{i \in J} v_i\right) \mid A \Rightarrow \lambda \subseteq \bigcup_{i \in J} (v_i \mid A) \Rightarrow$ $\lambda \subseteq \bigcup_{i \in J} u_i$. But $u_i \in t_A$, so $\{ u_i : i \in J \}$ is an open cover of λ in (A, t_A). Therefore $\left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$ is also an open cover of λ in (A, t_A). We have $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and since λ is almost compact in (A, t_A), then $\left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$ has a finite subfamily, say $\left\{\left(\overline{u_{i_k}}\right)^0 : k \in J_n\right\}$ such that $\lambda \subseteq \bigcup_{k \in J_n} \overline{u_{i_k}}$ i.e. $\lambda \subseteq \overline{u_{i_1}} \cup \overline{u_{i_2}} \cup \ldots\ldots \cup \overline{u_{i_n}}$. But we have $\overline{u_i} = \overline{v_i \mid A} \subseteq \overline{v_i} \mid A$ $\subseteq \overline{v_i}$. Therefore $\lambda \subseteq \overline{u_{i_1}} \cup \overline{u_{i_2}} \cup \ldots\ldots \cup \overline{u_{i_n}} \Rightarrow \lambda \subseteq (\overline{v_{i_1}} \mid A) \cup (\overline{v_{i_2}} \mid A) \cup \ldots\ldots \cup$ $(\overline{v_{i_n}} \mid A) \Rightarrow \lambda \subseteq \overline{v_{i_1}} \cup \overline{v_{i_2}} \cup \ldots\ldots \cup \overline{v_{i_n}}$, as $\lambda_0 \subseteq A$. Therefore $\{ \overline{v_{i_k}} : k \in J_n \}$ is a finite proximate subcover of $\{ v_i : i \in J \}$. Hence λ is almost compact in (X, t).

Corollary 7.4: Let (Y, t^*) be a fuzzy subspace of an fts (X, t) and $A \subset Y \subset X$. Let λ be a fuzzy set in X with $\lambda_0 \subseteq A$. Then λ is almost compact in (X, t) iff λ is almost compact in (Y, t^*).

Proof: Let t_A and t_A^* be the subspace fuzzy topologies on A. Then preceding theorem (7.3), λ is almost compact in (X, t) or (Y, t^*) iff λ is almost compact in (A, t_A) or (A, t_A^*). But $t_A = t_A^*$.

Theorem 7.5: Let (X, t) and (Y, s) be two fts's and $f : (X, t) \to (Y, s)$ be fuzzy continuous and surjective mapping. If λ is almost compact fuzzy set in (X, t), then $f(\lambda)$ is almost compact in (Y, s).

Proof: Let $\{ u_i : i \in J \}$ be an open cover of $f(\lambda)$ in (Y, s), then $\{ (\overline{u_i})^0 : i \in J \}$ is also an open cover of $f(\lambda)$ in (Y, s). As f is fuzzy continuous, then $f^{-1}(\overline{u_i})^0 \in t$ and hence $\{ f^{-1}(\overline{u_i})^0 : i \in J \}$ is an open cover of λ in (X, t). Since λ is almost compact in (X, t), then $\{ f^{-1}(\overline{u_i})^0 : i \in J \}$ has a finite subfamily, say $\{ f^{-1}(\overline{u_{i_k}})^0 : k \in J_n \}$ such that $\lambda \subseteq \bigcup_{k \in J_n} \overline{f^{-1}(\overline{u_{i_k}})^0}$ i.e. $\lambda \subseteq \overline{f^{-1}(\overline{u_{i_1}})^0} \cup \overline{f^{-1}(\overline{u_{i_2}})^0} \cup \ldots\ldots \cup \overline{f^{-1}(\overline{u_{i_n}})^0}$. But from $(\overline{u_i})^0 \subseteq \overline{u_i}$ and f is fuzzy continuous and surjective, $f^{-1}(\overline{u_i})$ must be a closed fuzzy set in X such that $f^{-1}(\overline{u_i})^0 \subseteq f^{-1}(\overline{u_i})$ and then $\overline{f^{-1}(\overline{u_i})^0} \subseteq f^{-1}(\overline{u_i})$. Therefore $f\left(\overline{f^{-1}(\overline{u_i})^0} \right) \subseteq \overline{u_i}$ for each $i \in J$. Hence $f(\lambda) \subseteq f\left(\overline{f^{-1}(\overline{u_{i_1}})^0} \right) \cup f\left(\overline{f^{-1}(\overline{u_{i_2}})^0} \right) \cup \ldots\ldots \cup f\left(\overline{f^{-1}(\overline{u_{i_n}})^0} \right)$

$\Rightarrow f(\lambda) \subseteq \overline{u_{i_1}} \cup \overline{u_{i_2}} \cup \ldots\ldots \cup \overline{u_{i_n}}$. Thus $f(\lambda)$ is almost compact in (Y, s).

Theorem 7.6: Let (X, t) and (Y, s) be two fts's and $f : (X, t) \to (Y, s)$ be fuzzy open, fuzzy closed and bijective mapping. If λ is almost compact fuzzy set in (Y, s), then $f^{-1}(\lambda)$ is almost compact in (X, t).

Proof: Let $\{ u_i : i \in J \}$ be an open cover of $f^{-1}(\lambda)$ in (X, t), then $\{ (\overline{u_i})^0 : i \in J \}$ is also an open cover of $f^{-1}(\lambda)$ in (X, t). As f is fuzzy open, then $f(\overline{u_i})^0 \in s$ and hence $\{ f(\overline{u_i})^0 : i \in J \}$ is an open cover of λ in (Y, s). Since λ is almost compact in (Y, s), then $\{ f(\overline{u_i})^0 : i \in J \}$ has a finite subfamily, say $\{ f(\overline{u_{i_k}})^0 : k \in J_n \}$ such that $\lambda \subseteq \bigcup_{k \in J_n} \overline{f(\overline{u_{i_k}})^0}$ i.e. $\lambda \subseteq \overline{f(\overline{u_{i_1}})^0} \cup \overline{f(\overline{u_{i_2}})^0} \cup \ldots\ldots \cup \overline{f(\overline{u_{i_n}})^0}$. But from $(\overline{u_i})^0 \subseteq \overline{u_i}$ and f is closed, $f(\overline{u_i})$ must be a closed fuzzy set in Y such that $f(\overline{u_i})^0 \subseteq f(\overline{u_i})$ and then $\overline{f(\overline{u_i})^0} \subseteq f(\overline{u_i})$. Therefore $f^{-1}\left(\overline{f(\overline{u_i})^0}\right) \subseteq \overline{u_i}$ for each $i \in J$. Hence $f^{-1}(\lambda) \subseteq f^{-1}\left(\overline{f(\overline{u_{i_1}})^0} \cup \overline{f(\overline{u_{i_2}})^0} \cup \ldots\ldots\ldots \cup \overline{f(\overline{u_{i_n}})^0}\right) \Rightarrow f^{-1}(\lambda) \subseteq f^{-1}\left(\overline{f(\overline{u_{i_1}})^0}\right) \cup f^{-1}\left(\overline{f(\overline{u_{i_2}})^0}\right) \cup \ldots\ldots \cup f^{-1}\left(\overline{f(\overline{u_{i_n}})^0}\right) \Rightarrow f^{-1}(\lambda) \subseteq \overline{u_{i_1}} \cup \overline{u_{i_2}} \cup \ldots\ldots \cup \overline{u_{i_n}}$. Hence $f^{-1}(\lambda)$ is almost compact in (X, t).

Theorem 7.7: Let (X, t) be an fts and let every family of closed fuzzy sets in X with empty intersection has a finite subfamily with empty intersection. Then any fuzzy set λ in X is almost compact. The converse is not true in general.

Proof: Let λ be any fuzzy set in X and let $\{ u_i : i \in J \}$ be an open cover of λ, then $\{ (\overline{u_i})^0 : i \in J \}$ is also an open cover of λ. From the first condition of the theorem, we have $\bigcap_{i\in J} u_i^c = 0_X$. Therefore $\bigcup_{i\in J} u_i = 1_X$ and hence $\bigcup_{i\in J} \left(\overline{u_i}\right)^0 = 1_X$, as $u_i \subseteq \left(\overline{u_i}\right)^0$. Again, by the second condition of the theorem, we get $\bigcap_{k\in J_n} u_{i_k}^c = 0_X$. Thus we have $\bigcup_{k\in J_n} u_{i_k} = 1_X$ and hence $\bigcup_{k\in J_n} \left(\overline{u_{i_k}}\right)^0 = 1_X$, as $u_i \subseteq \left(\overline{u_i}\right)^0$. But from $u_i \subseteq \left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$, then we get $\bigcup_{k\in J_n} \overline{u_{i_k}} = 1_X$ and consequently we have $\lambda \subseteq \bigcup_{k\in J_n} \overline{u_{i_k}}$ i.e. $\lambda \subseteq \overline{u_{i_1}} \cup \overline{u_{i_2}} \cup \ldots\ldots \cup \overline{u_{i_n}}$. Therefore $\{\, \overline{u_{i_k}} : k \in J_n \,\}$ is a finite proximate subcover of $\{\, u_i : i \in J \,\}$. Hence λ is almost compact.

For the converse, consider the following example.

Let $X = \{a, b\}$ and $I = [0, 1]$. Let u, $v \in I^X$ defined by $u(a) = 0.3$, $u(b) = 0.2$ and $v(a) = 0.4$, $v(b) = 0.3$. Choose $t = \{\, 0, u, v, 1 \,\}$, then (X, t) is an fts. Now, $0^c(a) = 1$, $0^c(b) = 1$; $u^c(a) = 0.7$, $u^c(b) = 0.8$ and $v^c(a) = 0.6$, $v^c(b) = 0.7$. So we have $\bar{u} = \bigcap\{\, 0^c, u^c, v^c \,\} = v^c$ i.e. $\bar{u}(a) = 0.6$, $\bar{u}(b) = 0.7$ and $\bar{v} = \bigcap\{\, 0^c, u^c, v^c \,\} = v^c$ i.e. $\bar{v}(a) = 0.6$, $\bar{v}(b) = 0.7$. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.6$, $\lambda(b) = 0.4$. Then clearly λ is almost compact in (X, t). But $u^c \cap v^c \neq 0$. Therefore the converse of the theorem is not true in general.

The following example will show that the almost compact fuzzy sets in an fts need not be closed.

Example 7.8: Consider the fts (X, t) in the example of the theorem (7.7). Again, let $\lambda \in I^X$ with $\lambda(a) = 0.5$, $\lambda(b) = 0.6$. Then clearly λ is almost compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 7.9: Let λ and μ be almost compact fuzzy sets in an fts (X,t). Then $\lambda \cup \mu$ is also almost compact in (X,t).

Proof: Let $\{ u_i : i \in J \}$ be an open cover of $\lambda \cup \mu$, then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open cover of $\lambda \cup \mu$. Therefore $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is any open cover of both λ and μ respectively. But we have $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and since λ is almost compact, so $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ has a finite proximate subcover, say $\{ \overline{u_{i_k}} : k \in J_n \}$ such that $\lambda \subseteq \overline{u_{i_1}} \cup \overline{u_{i_2}} \cup \ldots\ldots \cup \overline{u_{i_n}}$. Similarly, we can find $\{ \overline{u_{i_r}} : r \in J_n \}$ is a finite proximate subcover of $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$. Therefore $\{ \overline{u_{i_k}} , \overline{u_{i_r}} \}$ is a finite proximate subcover of $\{ u_i : i \in J \}$. Hence $\lambda \cup \mu$ is also almost compact.

Theorem 7.10: Let λ and μ be almost compact fuzzy sets in an fts (X,t). Then $\lambda \cap \mu$ is also almost compact in (X,t).

Proof: We have $\lambda \cap \mu \subseteq \lambda$ and $\lambda \cap \mu \subseteq \mu$. As λ and μ are almost compact, it is clear that $\lambda \cap \mu$ is almost compact.

Theorem 7.11: Let (X,t) be a fuzzy T_1-space (as def. 1.45) and λ be an almost compact fuzzy set in X with $\lambda_0 \subset X$. Let $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u , $v \in t$ such that $\bar{u}(x) = 1$ and $\lambda_0 \subseteq \left(\bar{v}\right)^{-1}(0,1]$.

Proof: Suppose $y \in \lambda_0$. Then clearly $x \neq y$. As (X,t) is fuzzy T_1-space, there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $u_y(y) = 0$ and $v_y(x) = 0$, $v_y(y) = 1$. Hence we observe that $\lambda \subseteq \bigcup \{ v_y : y \in \lambda_0 \}$ i.e. $\{ v_y : y \in \lambda_0 \}$ is an open cover of λ. Thus we have $\left(\overline{u_y}\right)^0 (x) = 1$, $\left(\overline{v_y}\right)^0 (y) = 1$, as $u_y \subseteq \left(\overline{u_y}\right)^0$ and $v_y \subseteq \left(\overline{v_y}\right)^0$. Then $\left\{ \left(\overline{v_y}\right)^0 : y \in \lambda_0 \right\}$ is also an open cover of λ. Since λ is almost compact, then $\left\{\left(\overline{v_y}\right)^0 : y \in \lambda_0\right\}$ has a finite proximate subcover, say $\{\overline{v_{y_k}} : k \in J_n\}$ such that $\lambda \subseteq \bigcup_{k \in J_n} \overline{v_{y_k}}$ i.e. $\lambda \subseteq \overline{v_{y_1}} \cup \overline{v_{y_2}} \cup \ldots\ldots \cup \overline{v_{y_n}}$.

Now, let $\left(\bar{v}\right)^0 = \left(\overline{v_{y_1}}\right)^0 \cup \left(\overline{v_{y_2}}\right)^0 \cup \ldots\ldots \cup \left(\overline{v_{y_n}}\right)^0$ and $\left(\bar{u}\right)^0 = \left(\overline{u_{y_1}}\right)^0 \cap \left(\overline{u_{y_2}}\right)^0 \cap \ldots\ldots \cap \left(\overline{u_{y_n}}\right)^0$.

Hence $\left(\bar{v}\right)^0$ and $\left(\bar{u}\right)^0$ are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $\left(\bar{v}\right)^0$, $\left(\bar{u}\right)^0 \in t$. But $\left(\overline{v_y}\right)^0 \subseteq \overline{v_y}$ and $\left(\overline{u_y}\right)^0 \subseteq \overline{u_y}$. Moreover, $\lambda_0 \subseteq \left(\bar{v}\right)^{-1}(0,1]$ and $\bar{u}(x) = 1$, as $\overline{u_{y_k}}(x) = 1$ for each k.

Theorem 7.12: Let (X, t) be a fuzzy T_1-space (as def. 1.45) and λ, μ be disjoint almost compact fuzzy sets in X with λ_0, $\mu_0 \subset X$. Then there exist u, $v \in t$ such that $\lambda_0 \subseteq \left(\bar{u}\right)^{-1}(0,1]$ and $\mu_0 \subseteq \left(\bar{v}\right)^{-1}(0,1]$.

Proof: Suppose $y \in \lambda_0$. Then we have $y \notin \mu_0$, as λ and μ are disjoint. As μ is almost compact, then by theorem (7.11), there exist u_y, $v_y \in t$ such that $\overline{u_y}(y) = 1$ and $\mu_0 \subseteq \left(\overline{v_y}\right)^{-1}(0,1]$. Since $\overline{u_y}(y) = 1$, then we have $\left\{\left(\overline{u_y}\right)^0 : y \in \lambda_0\right\}$ is also an open cover of λ. But λ is almost compact, then $\left\{\left(\overline{u_y}\right)^0 : y \in \lambda_0\right\}$ has a finite proximate subcover, say $\{\overline{u_{y_k}} : k \in J_n\}$ such that $\lambda \subseteq \bigcup_{k \in J_n} \overline{u_{y_k}}$ i.e. $\lambda \subseteq \overline{u_{y_1}} \cup \overline{u_{y_2}} \cup \ldots\ldots \cup \overline{u_{y_n}}$. Furthermore, $\mu \subseteq \overline{v_{y_1}} \cap \overline{v_{y_2}} \cap \ldots\ldots \cap \overline{v_{y_n}}$, as $\mu_0 \subseteq \left(\overline{v_{y_k}}\right)^{-1}(0,1]$ for each k. Now, let $\left(\bar{u}\right)^0 = \left(\overline{u_{y_1}}\right)^0 \cup \left(\overline{u_{y_2}}\right)^0 \cup \ldots\ldots \cup \left(\overline{u_{y_n}}\right)^0$ and $\left(\bar{v}\right)^0 = \left(\overline{v_{y_1}}\right)^0 \cap \left(\overline{v_{y_2}}\right)^0 \cap \ldots\ldots \cap \left(\overline{v_{y_n}}\right)^0$. But $\left(\overline{u_y}\right)^0 \subseteq \overline{u_y}$ and $\left(\overline{v_y}\right)^0 \subseteq \overline{v_y}$, we see that $\lambda_0 \subseteq \left(\bar{u}\right)^{-1}(0,1]$ and $\mu_0 \subseteq \left(\bar{v}\right)^{-1}(0,1]$. Also $\left(\bar{u}\right)^0$ and $\left(\bar{v}\right)^0$ are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $(\bar{u})^0$, $(\bar{v})^0 \in t$.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (7.11) and (7.12) are not at all true.

The following example will show that the almost compact fuzzy sets in fuzzy T_1-space (as def. 1.45) need not be closed.

Example 7.13: Let $X = \{a, b\}$ and $I = [0, 1]$. Let u , $v \in I^X$ defined by $u(a) = 1$, $u(b) = 0$ and $v(a) = 0$, $v(b) = 1$. Take $t = \{ 0 , u , v , 1\}$, then (X, t) is a fuzzy T_1-space. Now, $0^c(a) = 1$, $0^c(b) = 1$; $u^c(a) = 0$, $u^c(b) = 1$ and $v^c(a) = 1$, $v^c(b) = 0$. So we have $\bar{u} = \bigcap\{ 0^c , v^c \} = v^c$ i.e. $\bar{u}(a) = 1$, $\bar{u}(b) = 0$ and $\bar{v} = \bigcap\{ 0^c , u^c \} = u^c$ i.e. $\bar{v}(a) = 0$, $\bar{v}(b) = 1$. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.4$, $\lambda(b) = 0.7$. Clearly λ is almost compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 7.14: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is almost compact in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u , $v \in t$ such that $\bar{u}(x) > 0$ and $\lambda_0 \subseteq (\bar{v})^{-1}(0, 1]$. The converse is not true in general.

The proof is similar as that of theorem (7.11).

Now, for the converse, we give the following example.

Let $X = \{a, b\}$ and $I = [0, 1]$. Let u_1, u_2 , $u_3 \in I^X$ defined by $u_1(a) = 0.2$, $u_1(b) = 0$; $u_2(a) = 0$, $u_2(b) = 0.3$ and $u_3(a) = 0.2$, $u_3(b) = 0.3$. Now, put $t = \{ 0 , u_1 , u_2 , u_3 ,$

$1\}$, then we see that (X,t) is a fuzzy T_1-space. Now, we have $0^c(a)=1$, $0^c(b)=1$; $u_1^c(a)=0.8$, $u_1^c(b)=1$; $u_2^c(a)=1$, $u_2^c(b)=0.7$ and $u_3^c(a)=0.8$, $u_3^c(b)=0.7$. Therefore, $\overline{u_1}=\bigcap\{\, 0^c,\ u_1^c,\ u_2^c,\ u_3^c\,\}=u_3^c$ i.e. $\overline{u_1}(a)=0.8$, $\overline{u_1}(b)=0.7$; $\overline{u_2}=\bigcap\{\, 0^c,\ u_1^c,\ u_2^c,\ u_3^c\,\}=u_3^c$ i.e. $\overline{u_2}(a)=0.8$, $\overline{u_2}(b)=0.7$ and $\overline{u_3}=\bigcap\{\, 0^c,\ u_1^c,\ u_2^c,\ u_3^c\,\}=u_3^c$ i.e. $\overline{u_3}(a)=0.8$, $\overline{u_3}(b)=0.7$. Again, let $\lambda\in I^X$ defined by $\lambda(a)=0$, $\lambda(b)=0.9$. Hence we observe that $\lambda_0=\{b\}$ and $a\notin\lambda_0$. Here u_1, $u_2\in t$ where $\overline{u_1}(a)=0.8>0$ and $\left(\overline{u_2}\right)^{-1}(0,1]=\{a,b\}$. Hence $\lambda_0\subset\left(\overline{u_2}\right)^{-1}(0,1]$. Thus we see that λ is not almost compact in (X,t), as there do not exist $\overline{u_k}$ such that $\lambda\subseteq\bigcup_{k=1}^{3}\overline{u_k}$. Thus the converse of the theorem is not true in general.

Theorem 7.15: Let (X,t) be a fuzzy T_1-space (as def. 1.46) and λ, μ be fuzzy sets in X with λ_0, $\mu_0\subset X$. If λ and μ are disjoint almost compact fuzzy sets in (X,t), then there exist u, $v\in t$ such that $\lambda_0\subseteq\left(\overline{u}\right)^{-1}(0,1]$ and $\mu_0\subseteq\left(\overline{v}\right)^{-1}(0,1]$.

The work is similar as that of theorem (7.12).

Now, for the converse, consider the fuzzy T_1-space (X,t) in the example of the theorem (7.14). Let λ, $\mu\in I^X$ with $\lambda(a)=0.9$, $\lambda(b)=0$ and $\mu(a)=0$, $\mu(b)=0.8$. Thus we see that $\lambda_0=\{a\}$ and $\mu_0=\{b\}$. Now u_1, $u_2\in t$ where $\left(\overline{u_1}\right)^{-1}(0,1]=\{a,b\}$ and $\left(\overline{u_2}\right)^{-1}(0,1]=\{a,b\}$. Hence we observe that $\lambda_0\subset\left(\overline{u_1}\right)^{-1}(0,1]$ and $\mu_0\subset\left(\overline{u_2}\right)^{-1}(0,1]$, where λ and μ are disjoint. But we see that λ and μ are not almost compact in (X,t), as there do not exist $\overline{u_k}$ such tht $\lambda\subseteq\bigcup_{k=1}^{3}\overline{u_k}$ and $\mu\subseteq\bigcup_{k=1}^{3}\overline{u_k}$. Thus the converse of the theorem is not true in general.

The following example will show that the almost compact fuzzy sets in fuzzy T_1-space (as def. 1.46) need not be closed.

Example 7.16: Consider the fuzzy T_1-space in the example of the theorem (7.14). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.5$, $\lambda(b) = 0.6$. Then clearly λ is almost compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 7.17: An almost compact fuzzy sets in fuzzy regular space (X, t) (as def. 1.52) is compact.

Proof: Let $\{ u_i : i \in J \}$ be an open cover of a fuzzy set λ in X i.e. $\lambda \subseteq \bigcup_{i \in J} u_i$. As (X, t) is fuzzy regular, then we have $u_i = \bigcup_{i \in J} v_{ij}$, where v_{ij} is an open fuzzy set such that $\overline{v_{ij}} \subseteq u_i$ for each i. Since $\lambda \subseteq \bigcup_{i \in J} u_i = \bigcup_{i \in J} v_{ij}$, then $\{ v_{ij} : i \in J \}$ is an open cover of λ. As λ is almost compact, then $\{ v_{ij} : i \in J \}$ has a finite proximate subcover, say $\left\{ \overline{v_{i_k j}} : k \in J_n \right\}$ such that $\lambda \subseteq \bigcup_{k \in J_n} \overline{v_{i_k j}}$. But $\overline{v_{i_k j}} \subseteq u_{i_k}$, so $\lambda \subseteq \bigcup_{k \in J_n} \overline{v_{i_k j}} \subseteq \bigcup_{k \in J_n} u_{i_k}$. Hence $\{ u_{i_k} : k \in J_n \}$ is a finite subcover of $\{ u_i : i \in J \}$. Therefore λ is compact.

The following example will show that the "good extension" property does not hold for almost compact fuzzy sets.

Example 7.18: Let $X = \{a, b, c\}$ and $T = \{ \phi, \{a\}, \{b\}, \{a, b\}, X \}$. Then (X, T) is a topological space. Again, let $u_1, u_2, u_3 \in I^X$ with $u_1(a) = 0.2$, $u_1(b) = 0$, $u_1(c) = 0$; $u_2(a) = 0$, $u_2(b) = 0.4$, $u_2(c) = 0$ and $u_3(a) = 0.2$, $u_3(b) = 0.4$, $u_3(c) = 0$. Then $\omega(T) = \{ 0, u_1, u_2, u_3, 1\}$ and $(X, \omega(T))$ is an fts. Now, $0^c(a) = 1$, $0^c(b) = 1$, $0^c(c) = 1$; $u_1^c(a) = 0.8$, $u_1^c(b) = 1$, $u_1^c(c) = 1$; $u_2^c(a) = 1$, $u_2^c(b) = 0.6$,

$u_2^c(c) = 1$ and $u_3^c(a) = 0.8$, $u_3^c(b) = 0.6$, $u_3^c(c) = 1$. So we have $\overline{u_1} = \bigcap\{\ 0^c,\ u_1^c,\ u_2^c,\ u_3^c\ \} = u_3^c$ i.e. $\overline{u_1}(a) = 0.8$, $\overline{u_1}(b) = 0.6$, $\overline{u_1}(c) = 1$; $\overline{u_2} = \bigcap\{\ 0^c,\ u_1^c,\ u_2^c,\ u_3^c\ \} = u_3^c$ i.e. $\overline{u_2}(a) = 0.8$, $\overline{u_2}(b) = 0.6$, $\overline{u_2}(c) = 1$ and $\overline{u_3} = \bigcap\{\ 0^c,\ u_1^c,\ u_2^c,\ u_3^c\ \} = u_3^c$ i.e. $\overline{u_3}(a) = 0.8$, $\overline{u_3}(b) = 0.6$, $\overline{u_3}(c) = 1$. Now, let $\lambda \in I^X$ defined by $\lambda(a) = 0.9$, $\lambda(b) = 0.8$, $\lambda(c) = 0$. Then we have $\lambda_0 = \{a, b\}$. Clearly λ_0 is compact in (X, T). But λ is not almost compact in $(X, \omega(T))$, as there don't exist $u_k \in \omega(T)$ for $k = 1, 2, 3$ such that $\lambda \subseteq \bigcup_{k=1}^{3} \overline{u_k}$. Again, let $\mu \in I^X$ defined by $\mu(a) = 0$, $\mu(b) = 0.3$, $\mu(c) = 0.8$. Then we have $\mu_0 = \{b, c\}$. Clearly μ is almost compact in $(X, \omega(T))$. But μ_0 is not compact in (X, T).

Theorem 7.19: Let λ and μ be almost compact fuzzy sets in an fts (X, t). Then $(\lambda \times \mu)$ is also almost compact in $(X \times X, t \times t)$.

Proof: Let $\{\ u_i \times v_i :\ i \in J\ \}$ be an open cover of $(\lambda \times \mu)$ in $(X \times X, t \times t)$ i.e. $(\lambda \times \mu) \subseteq \bigcup_{i \in J} (\ u_i \times v_i\ )$. Hence it can be easily shown that, $\min(\ \lambda(x),\ \mu(y)\ ) \subseteq \bigcup_{i \in J} \min(\ u_i(x),\ v_i(y)\ )$ for every $(x, y) \in X \times X$. So it is clear that $\lambda \subseteq \bigcup_{i \in J} u_i$ and $\mu \subseteq \bigcup_{i \in J} v_i$. Therefore $\{\ u_i :\ i \in J\ \}$ and $\{\ v_i :\ i \in J\ \}$ are open cover of λ and μ respectively. Thus $\left\{ \left(\overline{u_i}\right)^0 :\ i \in J \right\}$ and $\left\{ \left(\overline{v_i}\right)^0 :\ i \in J \right\}$ are also open cover of λ and μ respectively. Now, we have $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and $\left(\overline{v_i}\right)^0 \subseteq \overline{v_i}$. As λ and μ are almost compact, then $\left\{ \left(\overline{u_i}\right)^0 :\ i \in J \right\}$ and $\left\{ \left(\overline{v_i}\right)^0 :\ i \in J \right\}$ have finite proximate subcover, say $\{\ \overline{u_{i_k}} :\ k \in J_n\ \}$ and $\{\ \overline{v_{i_k}} :\ k \in J_n\ \}$ such that $\lambda \subseteq \bigcup_{k \in J_n} \overline{u_{i_k}}$ and $\mu \subseteq \bigcup_{k \in J_n} \overline{v_{i_k}}$ i.e.

$\lambda \subseteq \overline{u_{i_1}} \cup \overline{u_{i_2}} \cup \ldots\ldots \cup \overline{u_{i_n}}$ and $\mu \subseteq \overline{v_{i_1}} \cup \overline{v_{i_2}} \cup \ldots\ldots \cup \overline{v_{i_n}}$ respectively. Hence we can write $(\lambda \times \mu) \subseteq \bigcup_{k \in J_n} (\overline{u_{i_k}} \times \overline{v_{i_k}})$. Therefore $\{ \overline{u_{i_k}} \times \overline{v_{i_k}} : k \in J_n \}$ is a finite proximate subcover of $\{ u_i \times v_i : i \in J \}$. Thus $(\lambda \times \mu)$ is almost compact in $(X \times X, t \times t)$.

Definition 7.20: Let $M = \{ u_i : i \in J \}$ be a family of δ-open fuzzy sets and λ be a fuzzy set in X. Then M is said to be proximate δ-cover of λ when $\{ \overline{u_i} : i \in J \}$ is a δ-cover of λ i.e. $\lambda \subseteq \bigcup_{i \in J} \overline{u_i}$. A subfamily of $\{ u_i : i \in J \}$ which is also a proximate δ-cover of λ is said to be proximate δ-subcover of λ.

Definition 7.21: A fuzzy set λ is said to be almost δ-compact iff every δ-cover of λ has a finite subfamily whose closures is δ-cover of λ or equivalently, every δ-cover of λ has a finite proximate δ-subcover.

Every fuzzy subsets of an almost δ-compact fuzzy set is also almost δ-compact.

Theorem 7.22: Any almost δ-compact fuzzy set in an fts is almost compact. The converse is not true in general.

The proof of the theorem is straightforward.

Now, for the converse, consider the following example.

Let $X = \{a, b\}$, $I = [0, 1]$ and $0 < \delta \leq 1$. Let u_1, $u_2 \in I^X$ defined by $u_1(a) = 0.3$, $u_1(b) = 0.2$ and $u_2(a) = 0.4$, $u_2(b) = 0.5$. Now, take $t = \{ 0, u_1, u_2, 1\}$, then we see that (X, t) is an fts. Now, $0^c(a) = 1$, $0^c(b) = 1$; $u_1^c(a) = 0.7$, $u_1^c(b) = 0.8$ and $u_2^c(a) = 0.6$, $u_2^c(b) = 0.5$. So we have $\overline{u_1} = \bigcap \{ 0^c, u_1^c, u_2^c \} = u_2^c$ i.e. $\overline{u_1}(a) = 0.6$, $\overline{u_1}(b) = 0.5$ and $\overline{u_2} = \bigcap \{ 0^c, u_1^c, u_2^c \} = u_2^c$ i.e. $\overline{u_2}(a) = 0.6$, $\overline{u_2}(b) = 0.5$. Again, let

$\lambda \in I^X$ defined by $\lambda(a) = 0.6$, $\lambda(b) = 0.3$. Clearly λ is almost compact in (X, t). Take $\delta = 0.9$. Then we observe that there is no finite proximate δ-subcover of λ. Hence λ is not almost δ-compact in (X, t). Thus the converse of theorem is not necessarily true.

Chapter Eight

Almost α-Compact Spaces

Almost α-compact spaces was first introduced by Mukherjee and Bhattacharyya [130] which is global property. We aim to discuss several other characterizations of this concept and established some theorems, corollary and examples. Also we have defined almost δ-α-compact spaces and found different characterizations between almost α-compact and almost δ-α-compact spaces.

Definition 8.1[130]: A family $\{ u_i : i \in J \}$, $u_i \in I^X$ is a proximate α-shading of X when $\{ \overline{u_i} : i \in J \}$ is an α-shading of X i.e. $\overline{u_i}(x) > \alpha$ for each $x \in X$.

A subfamily of $\{ u_i : i \in J \}$ which is also a proximate α-shading of X is called a proximate α-subshading of X.

Definition 8.2[130]: An fts (X, t) is said to be almost α-compact iff every open α-shading of X has a finite subfamily whose closures is an α-shading or equivalently, every open α-shading of X has a finite proximate α-subshading.

Theorem 8.3: Let (X, t) be an fts and $A \subset X$. Then 1_A is almost α-compact in (X, t) iff 1_A is almost α-compact in (A, t_A).

Proof: Suppose 1_A is almost α-compact in (X, t). Let $\{ u_i : i \in J \}$ be an open α-shading of 1_A in (A, t_A), then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open α-shading of 1_A in (A, t_A). Then there exists $v_i \in t$ such that $u_i = v_i \mid A \subseteq v_i$. Therefore $\{ v_i : i \in J \}$ be an open α -shading of 1_A in (X, t) and so $\left\{\left(\overline{v_i}\right)^0 : i \in J\right\}$ is also an open α -shading of 1_A in (X, t). But 1_A is almost α -compact in (X, t), then $\left\{\left(\overline{v_i}\right)^0 : i \in J\right\}$ has a finite proximate α -subshading, say $\{ \overline{v_{i_k}} : k \in J_n \}$ such that $\overline{v_{i_k}}(x) > \alpha$ for each $x \in A$. We have $\overline{u_i} = \overline{v_i \mid A} \subseteq \overline{v_i} \mid A \subseteq \overline{v_i}$. Now, $\left(\left(\bigcup_{k=1}^{n} \overline{v_{i_k}}\right) \mid A\right)(x) > \alpha \Rightarrow \bigcup_{k=1}^{n} (\overline{v_{i_k}} \mid A)(x) > \alpha \Rightarrow$ $\bigcup_{k=1}^{n} \overline{u_{i_k}}(x) > \alpha$, as $A \subset X$ and consequently $\{ \overline{u_{i_k}} : k \in J_n \}$ is a finite proximate α -subshading of $\{ u_i : i \in J \}$. Hence 1_A is almost α -compact in (A, t_A).

Conversely, suppose 1_A is almost α -compact in (A, t_A). Let $\{ v_i : i \in J \}$ be an open α -shading of 1_A in (X, t), then $\left\{\left(\overline{v_i}\right)^0 : i \in J\right\}$ is also an open α -shading of 1_A in (X, t). Put $u_i = v_i \mid A$. Then $\left(\bigcup_{i \in J} v_i\right) \mid A = \bigcup_{i \in J} (v_i \mid A) = \bigcup_{i \in J} u_i$. But $u_i \in t_A$ and so $\{ u_i : i \in J \}$ is an open α -shading of 1_A in (A, t_A). Therefore $\left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$ is also an open α -shading of 1_A in (A, t_A). Since 1_A is almost α -compact in (A, t_A), then $\left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$ has a finite proximate α -subshading, say $\{ \overline{u_{i_k}} : k \in J_n \}$ such that $\overline{u_{i_k}}(x) > \alpha$ for each $x \in A$. But $\overline{u_i} = \overline{v_i \mid A} \subseteq \overline{v_i} \mid A \subseteq \overline{v_i}$ and consequently $\{ \overline{v_{i_k}} : k \in J_n \}$ is a finite proximate α -subshading of $\{ v_i : i \in J \}$. Therefore 1_A is almost α -compact in (X, t).

Corollary 8.4: Let (Y, t^*) be a fuzzy subspace of (X, t) and $A \subset Y \subset X$. Then 1_A is almost α -compact in (X, t) iff 1_A is almost α -compact in (Y, t^*).

Proof: Let t_A and t_A^* be the subspace fuzzy topologies on A. Then by preceding theorem (8.3), 1_A is almost α -compact in (X, t) or (Y, t^*) iff 1_A is almost α -compact in (A, t_A) or (A, t_A^*). But $t_A = t_A^*$.

Theorem 8.5: Let (X, t) be an fts and 1_A be a closed subset of X ($A \subset X$). If (X, t) is almost α -compact, then so also is (A, t_A).

Proof: Let $M = \{ u_i : i \in J \}$ be an open α -shading of 1_A in (A, t_A), then $\{(\overline{u_i})^0 : i \in J \}$ is also an open α-shading of 1_A in (A, t_A). Then there exist $v_i \in t$ such that $u_i = v_i \mid A \subseteq v_i$. Let $H = \{ v_i \in t : v_i \mid A \in M \}$. Then $\{v_i\} \cup \{1_{X-A}\}$ is an open α -shading of 1_X. To show this, let $x \in X$. Now if $x \in A$, there exist some $u_i \in M$ such that $u_i(x) > \alpha$. Let $g_i \in t$ such that $g_i \mid A = u_i$. So $g_i \in H$ and we have $g_i(x) > \alpha$. Again if $x \in X - A$, then $(1_{X-A})(x) = 1 > \alpha$. But $v_i \subseteq (\overline{v_i})^0 \subseteq \overline{v_i}$ and since (X, t) is almost α -compact, then $\{v_i\} \cup \{1_{X-A}\}$ has a finite proximate α -subshading, say $\{ \overline{v_{i_k}} : k \in J_n \}$ such that $\overline{v_{i_k}}(x) > \alpha$. Now, we have $\overline{u_i} = \overline{v_i \mid A} \subseteq \overline{v_i} \mid A \subseteq \overline{v_i}$. Then $\{ \overline{v_{i_k}} \mid A : k \in J_n \}$, as $A \subset X$ and hence $\{ \overline{u_{i_k}} : k \in J_n \}$ is a finite proximate α -subshading of M. Therefore 1_A is almost α -compact in (A, t_A).

Theorem 8.6: Let (X, t) be an fts and $A, B \subset X$. If 1_A and 1_B are almost α -compact, then $1_{A \cup B}$ is also almost α -compact.

Proof: Let $\{ u_i : i \in J \}$ be an open α -shading of $1_{A \cup B}$, then $\{(\overline{u_i})^0 : i \in J \}$ is also an open α -shading of $1_{A \cup B}$. Hence $\{ u_i : i \in J \}$ is any open α -shading of both 1_A and 1_B respectively. Thus $\{(\overline{u_i})^0 : i \in J \}$ is also any open α -shading of both 1_A and 1_B

respectively. But $u_i \subseteq (\overline{u_i})^0 \subseteq \overline{u_i}$ and 1_A is almost α -compact, then $\{(\overline{u_i})^0 : i \in J\}$ has a finite proximate α -subshading, say $\{\overline{u_{i_k}} : k \in J_n\}$ such that $\overline{u_{i_k}}(x) > \alpha$ for all $x \in A$. Similarly, we can find $\{\overline{u_{i_r}} : r \in J_n\}$ is a finite proximate α -subshading of $\{(\overline{u_i})^0 : i \in J\}$. Therefore $\{\overline{u_{i_k}}, \overline{u_{i_r}}\}$ is a finite proximate α -subshading of $\{u_i : i \in J\}$. Thus $1_{A \cup B}$ is also almost α -compact.

Theorem 8.7: Let (X, t) be an fts and A , $B \subset X$ ($A \cap B \neq \phi$). If 1_A and 1_B are almost α -compact, then $1_{A \cap B}$ is also almost α -compact.

Proof: We have $A \cap B \subseteq A$ and $A \cap B \subseteq B$. As 1_A and 1_B are almost α -compact, then it is clear that $1_{A \cap B}$ is also almost α -compact.

Theorem 8.8: Let (X, t) be an fts and if t_α becomes a cofinite topology on X . Then (X, t) is almost α -compact.

Proof: Let $M = \{u_i : i \in J\}$ be an open α -shading of (X, t), then $\{(\overline{u_i})^0 : i \in J\}$ is also an open α -shading of (X, t). Now, we have $t_\alpha = \{\alpha(u_i) : u_i \in t\}$, where $\alpha(u_i) = \{x \in X : u_i(x) > \alpha\}$ and by the theorem t_α is a cofinite topology on X . Hence we see that $\{\alpha(u_i) : i \in J\}$ is an open cover of (X, t_α), then $\{\alpha(\overline{u_i})^0 : i \in J\}$ is also an open cover of (X, t_α). For let, $x \in X$, then there exists $u_{i_0} \in M$ such that $u_{i_0}(x) > \alpha \Rightarrow (\overline{u_{i_0}})^0(x) > \alpha$, as $u_i \subseteq (\overline{u_i})^0$. Therefore, $x \in \alpha(u_{i_0})$ and $\alpha(u_{i_0}) \in \{\alpha(u_i)\} \Rightarrow x \in \alpha(\overline{u_{i_0}})^0$ and $\alpha(\overline{u_{i_0}})^0 \in \{\alpha(\overline{u_i})^0 : i \in J\}$. Since (X, t_α) is cofinite, hence compact, then $\{\alpha(u_i) : i \in J\}$ has a finite subcover, say $\{\alpha(u_{i_k}) : k \in J_n\}$, where $u_{i_k} \in t$ and

$\alpha(u_{i_k}) \in t_\alpha \Rightarrow \{\alpha(\overline{u_{i_k}})^0 : k \in J_n\}$ is also forms a finite subcover of $\{\alpha(\overline{u_i})^0 : i \in J\}$. But $u_i \subseteq (\overline{u_i})^0 \subseteq \overline{u_i}$, the family $\{\overline{u_{i_k}} : k \in J_n\}$ forms a finite proximate α -subshading of M. Hence (X, t) is almost α -compact.

Theorem 8.9: Let $f : (X, t) \to (Y, s)$ be fuzzy continuous and surjective mapping. If 1_X is almost α -compact, then $f(1_X)$ is almost α -compact as a subspace of Y.

Proof: We have $f(X) = Y$, as f is surjective. Let $M = \{u_i : i \in J\}$ be an open α -shading of 1_Y. Then $\{(\overline{u_i})^0 : i \in J\}$ is also an open α -shading of 1_Y. Since f is fuzzy continuous, then $f^{-1}(\overline{u_i})^0 \in t$ and hence $\{f^{-1}(\overline{u_i})^0 : i \in J\}$ is open α -shading of 1_X. For, let $x \in X$, then $f(x) \in Y$. So there exists some $(\overline{u_{i_0}})^0 \in \{(\overline{u_i})^0 : i \in J\}$ such that $(\overline{u_{i_0}})^0 (f(x)) > \alpha \Rightarrow f^{-1}(\overline{u_{i_0}})^0(x) > \alpha$. As 1_X is almost α -compact, then there exists $f^{-1}(\overline{u_{i_k}})^0 \in \{f^{-1}(\overline{u_i})^0 : i \in J\}$ ($k \in J_n$) such that $\overline{f^{-1}(\overline{u_{i_k}})^0}(x) > \alpha$ for each $x \in X$. But from $(\overline{u_i})^0 \subseteq \overline{u_i}$ and fuzzy continuity of f, $f^{-1}(\overline{u_i})$ must be a closed fuzzy set in X such that $f^{-1}(\overline{u_i})^0 \subseteq f^{-1}(\overline{u_i})$ and then $\overline{f^{-1}(\overline{u_i})^0} \subseteq f^{-1}(\overline{u_i})$. Therefore $f\left(\overline{f^{-1}(\overline{u_i})^0}\right) \subseteq \overline{u_i}$ for each $i \in J$. For if $y \in Y$, then $y = f(x)$ for some $x \in X$, as f is surjective. Then there exist some k such that $\overline{u_{i_k}}(f(x)) > \alpha \Rightarrow \overline{u_{i_k}}(y) > \alpha$ for each $y \in Y$. Therefore $f(1_X)$ is almost α -compact.

Theorem 8.10: Let $f : (X, t) \to (Y, s)$ be fuzzy open, fuzzy closed and bijective mapping . If (Y, s) is almost α -compact, then (X, t) is also almost α -compact.

Proof: Let $\{ u_i : i \in J \}$ be an open α -shading of 1_X, then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open α -shading of 1_X. As f is fuzzy open, then $f\left(\overline{u_i}\right)^0 \in s$ and hence it follows that $\left\{ f\left(\overline{u_i}\right)^0 : i \in J \right\}$ is an open α -shading of 1_Y. For let, $y \in Y$, then $f^{-1}(y) \in X$. So there exists some $\left(\overline{u_{i_0}}\right)^0 \in \left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ such that $\left(\overline{u_{i_0}}\right)^0 \left(f^{-1}(y)\right) > \alpha \Rightarrow f\left(\overline{u_{i_0}}\right)^0 (y) > \alpha$. Since 1_Y is almost α -compact, then there exists $f\left(\overline{u_{i_k}}\right)^0 \in \left\{ f\left(\overline{u_i}\right)^0 : i \in J \right\}$ ($k \in J_n$) such that $\overline{f\left(\overline{u_{i_k}}\right)^0} (y) > \alpha$ for all $y \in Y$. But from $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and f is fuzzy closed, $f\left(\overline{u_i}\right)$ must be a closed fuzzy set in Y such that $f\left(\overline{u_i}\right)^0 \subseteq f\left(\overline{u_i}\right)$ and then $\overline{f\left(\overline{u_i}\right)^0} \subseteq f\left(\overline{u_i}\right)$. Therefore $f^{-1}\left(\overline{f\left(\overline{u_i}\right)^0}\right) \subseteq \overline{u_i}$ for each $i \in J$. Since f is bijective, we have for each $x \in X$, there exists a $y \in Y$ such that $x = f^{-1}(y)$. So, we can obtain some k such that $f\left(\overline{u_{i_k}}\right)(y) > \alpha$ $\Rightarrow \overline{u_{i_k}}\left(f^{-1}(y)\right) > \alpha \Rightarrow \overline{u_{i_k}}(x) > \alpha$ for each $x \in X$. Therefore (X, t) is almost α -compact.

Theorem 8.11: Let (X, t) be an fts. If every family of closed fuzzy sets which has empty intersection has a finite subfamily with empty intersection, then (X, t) is almost α -compact. The converse is not true in general.

Proof: Let $\{ u_i : i \in J \}$ be an open α -shading of 1_X. By the first condition of the theorem, we have $\bigcap_{i \in J} u_i^c = 0_X$. Thus $\bigcup_{i \in J} u_i = 1_X$ and so $\bigcup_{i \in J} \left(\overline{u_i}\right)^0 = 1_X$, as $u_i \subseteq \left(\overline{u_i}\right)^0$. Therefore $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open α -shading of 1_X. Again from the second condition of the theorem, we get $\bigcap_{k \in J_n} u_{i_k}^c = 0_X$. So, we have $\bigcup_{k \in J_n} u_{i_k} = 1_X$ and hence

$\bigcup_{k\in J_n}\left(\overline{u_{i_k}}\right)^0 = 1_X$, as $u_i \subseteq \left(\overline{u_i}\right)^0$. But $u_i \subseteq \left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$, then $\bigcup_{k\in J_n}\overline{u_{i_k}} = 1_X$ and consequently $\{\ \overline{u_{i_k}}\ :\ k \in J_n\ \}$ is a finite proximate α -subshading of $\{\ u_i :\ i \in J\ \}$. Thus (X, t) is almost α -compact.

Now, for the converse, we consider the following example.

Let $X = \{a, b\}$, $I = [0, 1]$ and $0 \le \alpha < 1$. Again, let u, $v \in I^X$ defined by $u(a) = 0.1$, $u(b) = 0.2$ and $v(a) = 0.3$, $v(b) = 0.4$. Put $t = \{\ 0\ ,\ u\ ,\ v\ ,\ 1\}$, then (X, t) is an fts. Now $0^c(a) = 1$, $0^c(b) = 1$; $u^c(a) = 0.9$, $u^c(b) = 0.8$ and $v^c(a) = 0.7$, $v^c(b) = 0.6$. So, $\overline{u} = \bigcap\{\ 0^c\ ,\ u^c\ ,\ v^c\ \} = v^c$ i.e. $\overline{u}(a) = 0.7$, $\overline{u}(b) = 0.6$ and $\overline{v} = \bigcap\{\ 0^c\ ,\ u^c\ ,\ v^c\ \} = v^c$ i.e. $\overline{v}(a) = 0.7$, $\overline{v}(b) = 0.6$. Take $\alpha = 0.4$. Clearly (X, t) is almost α -compact. But $u^c \cap v^c \ne 0$. Therefore the converse of the theorem is not true in general.

The following example will show that the almost α -compact subsets in an fts need not be closed.

Example 8.12: Consider the fts in the example of the theorem (8.11). Again, let $1_A \in I^X$ defined by $1_A(a) = 1$, $1_A(b) = 0$. Hence we have $A = \{a\}$ and $A \subset X$. Take $\alpha = 0.5$. Then clearly 1_A is almost α -compact in (X, t). But 1_A is not closed in (X, t), as its complement 1_{A^c} is not open in (X, t).

Theorem 8.13: Let (X, t) be a fuzzy T_1-space (as def. 1.45), $A \subset X$ and 1_A be an almost α -compact subset in (X, t). Suppose $x \in A^c$, then there exist u, $v \in t$ such that $\overline{u}(x) = 1$ and $A \subseteq \left(\overline{v}\right)^{-1}(0, 1]$.

Proof: Let $y \in A$. Then clearly $x \ne y$. As (X, t) is fuzzy T_1-space, then there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $u_y(y) = 0$ and $v_y(x) = 0$, $v_y(y) = 1$. Let us take

$0 \le \alpha < 1$ such that $v_y(y) > \alpha > 0$, as $v_y(y) = 1$. Then $\{ v_y : y \in A \}$ is an open α-shading of 1_A. Hence we have $(\overline{u_y})^0 (x) = 1$, $(\overline{v_y})^0 (y) = 1$, as $u_y \subseteq (\overline{u_y})^0$ and $v_y \subseteq (\overline{v_y})^0$. Thus $\{(\overline{v_y})^0 : y \in A\}$ is also an open α-shading of 1_A. Since 1_A is almost α-compact, then $\{(\overline{v_y})^0 : y \in A\}$ has a finite proximate α-subshading , say $\{ \overline{v_{y_k}} : k \in J_n \}$ such that $\overline{v_{y_k}}(y) > \alpha$ for each $y \in A$. Now, let $(\overline{v})^0 = (\overline{v_{y_1}})^0 \cup (\overline{v_{y_2}})^0 \cup \ldots\ldots \cup (\overline{v_{y_n}})^0$ and $(\overline{u})^0 = (\overline{u_{y_1}})^0 \cap (\overline{u_{y_2}})^0 \cap \ldots\ldots \cap (\overline{u_{y_n}})^0$. Hence $(\overline{v})^0$ and $(\overline{u})^0$ are open fuzzy sets , as they are the union and finite intersection of open fuzzy sets respectively i.e. $(\overline{v})^0$, $(\overline{u})^0 \in t$. But we have $(\overline{v_y})^0 \subseteq \overline{v_y}$ and $(\overline{u_y})^0 \subseteq \overline{u_y}$. Moreover, $A \subseteq (\overline{v})^{-1}(0,1]$ and $\overline{u}(x) = 1$, as $\overline{u_{y_k}}(x) = 1$ for each k.

Theorem 8.14: Let (X, t) be a fuzzy T_1-space (as def. 1.45) and 1_A , 1_B be disjoint almost α-compact subsets in (X, t) (A , $B \subset X$). Then there exist u , $v \in t$ such that $A \subseteq (\overline{u})^{-1}(0,1]$ and $B \subseteq (\overline{v})^{-1}(0,1]$.

Proof: Let $y \in A$. Then $y \notin B$, as 1_A and 1_B are disjoint . Since 1_B is almost α-compact, then by theorem (8.13), there exist u_y , $v_y \in t$ such that $\overline{u_y}(y) = 1$, $B \subseteq (\overline{v_y})^{-1}(0,1]$. Assume that $0 \le \alpha < 1$ such that $u_y(y) > \alpha > 0$. As $\overline{u_y}(y) = 1$, then we we have $\{(\overline{u_y})^0 : y \in A\}$ is an open α-shading of 1_A. But 1_A is almost α-compact, then $\{(\overline{u_y})^0 : y \in A\}$ has a finite proximate α-subshading, say $\{ \overline{u_{y_k}} : k \in J_n \}$ such that $\overline{u_{y_k}}(y) > \alpha$ for all $y \in A$. Again, 1_B is almost α-compact, then $\{(\overline{v_y})^0 : x \in B\}$ has a finite proximate α-subshading, say $\{ \overline{v_{y_k}} : k \in J_n \}$ such that $\overline{v_{y_k}}(x) > \alpha$ for all $x \in B$

and $B \subseteq \left(\overline{v_{y_k}}\right)^{-1}(0,1]$ for each k. Now, let $\left(\overline{u}\right)^0 = \left(\overline{u_{y_1}}\right)^0 \cup \left(\overline{u_{y_2}}\right)^0 \cup \ldots\ldots \cup \left(\overline{u_{y_n}}\right)^0$ and $\left(\overline{v}\right)^0 = \left(\overline{v_{y_1}}\right)^0 \cap \left(\overline{v_{y_2}}\right)^0 \cap \ldots\ldots \cap \left(\overline{v_{y_n}}\right)^0$. But we have $\left(\overline{v_y}\right)^0 \subseteq \overline{v_y}$ and $\left(\overline{u_y}\right)^0 \subseteq \overline{u_y}$. Thus we see that $A \subseteq \left(\overline{u}\right)^{-1}(0,1]$ and $B \subseteq \left(\overline{v}\right)^{-1}(0,1]$. Hence $\left(\overline{u}\right)^0$ and $\left(\overline{v}\right)^0$ are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $\left(\overline{u}\right)^0$, $\left(\overline{v}\right)^0 \in t$.

Theorem 8.15: Let (X, t) be a fuzzy T_1-space (as def. 1.45), $A \subset X$ and 1_A be an almost α -compact subset in (X, t). Then 1_A is closed.

Proof: Let $x \in A^c$. We have to show that, there exists $u \in t$ such that $\overline{u}(x) = 1$ and $u \subseteq A^p$, where A^p is the characteristic function of A^c. If $y \in A$, then $x \neq y$ and hence there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $u_y(y) = 0$ and $v_y(x) = 0$, $v_y(y) = 1$. Let us take $0 \le \alpha < 1$ such that $v_y(y) > \alpha > 0$. Thus { $v_y :\ y \in A$ } is an open α -shading of 1_A. Hence we have $\left(\overline{u_y}\right)^0(x) = 1$, $\left(\overline{v_y}\right)^0(y) = 1$, as $u_y \subseteq \left(\overline{u_y}\right)^0$ and $v_y \subseteq \left(\overline{v_y}\right)^0$. Thus $\left\{\left(\overline{v_y}\right)^0 :\ y \in A\right\}$ is also an open α -shading of 1_A. Since 1_A is almost α -compact, then $\left\{\left(\overline{v_y}\right)^0 :\ y \in A\right\}$ has a finite proximate α -subshading, say { $\overline{v_{y_k}} :\ k \in J_n$ } such that $\overline{v_{y_k}}(y) > \alpha$ for each $y \in A$. Now, let $\left(\overline{u}\right)^0 = \left(\overline{u_{y_1}}\right)^0 \cap \left(\overline{u_{y_2}}\right)^0 \cap \ldots\ldots \cap \left(\overline{u_{y_n}}\right)^0$. But we have $\left(\overline{u_y}\right)^0 \subseteq \overline{u_y}$, then $\overline{u}(x) = 1$, as $\overline{u_{y_k}}(x) = 1$ for each k. Again, if $z \in A$, there exists r such that $\overline{v_{y_r}}(z) > \alpha \ge 0$ and clearly $u(z) = 0$. Hence $u \subseteq A^p$. Therefore, 1_{A^c} is open in (X, t) and consequently 1_A is closed.

Theorem 8.16: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and $A \subset X$. If 1_A is almost α -compact subset in (X, t) and $x \in A^c$, then there exist u , $v \in t$ such that $\overline{u}(x) > 0$ and $A \subseteq \left(\overline{v}\right)^{-1}(0,1]$. The converse is not true in general.

The proof is similar as that of theorem (8.13).

Now, for the converse, we give the following example.

Let $X = \{a, b\}$, $I = [0,1]$ and $0 \le \alpha < 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 0.2$, $u_1(b) = 0$; $u_2(a) = 0$, $u_2(b) = 0.3$ and $u_3(a) = 0.2$, $u_3(b) = 0.3$. Now, put $t = \{ 0 , u_1, u_2 , u_3 , 1\}$, then we see that (X, t) is a fuzzy T_1-space. Now, we have $0^c(a) = 1$, $0^c(b) = 1$; $u_1^c(a) = 0.8$, $u_1^c(b) = 1$; $u_2^c(a) = 1$, $u_2^c(b) = 0.7$ and $u_3^c(a) = 0.8$, $u_3^c(b) = 0.7$. Therefore $\overline{u_1} = \bigcap\{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c$ i.e. $\overline{u_1}(a) = 0.8$, $\overline{u_1}(b) = 0.7$; $\overline{u_2} = \bigcap\{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c$ i.e. $\overline{u_2}(a) = 0.8$, $\overline{u_2}(b) = 0.7$ and $\overline{u_3} = \bigcap\{ 0^c , u_1^c$, u_2^c , $u_3^c \} = u_3^c$ i.e. $\overline{u_3}(a) = 0.8$, $\overline{u_3}(b) = 0.7$. Again, let $1_A \in I^X$ defined by $1_A(a) = 0$, $1_A(b) = 1$. Hence we observe that $A = \{b\}$ and $a \in A^c$. Here u_1, $u_2 \in t$ where $\overline{u_1}(a) = 0.8 > 0$ and $\left(\overline{u_2}\right)^{-1}(0,1] = \{a, b\}$. Hence $A \subset \left(\overline{u_2}\right)^{-1}(0,1]$. Take $\alpha = 0.9$. Thus we see that 1_A is not almost α -compact in (X, t), as $\overline{u_k}(a) < \alpha$ for $k = 1, 2, 3$. Thus the converse of the theorem is not true in general.

Theorem 8.17: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and A , $B \subset X$. If 1_A and 1_B are disjoint almost α -compact subsets in (X, t), then there exist u , $v \in t$ such that $A \subseteq \left(\overline{u}\right)^{-1}(0,1]$ and $B \subseteq \left(\overline{v}\right)^{-1}(0,1]$.

Similar proof as theorem (8.14).

Now, for the converse, consider the fuzzy T_1-space (X, t) in the example of the theorem (8.16). Let 1_A , $1_B \in I^X$ with $1_A(a) = 1$, $1_A(b) = 0$ and $1_B(a) = 0$, $1_B(b) = 1$. Thus we see that $A = \{a\}$ and $B = \{b\}$. Now u_1 , $u_2 \in t$ where $\left(\overline{u_1}\right)^{-1}(0,1] = \{a, b\}$ and $\left(\overline{u_2}\right)^{-1}(0,1] = \{a, b\}$. Hence we observe that $A \subset \left(\overline{u_1}\right)^{-1}(0,1]$ and $B \subset \left(\overline{u_2}\right)^{-1}(0,1]$, where 1_A and 1_B are disjoint. Take $\alpha = 0.9$. Hence we see that 1_A and 1_B are not almost α -compact in (X, t), as $\overline{u_k}(a) < \alpha$ and $\overline{u_k}(b) < \alpha$, for $k = 1$, 2 , 3 respectively. Thus the converse of the theorem is not true in general.

The following example will show that the almost α -compact subsets in fuzzy T_1-space (as def. 1.46) need not be closed.

Example 8.18: Consider the fuzzy T_1-space in the example of the theorem (8.16). Again, let $1_A \in I^X$ defined by $1_A(a) = 1$, $1_A(b) = 0$. Take $\alpha = 0.6$. Then clearly 1_A is almost α -compact in (X, t). But 1_A is not closed in (X, t), as its complement 1_{A^c} is not open in (X, t).

Theorem 8.19: An almost α -compact fuzzy regular topological space (X, t) (as def. 1.52) is α -compact.

Proof: Let $M = \{ u_i : \ i \in J \}$ be an open α -shading of 1_X i.e. $u_i(x) > \alpha$ for every $x \in X$. By fuzzy regularity of X, we have $u_i = \bigcup_{i \in J} v_{ij}$, where v_{ij} is an open fuzzy set such that $\overline{v_{ij}} \subseteq u_i$ for each i . As $u_i(x) > \alpha \ \Rightarrow \bigcup_{i \in J} v_{ij}(x) > \alpha$ for each $x \in X$. So $v_{ij}(x) > \alpha$ for all $x \in X$ and for $i \in J$. Therefore we have $\{ v_{ij} : \ i \in J \}$ an open α -shading of 1_X . Since 1_X is almost α -compact, then there exist $v_{i_k j} \in \{ v_{ij} \}$ ($k \in J_n$) such that

$\overline{v_{i_k j}}(x) > \alpha$ for each $x \in X$. But we have $\overline{v_{i_k j}} \subseteq u_{i_k} \Rightarrow u_{i_k}(x) > \alpha$ for each $x \in X$. Therefore 1_X is α -compact.

Theorem 8.20: An fts (X, t) is almost α -compact iff (X, t_α) is compact topological space.

Proof: Suppose (X, t) is almost α -compact. Let $W = \{ U_i : i \in J \}$ be an open cover of (X, t_α). Then for each U_i, there exists a $v_i \in t$ such that $U_i = \alpha(v_i)$. Thus we have $W = \{ \alpha(v_i) : i \in J \}$. So the family $M = \{ v_i : i \in J \}$ is an open α -shading of (X, t). Then $\left\{ \left(\overline{v_i}\right)^0 : i \in J \right\}$ is also an open α -shading of (X, t). To see this, let $x \in X$. Since W is an open cover of (X, t_α), there is an $U_{i_0} \in W$ such that $x \in U_{i_0}$. But $U_{i_0} = \alpha(v_{i_0})$ for some $v_{i_0} \in t$. Therefore $x \in \alpha(v_{i_0})$ which implies that $v_{i_0}(x) > \alpha$. Since (X, t) is almost α -compact, then M has a proximate α -subshading, say $v_{i_k} \in M$ ($k \in J_n$) such that $\overline{v_{i_k}}(x) > \alpha$. Since $v_i \subseteq \overline{v_i}$, then $\{ \alpha\left(\overline{v_{i_k}}\right) : k \in J_n \}$ forms a finite subcover of W and thus (X, t_α) is compact.

Conversely, suppose that (X, t_α) is compact. Let $M = \{ u_i : i \in J \}$ be an open α -shading of (X, t), then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open α -shading of (X, t). Therefore we have the family $W = \{ \alpha(u_i) : i \in J \}$ is an open cover of (X, t_α). Now, for $x \in X$, there exists a $u_{i_0} \in M$ such that $u_{i_0}(x) > \alpha$. So $x \in \alpha(u_{i_0})$ and $\alpha(u_{i_0}) \in W$. Since (X, t_α) is compact, then W has a finite subcover, say $\alpha(u_{i_k}) \in W$ ($k \in J_n$) such that $X = \alpha(u_{i_1}) \cup \alpha(u_{i_2}) \cup \ldots\ldots \cup \alpha(u_{i_n})$. But $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$, so $\{ \overline{u_{i_k}} : k \in J_n \}$ forms finite proximate α -subshading of M. Hence (X, t) is almost α -compact.

Theorem 8.21: Let (X, t) be an fts and (X, t_α) be a α -level topological space. Let $f : (X, t_\alpha) \to (X, t)$ be α -level continuous and bijective mapping. If (X, t_α) is compact, then (X, t) is almost α -compact.

Proof: Let $\{ u_i : i \in J \}$ be an open α -shading of (X, t), then $\left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$ is also an open α -shading of (X, t). Since f is α -level continuous, then $\alpha(f^{-1}(u_i)) \in t_\alpha \Rightarrow$ $\alpha\left(f^{-1}\left(\overline{u_i}\right)^0 \right) \in t_\alpha$ and hence $\left\{\alpha\left(f^{-1}\left(\overline{u_i}\right)^0 \right) : i \in J\right\}$ is an open cover of (X, t_α). As (X, t_α) is compact, then $\left\{\alpha\left(f^{-1}\left(\overline{u_i}\right)^0 \right) : i \in J\right\}$ has a finite subcover, say $\left\{\alpha\left(f^{-1}\left(\overline{u_{i_k}}\right)^0 \right) : k \in J_n\right\}$. Now, we have $f(x) = y$ for $y \in X$, as f is bijective. Since $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and $\left\{\alpha\left(f^{-1}\left(\overline{u_{i_k}}\right)^0 \right) : k \in J_n\right\}$ is a finite subcober of $\left\{\alpha\left(f^{-1}\left(\overline{u_i}\right)^0 \right) : i \in J\right\}$, then there exist some k such that $\overline{u_{i_k}}\left(f(x)\right) > \alpha \Rightarrow \overline{u_{i_k}}(y) > \alpha$ for every $y \in X$. Therefore $\{ \overline{u_{i_k}} : k \in J_n \}$ forms a finite proximate α -subshading of $\{ u_i : i \in J \}$. Hence (X, t) is almost α -compact.

Theorem 8.22: A topological space (X, T) is compact iff $(X, \omega(T))$ is almost α -compact.

Proof: Suppose (X, T) is compact. Let $\{ u_i : i \in J \}$ be an open α -shading of $(X, \omega(T))$. Then $\left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$ is also an open α -shading $(X, \omega(T))$. Therefore we can write $u_i^{-1}(a, 1] \in T$ and hence $\{ u_i^{-1}(a, 1] : u_i^{-1}(a, 1] \in T \}$ is an open cover of (X, T). As (X, T) is compact, then $\{ u_i^{-1}(a, 1] : u_i^{-1}(a, 1] \in T \}$ has a finite subcover, say $u_{i_k}^{-1}(a, 1] \in \{ u_i^{-1}(a, 1] \}$ $(k \in J_n)$ such that $X = u_{i_1}^{-1}(a, 1] \cup u_{i_2}^{-1}(a, 1] \cup \ldots\ldots \cup$

$u_{i_n}^{-1}(a,1]$. But from $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$, we observe that there exists $\left(\overline{u_{i_k}}\right)^0 \in \left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$ ($k \in J_n$) such that $\overline{u_{i_k}}(x) > \alpha$ for all $x \in X$. Hence it is observe that { $\overline{u_{i_k}}$: $k \in J_n$ } is a finite proximate α-subshading of { u_i : $i \in J$ }. Therefore $\left(X, \omega(T)\right)$ is almost α-compact .

Conversely, suppose that $\left(X, \omega(T)\right)$ is almost α-compact. Let { V_j : $j \in J$ } be open cover of $\left(X, T\right)$ i.e. $X = \bigcup_{j \in J}\{ V_j : V_j \in T \}$. As 1_{V_j} is l. s. c., then $1_{V_j} \in \omega(T)$ and we have { 1_{V_j} : $1_{V_j} \in \omega(T)$ } is an open α-shading of $\left(X, \omega(T)\right)$. Then $\left\{\left(\overline{1_{V_j}}\right)^0 : 1_{V_j} \in \omega(T)\right\}$ is also an open α-shading of $\left(X, \omega(T)\right)$. Since $\left(X, \omega(T)\right)$ is almost α-compact, then $\left\{\left(\overline{1_{V_j}}\right)^0 : 1_{V_j} \in \omega(T)\right\}$ has a finite proximate α-subshading, say $\left(\overline{1_{V_{jk}}}\right)^0 \in \left\{\left(\overline{1_{V_j}}\right)^0 : 1_{V_j} \in \omega(T)\right\}$ ($k \in J_n$) such that $\overline{1_{V_{jk}}}(x) > \alpha$ for all $x \in X$. As $1_{V_j} \in \omega(T)$ and $1_{V_j} \subseteq \overline{1_{V_j}}$, then we can write $X = V_{j_1} \cup V_{j_2} \cup \ldots\ldots\cup V_{j_n}$ and hence it is clear that { V_{j_k} } ($k \in J_n$) is a finite subcover of { V_j : $j \in J$ }. Hence $\left(X, T\right)$ is compact.

Definition 8.23: Let $\left(X, t\right)$ be an fts and $0 < \delta \leq 1$, $0 < \alpha \leq 1$. Let { u_i : $i \in J$ } be a family of δ-open fuzzy sets in $\left(X, t\right)$. Then { u_i : $i \in J$ } is a proximate δ-α-shading of X when { $\overline{u_i}$: $i \in J$ } is a δ-α-shading of X i.e. $\overline{u_i}(x) > \alpha$ for all $x \in X$.

A subfamily of { u_i : $i \in J$ } which is also a proximate δ-α-shading of X is said to be proximate δ-α-subshading of X .

Definition 8.24: Let $0 < \delta \leq 1, \alpha \in I$. An fts (X, t) is said to be almost δ-α-compact, $0 \leq \alpha < 1$ iff every δ-α-shading of X has a finite subfamily whose closures is δ-α-shading of X or equivalently, every δ-α-shading of X has a finite proximate δ-α-subshading.

Theorem 8.25: Every almost δ-α-compact space is almost α-compact. But the converse is not true.

The proof is straightforward.

For the converse, we consider the following example.

Let $X = [0, 1]$, $I = [0, 1]$ and $0 < \delta \leq 1$, $0 \leq \alpha < 1$. Let u_1, u_2, $u_3 \in I^X$ defined by

$$u_1(x) = \begin{cases} 0 & \text{for } 0 \leq x < 0.6 \\ 0 & \text{for x} = 0.6 \\ 0.3 & \text{for } 0.6 < x \leq 1 \end{cases} , \quad u_2(x) = \begin{cases} 0.4 & \text{for } 0 \leq x < 0.6 \\ 0 & \text{for x} = 0.6 \\ 0 & \text{for } 0.6 < x \leq 1 \end{cases} \text{ and}$$

$$u_3(x) = \begin{cases} 0.4 & \text{for } 0 \leq x < 0.6 \\ 0 & \text{for x} = 0.6 \\ 0.3 & \text{for } 0.6 < x \leq 1 \end{cases} .$$ Put $t = \{ 0 , u_1 , u_2 , u_3 , 1 \}$, then we see that (X, t) is an fts. Now,

$$u_1^c(x) = \begin{cases} 1 & \text{for } 0 \leq x < 0.6 \\ 1 & \text{for x} = 0.6 \\ 0.7 & \text{for } 0.6 < x \leq 1 \end{cases} , \quad u_2^c(x) = \begin{cases} 0.6 & \text{for } 0 \leq x < 0.6 \\ 1 & \text{for x} = 0.6 \\ 1 & \text{for } 0.6 < x \leq 1 \end{cases} \text{ and}$$

$$u_3^c(x) = \begin{cases} 0.6 & \text{for } 0 \leq x < 0.6 \\ 1 & \text{for x} = 0.6 \\ 0.7 & \text{for } 0.6 < x \leq 1 \end{cases} .$$ So we have $\overline{u_1} = \bigcap \{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c$ i.e.

$$\overline{u_1}(x) = \begin{cases} 0.6 & \text{for } 0 \leq x < 0.6 \\ 1 & \text{for x} = 0.6 \\ 0.7 & \text{for } 0.6 < x \leq 1 \end{cases} ; \quad \overline{u_2} = \bigcap \{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c \text{ i.e.}$$

$$\overline{u_2}(x) = \begin{cases} 0.6 & \text{for } 0 \leq x < 0.6 \\ 1 & \text{for x} = 0.6 \\ 0.7 & \text{for } 0.6 < x \leq 1 \end{cases} \text{ and } \overline{u_3} = \bigcap \{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c \text{ i.e.}$$

$$\overline{u_3}(x) = \begin{cases} 0.6 & \text{for } 0 \le x < 0.6 \\ 1 & \text{for } x = 0.6 \\ 0.7 & \text{for } 0.6 < x \le 1 \end{cases}$$

. Take $\alpha = 0.4$. Clearly (X, t) is almost α -compact. Again, take $\delta = 0.9$. Then we observe that there is no finite proximate δ - α -subshading of X. Hence (X, t) is not almost δ - α -compact. Thus the converse of theorem is not necessarily true.

Chapter Nine

Almost Partially α-Compact Fuzzy Sets

In this chapter, we have introduced almost partially α-compact fuzzy sets. Furthermore, we have established some theorems, corollary and examples about almost partially α-compact fuzzy sets. Also we have defined almost partially δ-α-compact fuzzy sets and found different characterizations between almost partially α-compact and almost partially δ-α-compact fuzzy sets.

Definition 9.1: A family $\{ u_i : i \in J \}$ is a proximate partial α-shading, in short $pp\alpha$-shading of a fuzzy set λ in X when $\{ \overline{u_i} : i \in J \}$ is a $p\alpha$-shading of λ i.e. $\overline{u_i}(x) > \alpha$ for each $x \in \lambda_0$.

A subfamily of $\{ u_i : i \in J \}$ which is also a $pp\alpha$-shading of λ is said to be $pp\alpha$-subshading of λ.

Definition 9.2: Let (X, t) be an fts and $\alpha \in I$. A fuzzy set λ is said to be almost partially α-compact, $0 \le \alpha < 1$, in short, $ap\alpha$-compact iff every open $p\alpha$-shading of λ has a finite subfamily whose closures is $p\alpha$-shading of λ or equivalently, every open $p\alpha$-shading of λ has a finite $pp\alpha$-subshading.

Theorem 9.3: Let (X, t) be an fts, $A \subset X$ and λ be a fuzzy set in X with $\lambda_0 \subseteq A$. Then λ is $ap\alpha$-compact in (X, t) iff λ is $ap\alpha$-compact in (A, t_A).

Proof: Suppose λ is $ap\alpha$-compact in (X, t). Let $\{ u_i : i \in J \}$ be an open $p\alpha$-shading of λ in (A, t_A), then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open $p\alpha$-shading of λ in (A, t_A). So there exist $v_i \in t$ such that $u_i = v_i \mid A \subseteq v_i$. Therefore $\{ v_i : i \in J \}$ is an open $p\alpha$-shading of λ in (X, t) and so $\left\{ \left(\overline{v_i}\right)^0 : i \in J \right\}$ is also an open $p\alpha$-shading of λ in (X, t). Since $\left(\overline{v_i}\right)^0 \subseteq \overline{v_i}$ and λ is $ap\alpha$-compact in (X, t), then $\left\{ \left(\overline{v_i}\right)^0 : i \in J \right\}$ has a finite $pp\alpha$-subshading, say $\{ \overline{v_{i_k}} : k \in J_n \}$ such that $\overline{v_{i_k}}(x) > \alpha$ for each $x \in \lambda_0$. But $\overline{u_i} = \overline{v_i \mid A} \subseteq \overline{v_i} \mid A \subseteq \overline{v_i}$. Now, $\left(\left(\bigcup_{k \in J_n} \overline{v_{i_k}} \right) \mid A \right)(x) > \alpha \quad \Rightarrow \quad \bigcup_{k \in J_n} \left(\overline{v_{i_k}} \mid A \right)(x) > \alpha \quad \Rightarrow$ $\bigcup_{k \in J_n} \overline{u_{i_k}}(x) > \alpha$, as $\lambda_0 \subseteq A$ and hence it shows that $\{ \overline{u_{i_k}} : k \in J_n \}$ is a finite $pp\alpha$-subshading of $\{ u_i : i \in J \}$. Therefore λ is $ap\alpha$-compact in (A, t_A).

Conversely, suppose λ is $ap\alpha$-compact in (A, t_A). Let $\{ v_i : i \in J \}$ be an open $p\alpha$-shading of λ in (X, t), then $\left\{ \left(\overline{v_i}\right)^0 : i \in J \right\}$ is also an open $p\alpha$-shading of λ in (X, t). Put $u_i = v_i \mid A$, then $\left(\bigcup_{i \in J} v_i \right) \mid A = \bigcup_{i \in J} (v_i \mid A) = \bigcup_{i \in J} u_i$. But $u_i \in t_A$, so $\{ u_i : i \in J \}$ is an open $p\alpha$-shading of λ in (A, t_A). Therefore $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open $p\alpha$-shading of λ in (A, t_A). As $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and λ is $ap\alpha$-compact in (A, t_A), then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ has a finite $pp\alpha$-subshading, say $\{ \overline{u_{i_k}} : k \in J_n \}$ such that $\overline{u_{i_k}}(x) > \alpha$ for each $x \in \lambda_0$. But $\overline{u_i} = \overline{v_i \mid A} \subseteq \overline{v_i} \mid A \subseteq \overline{v_i}$, then $\{ \overline{v_{i_k}} : k \in J_n \}$ is a finite $pp\alpha$-subshading of $\{ v_i : i \in J \}$. Thus λ is $ap\alpha$-compact in (X, t).

Corollary 9.4: Let (Y, t^*) be a fuzzy subspace of an fts (X, t) and $A \subset Y \subset X$. Let λ be a fuzzy set in X with $\lambda_0 \subseteq A$. Then λ is $ap\alpha$ -compact in (X, t) iff λ is $ap\alpha$ -compact in (Y, t^*).

Proof: Let t_A and t_A^* be the subspace fuzzy topologies on A. Then by preceding theorem (9.3), λ is $ap\alpha$ -compact in (X, t) or (Y, t^*) iff λ is $ap\alpha$ -compact in (A, t_A) or (A, t_A^*). But $t_A = t_A^*$.

Theorem 9.5: Let (X, t) and (Y, s) be two fts's and $f : (X, t) \to (Y, s)$ be fuzzy continuous and surjective mapping. If λ is $ap\alpha$ -compact in (X, t), then $f(\lambda)$ is $ap\alpha$ -compact in (Y, s).

Proof: Let $\{ u_i : i \in J \}$ be an open $p\alpha$ -shading of $f(\lambda)$ in (Y, s), then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open $p\alpha$ -shading of $f(\lambda)$ in (Y, s). As f is fuzzy continuous, then $f^{-1}\left(\overline{u_i}\right)^0 \in t$ and hence $\left\{ f^{-1}\left(\overline{u_i}\right)^0 : i \in J \right\}$ is an open $p\alpha$ -shading of λ in (X, t). For, let $x \in \lambda_0$, then $f(x) \in \left(f(\lambda)\right)_0$. So there exists $\left(\overline{u_{i_0}}\right)^0 \in \left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ such that $\left(\overline{u_{i_0}}\right)^0 \left(f(x)\right) > \alpha \Rightarrow f^{-1}\left(\overline{u_{i_0}}\right)^0 (x) > \alpha$. As λ is $ap\alpha$ -compact, then $\left\{ f^{-1}\left(\overline{u_i}\right)^0 : i \in J \right\}$ has a finite subfamily, say $\left\{ f^{-1}\left(\overline{u_{i_k}}\right)^0 : k \in J_n \right\}$ such that $\overline{f^{-1}\left(\overline{u_{i_k}}\right)^0}(x) > \alpha$ for each $x \in \lambda_0$. But $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and fuzzy continuity of f, $f^{-1}\left(\overline{u_i}\right)$ must be a closed fuzzy set in X such that $f^{-1}\left(\overline{u_i}\right)^0 \subseteq f^{-1}\left(\overline{u_i}\right)$ and then $\overline{f^{-1}\left(\overline{u_i}\right)^0} \subseteq f^{-1}\left(\overline{u_i}\right)$. Therefore $f\left(\overline{f^{-1}\left(\overline{u_i}\right)^0}\right) \subseteq \overline{u_i}$ for each $i \in J$. Now, if $y \in \left(f(\lambda)\right)_0$, then $y = f(x)$ for some $x \in \lambda_0$, as f is surjective. So there exists k such that $f^{-1}\left(\overline{u_{i_k}}\right)(x) > \alpha \Rightarrow \overline{u_{i_k}}\left(f(x)\right) > \alpha \Rightarrow \overline{u_{i_k}}(y) > \alpha$. Hence $f(\lambda)$ is $ap\alpha$ -compact in (Y, s).

Theorem 9.6: Let (X, t) and (Y, s) be two fts's and $f : (X, t) \to (Y, s)$ be fuzzy open, fuzzy closed and bijective mapping. If λ is $ap\alpha$ -compact in (Y, s), then $f^{-1}(\lambda)$ is $ap\alpha$ -compact in (X, t).

Proof: Let $\{ u_i : i \in J \}$ be an open $p\alpha$ -shading of $f^{-1}(\lambda)$ in (X, t), then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open $p\alpha$ -shading of $f^{-1}(\lambda)$ in (X, t). Since f is fuzzy open, then $f\left(\overline{u_i}\right)^0 \in s$ and hence $\left\{ f\left(\overline{u_i}\right)^0 : i \in J \right\}$ is an open $p\alpha$ -shading of λ in (Y, s). For, let $y \in \lambda_0$, then $f^{-1}(y) \in \left(f^{-1}(\lambda)\right)_0$. So there exists $\left(\overline{u_{i_0}}\right)^0 \in \left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ such that $\left(\overline{u_{i_0}}\right)^0 \left(f^{-1}(y)\right) > \alpha \Rightarrow$ $f\left(\overline{u_{i_0}}\right)^0 (y) > \alpha$. As λ is $ap\alpha$ -compact in (Y, s), then $\left\{ f\left(\overline{u_i}\right)^0 : i \in J \right\}$ has a finite subfamily, say $\left\{ f\left(\overline{u_{i_k}}\right)^0 : k \in J_n \right\}$ such that $\overline{f\left(\overline{u_{i_k}}\right)^0} (y) > \alpha$ for each $y \in \lambda_0$. But $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and f is fuzzy closed, $f\left(\overline{u_i}\right)$ must be a closed fuzzy set in Y such that $f\left(\overline{u_i}\right)^0 \subseteq f\left(\overline{u_i}\right)$ and then $\overline{f\left(\overline{u_i}\right)^0} \subseteq f\left(\overline{u_i}\right)$. Therefore $f^{-1}\left(\overline{f\left(\overline{u_i}\right)^0} \right) \subseteq \overline{u_i}$ for each $i \in J$. For, if $x \in \left(f^{-1}(\lambda)\right)_0$, then $x = f^{-1}(y)$ for $y \in \lambda_0$, as f is bijective. So we can obtain, there exists k such that $f\left(\overline{u_{i_k}}\right)(y) > \alpha \Rightarrow \overline{u_{i_k}}\left(f^{-1}(y)\right) > \alpha \Rightarrow \overline{u_{i_k}}(x) > \alpha$. Hence $f^{-1}(\lambda)$ is $ap\alpha$ -compact in (X, t).

Theorem 9.7: Let (X, t) be an fts and let every family of closed fuzzy sets in X with empty intersection has a finite subfamily with empty intersection. Then any fuzzy set λ in X is $ap\alpha$ -compact. The converse is not true in general.

Proof: Let λ be any fuzzy set in X and let $\{ u_i : i \in J \}$ be an open $p\alpha$ -shading of λ, then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open $p\alpha$ -shading of λ. By the first condition of the theorem, we have $\bigcap_{i \in J} u_i^c = 0_X$. Therefore $\bigcup_{i \in J} u_i = 1_X$ and hence $\bigcup_{i \in J} \left(\overline{u_i}\right)^0 = 1_X$, as $u_i \subseteq \left(\overline{u_i}\right)^0$.

Again, by the second condition of the theorem, we get $\bigcap_{k \in J_n} u_{i_k}^c = 0_X$. So we have $\bigcup_{k \in J_n} u_{i_k} = 1_X$ and hence $\bigcup_{k \in J_n} \left(\overline{u_{i_k}}\right)^0 = 1_X$, as $u_i \subseteq \left(\overline{u_i}\right)^0$. But $u_i \subseteq \left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$, then we get $\bigcup_{k \in J_n} \overline{u_{i_k}} = 1_X$ and consequently we have $\overline{u_{i_k}}(x) > \alpha$ for each $x \in \lambda_0$. Therefore $\{ \overline{u_{i_k}} : k \in J_n \}$ is a finite $pp\alpha$-subshading of $\{ u_i : i \in J \}$. Hence λ is $ap\alpha$-compact.

Now, for the converse, we consider the following example.

Let $X = \{a, b, c\}$, $I = [0, 1]$ and $0 \le \alpha < 1$. Let u, $v \in I^X$ defined by $u(a) = 0.3$, $u(b) = 0.2$, $u(c) = 0.4$ and $v(a) = 0.4$, $v(b) = 0.3$, $v(c) = 0.5$. Choose $t = \{ 0 , u , v , 1\}$, then (X, t) is an fts. Now, $0^c(a) = 1$, $0^c(b) = 1$, $0^c(c) = 1$; $u^c(a) = 0.7$, $u^c(b) = 0.8$, $u^c(c) = 0.6$ and $v^c(a) = 0.6$, $v^c(b) = 0.7$, $v^c(c) = 0.5$. So we have $\bar{u} = \bigcap\{ 0^c , u^c , v^c \} = v^c$ i.e. $\bar{u}(a) = 0.6$, $\bar{u}(b) = 0.7$, $\bar{u}(c) = 0.5$ and $\bar{v} = \bigcap\{ 0^c , u^c , v^c \} = v^c$ i.e. $\bar{v}(a) = 0.6$, $\bar{v}(b) = 0.7$, $\bar{v}(c) = 0.5$. Again, let $\lambda \in I^X$ with $\lambda(a) = 0$, $\lambda(b) = 0.3$, $\lambda(c) = 0.8$. Take $\alpha = 0.3$. Then clearly λ is $ap\alpha$-compact in (X, t). But $u^c \cap v^c \neq 0$. Therefore the converse is not true in general.

The following example will show that the $ap\alpha$-compact fuzzy sets in an fts need not be closed.

Example 9.8: Consider the fts (X, t) in the example of the theorem (9.7). Again, let $\lambda \in I^X$ with $\lambda(a) = 0.5$, $\lambda(b) = 0.6$, $\lambda(c) = 0$. Take $\alpha = 0.5$. Then clearly λ is almost $ap\alpha$-compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 9.9: Let (X,t) be a fuzzy T_1-space (as def. 1.45) and λ be an $ap\alpha$ -compact fuzzy set in X with $\lambda_0 \subset X$. Let $x \notin \lambda_0$ ($\lambda(x)=0$), then there exist u , $v \in t$ such that $\bar{u}(x)=1$ and $\lambda_0 \subseteq (\bar{v})^{-1}(0,1]$.

Proof: Let $y \in \lambda_0$. So clearly we have $x \neq y$. As (X,t) is fuzzy T_1-space, there exist u_y , $v_y \in t$ such that $u_y(x)=1$, $u_y(y)=0$ and $v_y(x)=0$, $v_y(y)=1$. Let us assume that $0 \le \alpha < 1$ such that $v_y(y) > \alpha > 0$ (as $v_y(y)=1$). Thus we see that { v_y : $y \in \lambda_0$ } is an open $p\alpha$-shading of λ. Also we have $(\overline{u_y})^0(x)=1$, $(\overline{v_y})^0(y)=1$, as $u_y \subseteq (\overline{u_y})^0$, $v_y \subseteq (\overline{v_y})^0$ and then $\{(\overline{v_y})^0 : y \in \lambda_0\}$ is also an open $p\alpha$-shading of λ. Since λ is $ap\alpha$ -compact, then $\{(\overline{v_y})^0 : y \in \lambda_0\}$ has a finite $pp\alpha$-subshading, say { $\overline{v_{y_k}}$: $k \in J_n$ } such that $\overline{v_{y_k}}(y) > \alpha$ for each $y \in \lambda_0$. Now, let $(\bar{v})^0 = (\overline{v_{y_1}})^0 \cup (\overline{v_{y_2}})^0 \cup \ldots\ldots \cup (\overline{v_{y_n}})^0$ and $(\bar{u})^0 = (\overline{u_{y_1}})^0 \cap (\overline{u_{y_2}})^0 \cap \ldots\ldots \cap (\overline{u_{y_n}})^0$. Hence $(\bar{v})^0$ and $(\bar{u})^0$ are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $(\bar{v})^0$, $(\bar{u})^0 \in t$. But we have $(\overline{v_y})^0 \subseteq \overline{v_y}$ and $(\overline{u_y})^0 \subseteq \overline{u_y}$. Moreover, $\lambda_0 \subseteq (\bar{v})^{-1}(0,1]$ and $\bar{u}(x)=1$, as $\overline{u_{y_k}}(x)=1$ for each k.

Theorem 9.10: Let (X,t) be a fuzzy T_1-space (as def. 1.45) and λ , μ be disjoint $ap\alpha$ -compact fuzzy sets in X with λ_0 , $\mu_0 \subset X$. Then there exist u , $v \in t$ such that $\lambda_0 \subseteq (\bar{u})^{-1}(0,1]$ and $\mu_0 \subseteq (\bar{v})^{-1}(0,1]$.

Proof: Let $y \in \lambda_0$. Then we have $y \notin \mu_0$, as λ and μ are disjoint. As μ is $ap\alpha$ -compact, then by theorem (9.9), there exist u_y , $v_y \in t$ such that $\overline{u_y}(y)=1$ and

$\mu_0 \subseteq \left(\overline{v_y}\right)^{-1}(0,1]$. Assume that $0 \le \alpha < 1$ such that $\overline{u_y}(y) > \alpha > 0$. Since $\overline{u_y}(y) = 1$, then we have $\left\{\left(\overline{u_y}\right)^0 : y \in \lambda_0\right\}$ is an open $p\alpha$-shading of λ. But λ is $ap\alpha$-compact, so $\left\{\left(\overline{u_y}\right)^0 : y \in \lambda_0\right\}$ has a finite $pp\alpha$-subshading, say $\{\overline{u_{y_k}} : k \in J_n\}$ such that $\overline{u_{y_k}}(y) > \alpha$ for each $y \in \lambda_0$. Again, μ is $ap\alpha$-compact, then $\left\{\left(\overline{v_y}\right)^0 : x \in \mu_0\right\}$ has a finite $pp\alpha$-subshading, say $\{\overline{v_{y_k}} : k \in J_n\}$ such that $\overline{v_{y_k}}(x) > \alpha$ for each $x \in \mu_0$ and $\mu_0 \subseteq \left(\overline{v_{y_k}}\right)^{-1}(0,1]$ for each k. Now, let $\left(\overline{u}\right)^0 = \left(\overline{u_{y_1}}\right)^0 \cup \left(\overline{u_{y_2}}\right)^0 \cup \ldots\ldots \cup \left(\overline{u_{y_n}}\right)^0$ and $\left(\overline{v}\right)^0 = \left(\overline{v_{y_1}}\right)^0 \cap \left(\overline{v_{y_2}}\right)^0 \cap \ldots\ldots \cap \left(\overline{v_{y_n}}\right)^0$. But we have $\left(\overline{u_y}\right)^0 \subseteq \overline{u_y}$ and $\left(\overline{v_y}\right)^0 \subseteq \overline{v_y}$, we see that $\lambda_0 \subseteq \left(\overline{u}\right)^{-1}(0,1]$ and $\mu_0 \subseteq \left(\overline{v}\right)^{-1}(0,1]$. Also $\left(\overline{u}\right)^0$ and $\left(\overline{v}\right)^0$ are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $\left(\overline{u}\right)^0$, $\left(\overline{v}\right)^0 \in t$.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (9.9) and (9.10) are not at all true.

The following example will show that the $ap\alpha$-compact fuzzy sets in fuzzy T_1-pace (as def. 1.45) need not be closed.

Example 9.11: Let $X = \{a, b\}$, $I = [0,1]$ and $0 \le \alpha < 1$. Let u, $v \in I^X$ defined by $u(a) = 1$, $u(b) = 0$ and $v(a) = 0$, $v(b) = 1$. Take $t = \{0, u, v, 1\}$, then (X, t) is a fuzzy T_1-space. Now, $0^c(a) = 1$, $0^c(b) = 1$; $u^c(a) = 0$, $u^c(b) = 1$ and $v^c(a) = 1$, $v^c(b) = 0$. So we have $\overline{u} = \bigcap\{0^c, v^c\} = v^c$ i.e. $\overline{u}(a) = 1$, $\overline{u}(b) = 0$ and $\overline{v} = \bigcap\{0^c, u^c\} = u^c$ i.e. $\overline{v}(a) = 0$, $\overline{v}(b) = 1$. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.4$, $\lambda(b) = 0$. Take $\alpha = 0.8$. Then clearly λ is $ap\alpha$-compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 9.12: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is $ap\alpha$-compact in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u , $v \in t$ such that $\overline{u}(x) > 0$ and $\lambda_0 \subseteq (\overline{v})^{-1}(0,1]$. The converse is not true in general

The proof is similar as that of theorem (9.9).

Now, for the converse, we give the following example.

Let $X = \{a, b\}$, $I = [0,1]$ and $0 \le \alpha < 1$. Let u_1, u_2, $u_3 \in I^X$ defined by $u_1(a) = 0.2$, $u_1(b) = 0$; $u_2(a) = 0$, $u_2(b) = 0.3$ and $u_3(a) = 0.2$, $u_3(b) = 0.3$. Now, put $t = \{ 0 , u_1 , u_2 , u_3 , 1 \}$, then we see that (X, t) is a fuzzy T_1-space. Now, we have $0^c(a) = 1$, $0^c(b) = 1$; $u_1^c(a) = 0.8$, $u_1^c(b) = 1$; $u_2^c(a) = 1$, $u_2^c(b) = 0.7$ and $u_3^c(a) = 0.8$, $u_3^c(b) = 0.7$. Therefore, $\overline{u_1} = \bigcap\{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c$ i.e. $\overline{u_1}(a) = 0.8$, $\overline{u_1}(b) = 0.7$; $\overline{u_2} = \bigcap\{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c$ i.e. $\overline{u_2}(a) = 0.8$, $\overline{u_2}(b) = 0.7$ and $\overline{u_3} = \bigcap\{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c$ i.e. $\overline{u_3}(a) = 0.8$, $\overline{u_3}(b) = 0.7$. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.6$. Hence we observe that $\lambda_0 = \{b\}$ and $a \notin \lambda_0$. Here u_1, $u_2 \in t$ where $\overline{u_1}(a) = 0.8 > 0$ and $(\overline{u_2})^{-1}(0,1] = \{a, b\}$. Hence $\lambda_0 \subset (\overline{u_2})^{-1}(0,1]$. Take $\alpha = 0.9$. Thus we see that λ is not almost $ap\alpha$-compact in (X, t), as $\overline{u_k}(b) < \alpha$ for $k = 1, 2, 3$ and $b \in \lambda_0$. Thus the converse of the theorem is not true in general.

Theorem 9.13: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and λ , μ be disjoint fuzzy sets in X with λ_0 , $\mu_0 \subset X$. If λ and μ are $ap\alpha$-compacts in (X, t), then there exist u , $v \in t$ such that $\lambda_0 \subseteq (\overline{u})^{-1}(0,1]$ and $\mu_0 \subseteq (\overline{v})^{-1}(0,1]$. The converse is not true in general.

The proof is similar as that of theorem (9.10).

Now, for the converse, consider the fuzzy T_1-space (X, t) in the example of the theorem (9.12). Let λ , $\mu \in I^X$ with $\lambda(a) = 0.5$, $\lambda(b) = 0$ and $\mu(a) = 0$, $\mu(b) = 0.4$. Thus we see that $\lambda_0 = \{a\}$ and $\mu_0 = \{b\}$. Now u_1 , $u_2 \in t$ where $\left(\overline{u_1}\right)^{-1}(0, 1] = \{a, b\}$ and $\left(\overline{u_2}\right)^{-1}(0, 1] = \{a, b\}$. Hence we observe that $\lambda_0 \subset \left(\overline{u_1}\right)^{-1}(0, 1]$ and $\mu_0 \subset \left(\overline{u_2}\right)^{-1}(0, 1]$, where λ and μ are disjoint. Take $\alpha = 0.9$. Hence we see that λ and μ are not almost $ap\alpha$ -compact in (X, t), as $\overline{u_k}(a) < \alpha$ where $a \in \lambda_0$ and $\overline{u_k}(b) < \alpha$ where $b \in \mu_0$, for $k = 1$, 2 , 3 respectively. Thus the converse of the theorem is not true in general.

The following example will show that the $ap\alpha$ -compact fuzzy sets in fuzzy T_1-space (as def. 1.46) need not be closed.

Example 9.14: Consider the fuzzy T_1-space in the example of the theorem (9.12). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.7$, $\lambda(b) = 0$. Take $\alpha = 0.6$. Then clearly λ is $ap\alpha$ -compact in (X, t). But λ is not closed in (X, t), as its complement λ^c is not open in (X, t).

Theorem 9.15: An $ap\alpha$ -compact fuzzy sets in fuzzy regular space (X, t) (as def. 1.52) is $p\alpha$ -compact.

Proof: Let $\{ u_i : i \in J \}$ be an open $p\alpha$ -shading of a fuzzy set λ in X i.e. $u_i(x) > \alpha$ for each $x \in \lambda_0$. Since (X, t) is fuzzy regular, then we have $u_i = \bigcup_{i \in J} v_{ij}$, where v_{ij} is an open fuzzy set such that $\overline{v_{ij}} \subseteq u_i$ for each i. But $u_i(x) > \alpha \Rightarrow \bigcup_{i \in J} v_{ij}(x) > \alpha$ for each $x \in \lambda_0$. Therefore $v_{ij}(x) > \alpha$ for each $x \in \lambda_0$ and for some $i \in J$. So $\{ v_{ij} : i \in J \}$ is an open $p\alpha$-shading of λ. As λ is $ap\alpha$ -compact, then $\{ v_{ij} : i \in J \}$ has a finite

$pp\alpha$ -subshading, say $\{ \overline{v_{i_k j}} : k \in J_n \}$ such that such that $\overline{v_{i_k j}}(x) > \alpha$ for each $x \in \lambda_0$. But we have $\overline{v_{i_k j}} \subseteq u_{i_k}$, then $u_{i_k}(x) > \alpha$ for each $x \in \lambda_0$. Thus we see that $\{ u_{i_k} : k \in J_n \}$ is a finite $p\alpha$ -subshading of $\{ u_i : i \in J \}$ and hence λ is $p\alpha$ -compact.

Theorem 9.16: Let (X, t) be an fts and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ_0 is compact in (X, t_α), then λ is $ap\alpha$ -compact in (X, t). The converse is not true in general.

Proof: Suppose λ_0 is compact in (X, t_α). Let $\{ u_i : i \in J \}$ be an open $p\alpha$ -shading λ in (X, t), then $\{ (\overline{u_i})^0 : i \in J \}$ is also an open $p\alpha$ -shading of λ in (X, t). So the family $\{ \alpha(\overline{u_i})^0 : i \in J \}$ is an open cover of λ_0 in (X, t_α). For let $x \in \lambda_0$, so there exists a $(\overline{u_{i_0}})^0 \in \{ (\overline{u_i})^0 : i \in J \}$ such that $(\overline{u_{i_0}})^0(x) > \alpha$. Hence $x \in \alpha(\overline{u_{i_0}})^0$ and thus $\alpha(\overline{u_{i_0}})^0 \in \{ \alpha(\overline{u_i})^0 : i \in J \}$. But λ_0 is compact in (X, t_α), so $\{ \alpha(\overline{u_i})^0 : i \in J \}$ has a finite subcover, say $\{ \alpha(\overline{u_{i_k}})^0 : k \in J_n \}$. So $\{ (\overline{u_{i_k}})^0 : k \in J_n \}$ forms a finite subfamily of $\{ (\overline{u_i})^0 : i \in J \}$ such that $\overline{u_{i_k}}(x) > \alpha$ for each $x \in \lambda_0$ i.e. $\{ \overline{u_{i_k}} : k \in J_n \}$ is a finite $pp\alpha$ -subshading of $\{ u_i : i \in J \}$. Hence λ is $ap\alpha$ -compact in (X, t).

Now, for the converse, we consider the following example.

Let $X = \{a, b, c\}$, $I = [0, 1]$ and $0 \le \alpha < 1$. Let u, $v \in I^X$ defined by $u(a) = 0.2$, $u(b) = 0.3$, $u(c) = 0.4$ and $v(a) = 0.3$, $v(b) = 0.4$, $v(c) = 0.5$. Put $t = \{ 0, u, v, 1 \}$, then (X, t) is an fts. Again, let $\lambda \in I^X$ with $\lambda(a) = 0$, $\lambda(b) = 0.6$, $\lambda(c) = 0.8$. Then $\lambda_0 = \{b, c\}$. Now $0^c(a) = 1$, $0^c(b) = 1$, $0^c(c) = 1$; $u^c(a) = 0.8$, $u^c(b) = 0.7$, $u^c(c) = 0.6$ and $v^c(a) = 0.7$, $v^c(b) = 0.6$, $v^c(c) = 0.5$. So we have

$\overline{u} = \bigcap\{\ 0^c\ ,\ u^c\ ,\ v^c\ \} = v^c$ i.e. $\overline{u}(a) = 0.7$, $\overline{u}(b) = 0.6$, $\overline{u}(c) = 0.5$ and $\overline{v} = \bigcap\{\ 0^c\ ,\ u^c\ ,\ v^c\ \} = v^c$ i.e. $\overline{v}(a) = 0.7$, $\overline{v}(b) = 0.6$, $\overline{v}(c) = 0.5$. Take $\alpha = 0.4$. Then clearly λ is $ap\alpha$ -compact in (X, t). Now, we have $t_{0.4} = \{\ \phi\ ,\ \{c\}\ ,\ X\ \}$. Hence it is clear that λ_0 is not compact in $(X, t_{0.4})$.

Theorem 9.17: Let $f : (X, t_\alpha) \to (X, t)$ be α -level continuous, bijective and λ be a fuzzy set in X . If λ_0 is compact in (X, t_α), then $f(\lambda)$ is $ap\alpha$ -compact in (X, t).

Proof: Suppose λ_0 is compact in (X, t_α). Let $\{\ u_i :\ i \in J\ \}$ be an open $p\alpha$ -shading $f(\lambda)$ in (X, t), then $\left\{\left(\overline{u_i}\right)^0 :\ i \in J\right\}$ is also an open $p\alpha$ -shading of $f(\lambda)$ in (X, t). Since f is α -level continuous, then $\alpha(f^{-1}(u_i)) \in t_\alpha \Rightarrow \alpha\left(f^{-1}\left(\overline{u_i}\right)^0\right) \in t_\alpha$ and hence $\left\{\alpha\left(f^{-1}\left(\overline{u_i}\right)^0\right) :\ i \in J\right\}$ is an open cover of λ_0 in (X, t_α). As λ_0 is compact in (X, t_α), then $\left\{\alpha\left(f^{-1}\left(\overline{u_i}\right)^0\right) :\ i \in J\right\}$ has a finite subcover, say $\left\{\alpha\left(f^{-1}\left(\overline{u_{i_k}}\right)^0\right) :\ k \in J_n\right\}$. Now, we have $f(x) = y$ for $y \in f(\lambda)_0$, since f is bijective. As $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and $\left\{\alpha\left(f^{-1}\left(\overline{u_{i_k}}\right)^0\right) :\ k \in J_n\right\}$ is a finite subcober of $\left\{\alpha\left(f^{-1}\left(\overline{u_i}\right)^0\right) :\ i \in J\right\}$, there exist some k such that $\overline{u_{i_k}}\left(f(x)\right) > \alpha \Rightarrow \overline{u_{i_k}}(y) > \alpha$ for each $y \in f(\lambda)_0$. Thus $\{\ \overline{u_{i_k}} :\ k \in J_n\}$ is a finite $pp\alpha$ -subshading of $\{\ u_i :\ i \in J\ \}$. Therefore, $f(\lambda)$ is $ap\alpha$ -compact in (X, t).

The following example will show that the "good extension" property does not hold for $ap\alpha$ -compact fuzzy sets.

Example 9.18: Let $X = \{a, b, c\}$ and $T = \{ \phi , \{b\} , \{c\} , \{b, c\} , X \}$. Then (X, T) is a topological space. Let u_1, u_2, $u_3 \in I^X$ with $u_1(a) = 0$, $u_1(b) = 0.3$, $u_1(c) = 0$; $u_2(a) = 0$, $u_2(b) = 0$, $u_2(c) = 0.4$ and $u_3(a) = 0$, $u_3(b) = 0.3$, $u_3(c) = 0.4$. Then $\omega(T) = \{ 0 , u_1 , u_2 , u_3 , 1\}$ and $(X, \omega(T))$ is an fts. Now, $0^c(a) = 1$, $0^c(b) = 1$, $0^c(c) = 1$; $u_1^c(a) = 1$, $u_1^c(b) = 0.7$, $u_1^c(c) = 1$; $u_2^c(a) = 1$, $u_2^c(b) = 1$, $u_2^c(c) = 0.6$ and $u_3^c(a) = 1$, $u_3^c(b) = 0.7$, $u_3^c(c) = 0.6$. So we have $\overline{u_1} = \bigcap\{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c$ i.e. $\overline{u_1}(a) = 1$, $\overline{u_1}(b) = 0.7$, $\overline{u_1}(c) = 0.6$; $\overline{u_2} = \bigcap\{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c$ i.e. $\overline{u_2}(a) = 1$, $\overline{u_2}(b) = 0.7$, $\overline{u_2}(c) = 0.6$ and $\overline{u_3} = \bigcap\{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c$ i.e. $\overline{u_3}(a) = 1$, $\overline{u_3}(b) = 0.7$, $\overline{u_3}(c) = 0.6$. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.6$, $\lambda(c) = 0.5$. Then we have $\lambda_0 = \{b, c\}$. Clerly λ_0 is compact in (X, T). Take $\alpha = 0.9$. Then λ is not $ap\alpha$-compact in $(X, \omega(T))$, as there do not exist $\overline{u_k}$ for $k = 1, 2, 3$ such that $\overline{u_k}(b) > \alpha$ for $b \in \lambda_0$. Again, let $\mu \in I^X$ defined by $\mu(a) = 0.2$, $\mu(b) = 0.2$, $\mu(c) = 0$. Then we have $\mu_0 = \{a, b\}$. Take $\alpha = 0.3$. Then clearly μ is $ap\alpha$-compact in $(X, \omega(T))$. But μ_0 is not compact in (X, T), as there is no finite subcover of μ_0 in (X, T).

Theorem 9.19: Let λ and μ be $ap\alpha$-compact fuzzy sets in an fts (X, t). Then $(\lambda \times \mu)$ is also $ap\alpha$-compact in $(X \times X, t \times t)$.

Proof: Let $\{ u_i \times v_i : i \in J \}$ be an open $p\alpha$-shading of $(\lambda \times \mu)$ in $(X \times X, t \times t)$ i.e. $(u_i \times v_i)\ (x, y) > \alpha$ for each $(x, y) \in (\lambda \times \mu)_0$. Therefore we have $u_i(x) > \alpha$ for each $x \in \lambda_0$ and $v_i(y) > \alpha$ for each $y \in \mu_0$. Hence $\{ u_i : i \in J \}$ and $\{ v_i : i \in J \}$ are open $p\alpha$-shadings of λ and μ respectively. Thus $\left\{\left(\overline{u_i}\right)^0 : i \in J \right\}$ and $\left\{\left(\overline{v_i}\right)^0 : i \in J \right\}$ are also open $p\alpha$-shading of λ and μ respectively. Now we have $(\overline{u_i})^0 \subseteq \overline{u_i}$ and $(\overline{v_i})^0 \subseteq \overline{v_i}$. As λ and μ are $ap\alpha$-compact, then $\{(\overline{u_i})^0 : i \in J\}$ and $\{(\overline{v_i})^0 : i \in J\}$ have finite $pp\alpha$-subshading, say $\{\overline{u_{i_k}} : k \in J_n\}$ and $\{\overline{v_{i_k}} : k \in J_n\}$ such that $\overline{u_{i_k}}(x) > \alpha$ for each $x \in \lambda_0$ and $\overline{v_{i_k}}(y) > \alpha$ for each $y \in \mu_0$ respectively. Hence we can write $(\overline{u_{i_k}} \times \overline{v_{i_k}})(x, y) > \alpha$ for each $(x, y) \in (\lambda \times \mu)_0$. Therefore $(\lambda \times \mu)$ is $ap\alpha$-compact in $(X \times X, t \times t)$.

Definition 9.20: Let (X, t) be an fts, λ be a fuzzy set in X and $0 < \delta \le 1$, $0 < \alpha < 1$. Let $\{u_i : i \in J\}$ be a family of δ-open fuzzy sets in (X, t). Then $\{u_i : i \in J\}$ is a proximate partial δ-α-shading of λ, in short, $pp\delta\alpha$-shading, when $\{\overline{u_i} : i \in J\}$ is a $p\delta\alpha$-shading of λ i.e. $\overline{u_i}(x) > \alpha$ for all $x \in \lambda_0$.

A subfamily of $\{u_i : i \in J\}$ which is also a $pp\delta\alpha$-shading of λ is said to be $pp\delta\alpha$-subshading of λ.

Definition 9.21: Let (X, t) be an fts and $0 < \delta \le 1$, $\alpha \in I$. A fuzzy set λ in X is said to be almost partially δ-α-compact, $0 \le \alpha < 1$, in short, $ap\delta\alpha$-compact iff every $p\delta\alpha$-shading of λ has a finite subfamily whose closures is $p\delta\alpha$-shading of λ or equivalently, every $p\delta\alpha$-shading of λ has a finite $pp\delta\alpha$-subshading.

Theorem 9.22: Every $ap\delta\alpha$-compact fuzzy set in an fts is $ap\alpha$-compact. But the converse is not true.

The proof is straightforward.

Now, for the converse, we consider the following example.

Let $X = \{a, b, c\}$, $I = [0, 1]$ and $0 < \delta \leq 1$, $0 \leq \alpha < 1$. Let u_1, $u_2 \in I^X$ defined by $u_1(a) = 0.2$, $u_1(b) = 0.4$, $u_1(c) = 0.3$ and $u_2(a) = 0.3$, $u_2(b) = 0.5$, $u_2(c) = 0.4$. Put $t = \{ 0, u_1, u_2, 1\}$, then (X, t) is an fts. Now, $0^c(a) = 1$, $0^c(b) = 1$, $0^c(c) = 1$; $u_1^c(a) = 0.8$, $u_1^c(b) = 0.6$, $u_1^c(c) = 0.7$ and $u_2^c(a) = 0.7$, $u_2^c(b) = 0.5$, $u_2^c(c) = 0.6$. So we have $\overline{u_1} = \bigcap\{ 0^c, u_1^c, u_2^c \} = u_2^c$ i.e. $\overline{u_1}(a) = 0.7$, $\overline{u_1}(b) = 0.5$, $\overline{u_1}(c) = 0.6$ and $\overline{u_2} = \bigcap\{ 0^c, u_1^c, u_2^c \} = u_2^c$ i.e. $\overline{u_2}(a) = 0.7$, $\overline{u_2}(b) = 0.5$, $\overline{u_2}(c) = 0.6$. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.9$, $\lambda(b) = 0.7$, $\lambda(c) = 0$. So we have $\lambda_0 = \{a, b\}$. Take $\alpha = 0.4$. Then clearly λ is $ap\alpha$-compact in (X, t). Again, take $\delta = 0.9$. Then we observe that there is finite $pp\delta\alpha$-subshading of λ. Hence λ is not $ap\delta\alpha$-compact in (X, t). Thus the converse of theorem is not necessarily true.

Chapter Ten

Almost $Q\alpha$-Compact Fuzzy Sets

In this chapter, we have introduced almost $Q\alpha$-compact fuzzy sets. Furthermore, we have established several theorems, corollary and examples of almost $Q\alpha$-compact fuzzy sets. Also we have defined almost δ-$Q\alpha$-compact fuzzy sets and identified different characterizations between almost $Q\alpha$-compact and almost δ-$Q\alpha$-compact fuzzy sets.

Definition 10.1: A family $\{ u_i : i \in J \}$ is said to be proximate $Q\alpha$-cover of a fuzzy set λ in X when $\{ \overline{u_i} : i \in J \}$ is $Q\alpha$-cover of λ i.e. $\lambda(x) + \overline{u_i}(x) \geq \alpha$ for each $x \in X$ and for some u_i, where $\alpha \in I_0$.

A subfamily of $\{ u_i : i \in J \}$ which is also a proximate $Q\alpha$-cover of λ is called a proximate $Q\alpha$-subcover of λ.

Definition 10.2: A fuzzy set λ is said to be almost $Q\alpha$-compact iff every open $Q\alpha$-cover of λ has a finite subfamily whose closures is $Q\alpha$-cover of λ or equivalently, every open $Q\alpha$-cover of λ has a finite proximate $Q\alpha$-subcover.

Every super sets of an almost $Q\alpha$-compact fuzzy set is also almost $Q\alpha$-compact.

Theorem 10.3: Let (X, t) be an fts, $A \subset X$ and λ be a fuzzy set in A. Then λ is almost $Q\alpha$-compact in (X, t) iff λ is almost $Q\alpha$-compact in (A, t_A).

Proof: Suppose λ is almost $Q\alpha$ -compact in (X, t). Let $\{ u_i : i \in J \}$ be an open $Q\alpha$ -cover of λ in (A, t_A), then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open $Q\alpha$ -cover of λ in (A, t_A). Then there exist $v_i \in t$ such that $u_i = v_i \mid A \subseteq v_i$. Therefore $\{ v_i : i \in J \}$ is an open $Q\alpha$ -cover of λ in (X, t) and so $\left\{ \left(\overline{v_i}\right)^0 : i \in J \right\}$ is also an open $Q\alpha$ -cover of λ in (X, t). But $\left(\overline{v_i}\right)^0 \subseteq \overline{v_i}$ and λ is almost $Q\alpha$ -compact in (X, t), then $\left\{ \left(\overline{v_i}\right)^0 : i \in J \right\}$ has a finite proximate $Q\alpha$ -subcover, say $\{ \overline{v_{i_k}} : k \in J_n \}$ such that $\lambda(x) + \overline{v_{i_k}}(x) \geq \alpha$ for each $x \in A$. Hence $\lambda(x) + \left(\overline{v_{i_k}} \mid A\right)(x) \geq \alpha$ for each $x \in A$ and consequently $\lambda(x) + \overline{u_{i_k}}(x) \geq \alpha$ for each $x \in A$. Therefore $\{ \overline{u_{i_k}} : k \in J_n \}$ is a finite proximate $Q\alpha$ -subcover of $\{ u_i : i \in J \}$. Thus λ is almost $Q\alpha$ -compact in (A, t_A).

Conversely, suppose λ is almost $Q\alpha$ -compact in (A, t_A). Let $\{ v_i : i \in J \}$ be an open $Q\alpha$ -cover of λ in (X, t), then $\left\{ \left(\overline{v_i}\right)^0 : i \in J \right\}$ is also an open $Q\alpha$ -cover of λ in (X, t). Put $u_i = v_i \mid A$. Then $\lambda(x) + v_i(x) \geq \alpha$ for all $x \in A \Rightarrow \lambda(x) + (v_i \mid A)(x) \geq \alpha$ for each $x \in A \Rightarrow \lambda(x) + u_i(x) \geq \alpha$ for each $x \in A$. Since $u_i \in t_A$, then $\{ u_i : i \in J \}$ is an open $Q\alpha$ -cover of λ in (A, t_A). Therefore $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open $Q\alpha$ -cover of λ in (A, t_A). But from $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and λ is almost $Q\alpha$ -compact in (A, t_A), then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ has a finite proximate $Q\alpha$ -subcover, say $\{ \overline{u_{i_k}} : k \in J_n \}$ such that $\lambda(x) + \overline{u_{i_k}}(x) \geq \alpha$ for each $x \in A$. But $\overline{u_i} = \overline{v_i \mid A} \subseteq \overline{v_i} \mid A \subseteq \overline{v_i}$, then $\lambda(x) + \left(\overline{v_{i_k} \mid A}\right)(x) \geq \alpha$ for each $x \in A$ $\Rightarrow \lambda(x) + \left(\overline{v_{i_k}} \mid A\right)(x) \geq \alpha$ for each $x \in A$ and consequently $\lambda(x) + \overline{v_{i_k}}(x) \geq \alpha$ for each $x \in A$. Therefore $\{ \overline{v_{i_k}} : k \in J_n \}$ is a finite proximate $Q\alpha$ -subcover of $\{ v_i : i \in J \}$. Therefore λ is almost $Q\alpha$ -compact in (X, t).

Corollary 10.4: Let (Y, t^*) be a fuzzy subspace of (X, t) and $A \subset Y \subset X$. Let λ be a fuzzy set in A. Then λ is almost $Q\alpha$-compact in (X, t) iff λ is almost $Q\alpha$ -compact in (Y, t^*).

Proof: Let t_A and t_A^* be the subspace fuzzy topologies on A. Then by theorem (10.3), λ is almost $Q\alpha$ -compact in (X, t) or (Y, t^*) iff λ is almost $Q\alpha$ -compact in (A, t_A) or (A, t_A^*). But $t_A = t_A^*$.

Theorem 10.5: Let λ be an almost $Q\alpha$ -compact fuzzy set in an fts (X, t). If $\mu \subseteq \lambda$ and $\mu \in t^c$, then μ is also almost $Q\alpha$ -compact.

Proof: Let $\{ u_i : i \in J \}$ be an open $Q\alpha$ -cover of μ, then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open $Q\alpha$ -cover of μ. So $\left\{ \left(\overline{u_i}\right)^0 \right\} \cup \left\{ \left(\overline{\mu^c}\right)^0 \right\}$ is an open $Q\alpha$ -cover of λ. As $\mu(x) + \left(\overline{u_i}\right)^0 (x) \geq \alpha$ for each $x \in X$, then we have $\lambda(x) + \max\left(\left(\overline{u_i}\right)^0 (x), \left(\overline{\mu^c}\right)^0 (x) \right) \geq \alpha$ for each $x \in X$. Hence $\mu(x) + \left(\overline{u_i}\right)^0 (x) \leq \lambda(x) + \left(\overline{u_i}\right)^0 (x) \geq \alpha$ for each $x \in X$. Since $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and λ is almost $Q\alpha$ -compact, then $\left\{ \left(\overline{u_i}\right)^0 \right\} \cup \left\{ \left(\overline{\mu^c}\right)^0 \right\}$ has a finite subcollection, say $\left\{ \left(\overline{u_{i_k}}\right)^0 : k \in J_n \right\} \cup \left\{ \left(\overline{\mu^c}\right)^0 \right\}$ such that $\lambda(x) + \max\left(\overline{u_{i_k}}(x), \overline{\mu^c}(x) \right) \geq \alpha$ for each $x \in X$. Therefore $\{ \overline{u_{i_k}} : k \in J_n \}$ is a finite proximate $Q\alpha$ -subcover of $\{ u_i : i \in J \}$. Hence μ is almost $Q\alpha$ -compact.

Theorem 10.6: Let (X, t) be an fts and λ be a fuzzy set in X. If every family of closed fuzzy sets having the empty intersection has a finite subfamily with empty intersection, then λ is almost $Q\alpha$ -compact. The converse is not true in general.

Proof: Let $\{ u_i : i \in J \}$ be an open $Q\alpha$-cover of λ, then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open $Q\alpha$-cover of λ. From the first condition of the theorem, we have $\bigcap_{i \in J} u_i^c = 0_X$. Thus $\bigcup_{i \in J} u_i = 1_X$ and so $\bigcup_{i \in J} \left(\overline{u_i}\right)^0 = 1_X$, as $u_i \subseteq \left(\overline{u_i}\right)^0$. Again by the second condition of the theorem, we get $\bigcap_{k \in J_n} u_{i_k}^c = 0_X$. So we have $\bigcup_{k \in J_n} u_{i_k} = 1_X$ and hence $\bigcup_{k \in J_n} \left(\overline{u_{i_k}}\right)^0 = 1_X$, as $u_i \subseteq \left(\overline{u_i}\right)^0$. But $u_i \subseteq \left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$, then $\bigcup_{k \in J_n} \overline{u_{i_k}} = 1_X$ and consequently $\lambda(x) + \overline{u_{i_k}}(x) \geq \alpha$ for each $x \in X$. Therefore $\{ \overline{u_{i_k}} : k \in J_n \}$ is a finite proximate $Q\alpha$-subcover of $\{ u_i : i \in J \}$. Thus λ is almost $Q\alpha$-compact.

Now, for the converse, we give the following example.

Let $X = \{a, b\}$, $I = [0, 1]$ and $\alpha \in I_0$. Again, let u, $v \in I^X$ defined by $u(a) = 0.2$, $u(b) = 0.4$ and $v(a) = 0.3$, $v(b) = 0.6$. Put $t = \{ 0, u, v, 1\}$, then (X, t) is an fts. Now, $0^c(a) = 1$, $0^c(b) = 1$; $u^c(a) = 0.8$, $u^c(b) = 0.6$ and $v^c(a) = 0.7$, $v^c(b) = 0.4$. So we have $\bar{u} = \bigcap \{ 0^c, u^c, v^c \} = v^c$ i.e. $\bar{u}(a) = 0.7$, $\bar{u}(b) = 0.4$ and $\bar{v} = \bigcap \{ 0^c, u^c \} = u^c$ i.e. $\bar{v}(a) = 0.8$, $\bar{v}(b) = 0.6$. Let $\lambda \in I^X$ with $\lambda(a) = 0.3$, $\lambda(b) = 0.7$. Take $\alpha = 0.9$. Then clearly λ is almost $Q\alpha$-compact in (X, t). But $u^c \cap v^c \neq 0$. Therefore the converse of the theorem is not true in general.

Theorem 10.7: Let λ and μ be almost $Q\alpha$-compact fuzzy sets in an fts (X, t). Then $\lambda \cap \mu$ is also almost $Q\alpha$-compact in (X, t).

Proof: Let $\{ u_i : i \in J \}$ be an open $Q\alpha$-cover of $\lambda \cap \mu$, then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open $Q\alpha$-cover of $\lambda \cap \mu$. Therefore $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is any open $Q\alpha$-cover of both λ

and μ respectively. But from $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and λ is almost $Q\alpha$ -compact in (X, t), then $\left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$ has a finite proximate $Q\alpha$ -subcover, say $\{ \overline{u_{i_k}} : k \in J_n \}$ such that $\lambda(x) + \overline{u_{i_k}}(x) \geq \alpha$ for each $x \in X$. Similarly, we can find $\{ \overline{u_{i_r}} : r \in J_n \}$ is a finite proximate $Q\alpha$ -subcover of $\left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$. Therefore $\{ \overline{u_{i_k}} , \overline{u_{i_r}} \}$ is a finite proximate $Q\alpha$ -subcover of $\{ u_i : i \in J \}$. Hence $\lambda \cap \mu$ is almost $Q\alpha$ -compact in (X, t).

Theorem 10.8: Let λ and μ be almost $Q\alpha$ -compact fuzzy sets in an fts (X, t). Then $\lambda \cup \mu$ is also almost $Q\alpha$ -compact in (X, t).

Proof: We have $\lambda \subseteq \lambda \cup \mu$, $\mu \subseteq \lambda \cup \mu$. As λ and μ are almost $Q\alpha$ -compact, then it is clear that $\lambda \cup \mu$ is almost $Q\alpha$ -compact in (X, t).

The following example will show that any other subsets of an almost $Q\alpha$ -compact fuzzy set in an fts need not be almost $Q\alpha$ -compact.

Example 10.9: Let $X = \{a, b\}$, $I = [0, 1]$ and $\alpha \in I_0$. Again, let u , $v \in I^X$ defined by $u(a) = 0.3$, $u(b) = 0.4$ and $v(a) = 0.4$, $v(b) = 0.5$. Consider $t = \{ 0 , u , v , 1\}$, then (X, t) is an fts. Now, $0^c(a) = 1$, $0^c(b) = 1$; $u^c(a) = 0.7$, $u^c(b) = 0.6$ and $v^c(a) = 0.6$, $v^c(b) = 0.5$. Therefore $\overline{u} = \bigcap \{ 0^c , u^c , v^c \} = v^c$ i.e. $\overline{u}(a) = 0.6$, $\overline{u}(b) = 0.5$ and $\overline{v} = \bigcap \{ 0^c , u^c , v^c \} = v^c$ i.e. $\overline{v}(a) = 0.6$, $\overline{v}(b) = 0.5$. Let λ , $\mu \in I^X$ with $\lambda(a) = 0.3$, $\lambda(b) = 0.7$ and $\mu(a) = 0.1$, $\mu(b) = 0.4$. We observe that $\mu \subset \lambda$. Take $\alpha = 0.8$. Clearly λ is almost $Q\alpha$ -compact in (X, t). But μ is not almost $Q\alpha$ -compact in (X, t), as μ have no finite proximate $Q\alpha$ -subcover in (X, t).

Note: The example (10.9) also shows that almost $Q\alpha$ -compact fuzzy sets in an fts need not be closed, as λ is almost $Q\alpha$ -compact in (X,t) but λ^c is not open in (X,t).

Theorem 10.10: Let (X,t) be a fuzzy T_1-space (as def. 1.45) and λ be an almost $Q\alpha$ -compact fuzzy set in X with $\lambda_0 \subset X$. Let $x \notin \lambda_0$ ($\lambda(x)=0$), then there exist u , $v \in t$ such that $\bar{u}(x)=1$ and $\lambda_0 \subseteq (\bar{v})^{-1}(0,1]$.

Proof: Let $y \in \lambda_0$. Then clearly we have $x \neq y$. Since (X,t) is fuzzy T_1-space, then there exist u_y , $v_y \in t$ such that $u_y(x)=1$, $u_y(y)=0$ and $v_y(x)=0$, $v_y(y)=1$. Let us assume that $\alpha \in I_0$ such that $\lambda(x)+u_y(x) \geq \alpha$, $x \in X$ and $\lambda(y)+v_y(y) \geq \alpha$, $y \in \lambda_0$ i.e. $\{ u_y , \ v_y : \ y \in \lambda_0 \}$ is an open $Q\alpha$ -cover of λ. Also we have $(\overline{u_y})^0(x)=1$, $(\overline{v_y})^0(y)=1$, as $u_y \subseteq (\overline{u_y})^0$, $v_y \subseteq (\overline{v_y})^0$ and say $M = \{(\overline{u_y})^0 , \ (\overline{v_y})^0 : \ y \in \lambda_0 \}$ is also an open $Q\alpha$ -cover of λ. But we have $(\overline{u_y})^0 \subseteq \overline{u_y}$, $(\overline{v_y})^0 \subseteq \overline{v_y}$ and since λ is almost $Q\alpha$ -compact, then M has a finite proximate $Q\alpha$ -subcover, say $\{ \overline{u_{y_k}} , \ \overline{v_{y_k}} : \ k \in J_n \}$ such that $\lambda(x)+\overline{u_{y_k}}(x) \geq \alpha$ for each $x \in X$ with $\lambda(x)=0$, for some $(\overline{u_{y_k}})^0 \in M$ and $\lambda(y)+\overline{v_{y_k}}(y) \geq \alpha$ for each $y \in X$ with $\lambda(y)>0$, for some $(\overline{v_{y_k}})^0 \in M$. Now, let $(\bar{v})^0 = (\overline{v_{y_1}})^0 \cup (\overline{v_{y_2}})^0 \cup \ldots.. \cup (\overline{v_{y_n}})^0$ and $(\bar{u})^0 = (\overline{u_{y_1}})^0 \cap (\overline{u_{y_2}})^0 \cap \ldots\ldots \cap (\overline{u_{y_n}})^0$. Thus we see that $(\bar{v})^0$ and $(\bar{u})^0$ are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $(\bar{v})^0$, $(\bar{u})^0 \in t$. Moreover, $\lambda_0 \subseteq (\bar{v})^{-1}(0,1]$ and $\bar{u}(x)=1$, as $\overline{u_{y_k}}(x)=1$ for each k.

Theorem 10.11: Let (X, t) be a fuzzy T_1-space (as def. 1.45) and λ , μ be disjoint almost $Q\alpha$ -compact fuzzy sets in X with λ_0 , $\mu_0 \subset X$. Then there exist u , $v \in t$ such that $\lambda_0 \subseteq (\overline{u})^{-1}(0, 1]$ and $\mu_0 \subseteq (\overline{v})^{-1}(0, 1]$.

Proof: Let $y \in \lambda_0$. Then we have $y \notin \mu_0$, as λ and μ are disjoint. Since μ is almost $Q\alpha$ -compact, then by theorem (10.10), there exist u_y , $v_y \in t$ such that $\overline{u_y}(y) = 1$ and $\mu_0 \subseteq (\overline{v_y})^{-1}(0, 1]$. Let us take $\alpha \in I_0$ such that $\lambda(x) + (\overline{v_y})^0(x) \geq \alpha$, $x \in X$ and $\lambda(y) + (\overline{u_y})^0(y) \geq \alpha$, $y \in \lambda_0$ i.e. say $M = \{(\overline{v_y})^0 , (\overline{u_y})^0 : y \in \lambda_0\}$ is an open $Q\alpha$ -cover of λ. But we have $(\overline{v_y})^0 \subseteq \overline{v_y}$ and $(\overline{u_y})^0 \subseteq \overline{u_y}$. As $\overline{u_y}(y) = 1$ and λ is almost $Q\alpha$ -compact in (X, t), then M has a finite proximate $Q\alpha$ -subcover, say $\{ \overline{v_{y_k}} , \overline{u_{y_k}} : k \in J_n \}$ such that $\lambda(x) + \overline{v_{y_k}}(x) \geq \alpha$ for each $x \in X$ with $\lambda(x) = 0$, for some $(\overline{v_{y_k}})^0 \in M$ and $\lambda(y) + \overline{u_{y_k}}(y) \geq \alpha$ for each $y \in X$ with $\lambda(y) > 0$, for some $(\overline{u_{y_k}})^0 \in M$. Again, since μ is almost $Q\alpha$ -compact in (X, t), then we have $\mu(x) + \overline{v_{y_k}}(x) \geq \alpha$ for each $x \in X$ with $\mu(x) > 0$, for some $(\overline{v_{y_k}})^0 \in M$ and $\mu(y) + \overline{u_{y_k}}(y) \geq \alpha$ for each $y \in X$ with $\mu(y) = 0$, for some $(\overline{u_{y_k}})^0 \in M$ and also $\mu_0 \subseteq (\overline{v_{y_k}})^{-1}(0, 1]$ for each k. Now, let $(\overline{u})^0 = (\overline{u_{y_1}})^0 \cup (\overline{u_{y_2}})^0 \cup \ldots\ldots \cup (\overline{u_{y_n}})^0$ and $(\overline{v})^0 = (\overline{v_{y_1}})^0 \cap (\overline{v_{y_2}})^0 \cap \ldots\ldots \cap (\overline{v_{y_n}})^0$. Thus we observe that $\lambda_0 \subseteq (\overline{u})^{-1}(0, 1]$ and $\mu_0 \subseteq (\overline{v})^{-1}(0, 1]$. Hence $(\overline{u})^0$ and $(\overline{v})^0$ are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $(\overline{u})^0$, $(\overline{v})^0 \in t$.

Remark: If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems (10.10) and (10.11) are not at all true.

The following example will show that the almost $Q\alpha$ -compact fuzzy sets in fuzzy T_1-space (as def. 1.45) need not be closed.

Example 10.12: Let $X = \{a, b\}$, $I = [0, 1]$ and $\alpha \in I_0$. Let u , $v \in I^X$ defined by $u(a) = 1$, $u(b) = 0$ and $v(a) = 0$, $v(b) = 1$. Put $t = \{ 0 , u , v , 1 \}$, then we see that (X, t) is a fuzzy T_1-space. Now, $0^c(a) = 1$, $0^c(b) = 1$; $u^c(a) = 0$, $u^c(b) = 1$ and $v^c(a) = 1$, $v^c(b) = 0$. So we have $\overline{u} = \bigcap \{ 0^c , v^c \} = v^c = u$ i.e. $\overline{u}(a) = 1$, $\overline{u}(b) = 0$ and $\overline{v} = \bigcap \{ 0^c , u^c \} = u^c = v$ i.e. $\overline{v}(a) = 0$, $\overline{v}(b) = 1$. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.3$, $\lambda(b) = 0.2$. Take $\alpha = 0.6$. Then clearly λ is almost $Q\alpha$ -compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 10.13: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and λ be a fuzzy set in X with $\lambda_0 \subset X$. If λ is almost $Q\alpha$ -compact in (X, t) and $x \notin \lambda_0$ ($\lambda(x) = 0$), then there exist u , $v \in t$ such that $\overline{u}(x) > 0$ and $\lambda_0 \subseteq \left(\overline{v}\right)^{-1}(0, 1]$. The converse is not true in general.

The proof is similar as that of theorem (10.10).

Now, for the converse, we give the following example.

Let $X = \{a, b\}$, $I = [0, 1]$ and $0 \le \alpha < 1$. Let u_1 , u_2 , $u_3 \in I^X$ defined by $u_1(a) = 0.2$, $u_1(b) = 0$; $u_2(a) = 0$, $u_2(b) = 0.3$ and $u_3(a) = 0.2$, $u_3(b) = 0.3$. Now, put $t = \{ 0 , u_1 , u_2 , u_3 , 1 \}$, then we see that (X, t) is a fuzzy T_1-space. Now we have, $0^c(a) = 1$, $0^c(b) = 1$; $u_1^c(a) = 0.8$, $u_1^c(b) = 1$; $u_2^c(a) = 1$, $u_2^c(b) = 0.7$ and $u_3^c(a) = 0.8$, $u_3^c(b) = 0.7$. Therefore $\overline{u_1} = \bigcap \{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c$ i.e. $\overline{u_1}(a) = 0.8$, $\overline{u_1}(b) = 0.7$; $\overline{u_2} = \bigcap \{ 0^c , u_1^c , u_2^c , u_3^c \} = u_3^c$ i.e. $\overline{u_2}(a) = 0.8$, $\overline{u_2}(b) = 0.7$ and $\overline{u_3} = \bigcap \{ 0^c , u_1^c ,$

u_2^c , u_3^c } $= u_3^c$ i.e. $\overline{u_3}(a) = 0.8$, $\overline{u_3}(b) = 0.7$. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.3$. Hence we observe that $\lambda_0 = \{b\}$ and $a \notin \lambda_0$. Here u_1 , $u_2 \in t$ where $\overline{u_1}(a) = 0.8 > 0$ and $\left(\overline{u_2}\right)^{-1}(0, 1] = \{a, b\}$. Hence $\lambda_0 \subset \left(\overline{u_2}\right)^{-1}(0, 1]$. Take $\alpha = 0.9$. Thus we see that λ is not almost $Q\alpha$ -compact in (X, t), as $\lambda(a) + \overline{u_k}(a) < \alpha$ for $a \in X$ and $k = 1$, 2 , 3. Thus the converse of the theorem is not true in general.

Theorem 10.14: Let (X, t) be a fuzzy T_1-space (as def. 1.46) and λ , μ be fuzzy sets in X with λ_0 , $\mu_0 \subset X$. If λ and μ are disjoint almost $Q\alpha$ -compacts in (X, t), then there exist u , $v \in t$ such that $\lambda_0 \subseteq \left(\overline{u}\right)^{-1}(0, 1]$ and $\mu_0 \subseteq \left(\overline{v}\right)^{-1}(0, 1]$.

The similar work as that of theorem (10.11).

Now, for the converse, consider the fuzzy T_1-space (X, t) in the example of the theorem (10.13). Let λ , $\mu \in I^X$ with $\lambda(a) = 0.3$, $\lambda(b) = 0$ and $\mu(a) = 0$, $\mu(b) = 0.1$. Thus we see that $\lambda_0 = \{a\}$ and $\mu_0 = \{b\}$. Now u_1 , $u_2 \in t$ where $\left(\overline{u_1}\right)^{-1}(0, 1] = \{a, b\}$ and $\left(\overline{u_2}\right)^{-1}(0, 1] = \{a, b\}$. Hence we observe that $\lambda_0 \subset \left(\overline{u_1}\right)^{-1}(0, 1]$ and $\mu_0 \subset \left(\overline{u_2}\right)^{-1}(0, 1]$, where λ and μ are disjoint. Take $\alpha = 0.9$. Hence we see that λ and μ are not almost $Q\alpha$ -compact in (X, t), as $\lambda(b) + \overline{u_k}(b) < \alpha$ for $b \in X$ and $\mu(a) + \overline{u_k}(a) < \alpha$ for $a \in X$ where $k = 1$, 2 , 3. Thus the converse of the theorem is not true in general.

The following example will show that the almost $Q\alpha$ -compact fuzzy sets in fuzzy T_1-space (as def. 1.46) need not be closed.

Example 10.15: Consider the fuzzy T_1-space (X, t) in the example of the theorem (10.13). Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.4$, $\lambda(b) = 0.8$. Take $\alpha = 0.9$. Clearly λ

is almost $Q\alpha$ -compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).

Theorem 10.16: An almost $Q\alpha$ -compact fuzzy sets in fuzzy regular space (as def. 1.52) is $Q\alpha$ -compact.

Proof: Let $\{ u_i : i \in J \}$ be an open $Q\alpha$ -cover of λ i.e. $\lambda(x) + u_i(x) \geq \alpha$ for each $x \in X$. As (X, t) is fuzzy regular, then we have $u_i = \bigcup v_{ij}$, where v_{ij} is an open fuzzy set such that $\overline{v_{ij}} \subseteq u_i$ for each i. But $\lambda(x) + u_i(x) \geq \alpha$ for each $x \in X$ $\Rightarrow$ $\lambda(x) + \bigcup_{i \in J} v_{ij}(x) \geq \alpha$ for each $x \in X$. Then $\lambda(x) + v_{ij}(x) \geq \alpha$ for each $x \in X$ and for some $i \in J$. So $\{ v_{ij} : i \in J \}$ is an open $Q\alpha$ -cover of λ. Since λ is almost $Q\alpha$ -compact, then $\{ v_{ij} : i \in J \}$ has a finite proximate $Q\alpha$ -subcover, say $\{ \overline{v_{i_k j}} : k \in J_n \}$ such that $\lambda(x) + \overline{v_{i_k j}}(x) \geq \alpha$ for each $x \in X$. But we have $\overline{v_{i_k j}} \subseteq u_{i_k}$, then $\lambda(x) + u_{i_k}(x) \geq \alpha$ for each $x \in X$. Therefore $\{ u_{i_k} : k \in J_n \}$ is a finite $Q\alpha$ -subcover of $\{ u_i : i \in J \}$ and hence λ is $Q\alpha$ -compact.

Theorem 10.17: Let (X, t) be an fts and λ be a fuzzy set in X. If λ_0 is compact in (X, t_α), then λ is almost $Q\alpha$ -compact in (X, t). The converse is not true in general.

Proof: Suppose λ_0 is compact in (X, t_α). Let $\{ u_i : i \in J \}$ be an open $Q\alpha$ -cover of λ in (X, t), then $\left\{ \left(\overline{u_i}\right)^0 : i \in J \right\}$ is also an open $Q\alpha$ -cover of λ in (X, t). So the family $\left\{ \alpha\left(\overline{u_i}\right)^0 : i \in J \right\}$ is an open cover of λ_0 in (X, t_α). But λ_0 is compact in (X, t_α), so $\left\{ \alpha\left(\overline{u_i}\right)^0 : i \in J \right\}$ has a finite subcover, say $\left\{ \alpha\left(\overline{u_{i_k}}\right)^0 : k \in J_n \right\}$. Thus $\left\{ \left(\overline{u_{i_k}}\right)^0 : k \in J_n \right\}$

forms a finite subfamily of $\left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$ such that $\lambda(x) + \overline{u_{i_k}}(x) \geq \alpha$ for each $x \in X$ i.e. $\{\overline{u_{i_k}} : k \in J_n\}$ is a finite proximate $Q\alpha$-subcover of $\{u_i : i \in J\}$. Hence λ is almost $Q\alpha$-compact in (X, t).

Now, for the converse, consider the example.

Let $X = \{a, b\}$, $I = [0, 1]$ and $0 \leq \alpha < 1$. Let u, $v \in I^X$ defined by $u(a) = 0.3$, $u(b) = 0.4$ and $v(a) = 0.5$, $v(b) = 0.6$. Put $t = \{0, u, v, 1\}$, then (X, t) is an fts. Now, $0^c(a) = 1$, $0^c(b) = 1$; $u^c(a) = 0.7$, $u^c(b) = 0.6$ and $v^c(a) = 0.5$, $v^c(b) = 0.4$. So we have $\bar{u} = \bigcap\{0^c, u^c, v^c\} = v^c$ i.e. $\bar{u}(a) = 0.5$, $\bar{u}(b) = 0.4$ and $\bar{v} = \bigcap\{0^c, u^c\} = u^c$ i.e. $\bar{v}(a) = 0.7$, $\bar{v}(b) = 0.6$. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.1$, $\lambda(b) = 0$. Then $\lambda_0 = \{a\}$. Take $\alpha = 0.5$. Then clearly λ is almost $Q\alpha$-compact in (X, t). Now we have $t_{0.5} = \{\phi, \{b\}, X\}$ and $(X, t_{0.5})$ is a 0.5-level topological space. Hence we observe that λ_0 is not compact in $(X, t_{0.5})$, as there is no finite subcover of λ_0 in $(X, t_{0.5})$.

The "good extension property" does not remain valid for almost $Q\alpha$-compact fuzzy sets.

Example 10.18: Let $X = \{a, b, c\}$ and $T = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. Then (X, T) is a topological space. Let u_1, u_2, $u_3 \in I^X$ with $u_1(a) = 0$, $u_1(b) = 0.6$, $u_1(c) = 0$; $u_2(a) = 0$, $u_2(b) = 0$, $u_2(c) = 0.3$ and $u_3(a) = 0$, $u_3(b) = 0.6$, $u_3(c) = 0.3$. Then $\omega(T) = \{0, u_1, u_2, u_3, 1\}$ and $(X, \omega(T))$ is an fts. Now $0^c(a) = 1$, $0^c(b) = 1$, $0^c(c) = 1$; $u_1^c(a) = 1$, $u_1^c(b) = 0.4$, $u_1^c(c) = 1$; $u_2^c(a) = 1$, $u_2^c(b) = 1$, $u_2^c(c) = 0.7$ and $u_3^c(a) = 1$, $u_3^c(b) = 0.4$, $u_3^c(c) = 0.7$. So we have $\overline{u_1} = \bigcap\{0^c,$

$u_2^c\} = u_2^c$ i.e. $\overline{u_1}(a) = 1$, $\overline{u_1}(b) = 1$, $\overline{u_1}(c) = 0.7$; $\overline{u_2} = \bigcap\{0^c, u_1^c, u_2^c, u_3^c\} = u_3^c$ i.e. $\overline{u_2}(a) = 1$, $\overline{u_2}(b) = 0.4$, $\overline{u_2}(c) = 0.7$ and $\overline{u_3} = \bigcap\{0^c, u_2^c\} = u_2^c$ i.e. $\overline{u_3}(a) = 1$, $\overline{u_3}(b) = 1$, $\overline{u_3}(c) = 0.7$. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0$, $\lambda(b) = 0.4$, $\lambda(c) = 0.1$. Then we have $\lambda_0 = \{b, c\}$. Clearly λ_0 is compact in (X, T). Take $\alpha = 0.9$. Then λ is not almost $Q\alpha$-compact in $(X, \omega(T))$, as there do not exists $\overline{u_k}$ for $k = 1, 2, 3$ such that $\lambda(c) + \overline{u_k}(c) \geq \alpha$. Again, let $\mu \in I^X$ defined by $\mu(a) = 0.4$, $\mu(b) = 0$, $\mu(c) = 0.4$. So we have $\mu_0 = \{a, c\}$. Then clearly μ is almost $Q\alpha$-compact in $(X, \omega(T))$. But $\mu_0 = \{a, c\}$ is not compact in (X, T), as there do not exist a finite subcover of μ_0 in (X, T). It is, therefore, observed that "good extension property" does not hold good for almost $Q\alpha$-compact fuzzy sets.

Theorem 10.19: Let λ and μ be almost $Q\alpha$-compact fuzzy sets in an fts (X, t). Then $(\lambda \times \mu)$ is also almost $Q\alpha$-compact in $(X \times X, t \times t)$.

Proof: Let $\{u_i \times v_i : i \in J\}$ be an open $Q\alpha$-cover of $(\lambda \times \mu)$ in $(X \times X, t \times t)$ i.e. $(\lambda \times \mu)(x, y) + (u_i \times v_i)(x, y) \geq \alpha$ for each $(x, y) \in X \times X$. Then clearly we have $\lambda(x) + u_i(x) \geq \alpha$ for each $x \in X$ and $\mu(y) + v_i(y) \geq \alpha$ for each $y \in X$. Therefore $\{u_i : i \in J\}$ and $\{v_i : i \in J\}$ are open $Q\alpha$-cover of λ and μ respectively. Then $\left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$ and $\left\{\left(\overline{v_i}\right)^0 : i \in J\right\}$ are also open $Q\alpha$-cover of λ and μ respectively. Since $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$, $\left(\overline{v_i}\right)^0 \subseteq \overline{v_i}$ and λ, μ are almost $Q\alpha$-compact, then $\left\{\left(\overline{u_i}\right)^0 : i \in J\right\}$ and $\left\{\left(\overline{v_i}\right)^0 : i \in J\right\}$ have finite proximate $Q\alpha$-subcover, say $\{\overline{u_{i_k}} : k \in J_n\}$ and $\{\overline{v_{i_k}} : k \in J_n\}$ such that $\lambda(x) + \overline{u_{i_k}}(x) \geq \alpha$ for each $x \in X$ and $\mu(y) + \overline{v_{i_k}}(y) \geq \alpha$ for each $y \in X$ respectively. Hence we can write $(\lambda \times \mu)(x,y) + \left(\overline{u_{i_k}} \times \overline{v_{i_k}}\right)(x,y) \geq \alpha$ for each $(x,y) \in X \times X$. Hence $(\lambda \times \mu)$ is almost $Q\alpha$ -compact in $(X \times X, t \times t)$.

Definition 10.20: Let (X,t) be an fts, λ be a fuzzy set in X and $0 < \delta \leq 1$, $0 < \alpha \leq 1$. Let $\{ u_i : i \in J \}$ be a family of δ-open fuzzy sets in (X,t). Then $\{ u_i : i \in J \}$ is proximate δ - $Q\alpha$ -cover of λ when $\{ \overline{u_i} : i \in J \}$ is δ - $Q\alpha$ -cover of λ i.e. $\lambda(x) + \overline{u_i}(x) \geq \alpha$ for each $x \in X$. A subfamily of $\{ u_i : i \in J \}$ which is also a proximate δ - $Q\alpha$ -cover of λ is said to be proximate δ - $Q\alpha$ -subcover of λ.

Definition 10.21: A fuzzy set λ is said to be almost δ - $Q\alpha$ -compact iff every δ - $Q\alpha$ -cover of λ has a finite subfamily whose closures is δ - $Q\alpha$ -cover of λ or equivalently, every δ - $Q\alpha$ -cover of λ has a finite proximate δ - $Q\alpha$ -subcover.

Every fuzzy supersets of an almost δ - $Q\alpha$ -compact fuzzy set is also almost δ - $Q\alpha$ -compact.

Theorem 10.22: Any almost δ - $Q\alpha$ -compact fuzzy set in an fts is almost $Q\alpha$ -compact. The converse is not true in general.

The proof of the theorem is straightforward.

Now, for the converse, consider the following example.

Let $X = \{a, b\}$, $I = [0,1]$ and $0 < \delta \leq 1$, $0 < \alpha \leq 1$. Let u_1, $u_2 \in I^X$ defined by $u_1(a) = 0.4$, $u_1(b) = 0.3$ and $u_2(a) = 0.5$, $u_2(b) = 0.6$. Now, take $t = \{ 0 , u_1 , u_2 , 1\}$, then we see that (X,t) is an fts. Now, $0^c(a) = 1$, $0^c(b) = 1$; $u_1^c(a) = 0.6$, $u_1^c(b) = 0.7$ and $u_2^c(a) = 0.5$, $u_2^c(b) = 0.4$. So we have $\overline{u_1} = \bigcap\{ 0^c , u_1^c , u_2^c \} = u_2^c$ i.e. $\overline{u_1}(a) = 0.5$,

$\overline{u_1}(b) = 0.4$ and $\overline{u_2} = \bigcap\{ 0^c , u_1^c \} = u_1^c$ i.e. $\overline{u_2}(a) = 0.6$, $\overline{u_2}(b) = 0.7$. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.7$, $\lambda(b) = 0.2$. Take $\alpha = 0.9$. Clearly λ is almost $Q\alpha$ -compact in (X, t). Take $\delta = 0.8$. Then we observe that there is no finite proximate δ - $Q\alpha$ -subcover of λ . Hence λ is not almost δ - $Q\alpha$ -compact in (X, t). Thus the converse of theorem is not necessarily true.

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