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Investigation of the Soliton and Multi-soliton Solutions of nonlinear evolution equations In Mathematical Physics

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INVESTIGATION OF THE SOLITON AND MULTI-SOLITON SOLUTIONS OF NONLINEAR EVOLUTION EQUATIONS IN MATHEMATICAL PHYSICS

A Dissertation

Submitted to the Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh for the Degree of Master of Philosophy in Mathematics

Submitted By

MD. BELAL HOSSEN

Examination Roll No.: 1711121501

Session: 2016-2017

To The

Department of Mathematics

University of Rajshahi

Rajshahi-6205

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Dedicated

To

My Parents

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BELOVED SON

Author's Declaration

I do hereby declare that this thesis entitled "Investigation of the Soliton and Multi-Soliton Solutions of Nonlinear Evolution Equations in Mathematical Physics" under the supervision of Prof. Dr. Md. Zulfikar Ali, Department of Mathematics, University of Rajshahi, Rajshahi-6205 for the degree of Master of Philosophy (M.Phil.) in Mathematics is an original work of mine. No part of this thesis is any form has been submitted anywhere for the award of any degree or diploma.

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CERTIFICATE

It is certified that the work entitled, "Investigation of the Soliton and Multi-Soliton Solutions of Nonlinear Evolution Equations in Mathematical Physics" contained in this thesis is original and carried out by Md. Belal Hossen under my supervision. I believe that the research work is an original one and it has not been submitted elsewhere for any kind of degree.

I wish him every success in life.

Supervisor

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(Md. Belal Hossen)

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Abstract

Nonlinear evolution equations (NLEEs) play a noteworthy role in various scientific and engineering fields such as applied mathematics, plasma physics, fluid dynamics, optical fibers, biology, solid state physics, chemical physics, mechanics and geochemistry. Various effective procedure have been developed to solve NLEEs. In this work, we have discussed applications of two types methods: first type is modified double sub-equation (MDSE) method which is apply in the (1+1)-dimensional Burger equation, the (1+1)-dimensional Gardner equation and the (1+1)-dimensional Hirota-Ramani equation and secondly, Hirota's Bilinear method which is apply in (2+1)-dimensional Breaking Soliton, the (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov equations, and (3+1)-D generalized B-type Kadomtsev-Petviashvili equation.

Using Modified double sub-equation method, we have presented some complexiton solutions in terms of trigonometric, hyperbolic functions. Finally, the interaction phenomena of the achieved complexiton solutions between solitary waves and/or periodic waves are presented with in depth derivation.

Based on the bilinear formalism and with the aid of symbolic computation, we determine multi-solitons, breather solutions, rogue wave, lump soliton, lump-kink waves and multi lumps using various ansatze's function. We notice that multi-lumps in the form of breathers visualize as a straight line. Besides this, the breather wave degenerate into a single lump wave is determined by using parametric limit scheme. Also, we reflect a new interaction solution among lump, kink and periodic waves via 'rational-cosh-cos' type test function. To realize dynamics, we commit diverse graphical analysis on the presented solutions. Obtained solutions are reliable in the mathematical physics and engineering.

Chapter One

Introduction

Nonlinear phenomena have an extensive application in different branches of mathematical physics and engineering. Basically all the fundamental equations of physics are nonlinear and, generally, such types of nonlinear evolution equations (NLEEs) are often very tough to solve clearly. The explicit solutions of NLEEs play a prominent role in the study of nonlinear science. In recent years, both mathematicians and physicists have devoted considerable effort to study of soliton solutions of nonlinear partial differential equations (PDEs) and a number of powerful methods were presented. For instance the inverse scattering theory [1], Darboux transformation [2], the Hirota's bilinear method [3,4], the sech-function method [5], the homogeneous balance method [6], Bäcklund transformation method [7], the hyperbolic tangent function series method [8,9], the sine-cosine method [10], the (G'/G)-expansion method [11,12], the multiple exp-function method [13], the Jacobi elliptic function expansion method [14,15]. These algebraic methods have the power to give a clear picture of the relation between different terms of nonlinear wave equations and are to simplify the routine calculation of the method. One of the most effectively straightforward methods to constructing exact solutions of PDEs is the sub-equation method [16-19]. The complexiton solution, firstly introduced by Ma et al. [20], can be constructed by the multiple Riccati equations rational expansion method [21], which make use of two Riccati equations with the same variable. Chen [22] has presented the double sub-equation method using two ordinary differential equations (ODEs) with different independent variables. Complexiton solutions are obtained by combining elementary functions and the Jacobi elliptic functions using double sub-equation method [22].

Another effectively direct method is Hirota's bilinear method [23-25] which is one of the most direct and convenient method to obtain the exact soliton solution of NLEEs. If a NLEE can attain its bilinear form, Lax pairs, lump solutions, multiple soliton solutions of this equation can be obtained [26-32]. Lately, we have seen two types of phenomena, two or more solitons may fuse to a single soliton and at a specific time, a single soliton may fission to two or more solitons. These types of scenarios were called as soliton fission and soliton fusion respectively [33]. Indeed, people have observed these types of phenomena in many nonlinear science and engineering field such as the gas dynamics, laser, plasma physics, electromagnetic, and passive random walker dynamics [34-36].Therefore, it very necessary to discuss about the elastic interactions into the solitary waves in certain integrable or non-integrable system with a strong physical backgrounds.

Recently, researchers are highly impressed to rogue wave solutions [37-38] for it's engrossing class of lump-type solutions, which can be found in plasma, shallow-water waves, nonlinear optics and Bose-Einstein condensates [39]. In 2002, Lou et al. studied the lump solution with the variable separation method [40]. Very recently, Ma et al. proposed the positive quadratic function to get the lump solution. Special examples of lump type solutions have been found, such as the KPI equation [41], Boussinesq equation [42], BKP equation [43] and so on. Lump solution [44-45] is a kind of rational function solution which is localized in all directions in the space whereas lump-type [46-47] solutions are localized in almost all directions in the space. Rogue waves [47-50] are localized in both space and time, arise from nowhere and disappear without a trace [51], have taken the responsibility for unexpected disaster in the world.

In this work, we implement the Modified double sub-equation (MDSE) method and a direct method called Hirota's bilinear method to find new and more general traveling wave solutions to some NLEEs namely the Burger equation, the Gardner equation, the Hirota-Ramani (HR) equation, the Breaking Soliton (BS) equation, the asymmetric Nizhnik-Novikov-Veselov (ANNV) equation and the generalized B-type Kadomtsev-Petviashvili (gBKP) equation.

Outline of this work, In Chapter one, we introduce the application of NLEEs in different branches of mathematical physics and engineering.

In Chapter two, we included the historical background /of Burger equation, the Gardner equation, the HR equation, BS equation, ANNV equation and gBKP equation with the help different methods.

In Chapter three, we explain the MDSE method and a direct method step by step and also explain the working procedure of this method to solve different type's nonlinear evolution equation.

In chapter four, we implement the Burger equation, the Gardner equation, the HR equation, BS equation, ANNV equation and gBKP equation. We obtain some traveling wave solution such as exponential, hyperbolic function solutions and trigonometric function solutions etc.

In Chapter five, we have discussed about the nature of the obtained traveling wave solution of various equations which are mentioned above. With the aid of direct symbolic computation, we explain these natures with 2-D, 3-D, Density and Contour graph.

Finally, we give some concluding remarks in the Chapter six.

Chapter Two

Literature Review of Some PDEs

In this chapter, we will discuss the literature review of some nonlinear evolution equations (NLEEs) such as (1+1)-D Burger, Gardner and Hirota-Ramani equations, (2+1)-D Breaking Soliton and asymmetric Nizhnik-Novikov-Veselov equations, and (3+1)-D generalized B-type Kadomtsev-Petviashvili equation.

2.1 The (1+1)-dimensional Burger Equation

Nonlinear evolution equations (NLEEs) play a noteworthy role in various scientific and engineering fields such as applied mathematics, plasma physics, fluid dynamics, optical fibers, biology, solid state physics, chemical physics, mechanics and geochemistry. Burger equation is one kind of Diffusion reaction model.

Let us consider the (1+1)-dimensional Burger equation [52-54], in the following form,

$$u_t + 2uu_x - u_{xx} = 0, \tag{2.1}$$

Burgers equation (2.1) is a model for nonlinear wave propagation, especially in fluid mechanics. The equation arises in various characteristic areas of applied mathematics, such as modeling of gas dynamics and traffic flow.

Burger equation [52-54] are solved by many researcher for finding complexiton solutions. On the other hand, Burgers equation with space-and time-fractional order and and time-fractional Boussinesq–Burger's equations [55-57] are solved for soliton solutions which arise in propagation of shallow water waves.

In this section, the modified double sub-equation method is proposed for constructing complexiton solutions of nonlinear partial differential equations (PDEs). We apply this method to the Burger's equation [52-54].

2.2 The (1+1)-dimensional Gardner equation (or combined KdV-mKdV)

In this section, the modified double sub-equation method is proposed for constructing complexiton solutions of nonlinear partial differential equations (PDEs). We apply this method to the Gardner equation.

Let us consider the (1+1)-dimensional Gardner equation (or combined KdV-mKdV) [58-60], in the form

$$u_t + b_1 u u_x + b_2 u^2 u_x + b_3 u_{xxx} = 0, \qquad (2.2)$$

where $u = u(x,t)$ and b_1, b_2, b_3 are arbitrary constants. The Gardner equation has two nonlinear terms in the quadratic and cubic forms and the dissipative term is of third order. This is an significant model to realize the propagation of negative ion acoustic plasma waves [60] and can be derived from the structure of plasma motion equations in one dimension with arbitrarily charged cold ions and inertia neglected isothermal electrons. This equation can also be a good explanation of internal waves with large amplitudes [61].

2.3 The (1+1)-dimensional Hirota-Ramani equation

Nonlinear evolution equations (NLEEs) play a notable role in scientific and engineering fields such as mathematics, biology, mechanics, physics and geochemistry. Now a day's many mathematicians and physicists are engaged in the study of soliton solutions of nonlinear partial differential equations (PDEs).

We study the (1+1)-dimensional Hirota-Ramani equation [62-66], in the form

$$u_t - u_{xxt} + \alpha u_x (1 - u_t) = 0, \qquad (2.3)$$

where α is a nonzero real constant. There are many researchers discussed about Hirota-Ramani equation in diverse technique such as, Ji discussed above equation by Exp-function method [63], Konprasert et al discussed the various types exact solution of Hirota-Ramani equation using F-expansion process [64], Reza et al discussed some phenomena of above equation by (G′/G)-Expansion Technique [65]. Recently, Roshid et al studied above equation by direct rational exponential method to describe it's multi soliton phenomena [66].

2.4 The (2+1)-dimensional Breaking Soliton equation

In this section, we study the (2+1)-dimensional Breaking Soliton (BS) equation [67-69] reads as

$$\begin{cases} u_t + \alpha u_{xxy} + 4\alpha u v_x + 4\alpha u_x v = 0, \\ u_y = v_x. \end{cases} \tag{2.4}$$

where α is arbitrary constant. There are many researchers have been studied in Breaking soliton equation (BSE) in many ways such as: Zhang formed nontraveling wave solutions to BSE by a generalized auxiliary equation method [68], Mei investigated general solution of BSE using the projective Riccati equation expansion method [69], Peng solved BSE by the singular manifold method [70], and Dai derived BSE chaotic behaviors by the mapping method [71]. The structures of (2 + 1)-dimensional BSE are rich and there are still more structures to be discovered.

In this paper, we will focus on the (2+1)-dimensional Breaking Soliton (BS) equation to show the diversity of such interaction solutions aid of symbolic computation with Maple. The (2+1)-dimensional BS equation has a Hirota bilinear form, and so, we will do a search for positive quadratic function solutions to the corresponding (2+1)-dimensional bilinear BS equation. The obtained quadratic function solutions contain a set of free parameters, and taking special choices of parameters involved.

2.5 The (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov equation

In this part, we will consider the (2+1)-dimensional ANNV equation [72,73],

$$u_t + u_{xxx} + 3[uv]_x = 0; \qquad u_x = v_y. \tag{2.5}$$

where u and v are the components of the (dimensionless) velocity [74]. Eq. (2.5) is the only known isotropic Lax extension of the Korteweg-de Vries equation [75]. The ANNV equation has important applications in incompressible fluids, such as shallow-water waves, long internal waves and acoustic waves. There are many researchers have been studied in ANNV equation in many ways such as: Boiti et al. solved via the inverse scattering transformation [76]. Guo et al. discussed the N-soliton solution and Pfaffian expression by using a nonlinearized method of Lax pair [72], Osman et al. solved this system of equations via the unified and generalized unified method [77-80]. Also, ANNV equations can also be obtained from the inner parameter-dependent symmetry constraint of the KP equation [81].

The main purpose of this paper is to employ some proficient ansatzes to determine lump solution, lump-kink wave and multi-lump wave solutions and their dynamics for the above (2+1)-dimensional ANNV equation.

2.6 The (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili equation

Recently, finding accurate collision solutions of nonlinear partial differential equations (NLPDEs) is an essential issue in soliton theory. In recent years, scientists have been investing their research effort to study of soliton solution of NLEEs.

In this section, we study (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili (gBKP) equation is introduced to describe the dynamics of solitons and nonlinear waves in the field of fluid dynamics, plasma physics etc. Let us consider the (3+1) dimensional generalized B-type Kadomtsev-Petviashvili equation [84] in the following form:

$$u_{yt} + 3u_{xz} - 3u_x u_{xy} - 3u_{xx} u_y - u_{xxxy} = 0. \tag{2.6}$$

Several researchers studied on the gBKP equation (2.6) in many ways such as: Ma and Zhu [85] explored multiple wave solutions of Eq. (2.6) via the multiple exp-function scheme. Liu et. al. [86] presented new exact non-traveling wave solutions by exploitation of the generalized (G'/G)-expansion method. Ma [87] construct N-soliton solutions of Eq. (2.6) via the Hirota method. Recently, Cao [84] presented only lump wave solutions of the model.

Chapter Three

ALGORITHMS

In this Chapter, we will give a short overview of the Modified Double Sub-Equation Method and Hirota's bilinear method.

3.1 Description of the Modified Double Sub-Equation Method

In the following, we described the main steps of modified double sub-equation method.

Step 1: Consider a nonlinear partial differential equation (NLPDE), say in two independent variables x and t, is given by

$$\Re(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, ...) = 0 \tag{3.1}$$

where $u = u(x,t)$ is an unknown function, $\Re$ is a polynomial of $u = u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved.

Step 2: For the suggested method, we assume that the solutions of Eq. (3.1) are as follows:

$$u(x,t) = a_0 + \frac{a_1\varphi(\xi) + a_2\psi(\eta)}{\lambda_0 + \lambda_1\varphi(\xi)\psi(\eta)} \tag{3.2}$$

where $a_i(i = 0,1,2), \xi$ and η are all functions of x and t, λ_0 and λ_1 are arbitrary nonzero constants to be determined later. The new functions $\varphi(\xi)$ and $\psi(\eta)$ satisfy

$$\varphi'(\xi) = \frac{d\varphi(\xi)}{d\xi} = q_1 + p_1\varphi^2(\xi) \tag{3.3}$$

and $\psi'(\eta) = \dfrac{d\psi(\eta)}{d\eta} = q_2 + p_2\psi^2(\eta)$, (3.4)

where $\xi = k_1 x + w_1 t$ and $\eta = k_2 x + w_2 t$ respectively, which are known as wave transformation of Eq. (1).

Step 3: The general solutions of the Riccati Eq. (3.3,3.4) [21] are as follows:

$$\frac{d\varphi(\xi)}{d\xi} = q_1 + p_1\varphi^2(\xi)$$

i. When $q_1 = 1, p_1 = -1,$

$$\varphi(\xi) = \tanh(\xi), \varphi(\xi) = \coth(\xi), \tag{3.5}$$

ii. When $q_1 = p_1 = \pm\dfrac{1}{2},$

$$\varphi(\xi)=\sec(\xi)\pm\tan(\xi), \varphi(\xi)=\csc(\xi)\pm\cot(\xi), \tag{3.6}$$

iii. When $q_1 = p_1 = 1,$

$$\varphi(\xi)=\tan(\xi), \tag{3.7}$$

iv. When $q_1 = p_1 = -1,$

$$\varphi(\xi)=\cot(\xi) \tag{3.8}$$

v. When $q_1 = \frac{1}{2}, p_1 = -\frac{1}{2},$

$$\varphi(\xi)=\tanh(\xi)\pm i\,\mathrm{sec}h(\xi), \varphi(\xi)=\coth(\xi)\pm \mathrm{csc}h(\xi), \tag{3.9}$$

vi. When $q_1 = 0, p_1 = 1,$

$$\varphi(\xi)=-\frac{1}{p_1\xi+w}, \tag{3.10}$$

Step 4: By setting Eq. (3.2) into Eq. (3.1) along with Eq. (3.3) and Eq. (3.4) yields a system of equations with respect to $\varphi^m\psi^n, (m=0,1,2,\ldots,n=0,1,2,\ldots)$ then set all coefficients of $\varphi^m\psi^n$ in the obtained system of equations to be zero, we obtain a set of over-determined PDEs with respect to $a_0, a_1, a_2, k_1, w_1, k_2, w_2, \lambda_0$ and λ_1.

By solving the over-determined PDEs with the aid of symbolic computation system Maple, we obtain the subsequent solution in terms of $a_0, a_1, a_2, k_1, w_1, k_2, w_2, \lambda_0, \lambda_1$. Using the results obtained in the above steps and the various solutions of Eq. (3.3, 3.4), we can derive many solutions for Eq. (3.1).

3.2 Description of the Hirota's Bilinear Method

In this subsection, we briefly described the main features of Hirota's bilinear method that will be used in this work. Firstly, we substitute

$$\Omega(x,y,t)=e^{mx+ny+wt} \tag{3.11}$$

into the linear terms of any differential equation under discussion to determine the dispersion relation among m, n and w. Secondly, substitute the Cole–Hopf transformation

$$\Omega(x,y,t)=P\ln(\psi(x,y,t))_{xx}. \tag{3.12}$$

into the equation under discussion, where the auxiliary function $\psi(x,y,t)$ is given by

$$\psi(x,y,t)=1+B_1\psi_1(x,y,t)=1+B_1e^{\theta_i} \tag{3.13}$$

Where $\theta_i = m_i x + n_i y + w_i t, \quad i=1,2,3,\ldots\ldots.,N$

and solving the resulting equation with the aid of symbolic computation system Maple, to determine the numerical value for P. Notice that the N-soliton solutions can be gained by using the following forms for $\psi(x,y,t)$ into (3.12):

The steps of the Hirota's bilinear method [4] are as follows:

(i) For dispersion relation, we use

$$\Omega(x,y,t)=e^{\theta_i},\ \theta_i=m_i x+n_i y+w_i t. \tag{3.14}$$

(ii) For single soliton, we use

$$\psi(x,y,t)=1+e^{\theta_i} \tag{3.15}$$

(iii) For two-soliton solutions, we use

$$\psi(x,y,t)=1+e^{\theta_1}+e^{\theta_2}+B_{12}e^{\theta_i+\theta_2} \tag{3.16}$$

(iv) For three-soliton solutions, we use

$$\psi(x,y,t)=1+e^{\theta_1}+e^{\theta_2}+e^{\theta_3}+B_{12}e^{\theta_i+\theta_2}+B_{23}e^{\theta_2+\theta_3}+B_{13}e^{\theta_i+\theta_3}+B_{123}e^{\theta_i+\theta_2+\theta_3} \tag{3.17}$$

Notice that we use Eq. (3.14) to determine the dispersion relation, Eq. (3.16) to determine the phase shift B_{12} to be generalized for the other factors B_{ij}, and finally we use Eq. (3.17) to determine B_{123}, which is given by $B_{123}=B_{12}B_{23}B_{13}$ for completely integrable equations. The determination of three-soliton solutions confirms the fact that N-soliton solutions exist for any order.

Chapter Four

Applications of the Modified Double Sub-Equation and Direct method

In this chapter, we will discuss the applications of the Modified Double Sub-Equation (MDSE) method in (1+1)-D Burger, Gardner and Hirota-Ramani equations and Direct method named Hirota's Bilinear method in (2+1)-D Breaking Soliton and asymmetric Nizhnik-Novikov-Veselov equations, and (3+1)-D generalized B-type Kadomtsev-Petviashvili equation.

4.1 The (1+1)-dimensional Burger equation

In this sub-section, we will generate many new types of complexiton solutions combining elementary functions and the Jacobi elliptic functions using MDSE method. It makes the modified double sub-equation method more thoroughly.

To establish validity and effectiveness of our method, we handle this method in the (1+1)-dimensional Burger equation. Let us consider the Burger equation [52-54], in the following form:

$$u_t + 2uu_x - u_{xx} = 0, \tag{4.1}$$

According to the method, we assume that the solutions of Eq. (4.1) are as follows:

$$u(x,t) = b_0 + \frac{b_1\varphi(\xi) + b_2\psi(\eta)}{b_3 + b_4\varphi(\xi)\psi(\eta)}, \tag{4.2}$$

where b_i, $(i = 0,1,2,3,4)$, $\xi = k_1x + w_1t$ and $\eta = k_2x + w_2t$ are arbitrary nonzero constants. Substituting Eq. (4.2) into Eq. (4.1) along with Eq. (3.3) and Eq. (3.4) yields a system of equations with respect to $\varphi^m\psi^n, (m = 0,1,2,\ldots, n = 0,1,2,\ldots)$, then set all coefficients of $\varphi^m\psi^n$ in the obtained system of equations to be zero, we obtain a set of over-determined PDEs with respect to b_i, $(i = 0,1,2,3,4)$, k_1, w_1, k_2, w_2.

Solving the over-determined PDEs by use of Maple, we can obtain the following results.

Case 1.

$$\begin{cases} b_0 = b_0, b_1 = 0, b_2 = -b_4k_1q_1, b_3 = 0, b_4 = b_4, \\ w_1 = -2b_0k_1, w_2 = w_2. \end{cases} \tag{4.3}$$

Case 2.

$$\begin{cases} b_0 = b_0, b_1 = -b_4 k_2 q_2, b_2 = 0, b_3 = 0, b_4 = b_4, \\ w_1 = w_1, w_2 = -2b_0 k_2. \end{cases} \tag{4.4}$$

Case 3.

$$\begin{cases} b_0 = b_0, b_1 = 0, b_2 = b_3 k_2 p_2, b_3 = b_3, b_4 = 0, \\ w_1 = w_1, w_2 = -2b_0 k_2. \end{cases} \tag{4.5}$$

Case 4.

$$\begin{cases} b_0 = b_0, b_1 = b_1, b_2 = 0, b_3 = \dfrac{q_1}{k_1 p_1}, b_4 = 0, \\ w_1 = -2b_0 k_1, w_2 = w_2. \end{cases} \tag{4.6}$$

Case 5.

$$\begin{cases} b_0 = b_0, b_1 = b_2\gamma, b_2 = b_2, b_3 = \dfrac{b_2(k_2 q_2 + k_1 q_1 \gamma)}{p_1 q_1 k_1^2 + p_2 q_2 k_2^2 + 2k_1 q_1 k_2 p_2 \gamma}, \\ b_4 = \dfrac{p_1 b_2 (k_2 q_2 + k_1 q_1 \gamma)}{q_1 (p_1 q_1 k_1^2 \gamma + p_2 q_2 k_2^2 \gamma + 2k_1 p_1 k_2 q_2)}, w_1 = -2b_0 k_1, w_2 = -2b_0 k_2. \end{cases} \tag{4.7}$$

Case 6.

$$\begin{cases} b_0 = b_0, b_1 = b_2\gamma, b_2 = b_2, b_3 = \dfrac{b_2(\gamma k_1 q_1 + k_2 q_2)}{p_1 q_1 k_1^2 + p_2 q_2 k_2^2 + 2k_1 q_1 k_2 p_2 \gamma}, \\ b_4 = -\dfrac{p_1 b_2 (\gamma k_1 q_1 + k_2 q_2)}{q_1 (p_1 q_1 k_1^2 \gamma + p_2 q_2 k_2^2 \gamma + 2k_1 p_1 k_2 q_2)}, w_1 = w_1, \\ w_2 = -\dfrac{w_1 p_1 + 2b_0 \gamma k_2 p_2 + 2b_0 k_1 p_1}{p_2 \gamma}. \end{cases} \tag{4.8}$$

where $\gamma = \sqrt{\dfrac{p_1 q_2}{p_2 q_1}}$.

Note that: Since the solutions obtained here are so many with complexitons and without complexitons, we just write some new and complexiton solutions for the Burgers equation to demonstrate the effectiveness of our method.

Using (4.7), one can get various types of complexiton solutions of Eq. (4.1) as follows:

Family-1: When $b_0 = b_2 = const., q_1 = 1, p_1 = -1$, then we can get some complexiton solutions:

i. When $q_2 = p_2 = \dfrac{1}{2}$, then

$$u_{1,2} = a_0 \mp (\frac{1}{2}k_2 \pm Ik_1)\left\{\frac{I\tanh(\xi) \mp (\sec(\eta) - \tan(\eta)}{1 \pm I\tanh(\xi)(\sec(\eta) - \tan(\eta))}\right\}$$

$$u_{3,4} = a_0 \mp (\frac{1}{2}k_2 \pm Ik_1)\left\{\frac{I\tanh(\xi) \pm (\csc(\eta) - \cot(\eta)}{1 \mp I\tanh(\xi)(\csc(\eta) - \cot(\eta)}\right\}$$

$$u_{5,6} = a_0 \mp (\frac{1}{2}k_2 \pm Ik_1)\left\{\frac{I\coth(\xi) \mp (\sec(\eta) - \tan(\eta)}{1 \pm I\coth(\xi)(\sec(\eta) - \tan(\eta)}\right\}$$

$$u_{7,8} = a_0 \mp (\frac{1}{2}k_2 \pm Ik_1)\left\{\frac{I\coth(\xi) \pm (\csc(\eta) - \cot(\eta)}{1 \mp I\coth(\xi)(\csc(\eta) - \cot(\eta)}\right\}$$

ii. When $q_2 = p_2 = -\frac{1}{2}$, then

$$u_{9,10} = a_0 \pm (\frac{1}{2}k_2 \mp Ik_1)\left\{\frac{I\tanh(\xi) \mp (\sec(\eta) + \tan(\eta)}{1 \pm I\tanh(\xi)(\sec(\eta) + \tan(\eta)}\right\}$$

$$u_{11,12} = a_0 \pm (\frac{1}{2}k_2 \mp Ik_1)\left\{\frac{I\tanh(\xi) \pm (\csc(\eta) + \cot(\eta)}{1 \mp I\tanh(\xi)(\csc(\eta) + \cot(\eta)}\right\}$$

$$u_{13,14} = a_0 \pm (\frac{1}{2}k_2 \mp Ik_1)\left\{\frac{I\coth(\xi) \mp (\sec(\eta) + \tan(\eta)}{1 \pm I\coth(\xi)(\sec(\eta) + \tan(\eta)}\right\}$$

$$u_{15,16} = a_0 \pm (\frac{1}{2}k_2 \mp Ik_1)\left\{\frac{I\coth(\xi) \pm (\csc(\eta) + \cot(\eta)}{1 \mp I\coth(\xi)(\csc(\eta) + \cot(\eta)}\right\}$$

iii. When $q_2 = p_2 = 1$, then

$$u_{17,18} = a_0 \mp (k_2 \pm Ik_1)\left\{\frac{I\tanh(\xi) \pm \tan(\eta)}{1 \mp I\tanh(\xi)\tan(\eta)}\right\}$$

$$u_{19,20} = a_0 \mp (k_2 \pm Ik_1)\left\{\frac{I\coth(\xi) \pm \tan(\eta)}{1 \mp I\coth(\xi)\tan(\eta)}\right\}$$

iv. When $q_2 = p_2 = -1$, then

$$u_{21,22} = a_0 \pm (k_2 \mp Ik_1)\left\{\frac{I\tanh(\xi) \pm \cot(\eta)}{1 \mp I\tanh(\xi)\cot(\eta)}\right\}$$

$$u_{23,24} = a_0 \pm (k_2 \mp Ik_1)\left\{\frac{I\coth(\xi) \pm \cot(\eta)}{1 \mp I\coth(\xi)\cot(\eta)}\right\}$$

v. When $q_2 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$, then

$$u_{25,26}=a_0\pm(\frac{1}{2}k_2\pm k_1)\left\{\frac{\tanh(\xi)\pm\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}{1\pm\tanh(\xi)\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{27,28}=a_0\pm(\frac{1}{2}k_2\pm k_1)\left\{\frac{\tanh(\xi)\pm\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}{1\pm\tanh(\xi)\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{29,30}=a_0\pm(\frac{1}{2}k_2\pm k_1)\left\{\frac{\tanh(\xi)\pm\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}{1\pm\tanh(\xi)\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}\right\}$$

$$u_{31,32}=a_0\pm(\frac{1}{2}k_2\pm k_1)\left\{\frac{\tanh(\xi)\pm\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}{1\pm\tanh(\xi)\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}\right\}$$

$$u_{33,34}=a_0\pm(\frac{1}{2}k_2\pm k_1)\left\{\frac{\coth(\xi)\pm\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}{1\pm\coth(\xi)\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{35,36}=a_0\pm(\frac{1}{2}k_2\pm k_1)\left\{\frac{\coth(\xi)\pm\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}{1\pm\coth(\xi)\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{37,38}=a_0\pm(\frac{1}{2}k_2\pm k_1)\left\{\frac{\coth(\xi)\pm\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}{1\pm\coth(\xi)\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}\right\}$$

$$u_{39,40}=a_0\pm(\frac{1}{2}k_2\pm k_1)\left\{\frac{\coth(\xi)\pm\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}{1\pm\coth(\xi)\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}\right\}$$

where $\xi = k_1 x + w_1 t$ and $\eta = k_2 x + w_2 t,\ w_1 = -2b_0 k_1, w_2 = -2b_0 k_2$.

Family-2: When $b_0 = b_2 = const., q_1 = p_1 = \pm\frac{1}{2}$, then we can get some complexiton solutions:

i. When $q_2 = p_2 = 1$, then

$$u_{41,42} = a_0 \mp (\frac{1}{2}k_2 \pm k_1)\left\{\frac{\tan(\xi) \mp (\sec(\eta) - \tan(\eta))}{1 \pm \tan(\xi)(\sec(\eta) - \tan(\eta))}\right\}$$

$$u_{43,44} = a_0 \mp (\frac{1}{2}k_2 \pm k_1)\left\{\frac{\tan(\xi) \pm (\csc(\eta) - \cot(\eta))}{1 \mp \tan(\xi)(\csc(\eta) - \cot(\eta))}\right\}$$

$$u_{45,46} = a_0 \pm (\frac{1}{2}k_2 \mp k_1)\left\{\frac{\tan(\xi) \mp (\sec(\eta) + \tan(\eta))}{1 \pm \tan(\xi)(\sec(\eta) + \tan(\eta))}\right\}$$

$$u_{47,48} = a_0 \pm (\frac{1}{2}k_2 \mp k_1)\left\{\frac{\tan(\xi) \pm (\csc(\eta) + \cot(\eta))}{1 \mp \tan(\xi)(\csc(\eta) + \cot(\eta))}\right\}$$

where $\xi = k_1 x + w_1 t$ and $\eta = k_2 x + w_2 t,\ w_1 = -2b_0 k_1, w_2 = -2b_0 k_2$.

Family-3: When $b_0 = b_2 = const., q_1 = p_1 = 1$, then we can get some complexiton solutions:

i. When $q_2 = p_2 = -1$, then

$$u_{49,50} = a_0 \pm (k_2 \mp k_1)\left\{\frac{\tan(\xi) \pm \cot(\eta)}{1 \mp \tan(\xi)\cot(\eta)}\right\}$$

ii. When $q_2 = \frac{1}{2}, p_2 = -\frac{1}{2}$, then

$$u_{51,52} = a_0 \pm (\frac{1}{2}k_2 \pm Ik_1)\left\{\frac{I\tan(\xi) \pm \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}{1 \pm I\tan(\xi)\left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{53,54} = a_0 \pm (\frac{1}{2}k_2 \pm Ik_1)\left\{\frac{I\tan(\xi) \pm \left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}{1 \pm I\tan(\xi)\left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{55,56}=a_0\pm(\frac{1}{2}k_2\pm Ik_1)\left\{\frac{I\tan(\xi)\pm\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}{1\pm I\tan(\xi)\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}\right\}$$

$$u_{57,58}=a_0\pm(\frac{1}{2}k_2\pm Ik_1)\left\{\frac{I\tan(\xi)\pm\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}{1\pm I\tan(\xi)\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}\right\}$$

where $\xi=k_1x+w_1t$ and $\eta=k_2x+w_2t,\ w_1=-2b_0k_1, w_2=-2b_0k_2$.

Again, using (4.8), one can get various types of complexiton solutions of Eq. (4.1) as follows:

Family-1: When $b_0=b_2=w_1=const., q_1=1, p_1=-1$, then we can get some complexiton solutions:

i. When $q_2=p_2=\frac{1}{2}$, then

$$u_{59,60}=a_0\mp(\frac{1}{2}k_2\pm Ik_1)\left\{\frac{I\tanh(\xi)\mp(\sec(\eta)-\tan(\eta)}{1\pm I\tanh(\xi)(\sec(\eta)-\tan(\eta))}\right\}$$

$$u_{61,62}=a_0\mp(\frac{1}{2}k_2\pm Ik_1)\left\{\frac{I\tanh(\xi)\pm(\csc(\eta)-\cot(\eta)}{1\mp I\tanh(\xi)(\csc(\eta)-\cot(\eta)}\right\}$$

$$u_{63,64}=a_0\mp(\frac{1}{2}k_2\pm Ik_1)\left\{\frac{I\coth(\xi)\mp(\sec(\eta)-\tan(\eta)}{1\pm I\coth(\xi)(\sec(\eta)-\tan(\eta)}\right\}$$

$$u_{65,66}=a_0\mp(\frac{1}{2}k_2\pm Ik_1)\left\{\frac{I\coth(\xi)\pm(\csc(\eta)-\cot(\eta)}{1\mp I\coth(\xi)(\csc(\eta)-\cot(\eta)}\right\}$$

ii. When $q_2=p_2=-\frac{1}{2}$, then

$$u_{67,68}=a_0\mp(\frac{1}{2}k_2\pm Ik_1)\left\{\frac{I\tanh(\xi)\pm(\sec(\eta)+\tan(\eta)}{1\mp I\tanh(\xi)(\sec(\eta)+\tan(\eta)}\right\}$$

$$u_{69,70}=a_0\mp(\frac{1}{2}k_2\pm Ik_1)\left\{\frac{I\tanh(\xi)\mp(\csc(\eta)+\cot(\eta)}{1\pm I\tanh(\xi)(\csc(\eta)+\cot(\eta)}\right\}$$

$$u_{71,72}=a_0\mp(\frac{1}{2}k_2\pm Ik_1)\left\{\frac{I\coth(\xi)\pm(\sec(\eta)+\tan(\eta)}{1\mp I\coth(\xi)(\sec(\eta)+\tan(\eta)}\right\}$$

$$u_{73,74}=a_0\mp(\frac{1}{2}k_2\pm Ik_1)\left\{\frac{I\coth(\xi)\mp(\csc(\eta)+\cot(\eta))}{1\pm I\coth(\xi)(\csc(\eta)+\cot(\eta))}\right\}$$

iii. When $q_2=p_2=1$, then

$$u_{75,76}=a_0\mp(k_2\pm Ik_1)\left\{\frac{I\tanh(\xi)\pm\tan(\eta)}{1\mp I\tanh(\xi)\tan(\eta)}\right\}$$

$$u_{77,78}=a_0\mp(k_2\pm Ik_1)\left\{\frac{I\coth(\xi)\pm\tan(\eta)}{1\mp I\coth(\xi)\tan(\eta)}\right\}$$

iv. When $q_2=p_2=-1$, then

$$u_{79,80}=a_0\mp(k_2\pm Ik_1)\left\{\frac{I\tanh(\xi)\mp\cot(\eta)}{1\pm I\tanh(\xi)\cot(\eta)}\right\}$$

$$u_{81,82}=a_0\mp(k_2\pm Ik_1)\left\{\frac{I\coth(\xi)\mp\cot(\eta)}{1\pm I\coth(\xi)\cot(\eta)}\right\}$$

v. When $q_2=\frac{1}{2}, p_2=-\frac{1}{2}$, then

$$u_{83,84}=a_0\mp(\frac{1}{2}k_2\mp k_1)\left\{\frac{\tanh(\xi)\mp\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}{1\mp\tanh(\xi)\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{85,86}=a_0\mp(\frac{1}{2}k_2\mp k_1)\left\{\frac{\tanh(\xi)\mp\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}{1\mp\tanh(\xi)\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{87,88}=a_0\mp(\frac{1}{2}k_2\mp k_1)\left\{\frac{\tanh(\xi)\mp\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}{1\mp\tanh(\xi)\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}\right\}$$

$$u_{89,90}=a_0\mp(\frac{1}{2}k_2\mp k_1)\left\{\frac{\tanh(\xi)\mp\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}{1\mp\tanh(\xi)\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}\right\}$$

$$u_{91,92}=a_0\mp(\frac{1}{2}k_2\mp k_1)\left\{\frac{\coth(\xi)\mp\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}{1\mp\coth(\xi)\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{93,94}=a_0\mp(\frac{1}{2}k_2\mp k_1)\left\{\frac{\coth(\xi)\mp\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}{1\mp\coth(\xi)\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{95,96}=a_0\mp(\frac{1}{2}k_2\mp k_1)\left\{\frac{\coth(\xi)\mp\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}{1\mp\coth(\xi)\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}\right\}$$

$$u_{97,98}=a_0\mp(\frac{1}{2}k_2\mp k_1)\left\{\frac{\coth(\xi)\mp\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}{1\mp\coth(\xi)\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}\right\}$$

Where $\xi=k_1x+w_1t$ and $\eta=k_2x+w_2t,\ w_1=w_1, w_2=-\dfrac{w_1p_1+2b_0\left(\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}\right)k_2p_2+2b_0k_1p_1}{p_2\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}}$.

Family-2: When $b_0=b_2=w_1=const., q_1=p_1=\pm\frac{1}{2}$, then we can get some complexiton solutions:

i. when $q_2=p_2=1$, then

$$u_{99,100}=a_0\mp(\frac{1}{2}k_2\pm k_1)\left\{\frac{\tan(\xi)\mp(\sec(\eta)-\tan(\eta))}{1\pm\tan(\xi)(\sec(\eta)-\tan(\eta))}\right\}$$

$$u_{101,102}=a_0\mp(\frac{1}{2}k_2\pm k_1)\left\{\frac{\tan(\xi)\pm(\csc(\eta)-\cot(\eta))}{1\mp\tan(\xi)(\csc(\eta)-\cot(\eta))}\right\}$$

$$u_{103,104}=a_0\mp(\frac{1}{2}k_2\pm k_1)\left\{\frac{\tan(\xi)\pm(\sec(\eta)+\tan(\eta))}{1\mp\tan(\xi)(\sec(\eta)+\tan(\eta))}\right\}$$

$$u_{105,106} = a_0 \mp (\frac{1}{2}k_2 \pm k_1)\left\{\frac{\tan(\xi) \mp (\csc(\eta) + \cot(\eta))}{1 \pm \tan(\xi)(\csc(\eta) + \cot(\eta))}\right\}$$

ii. when $q_2 = p_2 = -1$, then

$$u_{107,108} = a_0 \mp (k_2 \pm \frac{1}{2}k_1)\left\{\frac{(\sec(\xi) + \tan(\xi)) \mp \cot(\eta)}{1 \pm (\sec(\xi) + \tan(\xi))\cot(\eta)}\right\}$$

iii. when $q_2 = \frac{1}{2}, p_2 = -\frac{1}{2}$, then

$$u_{109,110} = a_0 \mp (\frac{1}{2}k_2 \mp \frac{1}{2}Ik_1)\left\{\frac{I(\csc(\xi) - \cot(\xi)) \mp \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}{1 \mp I(\csc(\xi) - \cot(\xi))\left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}\right\}$$

Where $\xi = k_1 x + w_1 t$ and $\eta = k_2 x + w_2 t,\ w_1 = w_1, w_2 = -\dfrac{w_1 p_1 + 2b_0\left(\dfrac{\sqrt{p_1 q_1 p_2 q_2}}{p_2 q_1}\right)k_2 p_2 + 2b_0 k_1 p_1}{p_2 \dfrac{\sqrt{p_1 q_1 p_2 q_2}}{p_2 q_1}}$.

Family-3: When $b_0 = b_2 = w_1 = const., q_1 = p_1 = 1$, then we can get some complexiton solutions:

i. when $q_2 = p_2 = -1$, then

$$u_{111,112} = a_0 \mp (k_2 \pm k_1)\left\{\frac{\tan(\xi) \mp \cot(\eta)}{1 \pm \tan(\xi)\cot(\eta)}\right\}$$

ii. when $q_2 = \frac{1}{2}, p_2 = -\frac{1}{2}$, then

$$u_{113,114} = a_0 \mp (\frac{1}{2}k_2 \mp Ik_1)\left\{\frac{I\tan(\xi) \mp \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}{1 \mp I\tan(\xi)\left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{115,116} = a_0 \mp (\frac{1}{2}k_2 \mp Ik_1)\left\{\frac{I\tan(\xi) \mp \left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}{1 \mp I\tan(\xi)\left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{117,118} = a_0 \mp (\frac{1}{2}k_2 \mp Ik_1)\left\{\frac{I\tan(\xi) \mp \left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)}{1 \mp I\tan(\xi)\left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)}\right\}$$

$$u_{119,120} = a_0 \mp (\frac{1}{2}k_2 \mp Ik_1)\left\{\frac{I\tan(\xi) \mp \left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}{1 \mp I\tan(\xi)\left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}\right\}$$

where $\xi = k_1 x + w_1 t$ and $\eta = k_2 x + w_2 t,\ w_1 = w_1, w_2 = -\dfrac{w_1 p_1 + 2b_0\left(\dfrac{\sqrt{p_1 q_1 p_2 q_2}}{p_2 q_1}\right)k_2 p_2 + 2b_0 k_1 p_1}{p_2 \dfrac{\sqrt{p_1 q_1 p_2 q_2}}{p_2 q_1}}$.

Family-4: When $b_0 = b_2 = w_1 = const., q_1 = p_1 = -1$, and $q_2 = \frac{1}{2}, p_2 = -\frac{1}{2}$,then we can get a complexiton solution:

$$u_{121,122} = a_0 \pm (\frac{1}{2}k_2 \mp Ik_1)\left\{\frac{I\cot(\xi) \pm \left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}{1 \pm I\cot(\xi)\left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}\right\}$$

where $\xi = k_1 x + w_1 t$ and $\eta = k_2 x + w_2 t,\ w_1 = w_1, w_2 = -\dfrac{w_1 p_1 + 2b_0\left(\dfrac{\sqrt{p_1 q_1 p_2 q_2}}{p_2 q_1}\right)k_2 p_2 + 2b_0 k_1 p_1}{p_2 \dfrac{\sqrt{p_1 q_1 p_2 q_2}}{p_2 q_1}}$.

Similarly, we can write down the other complexiton solution of Eq. (4.1) which are omitted for convenience.

4.2 The (1+1)-dimensional Gardner equation (or combined KdV-mKdV)

In this sub-section, we will generate many new types of complexiton solutions combining elementary functions and the Jacobi elliptic functions using MDSE method.

To establish validity and effectiveness of our method, we handle this method in the (1+1)-dimensional

Gardner equation (or combined KdV-mKdV) equation. Let us consider the Gardner equation [58-60], in the following form:

$$u_t + b_1 u u_x + b_2 u^2 u_x + b_3 u_{xxx} = 0, \tag{4.9}$$

where $u = u(x,t)$ and b_1, b_2, b_3 are arbitrary nonzero constants.

According to the method, we assume that the solutions of Eq. (4.9) are as follows:

$$u(x,t) = a_0 + \frac{a_1 \varphi(\xi) + a_2 \psi(\eta)}{a_3 + a_4 \varphi(\xi)\psi(\eta)}, \tag{4.10}$$

where a_i, $(i = 0,1,2,3,4)$, $\xi = k_1 x + w_1 t$ and $\eta = k_2 x + w_2 t$ are arbitrary nonzero constants.

Substituting Eq. (4.10) into Eq. (4.9) along with Eq. (3.3) and Eq. (3.4) yields a system of equations with respect to $\varphi^m \psi^n, (m = 0,1,2,\ldots, n = 0,1,2,\ldots)$, then set all coefficients of $\varphi^m \psi^n$ in the obtained system of equations to be zero, we obtain a set of over-determined PDEs with respect to a_i, $(i = 0,1,2,3,4)$, k_1, w_1, k_2, w_2.

Solving the over-determined PDEs by use of Maple, we can obtain the following results.

Case 1.

$$\begin{cases} a_0 = -\dfrac{1}{2}\dfrac{b_1}{b_2}, a_1 = 0, a_2 = \Delta_1 q_1 k_1 a_4, a_3 = 0, a_4 = a_4, \\ w_1 = \dfrac{1}{4}\Delta_2, w_2 = w_2. \end{cases} \tag{4.11}$$

Case 2.

$$\begin{cases} a_0 = -\dfrac{1}{2}\dfrac{b_1}{b_2}, a_1 = 0, a_2 = \Delta_1 a_3 k_2 p_2, a_3 = a_3, a_4 = 0, \\ w_1 = w_1, w_2 = \dfrac{1}{4}\Delta_3. \end{cases} \tag{4.12}$$

Case 3.

$$\begin{cases} a_0 = -\dfrac{1}{2}\dfrac{b_1}{b_2}, a_1 = \Delta_1 k_2 q_2 a_4, a_2 = 0, a_3 = 0, a_4 = a_4, \\ w_1 = w_1, w_2 = \dfrac{1}{4}\Delta_3. \end{cases} \tag{4.13}$$

Case 4.

$$\begin{cases} a_0 = -\dfrac{1}{2}\dfrac{b_1}{b_2}, a_1 = \Delta_1 a_3 k_1 p_1, a_2 = 0, a_3 = a_3, a_4 = 0, \\ w_1 = \dfrac{1}{4}\Delta_2, w_2 = w_2. \end{cases} \tag{4.14}$$

Case 5.

$$\begin{cases} a_0 = -\dfrac{1}{2}\dfrac{b_1}{b_2}, a_1 = \gamma a_4, a_2 = \dfrac{-12\gamma a_4 p_2 b_3 q_2 k_1 q_1 k_2}{(6b_3 p_2 q_2^2 k_2^2 + p_2 b_2 \gamma^2 + 6q_2 q_1 k_1^2 p_1 b_3)}, \\ a_3 = \dfrac{12 b_3 a_4 q_2^2 k_1 q_1 k_2}{(6b_3 p_2 q_2^2 k_2^2 + p_2 b_2 \gamma + 6q_2 q_1 k_1^2 p_1 b_3)}, a_4 = a_4, \\ w_1 = \dfrac{1}{4}\dfrac{\begin{array}{l} k_1(-8p_2 b_2^2 k_1^2 \gamma^2 b_3 p_1 q_1 + p_2 b_2 \gamma^2 b_1^2 - 24 p_2^2 b_2^2 q_2 \gamma^2 b_3 k_2^2 + 48 b_3 q_2^2 p_2 b_2 k_2 w_2 \\ - 6b_3 q_2^2 p_2 b_1^2 k_2^2 - \\ 48 b_3^2 q_2^2 p_2^2 b_2 k_2^4 + 96 b_3^2 q_2^2 p_2 b_2 p_1 q_1 k_2^2 - 48 b_3^2 q_2 k_1^4 b_2 q_1^2 p_1^2 + 6 b_3 q_2 k_1^2 b_1^2 q_1 p_1) \end{array}}{b_2(6b_3 p_2 q_2^2 + p_2 b_2 \gamma^2 + 6q_1 k_1^2 p_1 q_2 b_3)}, \\ w_2 = w_2. \end{cases} \tag{4.15}$$

where, $\Delta_1 = \dfrac{\sqrt{-6b_2 b_3}}{b_2}, \Delta_2 = \dfrac{k_1(-8b_3 p_1 k_1^2 q_1 b_2 + b_1^2)}{b_2}, \Delta_3 = \dfrac{k_2(-8b_3 p_2 k_2^2 q_2 b_2 + b_1^2)}{b_2}$ and

$$\gamma = \frac{\sqrt{-6 p_2 b_2 b_3 q_2 (q_2 k_2^2 p_2 + q_1 k_1^2 p_1 + 2k_1 k_2 \sqrt{p_2 p_1 q_1 q_2})}}{p_2 b_2}.$$

Note that: Since the solutions obtained here are so many with complexitons and without complexitons, we just write complexiton solutions for the (1+1)-dimensional Gardner equation (or combined KdV-mKdV) equation.

Using (4.15), one can get various types of complexiton solutions of Eq. (4.9) as follows:

Family-1: When $b_0 = b_2 = w_1 = const., q_1 = 1, p_1 = -1$, then we can get some complexiton solutions:

i. When $q_2 = p_2 = \dfrac{1}{2}$, then

$$u_{1,2} = a_0 + \frac{\left\{\pm \dfrac{1}{b_2}\left(\alpha_1 a_4 \tanh(k_1 x + \dfrac{1}{4\beta_1}\delta_1 t) \mp \dfrac{3\alpha_1 a_4 b_3 k_1 k_2 (\sec(\eta) + \tan(\eta))}{b_2 \beta_1}\right)\right\}}{\left(\dfrac{3 b_3 a_4 k_1 k_2}{\dfrac{3}{4} b_3 k_2^2 - 3k_1^2 b_3 \pm \dfrac{1}{2}\alpha_1} + a_4 \tanh(k_1 x + \dfrac{1}{4\beta_1}\delta_1 t)\right)(\sec(\eta) + \tan(\eta))}$$

$$u_{3,4}=a_0+\frac{\left\{\pm\frac{1}{b_2}\left(\alpha_1a_4\tanh(k_1x+\frac{1}{4\beta_1}\delta_1t)\mp\frac{3\alpha_1a_4b_3k_1k_2\left(\csc(\eta)-\cot(\eta)\right)}{b_2\beta_1}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_1}+a_4\tanh(k_1x+\frac{1}{4\beta_1}\delta_1t)\right)\left(\csc(\eta)-\cot(\eta)\right)}$$

$$u_{5,6}=a_0+\frac{\left\{\pm\frac{1}{b_2}\left(\alpha_1a_4\coth(k_1x+\frac{1}{4\beta_1}\delta_1t)\mp\frac{3\alpha_1a_4b_3k_1k_2\left(\sec(\eta)+\tan(\eta)\right)}{b_2\beta_1}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_1}+a_4\coth(k_1x+\frac{1}{4\beta_1}\delta_1t)\right)\left(\sec(\eta)+\tan(\eta)\right)}$$

$$u_{7,8}=a_0+\frac{\left\{\pm\frac{1}{b_2}\left(\alpha_1a_4\coth(k_1x+\frac{1}{4\beta_1}\delta_1t)\mp\frac{3\alpha_1a_4b_3k_1k_2\left(\csc(\eta)-\cot(\eta)\right)}{b_2\beta_1}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_1}+a_4\coth(k_1x+\frac{1}{4\beta_1}\delta_1t)\right)\left(\csc(\eta)-\cot(\eta)\right)}$$

where, $a_o=-\frac{1}{2}\frac{b_1}{b_2},\alpha_1=\sqrt{-6b_2b_3\left(\frac{1}{2}k_2+Ik_1\right)^2},\beta_1=\frac{3}{4}b_3k_2^2-3k_1^2b_3-3b_3\left(\frac{1}{2}k_2+Ik_1\right)^2$ and

$$\delta_1=k_1(-24b_2k_1^2b_3^2\left(\frac{1}{2}k_2+Ik_1\right)^2-3b_3\left(\frac{1}{2}k_2+Ik_1\right)^2b_1^2+18b_2b_3^2\left(\frac{1}{2}k_2+Ik_1\right)^2k_2^2$$
$$+6b_3b_2k_2w_2-\frac{3}{4}b_3b_1^2k_2^2-\frac{3}{2}b_3^2b_2k_2^4-12b_3^2k_1^2b_2k_2^2-24b_3^2k_1^4b_2-3b_3k_1^2b_1^2)$$

ii. When $q_2=p_2=-\frac{1}{2}$, then

$$u_{9,10}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_2a_4\tanh(k_1x+\frac{1}{4\beta_2}\delta_2t)\pm\frac{3\alpha_2a_4b_3k_1k_2\left(\sec(\eta)-\tan(\eta)\right)}{b_2\beta_2}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{-\frac{3}{4}b_3k_2^2+3k_1^2b_3\pm\frac{1}{2}\alpha_2}+a_4\tanh(k_1x+\frac{1}{4\beta_1}\delta_1t)\right)\left(\sec(\eta)-\tan(\eta)\right)}$$

$$u_{11,12}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_2 a_4\tanh(k_1x+\frac{1}{4\beta_2}\delta_2 t)\pm\frac{3\alpha_2 a_4 b_3 k_1 k_2\left(\csc(\eta)+\cot(\eta)\right)}{b_2\beta_2}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{-\frac{3}{4}b_3k_2^2+3k_1^2b_3\pm\frac{1}{2}\alpha_2}+a_4\tanh(k_1x+\frac{1}{4\beta_1}\delta_1 t)\right)\left(\csc(\eta)+\cot(\eta)\right)}$$

$$u_{13,14}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_2 a_4\coth(k_1x+\frac{1}{4\beta_2}\delta_2 t)\pm\frac{3\alpha_2 a_4 b_3 k_1 k_2\left(\sec(\eta)-\tan(\eta)\right)}{b_2\beta_2}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{-\frac{3}{4}b_3k_2^2+3k_1^2b_3\pm\frac{1}{2}\alpha_2}+a_4\coth(k_1x+\frac{1}{4\beta_1}\delta_1 t)\right)\left(\sec(\eta)-\tan(\eta)\right)}$$

$$u_{15,16}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_2 a_4\coth(k_1x+\frac{1}{4\beta_2}\delta_2 t)\pm\frac{3\alpha_2 a_4 b_3 k_1 k_2\left(\csc(\eta)+\cot(\eta)\right)}{b_2\beta_2}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{-\frac{3}{4}b_3k_2^2+3k_1^2b_3\pm\frac{1}{2}\alpha_2}+a_4\coth(k_1x+\frac{1}{4\beta_1}\delta_1 t)\right)\left(\csc(\eta)+\cot(\eta)\right)}$$

where, $a_o=-\frac{1}{2}\frac{b_1}{b_2},\alpha_2=\sqrt{-6b_2b_3\left(\frac{1}{2}k_2+Ik_1\right)^2},\beta_2=-\frac{3}{4}b_3k_2^2+3k_1^2b_3+3b_3\left(\frac{1}{2}k_2+Ik_1\right)^2$ and

$$\delta_2=k_1(24b_2k_1^2b_3^2\left(\frac{1}{2}k_2+Ik_1\right)^2+3b_3\left(\frac{1}{2}k_2+Ik_1\right)^2b_1^2-18b_2b_3^2\left(\frac{1}{2}k_2+Ik_1\right)^2k_2^2$$
$$-6b_3b_2k_2w_2+\frac{3}{4}b_3b_1^2k_2^2+\frac{3}{2}b_3^2b_2k_2^4+12b_3^2k_1^2b_2k_2^2+24b_3^2k_1^4b_2+3b_3k_1^2b_1^2)$$

iii. When $q_2=p_2=1$, then

$$u_{17,18}=a_0+\frac{\left\{\pm\frac{1}{b_2}\left(\alpha_3 a_4\tanh(k_1x+\frac{1}{4\beta_3}\delta_3 t)\mp\frac{12\alpha_3 a_4 b_3 k_1 k_2\tan(\eta)}{b_2\beta_1}\right)\right\}}{\left(\frac{12b_3a_4k_1k_2}{6b_3k_2^2-6k_1^2b_3\pm\alpha_3}+a_4\tanh(k_1x+\frac{1}{4\beta_3}\delta_3 t)\right)\tan(\eta)}$$

$$u_{19,20}=a_0+\frac{\left\{\pm\frac{1}{b_2}\left(\alpha_3 a_4\coth(k_1x+\frac{1}{4\beta_3}\delta_3 t)\mp\frac{12\alpha_3 a_4 b_3 k_1 k_2\tan(\eta)}{b_2\beta_1}\right)\right\}}{\left(\frac{12b_3a_4k_1k_2}{6b_3k_2^2-6k_1^2b_3\pm\alpha_3}+a_4\coth(k_1x+\frac{1}{4\beta_3}\delta_3 t)\right)\tan(\eta)}$$

where, $a_o = -\frac{1}{2}\frac{b_1}{b_2}, \alpha_3 = \sqrt{-6b_2b_3(k_2+Ik_1)^2}, \beta_3 = 6b_3k_2^2 - 6k_1^2b_3 - 6b_3(k_2+Ik_1)^2$ and

$$\delta_3 = k_1(-48b_2k_1^2b_3^2(k_2+Ik_1)^2 - 6b_3(k_2+Ik_1)^2b_1^2 + 144b_2b_3^2(k_2+Ik_1)^2k_2^2 + 48b_3b_2k_2w_2 - 6b_3b_1^2k_2^2 - 48b_3^2b_2k_2^4 - 96b_3^2k_1^2b_2k_2^2 - 48b_3^2k_1^4b_2 - 6b_3k_1^2b_1^2)$$

iv. When $q_2 = p_2 = -1$, then

$$u_{21,22} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_4 a_4 \tanh(k_1x + \frac{1}{4\beta_4}\delta_4 t) \pm \frac{12\alpha_3 a_4 b_3 k_1 k_2 \cot(\eta)}{b_2\beta_1}\right)\right\}}{\left(\frac{12b_3a_4k_1k_2}{-6b_3k_2^2 + 6k_1^2b_3 \pm \alpha_4} + a_4\tanh(k_1x + \frac{1}{4\beta_4}\delta_4 t)\right)\cot(\eta)}$$

$$u_{23,24} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_4 a_4 \coth(k_1x + \frac{1}{4\beta_4}\delta_4 t) \pm \frac{12\alpha_3 a_4 b_3 k_1 k_2 \cot(\eta)}{b_2\beta_1}\right)\right\}}{\left(\frac{12b_3a_4k_1k_2}{-6b_3k_2^2 + 6k_1^2b_3 \pm \alpha_4} + a_4\coth(k_1x + \frac{1}{4\beta_4}\delta_4 t)\right)\cot(\eta)}$$

where, $a_0 = -\frac{1}{2}\frac{b_1}{b_2}, \alpha_4 = \sqrt{-6b_2b_3(k_2+Ik_1)^2}, \beta_4 = -6b_3k_2^2 + 6k_1^2b_3 + 6b_3(k_2+Ik_1)^2$ and

$$\delta_4 = k_1(48b_2k_1^2b_3^2(k_2+Ik_1)^2 + 6b_3(k_2+Ik_1)^2b_1^2 - 144b_2b_3^2(k_2+Ik_1)^2k_2^2 - 48b_3b_2k_2w_2 + 6b_3b_1^2k_2^2 + 48b_3^2b_2k_2^4 + 96b_3^2k_1^2b_2k_2^2 + 48b_3^2k_1^4b_2 + 6b_3k_1^2b_1^2)$$

v. When $q_2 = \frac{1}{2}, p_2 = -\frac{1}{2}$, then

$$u_{25,26} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4 \tanh(k_1x + \frac{1}{4\beta_5}\delta_5 t) \mp \frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{-\frac{3}{4}b_3k_2^2 - 3k_1^2b_3 \pm \frac{1}{2}\alpha_5} + a_4\tanh(k_1x + \frac{1}{4\beta_5}\delta_5 t)\right)\left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}$$

$$u_{27,28}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4\tanh(k_1x+\frac{1}{4\beta_5}\delta_5 t)\mp\frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{-\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_5}+a_4\tanh(k_1x+\frac{1}{4\beta_5}\delta_5 t)\right)\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}$$

$$u_{29,30}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4\tanh(k_1x+\frac{1}{4\beta_5}\delta_5 t)\mp\frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{-\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_5}+a_4\tanh(k_1x+\frac{1}{4\beta_5}\delta_5 t)\right)\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}$$

$$u_{31,32}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4\tanh(k_1x+\frac{1}{4\beta_5}\delta_5 t)\mp\frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{-\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_5}+a_4\tanh(k_1x+\frac{1}{4\beta_5}\delta_5 t)\right)\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}$$

$$u_{33,34}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4\coth(k_1x+\frac{1}{4\beta_5}\delta_5 t)\mp\frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{-\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_5}+a_4\coth(k_1x+\frac{1}{4\beta_5}\delta_5 t)\right)\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}$$

$$u_{35,36}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4\coth(k_1x+\frac{1}{4\beta_5}\delta_5 t)\mp\frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_5}+a_4\coth(k_1x+\frac{1}{4\beta_5}\delta_5 t)\right)\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}$$

$$u_{37,38}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4\coth(k_1x+\frac{1}{4\beta_5}\delta_5 t)\mp\frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_5}+a_4\coth(k_1x+\frac{1}{4\beta_5}\delta_5 t)\right)\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}$$

$$u_{39,40}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4\coth(k_1x+\frac{1}{4\beta_5}\delta_5 t)\mp\frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_5}+a_4\coth(k_1x+\frac{1}{4\beta_5}\delta_5 t)\right)\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}$$

where, $a_0=-\frac{1}{2}\frac{b_1}{b_2}, \alpha_5=\sqrt{-6b_2b_3\left(\frac{1}{2}k_2-k_1\right)^2}, \beta_5=-\frac{3}{4}b_3k_2^2-3k_1^2b_3+3b_3\left(\frac{1}{2}k_2-k_1\right)^2$ and

$$\delta_5=k_1(24b_2k_1^2b_3^2\left(\frac{1}{2}k_2-k_1\right)^2+3b_3\left(\frac{1}{2}k_2-k_1\right)^2b_1^2+18b_2b_3^2\left(\frac{1}{2}k_2-k_1\right)^2k_2^2$$

$$-6b_3b_2k_2w_2+\frac{3}{4}b_3b_1^2k_2^2-\frac{3}{2}b_3^2b_2k_2^4+12b_3^2k_1^2b_2k_2^2-24b_3^2k_1^4b_2-3b_3k_1^2b_1^2)$$

with $\eta=k_2x+w_2t$ and

$$w_1 = \frac{1}{4}\left(\frac{\begin{array}{l} k_1(-8p_2b_2^2k_1^2\gamma^2b_3p_1q_1 + p_2b_2\gamma^2b_1^2 - 24p_2^2b_2^2q_2\gamma^2b_3k_2^2 + 48b_3q_2^2p_2b_2k_2w_2 \\ -6b_3q_2^2p_2b_1^2k_2^2 - 48b_3^2q_2^2p_2^2b_2k_2^4 + \\ 96b_3^2q_2^2p_2b_2p_1q_1k_2^2 - 48b_3^2q_2k_1^4b_2q_1^2p_1^2 + 6b_3q_2k_1^2b_1^2q_1p_1) \end{array}}{b_2(6b_3p_2q_2^2 + p_2b_2\gamma^2 + 6q_1k_1^2p_1q_2b_3)} \right).$$

Similarly, we can write down the other complexiton solution of Eq. (4.9) which are omitted for convenience.

4.3 The (1+1)-dimensional Hirota-Ramani equation

Complexiton solution gives various types of wave speed which are produced by mix-up of trigonometric and hyperbolic functions. Modifed double sub-equation (MDSE) technique is a advantageous and practical tool to attain system of complexiton solutions of nonlinear evolution equations.

In this part, we have studied MDSE method to create a complexiton system solution of (1+1) Dimensional Hirota-Ramani equation [62-66], in the form

$$u_t - u_{xxt} + \alpha u_x(1-u_t) = 0, \tag{4.16}$$

where $u(x,t)$ is the amplitude of the relevant wave mode and a $\alpha \neq 0$ is a real constant. Hirota-Ramani equation is broadly used in several branches of physics, and such as plasma physics, fluid physics, and quantum field theory. It also pronounces a variation of wave phenomena in plasma and solid state [62].

According to the method, we assume that the solutions of Eq. (4.16) are as follows:

$$u(x,t) = a_0 + \frac{a_1\,\varphi(\xi) + a_2\,\psi(\eta)}{a_3 + a_4\,\varphi(\xi)\psi(\eta)} \tag{4.17}$$

where a_i, $(i = 0,1,2,3,4)$, $\xi = k_1x + w_1t$ and $\eta = k_2x + w_2t$ are arbitrary nonzero constants.

Substituting Eq. (4.17) into Eq. (4.16) along with Eq. (3.3) and (3.4) yields a system of equations with respect to $\varphi^m\psi^n$, $(m = 0,1,2,....,n = 0,1,2,....)$, then set all coefficients of $\varphi^m\psi^n$ in the obtained system of equations to be zero, we obtain a set of over-determined PDEs with respect to a_i, $(i = 0,1,2,3,4)$, k_1, w_1, k_2, w_2.

Solving the over-determined PDEs by use of Maple, we can obtain the following results.

Case 1.

$$\left\{\alpha=-\frac{w_2\left(4k_2^2p_2q_2+1\right)}{k_2},\ a_1=-\frac{6a_4k_2^2q_2}{w_2\left(4k_2^2p_2q_2+1\right)},\ a_2=0,\ a_3=0\right. \tag{4.18}$$

Case 2.

$$\left\{\alpha=-\frac{w_2\left(4k_2^2p_2q_2+1\right)}{k_2},\ a_1=0,\ a_2=\frac{6a_3k_2^2p_2}{w_2\left(4k_2^2p_2q_2+1\right)},\ a_4=0\right. \tag{4.19}$$

Case 3.

$$\left\{\begin{array}{l}\alpha=-\frac{6a_4\left(-k_2q_2+k_1p_1\sqrt{\Delta}\right)}{a_1},\ a_2=-\frac{p_2\sqrt{\Delta}\ a_1}{q_2},\ a_3=\sqrt{\Delta}\,a_4,\\ w_2=-\frac{\sigma\sqrt{\Delta}-8q_2q_1p_1p_2a_1w_1k_1k_2-6q_2a_4k_1^2p_1q_1-6a_4k_2^2q_2^2p_2}{q_2p_2a_1\left(8p_1p_2k_2k_1\sqrt{\Delta}-4p_2k_2^2q_2-1-4p_1k_1^2q_1\right)}\end{array}\right., \tag{4.20}$$

where $\Delta=\frac{q_1q_2}{p_1p_2}$ and $\sigma=p_1p_2\,a_1w_1+12k_1p_1p_2k_2a_4q_2+4p_1^2\,p_2a_1w_1k_1^2q_1+4p_1p_2^2a_1w_1k_2^2q_2$.

Using (4.20), one can get various types of complexiton solutions of Eq. (4.16) as follows:

Family-1: When $a_0=a_1=a_4=w_1=const.$, $q_1=1$, $p_1=-1$, then we can get some complexiton solutions:

i. When $q_2=p_2=\frac{1}{2}$, then

$$u_1=a_0-\frac{I\{\sec(2\vartheta_1t-k_2x)-I\tan(2\vartheta_1t-k_2x)\}-\tanh(xk_1+t)}{\{\sec(2\vartheta_1t-k_2x)-I\tan(2\vartheta_1t-k_2x)\}\tanh(xk_1+t)+I}$$

$$u_2=a_0+\frac{I\{\sec(2\vartheta_2t-k_2x)-I\tan(2\vartheta_2t-k_2x)\}-\tanh(xk_1+t)}{\{\sec(2\vartheta_2t-k_2x)-I\tan(2\vartheta_2t-k_2x)\}\tanh(xk_1+t)-I}$$

where $\vartheta_1=\frac{-I\left(1+6k_1k_2-4k_1^2+k_2^2\right)+4k_1k_2+6k_1^2-\frac{3}{2}k_2^2}{-4Ik_1k_2-k_2^2-1+4k_1^2}$ and

$$\vartheta_2=\frac{I\left(1+6k_1k_2-4k_1^2+k_2^2\right)+4k_1k_2+6k_1^2-\frac{3}{2}k_2^2}{4Ik_1k_2-k_2^2-1+4k_1^2}$$

ii. When $q_2=p_2=1$, then

$$u_3 = a_0 + \frac{2I\tan\left(\frac{1}{2}\frac{t\left(2I + 24Ik_1k_2 + 8Ik_1^2 + 8Ik_2^2 + 16k_1k_2 + 12k_1^2 - 12k_2^2\right)}{8Ik_1k_2 - 4k_2^2 - 1 + 4k_1^2} - k_2x\right) + 2\tanh(xk_1 + t)}{-2\tan\left(\frac{1}{2}\frac{t\left(2I + 24Ik_1k_2 + 8Ik_1^2 + 8Ik_2^2 + 16k_1k_2 + 12k_1^2 - 12k_2^2\right)}{8Ik_1k_2 - 4k_2^2 - 1 + 4k_1^2} - k_2x\right)\tanh(xk_1 + t) + 2I}$$

$$u_4 = a_0 + \frac{-2I\tan\left(\frac{1}{2}\frac{t\left(-2I - 24Ik_1k_2 - 8Ik_1^2 - 8Ik_2^2 + 16k_1k_2 + 12k_1^2 - 12k_2^2\right)}{-8Ik_1k_2 - 4k_2^2 - 1 + 4k_1^2} - k_2x\right) + 2\tanh(xk_1 + t)}{-2\tan\left(\frac{1}{2}\frac{t\left(-2I - 24Ik_1k_2 - 8Ik_1^2 - 8Ik_2^2 + 16k_1k_2 + 12k_1^2 - 12k_2^2\right)}{-8Ik_1k_2 - 4k_2^2 - 1 + 4k_1^2} - k_2x\right)\tanh(xk_1 + t) - 2I}$$

iii. When $q_2 = p_2 = -1$, then

$$u_5 = a_0 + \frac{2I\cot\left(-\frac{1}{2}\frac{t\left(2I + 24Ik_1k_2 + 8Ik_1^2 + 8Ik_2^2 + 16k_1k_2 - 12k_1^2 + 12k_2^2\right)}{8Ik_1k_2 - 4k_2^2 - 1 + 4k_1^2} + k_2x\right) + 2\tanh(xk_1 + t)}{2\cot\left(-\frac{1}{2}\frac{t\left(2I + 24Ik_1k_2 + 8Ik_1^2 + 8Ik_2^2 + 16k_1k_2 - 12k_1^2 + 12k_2^2\right)}{8Ik_1k_2 - 4k_2^2 - 1 + 4k_1^2} + k_2x\right)\tanh(xk_1 + t) + 2I}$$

$$u_6 = a_0 + \frac{-2I\cot\left(-\frac{1}{2}\frac{t\left(-2I - 24Ik_1k_2 - 8Ik_1^2 - 8Ik_2^2 + 16k_1k_2 - 12k_1^2 + 12k_2^2\right)}{-8Ik_1k_2 - 4k_2^2 - 1 + 4k_1^2} + k_2x\right) + 2\tanh(xk_1 + t)}{2\cot\left(-\frac{1}{2}\frac{t\left(-2I - 24Ik_1k_2 - 8Ik_1^2 - 8Ik_2^2 + 16k_1k_2 - 12k_1^2 + 12k_2^2\right)}{-8Ik_1k_2 - 4k_2^2 - 1 + 4k_1^2} + k_2x\right)\tanh(xk_1 + t) - 2I}$$

iv. When $q_2 = \frac{1}{2},\ p_2 = -\frac{1}{2}$, then

$$u_7 = a_0 + \frac{2\tanh(\Delta_1 t + k_2x) + 2I\operatorname{sec}h(\Delta_1 t + k_2x) + 2\tanh(xk_1 + t)}{2\left(\tanh(\Delta_1 t + k_2x) + I\operatorname{sec}h(\Delta_1 t + k_2x)\right) + \tanh(xk_1 + t) + 2}$$

$$u_8 = a_0 + \frac{2\tanh(xk_1 + t) - 2\tanh(\Delta_2 t + k_2x) - 2I\operatorname{sec}h(\Delta_2 t + k_2x)}{2\left(\tanh(\Delta_2 t + k_2x) + I\operatorname{sec}h(\Delta_2 t + k_2x)\right) + \tanh(xk_1 + t) - 2}$$

$$u_9 = a_0 + \frac{2\tanh(\Delta_1 t + k_2x) + 2I\operatorname{sec}h(\Delta_1 t + k_2x) + 2\tanh(xk_1 + t)}{2\left(\tanh(\Delta_1 t + k_2x) + I\operatorname{sec}h(\Delta_1 t + k_2x)\right) + \tanh(xk_1 + t) + 2}$$

$$u_{10} = a_0 + \frac{2\tanh(xk_1 + t) - 2\tanh(\Delta_2 t + k_2x) - 2I\operatorname{sec}h(\Delta_2 t + k_2x)}{2\left(\tanh(\Delta_2 t + k_2x) + I\operatorname{sec}h(\Delta_2 t + k_2x)\right)\tanh(xk_1 + t) - 2}$$

where $\Delta_1 = \frac{2\left(1+2k_1k_2+2k_1^2+\frac{1}{2}k_2^2\right)}{4k_1^2+4k_1k_2+k_2^2-1}$ and $\Delta_2 = \frac{2\left(-1-10k_1k_2+10k_1^2+\frac{5}{2}k_2^2\right)}{4k_1^2-4k_1k_2+k_2^2-1}$

Family-2: When $a_0 = a_1 = a_4 = w_1 = const.$, $q_1 = p_1 = \pm\frac{1}{2}$, then we can some complexion solutions:

i. When $q_2 = p_2 = 1$, then

$$u_{11} = a_0 + \frac{2\tan(xk_1+t) - 2\tan(\Delta_3 t + k_2 x) + 2\sec(xk_1+t)}{2\tan(\Delta_3 t + k_2 x)(\sec(xk_1+t)+\tan(xk_1+t)) + 2}$$

$$u_{12} = a_0 + \frac{2\tan(\Delta_4 t + k_2 x) + 2\sec(xk_1+t) + 2\tan(xk_1+t)}{2\tan(\Delta_4 t + k_2 x)(\sec(xk_1+t)+\tan(xk_1+t)) - 2}$$

$$u_{13} = a_0 - \frac{2\tan(\Delta_4 t + k_2 x) - 2\sec(xk_1+t) + 2\tan(xk_1+t)}{2\tan(\Delta_4 t + k_2 x)(\sec(xk_1+t)-\tan(xk_1+t)) + 2}$$

$$u_{14} = a_0 + \frac{2\tan(\Delta_3 t + k_2 x) + 2\sec(xk_1+t) - 2\tan(xk_1+t)}{2\tan(\Delta_3 t + k_2 x)(\sec(xk_1+t)-\tan(xk_1+t)) - 2}$$

where $\Delta_3 = -\frac{1}{2}\frac{\left(-2k_1^2+8k_1k_2-8k_2^2+1\right)}{-k_1^2+4k_1k_2-4k_2^2-1}$ and $\Delta_4 = -\frac{1}{2}\frac{\left(-4k_1^2-16k_1k_2-16k_2^2-1\right)}{-k_1^2-4k_1k_2-4k_2^2-1}$

ii. When $q_2 = p_2 = -1$, then

$$u_{15} = a_0 + \frac{2\cot(\Delta_5 t - k_2 x) + 2\sec(xk_1+t) + 2\tan(xk_1+t)}{-2\cot(\Delta_5 t - k_2 x)(\sec(xk_1+t)+\tan(xk_1+t)) + 2}$$

$$u_{16} = a_0 - \frac{2\cot(\Delta_6 t - k_2 x) - 2\sec(xk_1+t) - 2\tan(xk_1+t)}{-2\cot(\Delta_6 t - k_2 x)(\sec(xk_1+t)+\tan(xk_1+t)) - 2}$$

$$u_{17} = a_0 + \frac{2\cot(\Delta_6 t - k_2 x) + 2\sec(xk_1+t) - 2\tan(xk_1+t)}{-2\cot(\Delta_6 t - k_2 x)(\sec(xk_1+t)-\tan(xk_1+t)) + 2}$$

$$u_{18} = a_0 + \frac{-2\cot(\Delta_5 t - k_2 x) + 2\sec(xk_1+t) - 2\tan(xk_1+t)}{-2\cot(\Delta_5 t - k_2 x)(\sec(xk_1+t)-\tan(xk_1+t)) - 2}$$

where $\Delta_5 = \frac{1}{2}\frac{\left(2k_1^2+8k_1k_2+8k_2^2-1\right)}{-k_1^2-4k_1k_2-4k_2^2-1}$ and $\Delta_6 = \frac{1}{2}\frac{\left(4k_1^2-16k_1k_2+16k_2^2+1\right)}{-k_1^2+4k_1k_2-4k_2^2-1}$.

iii. When $q_2 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$, then

$$u_{19,20}=a_0+\frac{\{\pm 2I\{-\tanh(\Delta_7 t-k_2x)+I\sec h(\Delta_7 t-k_2x)\}+2\sec(xk_1+t)+2\tan(xk_1+t)\}}{2\{-\tanh(\Delta_7 t-k_2x)+I\sec(\Delta_7 t-k_2x)\}\{\sec(xk_1+t)+\tan(xk_1+t)\}\pm 2I}$$

$$u_{21,22}=a_0+\frac{\{\mp 2I(\tanh(\Delta_8 t-k_2x)-I\sec h(\Delta_8 t-k_2x))+2\sec(xk_1+t)+2\tan(xk_1+t)\}}{\{2(\tanh(\Delta_8 t-k_2x)-I\sec h(\Delta_8 t-k_2x))(\sec(xk_1+t)+\tan(xk_1+t))\pm 2I\}}$$

where $\Delta_7=-\dfrac{2\left(\pm\frac{1}{2}I\pm 3Ik_1k_2\pm\frac{1}{2}Ik_1^2\pm\frac{1}{2}Ik_2^2+k_1k_2-\frac{3}{2}k_1^2+\frac{3}{2}k_2^2\right)}{\pm 2Ik_1k_2+k_2^2-1-k_1^2}$ and

$$\Delta_8=-\frac{2\left(\mp\frac{1}{2}I\pm 3Ik_1k_2\pm\frac{1}{2}Ik_1^2\mp\frac{1}{2}Ik_2^2+k_1k_2-\frac{3}{2}k_1^2+\frac{3}{2}k_2^2\right)}{\mp 2Ik_1k_2-k_2^2-1+k_1^2}.$$

Family-3: When $a_0=a_1=a_4=w_1=const.,\ \ q_1=p_1=1$ then we can some complexion solutions:

i. When $q_2=p_2=-1$, then

$$u_{23,24}=a_0\mp\frac{2\cot\left(\frac{1}{2}\frac{t(4k_1^2+8k_1k_2+4k_2^2-2)}{-4k_1^2-8k_1k_2-4k_2^2-1}-k_2x\right)\pm 2\tan(xk_1+t)}{2\cot\left(\frac{1}{2}\frac{t(4k_1^2+8k_1k_2+4k_2^2-2)}{-4k_1^2-8k_1k_2-4k_2^2-1}-k_2x\right)\tan(xk_1+t)\mp 2}$$

ii. When $q_2=\frac{1}{2},\ \ p_2=-\frac{1}{2}$, then

$$u_{25}=a_0+\frac{\{2I(-\tanh(\Delta_9 t-k_2x)+I\sec h(\Delta_9 t-k_2x))+2\tan(xk_1+t)\}}{\{2(-\tanh(\Delta_9 t-k_2x)+I\sec h(\Delta_9 t-k_2x))\tan(xk_1+t)+2I\}}$$

$$u_{26}=a_0+\frac{\{-2I(-\tanh(\Delta_{10} t-k_2x)+I\sec h(\Delta_{10} t-k_2x))+2\tan(xk_1+t)\}}{\{2(-\tanh(\Delta_{10} t-k_2x)+I\sec h(\Delta_{10} t-k_2x))\tan(xk_1+t)-2I\}}$$

$$u_{27}=a_0+\frac{\{2I(-\tanh(\Delta_9 t-k_2x)-I\sec h(\Delta_9 t-k_2x))+2\tan(xk_1+t)\}}{\{2(-\tanh(\Delta_9 t-k_2x)-I\sec h(\Delta_9 t-k_2x))\tan(xk_1+t)+2I\}}$$

$$u_{28}=a_0+\frac{\{-2I(-\tanh(\Delta_{10} t-k_2x)-I\sec h(\Delta_{10} t-k_2x))+2\tan(xk_1+t)\}}{\{2(-\tanh(\Delta_{10} t-k_2x)-I\sec h(\Delta_{10} t-k_2x))\tan(xk_1+t)-2I\}}$$

where $\Delta_9=-\dfrac{2\left(I+6Ik_1k_2+4Ik_1^2+Ik_2^2+4k_1k_2-6k_1^2+\frac{3}{2}k_2^2\right)}{4Ik_1k_2+k_2^2-1-4k_1^2}$ and

$$\Delta_{10} = -\frac{2\left(-I - 6Ik_1k_2 - 4Ik_1^2 - Ik_2^2 + 4k_1k_2 - 6k_1^2 + \frac{3}{2}k_2^2\right)}{-4Ik_1k_2 + k_2^2 - 1 - 4k_1^2}$$

Family-4: When $a_0 = a_1 = a_4 = w_1 = const.$, $q_1 = p_1 = -1$ then we can some complexion solutions:

i. When $q_2 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$, then

$$u_{29,30} = a_0 + \frac{\{\pm 2I(-\tanh(\Delta_{11}t - k_2x) + I\,\mathrm{sec}h(\Delta_{11}t - k_2x)) + 2\cot(xk_1 + t)\}}{\{2(-\tanh(\Delta_{11}t - k_2x) + I\,\mathrm{sec}h(\Delta_{11}t - k_2x))\cot(xk_1 + t) \pm 2I\}}$$

$$u_{31,32} = a_0 + \frac{\{\pm 2I(-\tanh(\Delta_{12}t - k_2x) - I\,\mathrm{sec}h(\Delta_{12}t - k_2x)) + 2\cot(xk_1 + t)\}}{\{2(-\tanh(\Delta_{12}t - k_2x) - I\,\mathrm{sec}h(\Delta_{12}t - k_2x))\cot(xk_1 + t) \pm 2I\}}.$$

where $\Delta_{11} = -\frac{2\left(\pm I \pm 6Ik_1k_2 \pm 4Ik_1^2 \mp Ik_2^2 + 4k_1k_2 - 6k_1^2 + \frac{3}{2}k_2^2\right)}{\pm 4Ik_1k_2 + k_2^2 - 1 - 4k_1^2}$ and

$$\Delta_{12} = -\frac{2\left(\pm I \pm 6Ik_1k_2 \pm 4Ik_1^2 \pm Ik_2^2 + 4k_1k_2 - 6k_1^2 + \frac{3}{2}k_2^2\right)}{\pm 4Ik_1k_2 + k_2^2 - 1 - 4k_1^2}.$$

Similarly, we can obtain more complexiton solution of Eq. (4.16) using Eq. (4.18) and Eq. (4.19), which
are omitted for convenience.

4.4 The (2+1)-dimensional Breaking Soliton (BS) equation

In this section, we study (2+1)-D Breaking soliton equation via direct method called Hirota's bilinear method. With the assist of this method, we construct its rogue wave and solitary wave solutions using particular auxiliary function. Finally, the interactions between solitary waves and rogue waves are offered with a complete derivation.

The (2+1)-dimensional Breaking Soliton (BS) equation [67-69] reads as

$$P_{BS}(u,v) := u_t + \alpha u_{xxy} + 4\alpha u v_x + 4\alpha u_x v = 0, \tag{4.21}$$

where α is arbitrary constant and $u_y = v_x$. It is known that the BS equation above possesses a Hirota bilinear form:

$$B_{BS}(f) := \left(D_x D_t + \alpha D_y D_x^3\right)(f \cdot f) \\ = \left[f_{xt} f - f_t f_x + \alpha\left(f_{xxxy} f - 3 f_{xxy} f_x + 3 f_{xy} f_{xx} - f_y f_{xxx}\right)\right] = 0, \tag{4.22}$$

under the links from f to u and v are as follows:

$$u = 3(\ln f)_{xx} = \frac{3\left(f_{xx} f - f_x^2\right)}{f^2}, \tag{4.23}$$

and
$$v = 3(\ln f)_{xy} = \frac{3\left(f_{xy} f - f_x f_y\right)}{f^2}. \tag{4.24}$$

Such potential transformations used in Bell polynomial theories of soliton equations and a proper relation is

$$P_{BS}(u,v) = \left[\frac{B_{BS}(f)}{f^2}\right]_{xx}. \tag{4.25}$$

It is clear that, if f solves the bilinear breaking soliton equation (4.22), then $u = 3(\ln f)_{xx}$ and $v = 3(\ln f)_{xy}$ will solve the (2+1)-dimensional breaking soliton equation (4.21).

4.4.1 Rogue wave solutions

Let us adopt that Eq. (4.22) has a ansatz in the following form:

$$f = 1 + g^2 + h^2, \tag{4.26}$$

with

$$g(x,y,t) = a_1 x + a_2 y + a_3 t + a_4, \tag{4.27}$$

$$h(x,y,t) = a_5 x + a_6 y + a_7 t + a_8, \tag{4.28}$$

where a_i, $(1 \le i \le 8)$ are arbitrary constants. Setting Eq. (4.26) along with Eq. (4.27) and Eq. (4.28) into bilinear form Eq. (4.22), we obtain some polynomials which are functions of the variables x, y and t. Equating all the coefficient of x, y, t and the constant term to be zero, we can obtain the set of algebraic equations for a_i, $(1 \le i \le 8)$. Solving the system with the aid of symbolic computation system Maple, gives the following relations between the parameters a_i:

$$a_1 = a_4 = a_5 = a_6 = a_8 = const., a_2 = -\frac{a_5 a_6}{a_1}, a_3 = a_7 = 0. \tag{4.29}$$

Therefore, substituting Eq. (4.29) and Eq. (4.26) along with Eq. (4.27) and Eq. (4.28) into Eq. (4.22) yields the following rogue wave solution,

$$u=\frac{3\left(2a_1^2+2a_5^2\right)}{1+\vartheta_1^2+\vartheta_2^2}-\frac{3\left(2\vartheta_1 a_1+2\vartheta_2 a_5\right)^2}{\left(1+\vartheta_1^2+\vartheta_2^2\right)^2}, \tag{4.30}$$

with

$$\vartheta_1=a_1x-\frac{a_5a_6y}{a_1}+a_4 \text{ and } \vartheta_2=a_5x+a_6y+a_8.$$

where the parameters satisfy the constraints (4.29).

4.4.2 Solitary wave solutions

Here, we seek the solitary wave solutions of Eq. (4.21). We expand the test function f with small parameter λ

$$f(x,y,t)=1+\lambda f^{(1)}+\lambda^2 f^{(2)}, \tag{4.31}$$

with

$$f^{(1)}=\exp(k_1x+k_2y+k_3t),\ f^{(2)}=\exp(k_4x+k_5y+k_6t), \tag{4.32}$$

where k_i, $(1\le i\le 6)$ are arbitrary constants to be determined later. Setting Eq. (4.31) into bilinear form Eq. (4.22) and equating all the coefficient of exponential term to be zero, we can obtain the set of algebraic equations for k_i, $(1\le i\le 6)$. Solving the system with the aid of symbolic computation system Maple, gives the following relations between the parameters k_i:

$$\begin{cases} k_1=k_4=k_5=const., k_2=-\dfrac{k_1k_5(k_1-2k_4)}{k_4(2k_1-k_4)}, \\ k_3=-\dfrac{\mu k_1^3k_5(k_1-2k_4)}{k_4(2k_1-k_4)}, k_6=-\mu k_4^2k_5. \end{cases} \tag{4.33}$$

Setting Eq. (4.33) and Eq. (4.31) into Eq. (4.22) yields the following two-soliton solution

$$u=\frac{3\left(\lambda k_1^2e^{\rho}+\lambda^2k_4^2e^{\rho_1}\right)}{1+\lambda e^{\rho}+\lambda^2e^{\rho_1}}-\frac{\left(\lambda k_1e^{\rho}+\lambda^2k_4e^{\rho_1}\right)^2}{\left(1+\lambda e^{\rho}+\lambda^2e^{\rho_1}\right)^2}. \tag{4.34}$$

with

$$\rho=k_1x-\frac{k_1k_5(k_1-2k_4)}{k_4(2k_1-k_4)}y+\frac{\mu k_1^3k_5(k_1-2k_4)}{k_4(2k_1-k_4)}t, \rho_1=k_4x+k_5y-\mu k_4^2k_5t. \tag{4.35}$$

If we taking $f^{(2)}=0$ in Eq. (4.31), same as before we attain the following relations among the parameters k_i:

$$k_1=k_1, k_2=k_2, k_3=-\mu k_1^2k_2, \tag{4.36}$$

Inserting Eq. (4.36) and Eq. (4.31) into Eq. (4.22) yields the resulting one-soliton solution

$$u = \frac{3\lambda k_1^2 e^{\psi}}{1+\lambda e^{\psi}} - \frac{3\lambda^2 k_1^2 \left(e^{\psi}\right)^2}{\left(1+\lambda e^{\psi}\right)^2}, \tag{4.37}$$

where $\psi = k_1 x + k_2 y - \mu k_1^3 k_2 t.$

4.4.3 Interaction between rogue wave and solitary wave

In this sub-section, we will be discussed the interaction phenomena between rogue wave solution and solitary wave solution of a (2+1)-dimensional breaking soliton equation. We choose two different cases of stripe soliton named exponential and hyperbolic sine function respectively.

Case-1

In the first case, we choose $f(x, y, t)$ as a quadratic function with exponential part, that is,

$$f = 1 + g^2 + h^2 + \lambda \exp(\gamma), \tag{4.38}$$

where g and h are defined by Eq. (4.27) and Eq. (4.28), and $\gamma(x, y, t) = k_1 x + k_2 y + k_3 t,\ k_i,$ $(1 \le i \le 3)$ are the constant parameters which are determined later.

Substituting Eq. (4.38) into Eq. (4.22), with the help of symbolic computation system Maple, we get twenty number equations. After, solving these equations we find some relations one of them relation is:

$$\begin{cases} a_1 = a_4 = a_6 = k_1 = const., a_3 = a_7 = 0, a_2 = -a_6\sqrt{-1}, a_5 = a_1\sqrt{-1}, \\ a_8 = (a_4 - 1)\sqrt{-1}, k_2 = \dfrac{1}{2}\dfrac{k_1 a_6 \sqrt{-1}}{a_1}, k_3 = -\dfrac{1}{2}\dfrac{\mu k_1^3 a_6 \sqrt{-1}}{a_1}. \end{cases} \tag{4.39}$$

Setting Eq. (4.38) and Eq. (4.39) into Eq. (4.22) yields the resulting solution

$$u = \frac{3\lambda k_1^2 e^{\xi}}{1+\xi_1^2+\xi_2^2+\lambda e^{\xi}} - \frac{3\left(2\xi_1 a_1 + 2\sqrt{-1}\xi_2 a_1 + \lambda k_1 e^{\xi}\right)^2}{\left(1+\xi_1^2+\xi_2^2+\lambda e^{\xi}\right)^2}. \tag{4.40}$$

with

$$\begin{cases} \xi = k_1 x + \dfrac{\frac{1}{2}\sqrt{-1}k_1 a_6 y}{a_1} - \dfrac{\frac{1}{2}\sqrt{-1}\mu k_1^3 a_6 t}{a_1}, \xi_1 = a_1 x - \sqrt{-1}a_6 + a_4, \\ \xi_2 = \sqrt{-1}a_1 x + a_6 y + \sqrt{-1}(a_4 - 1). \end{cases} \tag{4.41}$$

where the parameters satisfy the constraints (4.39).

Case-2

Here, we choose $f(x,y,t)$ as a quadratic function with hyperbolic sine part, that is,

$$f = 1 + g^2 + h^2 + \lambda \sinh(\gamma), \tag{4.42}$$

where g, h and γ have been defined in the first case. Again, substituting Eq. (4.42) into Eq. (4.22), with the help of symbolic computation system Maple, gives the following equations for the parameters:

$$\begin{cases} a_1 = a_5\sqrt{-1}, a_3 = a_7\sqrt{-1}, a_4 = a_5 = a_7 = k_1 = const., \\ a_2 = -\dfrac{1}{3}\dfrac{a_7\left(-8\sqrt{-1}a_5^2a_8^2 + 8\sqrt{-1}a_5^2a_4^2 + 3\sqrt{-1}k_1^2\lambda^2 + 16a_5^2a_8a_4\right)}{\mu k_1^4\lambda^2}, \\ a_6 = \dfrac{1}{3}\dfrac{a_7\left(16\sqrt{-1}a_5^2a_8a_4 + 8a_5^2a_8^2 - 8a_5^2a_4^2 - 3k_1^2\lambda^2\right)}{\mu k_1^4\lambda^2}, \\ k_2 = -\dfrac{4}{3}\dfrac{a_7a_5\left(2\sqrt{-1}a_8a_4 + a_8^2 - a_4^2\right)}{\mu k_1^3\lambda^2}, k_3 = \dfrac{4}{3}\dfrac{a_7a_5\left(2\sqrt{-1}a_8a_4 + a_8^2 - a_4^2\right)}{k_1\lambda^2}. \end{cases} \tag{4.43}$$

Substituting these equations in g, h and γ which gives

$$\begin{cases} g(x,y,t) = a_5\sqrt{-1}x - \dfrac{1}{3}\dfrac{\psi_1}{\mu k_1^4\lambda^2}y + a_7\sqrt{-1}t + a_4, \\ h(x,y,t) = a_5x + \dfrac{1}{3}\dfrac{\psi_2}{\mu k_1^4\lambda^2}y + a_7t + a_8, \gamma(x,y,t) = k_1x - \dfrac{4}{3}\dfrac{\psi_3}{\mu k_1^3\lambda^2}y + \dfrac{4}{3}\dfrac{\psi_3}{k_1\lambda^2}t. \end{cases} \tag{4.44}$$

with

$$\begin{cases} \psi_1 = a_7\left(-8\sqrt{-1}a_5^2a_8^2 + 8\sqrt{-1}a_5^2a_4^2 + 3\sqrt{-1}k_1^2\lambda^2 + 16a_5^2a_8a_4\right), \\ \psi_2 = a_7\left(16\sqrt{-1}a_5^2a_8a_4 + 8a_5^2a_8^2 - 8a_5^2a_4^2 - 3k_1^2\lambda^2\right), \psi_3 = a_7a_5\left(2\sqrt{-1}a_8a_4 + a_8^2 - a_4^2\right). \end{cases} \tag{4.45}$$

Setting Eq. (4.44) into Eq. (4.42) along with Eq. (4.45), we obtain the expression of $f(x,y,t)$, which is

$$\begin{aligned} f = 1 &+ \left(a_5\sqrt{-1}x - \frac{1}{3}\frac{\psi_1}{\mu k_1^4\lambda^2}y + a_7\sqrt{-1}t + a_4\right)^2 + \left(a_5x\frac{1}{3}\frac{\psi_2}{\mu k_1^4\lambda^2}y + a_7t + a_8\right)^2 \\ &+ \lambda\sinh\left(k_1x - \frac{4}{3}\frac{\psi_3}{\mu k_1^3\lambda^2}y + \frac{4}{3}\frac{\psi_3}{k_1\lambda^2}t\right). \end{aligned} \tag{4.46}$$

Finally substitute Eq. (4.46) into Eq. (4.22), we obtain a new exact interaction solution of the (2+1)-dimensional breaking soliton equation

$$u=\frac{-3\lambda k_1^2\sinh\Delta_1}{\left(1+\Delta_2^2+\Delta_3^2+\lambda\sinh\Delta_1\right)}-\frac{3\left(2\sqrt{-1}\Delta_2 a_5+2\Delta_3 a_5+\lambda k_1\cosh\Delta_1\right)}{\left(1+\Delta_2^2+\Delta_3^2+\lambda\sinh\Delta_1\right)^2}, \tag{4.47}$$

with

$$\begin{cases}\Delta_1=k_1x-\frac{4}{3}\frac{\psi_3 y}{\mu k_1^3\lambda^2}+\frac{4}{3}\frac{\psi_3 t}{k_1\lambda^2},\Delta_2=a_5\sqrt{-1}x-\frac{1}{3}\frac{\psi_2 y}{\mu k_1^4\lambda^2}+\sqrt{-1}a_7t+a_4,\\ \Delta_3=a_5x+\frac{1}{3}\frac{\psi_1 y}{\mu k_1^4\lambda^2}+a_7t+a_8.\end{cases} \tag{4.48}$$

where the parameters satisfy the constraints (4.43).

4.5 The (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov (ANNV) equation

In this paper, we will consider the (2+1)-dimensional ANNV equation [72, 73],

$$u_t+u_{xxx}+3[uv]_x=0; \qquad u_x=v_y. \tag{4.49}$$

where u and v are the components of the (dimensionless) velocity [74]. Eq. (4.49) is the only known isotropic Lax extension of the Korteweg-de Vries equation [75]. The ANNV equation has important applications in incompressible fluids, such as shallow-water waves, long internal waves and acoustic waves.

Bilinear form

Let us introduce the following potential transformation

$$u=c(t)q_{xy} \text{ and } v=c(t)q_{xx} \tag{4.50}$$

in which $c=c(t)$ is a function to be known later. Substituting (4.50) into (4.49) and integrating the equation with respect to x once and taking $c=1$, we get

$$E(q)=q_{yt}+q_{xxxy}=0. \tag{4.51}$$

by choosing the integration constant as zero. Based on the results presented in Refs. [82-83], we obtain

$$E(q)=P_{yt}(q)+P_{xxxy}(q)=0. \tag{4.52}$$

with the help of the following two important transformations, we get

$$\begin{cases}q=2\ln f(x,y,t)\Leftrightarrow u=cq_{xy}=2[\ln f(x,y,t)]_{xy}\\ q=2\ln f(x,y,t)\Leftrightarrow v=cq_{xx}=2[\ln f(x,y,t)]_{xx}\end{cases}. \tag{4.53}$$

Substituting above transformations (4.53) into Eq. (4.49), (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov equation can be linearized into

$$\left(D_y D_t + D_y D_x^3\right) f \cdot f = 0, \tag{4.54}$$

4.5.1 Soliton solutions of the (2+1)-dimensional ANNV equation

The 1-soliton solution

To seek one-soliton solutions of Eq. (1), we suppose f is expressed in the following form

$$f = a_0 + a_1 e^{l_1 x + m_1 y + n_1 t} \tag{4.55}$$

where $l_1, m_1, n_1,\ a_i, (i = 0,1)$ are arbitrary constants to be determined later. Inserting Eq. (4.55) into Eq. (4.54) and after some simplification, equating all the coefficient of exponential term to be zero, we can obtain the set of algebraic equations for $l_1, m_1, n_1,\ a_i, (i = 0,1)$. Solving the system with the aid of symbolic computation system Maple, we obtain the subsequent solution:

$$a_0 = a_0, a_1 = a_1, l_1 = l_1, m_1 = m_1, n_1 = -l_1^3. \tag{4.56}$$

Therefore, setting Eq. (4.55) and Eq. (4.56) along with Eq. (4.53) into Eq. (4.54), yields the desired one-soliton solution of Eq. (4.49).

The 2-soliton solution

To seek two-soliton solutions of Eq. (4.49), we choose f is expressed as

$$f = a_0 + a_1 e^{l_1 x + m_1 y + n_1 t} + a_2 e^{l_2 x + m_2 y + n_2 t} + a_3 e^{l_1 x + m_1 y + n_1 t + l_2 x + m_2 y + n_2 t} \tag{4.57}$$

where $a_i, (i = 0,1),\ l_i, m_i, n_i (i = 1,2)$ are all real parameters to be determined. Substituting Eq. (4.57) into Eq. (4.54) and after some simplification, equating all the coefficient of exponential term to be zero, we can obtain

$$\left\{ a_0 = \frac{a_1 a_2 (l_1 - l_2)(m_1 - m_2)}{a_3 (l_1 + l_2)(m_1 + m_2)}, a_i = a_i, (1 \le i \le 3), l_1 = l_2 = m_1 = m_2 = const., n_1 = -l_1^3, n_2 = -l_2^3. \right. \tag{4.58}$$

which should satisfies the conditions $a_3 \neq 0$, $(l_1 + l_2) \neq 0$, and $(m_1 + m_2) \neq 0$.

Therefore, inserting Eq. (4.57) and Eq. (4.58) along with Eq. (4.53) into Eq. (4.54), yields the desired two- soliton solution. If we setting $a_1 = 1, a_2 = 1, a_3 = 10,\ l_1 = 2, l_2 = 2.5,\ m_1 = 1, m_2 = 3.5$, we can obtain a two-soliton solution of Eq. (4.49). If we setting $l_1 \neq -l_2 \in \Re$ and $m_1 = m_2 \in \Re$, then we obtain another type of two soliton solution. First type solution is elastic but second type is non-elastic solution.

Based on the above method Eq. (4.57) gives the breathers by asset of selecting suitable parameters. Breather solutions of Eq. (4.49) can be obtained in the (x, y) plane, where the parameters in Eq. (4.58) meeting the following conditions

$$l_1 = Ib_1,\ l_2 = -Ib_2,\ a_1 = k_1,\ a_2 = k_2,\ a_3 = k_3,\ m_1 = b + Ik,\ m_2 = b - Ik. \tag{4.59}$$

For instance, setting parameters as follows $l_1 = I,\ l_2 = -2I,\ m_1 = m_2^* = 1 + 2I,\ a_1 = 1.25,\ a_2 = 1.5,$ $a_3 = 2,$ we can obtain breather wave solution.

The 3-soliton solution

To seek three-soliton solutions of Eq. (4.49), we suppose f is expressed as

$$f = a_0 + e^{\varphi_1} + e^{\varphi_2} + e^{\varphi_3} + a_{12}e^{\varphi_1+\varphi_2} + a_{23}e^{\varphi_2+\varphi_3} + a_{13}e^{\varphi_1+\varphi_3} + a_{123}e^{\varphi_1+\varphi_2+\varphi_3} \tag{4.60}$$

with

$$\varphi_i = l_i x + m_i y + n_i t, i = 1,2,3 \tag{4.61}$$

where $a_0,\ a_{12}, a_{23}, a_{13}, a_{123},\ l_i, m_i, n_i (i = 1,2,3)$ are all real parameters to be determined. Based on above method, substituting Eq. (4.60) with Eq. (4.61) into Eq. (4.54), we can obtain the following relations among parameters

$$\begin{cases} a_0 = \dfrac{(l_1 - l_2)(2l_3 - l_2)}{a_{13}l_2(l_1 - l_2 + 2l_3)}, m_1 = a_{12} = a_{23} = a_{123} = 0, a_{13} = l_1 = l_2 = l_3 = m_2 = m_3 = const., \\ n_1 = -3l_1^2 l_3 - 3l_1 l_3^2 + 3l_1^2 l_2 - 3l_1 l_2^2 - l_1^3 + 6l_1 l_2 l_3, n_2 = -l_2^3 + 3l_2^2 l_3 - 3l_2 l_3^2, n_3 = -l_3^3. \end{cases} \tag{4.62}$$

which needs to satisfy the condition $a_{13}, l_2 \neq 0.$

Therefore, substituting Eqs. (4.60)- (4.62) along with Eq. (4.53) into Eq. (4.54), the three-soliton solution of Eq. (4.49) can be obtained.

4.5.2 Lump solutions of the (2+1)-dimensional ANNV equation

To seek lump solutions of Eq. (4.49), we suppose f is expressed in the following form:

$$f = g^2 + h^2 + p_1, \tag{4.63}$$

where,

$$g(x, y, t) = l_1 x + m_1 y + n_1 t, \text{ and } h(x, y, t) = l_2 x + m_2 y + n_2 t, \tag{4.64}$$

where $p_1, l_i, m_i, n_i\ (i = 1,2)$ are all real constants to be determined. A direct symbolic computation with f gives rises to the following relations:

$$p_1 = p_1, l_1 = -\frac{m_2 l_2}{m_1}, l_2 = l_2, m_1 = m_1, m_2 = m_2, n_1 = 0, n_2 = 0. \tag{4.65}$$

Therefore, substituting Eq. (4.65) with Eq. (4.64) into Eq. (4.63), we can get a class of quadratic function solutions Eq. (4.53). Then, the resulting exact rational solution for Eq. (4.49) are obtained through the transformation

$$u = 2(\ln f)_{xy} = \frac{4(l_1 m_1 + l_2 m_2)p_1 - 8gh(l_1 m_2 + l_2 m_1) + 4(l_1 m_1 - l_2 m_2)(-g^2 + h^2)}{(g^2 + h^2 + p_1)^2}, \tag{4.66}$$

and

$$v = 2(\ln f)_{xx} = \frac{4(l_1^2 + l_2^2)p_1 - 16 l_1 l_2 gh + 4(l_1^2 - l_2^2)(-g^2 + h^2)}{(g^2 + h^2 + p_1)^2}, \tag{4.67}$$

where $g(x,y,t) = l_1 x + m_1 y + n_1 t,\ h(x,y,t) = l_2 x + m_2 y + n_2 t,$ for example, the resulting solutions of Eq. (4.65) are as follows

$$u = -\frac{8\left(-\frac{g m_2 l_2}{m_1} + h l_2\right)(g m_1 + h m_2)}{(g^2 + h^2 + p_1)^2}, \quad v = \frac{4\left(\frac{m_2^2 l_2^2}{m_1^2} + l_2^2\right)}{g^2 + h^2 + p_1} - \frac{4\left(-\frac{g m_2 l_2}{m_1} + h l_2\right)^2}{(g^2 + h^2 + p_1)^2}, \tag{4.68}$$

with the function g and h are given as follows

$$g = -\frac{m_2 l_2 x}{m_1} + m_1 y, \text{ and } h = l_2 x + m_2 y. \tag{4.69}$$

For the exact solution $u(x,y,t)$ and $v(x,y,t)$ to Eq. (4.49) to be lump ones, it is observed that

$$\lim_{x^2+y^2\to\infty} u(x,y,t) = 0, \text{ and } \lim_{x^2+y^2\to\infty} v(x,y,t) = 0, \quad \forall t \in \Re. \tag{4.70}$$

It is easy to see that for any given time t, the lump solutions $u \to 0,\ v \to 0,$ if and only if the corresponding summation of squares $g^2 + h^2 \to \infty,$ which is equivalent to $x^2 + y^2 \to \infty.$ Substituting the noted values of $p_1, l_2, m_i\ (i = 1,2)$ into Eq. (4.68), then we can get abundant exact lump solutions of Eq. (4.49). We can notice that the solutions we obtained have a unified form of (4.67). If we taking the values of $t = t_0$, then the coordinates of the central point of the obtained lump solution is

$$\left(x = \frac{m_1 n_2 t_0 - m_2 n_1 t_0}{l_1 m_2 - l_2 m_1}, y = \frac{n_2 l_1 t_0 - n_1 l_2 t_0}{l_1 m_2 - l_2 m_1}\right) \tag{4.71}$$

where $l_1m_2 - l_2m_1 \neq 0$. Substituting Eq. (4.71) and $t = t_0$ into Eq. (4.67), the amplitude of v is attained $\mathrm{Max}(v) = \frac{4(l_1^2 + l_2^2)}{p_1} (p_1 \neq 0)$, from which we observe that the amplitude of the lump solution is depend on the values of l_1, l_2 and p_1. As we seen from Eq. (4.71) the lump soliton is centered at the origin when $t = 0$.

4.5.3 Interaction of lump waves with solitary waves

To get the interaction phenomena between lumps and solitary waves solutions of Eq. (4.49), assuming $f(x, y, t)$ in the following new form

$$f = g^2 + h^2 + p_1 + \lambda \exp(\eta), \tag{4.72}$$

with

$$g(x, y, t) = l_1x + m_1y + n_1t, h(x, y, t) = l_2x + m_2y + n_2t \text{ and } \eta(x, y, t) = l_3x + m_3y + n_3t, \tag{4.73}$$

where $p_1, l_i, m_i, n_i (1 \leq i \leq 3)$ are all real parameters to be determined. Substituting Eq. (4.72) along with Eq. (4.73) into Eq. (4.54) with the aid of symbolic computation system Maple, we can gain the following relations among parameters:

$$l_2 = -\frac{l_1m_1}{m_2}, n_3 = -l_3^3, m_3 = n_1 = n_2 = 0, p_1 = \lambda = l_1 = l_3 = m_1 = m_2 = const. \tag{4.74}$$

which should satisfy $m_2 \neq 0$.

Therefore, substituting Eq. (4.74) into Eq. (4.72), we can get a class of quadratic function solutions to the bilinear equation (4.54). Then, the resulting exact rational solution for Eq. (4.49) are obtained through the transformation,

$$u = 2(\ln f)_{xy} = \frac{\begin{array}{c} 4(l_1m_1 + l_2m_2)p_1 - 8gh(l_1m_2 + l_2m_1) + 4(l_1m_1 - l_2m_2)(-g^2 + h^2) + \\ 2\{(g^2 + h^2 + p_1)l_3m_3 + (l_1m_1 + l_2m_2) - (l_1m_3 + l_3m_1)g - (l_2m_3 + l_3m_2)h\}\lambda e^{\eta} \end{array}}{(g^2 + h^2 + p_1)^2}, \tag{4.75}$$

$$v = 2(\ln f)_{xx} = \frac{4(l_1^2 + l_2^2)p_1 - 16ghl_1l_2 + 4(l_1^2 - l_2^2)(-g^2 + h^2) + 2\left\{\begin{array}{l} (g^2 + h^2 + p_1)l_3^2 + \\ 2(l_1^2 + l_2^2) - 4(gl_1 + hl_2)l_3 \end{array}\right\}\lambda e^{\eta}}{(g^2 + h^2 + p_1)^2}, \tag{4.76}$$

where g, h and η are defined in Eq. (4.73).

for example, the resulting solutions of Eq. (4.74) are as follows

$$u=-\frac{8\left(gl_1-\frac{hl_1m_1}{m_2}+\frac{1}{2}\lambda l_3e^{\eta}\right)(gm_1+hm_2)}{\left(g^2+h^2+p_1+\lambda e^{\eta}\right)^2},\ v=\frac{4\left(l_1^2+\frac{l_1^2m_1^2}{m_2^2}+\frac{1}{2}\lambda l_3^2e^{\eta}\right)}{g^2+h^2+p_1+\lambda e^{\eta}}-\frac{4\left(gl_1-\frac{hl_1m_1}{m_2}+\frac{1}{2}\lambda l_3e^{\eta}\right)}{\left(g^2+h^2+p_1+\lambda e^{\eta}\right)^2} \quad (4.77)$$

where $g=l_1x+m_1y,\ h=-\frac{l_1m_1}{m_2}x+m_2y$ and $\eta=l_3x-l_3^3t.$ (4.78)

4.5.4 Multi lump solutions of (2+1)-dimensional ANNV equation

In this section, we will find the multi lump solution of Eq. (4.49). To this aim, the above function $f(x,y,t)$ can be taken as,

$$f=e^{-\psi_1}+h_1e^{\psi_1}+h_2\sin\psi_2, \quad (4.79)$$

with $\psi_1=p_1(x+n_1y-w_1t)$ and $\psi_2=p_2(x+n_2y-w_2t)$, (4.80)

where $p_i,n_i,w_i\,(i=1,2)$ are all real parameters to be determined. Substituting Eq. (4.79) along with Eq. (4.80) into Eq. (4.54) with the aid of symbolic computation system Maple, we can obtain the following relations among parameters

$$n_1=-\frac{1}{4}\frac{h_2^2p_2^2n_2}{h_1p_1^2},w_1=p_1^2-3p_2^2,w_2=-p_2^2+3p_1^2,h_1=h_2=n_2=p_1=p_2=const., \quad (4.81)$$

which should satisfy $h_1,p_1\neq 0.$

Under the transformation Eq. (4.53), we can get the periodic lump solutions of the (2+1)-dimensional ANNV equation as,

$$u=\frac{2\left(-p_1\zeta_1+p_1\zeta_2-h_2p_2^2n_2\sin\delta_2\right)}{\left(e^{-\delta_1}+h_1e^{\delta_1}+h_2\sin\delta_2\right)}-\frac{2\left(\zeta_1+\zeta_2+h_2p_2n_2\cos\delta_2\right)\left(\begin{array}{l}-p_1e^{-\delta_1}+h_1p_1e^{\delta_1}\\+h_2p_2\cos\delta_2\end{array}\right)}{\left(e^{-\delta_1}+h_1e^{\delta_1}+h_2\sin\delta_2\right)^2}, \quad (4.82)$$

and

$$v=\frac{2\left(p_1^2e^{-\delta_1}+h_1p_1^2e^{\delta_1}-h_2p_2^2\sin\delta_2\right)}{\left(e^{-\delta_1}+h_1e^{\delta_1}+h_2\sin\delta_2\right)}-\frac{2\left(-p_1e^{-\delta_1}+h_1p_1e^{\delta_1}+h_2p_2\cos\delta_2\right)^2}{\left(e^{-\delta_1}+h_1e^{\delta_1}+h_2\sin\delta_2\right)^2}, \quad (4.83)$$

where

$$\begin{cases}\zeta_1=\frac{h_2^2p_2^2n_2e^{-\delta_1}}{4h_1p_1},\zeta_2=-\frac{h_2^2p_2^2n_2e^{\delta_1}}{4p_1},\delta_1=p_1\left(x-\frac{1}{4}\frac{h_2^2p_2^2n_2y}{h_1p_1^2}-\left(p_1^2-3p_2^2\right)t\right)\\ \delta_2=p_2\left(x+n_2y-\left(-p_2^2+3p_1^2\right)t\right).\end{cases} \quad (4.84)$$

4.6 The (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili (gBKP) equation

Inspired by the mechanism of interaction solutions, we focus on the interaction solutions of the (3+1)-dimensional generalized B-type Kadomtsev-Petviashvili (gBKP) equation [84]

$$u_{yt}+3u_{xz}-3u_x u_{xy}-3u_{xx}u_y-u_{xxxy}=0. \tag{4.85}$$

Through the dependent variable transformation

$$u=2(\ln\phi)_x=\frac{2\phi_x}{\phi}, \tag{4.86}$$

the (3+1)-dimensional gBKP equation can be convert to the bilinear D-operator form

$$\left(D_y D_t+3D_x D_z-D_y D_x^3\right)(\phi\cdot\phi)=0, \tag{4.87}$$

where $\phi=\phi(x,y,z,t)$ and the derivatives D_x, D_y, D_z, D_t are the Hirota's bilinear operators [3] defined in

$$D_x^\alpha D_y^\beta D_z^\gamma D_t^\delta(\phi\cdot\varphi)=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^\alpha\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^\beta\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z'}\right)^\gamma\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^\delta \tag{4.88}$$

$$\times\phi(x,y,z,t)\cdot\varphi(x',y',z',t')\big|_{x=x',y=y',z=z',t=t'}$$

The chief aimed of this paper is to present mixed lump-stripe, breather and various dynamical of collision wave solutions for gBKP equation via suitable ansatzes approach.

2. Interaction phenomena between solitary wave and lump wave

In this section, we explore the dynamics of collisions between lump soliton and one stripe soliton of gBKP model (4.85). For this, we choose $\phi(x,y,z,t)$ as a combination of two positive quadratic functions and an exponential function as

$$\phi=g^2+h^2+a_{11}+\lambda\mu, \tag{4.89}$$

where

$$\begin{cases} g(x,y,z,t)=a_1x+a_2y+a_3z+a_4t+a_5, \\ h(x,y,z,t)=a_6x+a_7y+a_8z+a_9t+a_{10}, \mu=\exp(k_1x+k_2y+k_3z+k_4t), \end{cases} \tag{4.90}$$

where $a_i(i = 1,\dots,11)$, k and $k_i(i = 1,\dots,4)$ are real factors to be later calculated. Plugging Eq. (4.89) into Eq. (4.87), and with a direct symbol calculation, we acquire 6 classes of solutions. We only select one of them to analyze characters of the similar solutions.

$$a_2 = a_3 = a_6 = a_9 = k_2 = k_3 = 0,\ a_4 = -\frac{3a_1a_8}{a_7},\ k_4 = \frac{k_1\left(-3a_8 + a_7k_1^2\right)}{a_7}, \tag{4.91}$$

with $a_7 \neq 0$.

Combining Eq. (4.91) and Eq. (4.89), we obtain the expression of $\phi(x, y, z, t)$:

$$\phi = \left(a_1x - \frac{3ta_1a_8}{a_7} + a_5\right)^2 + \left(a_7y + a_8z + a_{10}\right)^2 + a_{11} + \lambda e^{k_1x + \frac{k_1\left(-3a_8 + a_7k_1^2\right)t}{a_7}}, \tag{4.92}$$

which, consecutively, produces the interaction of lump and stripe solitons to Eq. (4.85) through the transformation (4.86) as:

$$u_1 = \frac{2\left(2\left(a_1x - \frac{3ta_1a_8}{a_7} + a_5\right)a_1 + \lambda k_1 e^{k_1x + \frac{k_1\left(-3a_8 + a_7k_1^2\right)t}{a_7}}\right)}{\left(a_1x - \frac{3ta_1a_8}{a_7} + a_5\right)^2 + \left(a_7y + a_8z + a_{10}\right)^2 + a_{11} + \lambda e^{k_1x + \frac{k_1\left(-3a_8 + a_7k_1^2\right)t}{a_7}}}. \tag{4.93}$$

3. Breather-wave solutions

In this section, we spotlight on the breather-wave solutions of Eq. (4.85) that comes from the collisions between exponential and trigonometric functions.

Case-1: Here, we take $\phi(x, y, z, t)$ as a combination of a cosine function with two exponential functions:

$$\phi = e^{-\eta} + h_1e^{\eta} + h_2\cos(\xi), \tag{4.94}$$

with

$$\begin{aligned}\eta(x, y, z, t) &= p_1\left(r_1x + a_1y + b_1z + c_1t\right),\\ \xi(x, y, z, t) &= p_2\left(r_2x + a_2y + b_2z + c_2t\right),\end{aligned} \tag{4.95}$$

where $a_i, b_i, c_i, p_i, r_i, h_i, (i=1,2)$ are parameters to be designated later. Plugging Eq. (4.934) along with Eq. (4.95) into Eq. (4.87), with the help of symbolic computation system Maple, we achieve

$$r_1 = a_2 = 0, b_1 = -\frac{1}{3}\frac{a_1\left(r_2^3 p_2^2 + c_2\right)}{r_2},\ c_1 = \frac{3b_2 r_2 p_2^2}{a_1 p_1^2},\ h_1 = \frac{1}{4}h_2^2, \tag{4.96}$$

which needs to satisfy the following conditions $a_1 \neq 0,\ p_1 \neq 0$ and $r_2 \neq 0.$

Setting Eq. (4.96) along with Eq. (4.95) into Eq. (4.94), we obtain the expression of $\phi(x, y, z, t)$, which is

$$\phi = e^{-p_1\Delta} + \frac{1}{4}h_2^2 e^{p_1\Delta} + h_2 \cos\left(p_2\left(c_2 t + r_2 x + b_2 z\right)\right). \tag{4.97}$$

Finally, inserting Eq. (4.97) into Eq. (4.86), we attain a periodic lump solution of the (3+1)-D gBKP equation

$$u_2 = \frac{-2h_2 p_2 r_2 \sin\left(p_2\left(c_2 t + r_2 x + b_2 z\right)\right)}{\left(e^{-p_1\Delta} + \frac{1}{4}h_2^2 e^{p_1\Delta} + h_2 \cos\left(p_2\left(c_2 t + r_2 x + b_2 z\right)\right)\right)} \tag{4.98}$$

with $\Delta = \frac{3tb_2 r_2 p_2^2}{a_1 p_1^2} + a_1 y - \frac{1}{3}\frac{a_1 z\left(r_2^3 p_2^2 + c_2\right)}{r_2}.$

Putting $h_2 = 2, p_2 = p_1$ and taking limit as $p_1 \to 0,$ the equation (4.97) reduce to a perturbation solution

$$\phi = \left(\frac{3b_2 r_2 t}{a_1} + a_1 y - \frac{a_1 c_1 z}{r_2}\right)^2 + \left(c_2 t + r_2 x + b_2 z\right)^2. \tag{4.99}$$

Through the transformation (4.86), it reduces to a single lump wave solution as follows:

$$u_{Lump} = \frac{4r_2\left(c_2 t + r_2 x + b_2 z\right)}{\left(\frac{3b_2 r_2 t}{a_1} + a_1 y - \frac{a_1 c_1 z}{r_2}\right)^2 + \left(c_2 t + r_2 x + b_2 z\right)^2}. \tag{4.100}$$

Case-2: In this case, we consider $\phi(x, y, z, t)$ as a combination of a sine function with two exponential functions:

$$\phi = e^{-\eta} + h_1 e^{\eta} + h_2 \sin(\xi), \tag{4.101}$$

where η and ξ have been defined in the first case. Again, inserting Eq. (4.101) into Eq. (4.87), with the help of symbolic computation system Maple, gives the following solution.

$$a_1 = b_1 = r_2 = c_2 = 0, \quad c_1 = -\frac{1}{3}\frac{r_1\left(a_2 r_1^2 p_1^2 - 3b_2\right)}{a_2}, \tag{4.102}$$

which needs to satisfy the following condition $a_2 \neq 0$.

Setting Eq. (4.102) into Eq. (4.101), leads to the expression of $\phi(x, y, z, t)$:

$$\phi = e^{-p_1(\Delta_1 t + r_1 x)} + h_1 e^{p_1(\Delta_1 t + r_1 x)} + h_2 \sin\left(p_2\left(a_2 y + b_2 z\right)\right), \tag{4.103}$$

Finally, setting Eq. (4.103) into Eq. (4.86), we attain a periodic lump waves solution of the (3+1)-dimensional gBKP equation

$$u_3 = \frac{2\left(-p_1 r_1 e^{-p_1(\Delta_1 t + r_1 x)} + h_1 p_1 r e^{p_1(\Delta_1 t + r_1 x)}\right)}{e^{-p_1(\Delta_1 t + r_1 x)} + h_1 e^{p_1(\Delta_1 t + r_1 x)} + h_2 \sin\left(p_2\left(a_2 y + b_2 z\right)\right)}, \tag{4.104}$$

with $\Delta_1 = \dfrac{r_1\left(a_2 r_1^2 p_1^2 - 3b_2\right)}{a_2}$.

4. Interaction solutions with fission phenomena

In this section, we spotlight on a new interaction solutions of Eq. (4.85). For this aim adopt a different test function [44, 84, 86, 88, 89] as follows:

$$\phi = G^2 + H^2 + a_9 + p\cosh(\xi_1) + q\cos(\xi_2), \tag{4.105}$$

where

$$\begin{cases} G = a_1 x + a_2 y + a_3 z + a_4 t, H = a_6 x + a_7 y + a_8 z + a_9 t, \\ \xi_1 = m_1 x + m_2 y + m_3 t, \xi_2 = k_1 x + k_2 y + k_3 t, \end{cases} \tag{4.106}$$

Here, $a_i(i = 1,2,.....,9)$, $k_i(i = 1,2,3)$, p and q are real parameters while $m_i(i = 1,2,3)$ are real or imaginary constants. Plugging Eq. (4.105) into Eq. (4.87), via symbolic computation software Maple, we gain three sets of constraints. In the following, we analyze the three cases in details.

Case-1:

$$\begin{cases} a_1 = -\frac{a_6 a_5}{a_2}, a_3 = -\frac{1}{3}\frac{a_2 a_8}{a_5}, a_4 = -\frac{a_8 a_6}{a_2}, a_7 = -\frac{1}{3}\frac{a_6 a_8}{a_5}, k_3 = \frac{k_1\left(a_5 k_1^2 - a_8\right)}{a_5}, \\ m_3 = \frac{m_1\left(a_5 m_1^2 + a_8\right)}{a_5}, k_2 = m_2 = 0, \end{cases} \tag{4.107}$$

where $a_2 \neq 0$ and $a_5 \neq 0$. Inserting Eq. (4.107) along with Eq. (4.105) into the Eq. (4.86), we advance into the interaction solution of Eq. (4.85):

$$u_4 = \frac{2\left(-\frac{2\chi_1 a_6 a_5}{a_2} + 2\chi_2 a_5 + p m_1 \sinh(t\chi_3 + m_1 x) + q k_1 \sin(t\chi_4 - k_1 x)\right)}{\chi_1^2 + \chi_2^2 + a_9 + p\cosh(t\chi_3 + m_1 x) + q\cos(t\chi_4 - k_1 x)}, \tag{4.108}$$

where

$\chi_1 = -\frac{t a_8 a_6}{a_2} - \frac{x a_6 a_5}{a_2} + a_2 y - \frac{1}{3}\frac{z a_2 a_8}{a_5}, \chi_2 = a_8 t + a_5 x + a_6 y - \frac{1}{3}\frac{z a_8 a_6}{a_5}, \chi_3 = \frac{m_1\left(a_5 m_1^2 + a_8\right)}{a_5}$ and

$\chi_4 = \frac{k_1\left(a_5 k_1^2 - a_8\right)}{a_5}$.

Case-2:

$$\begin{cases} a_3 = -\frac{1}{3}\frac{a_2\left(k_1^3 + k_3\right)}{k_1}, a_7 = -\frac{1}{3}\frac{a_6\left(k_1^3 + k_3\right)}{k_1}, m_3 = \frac{m_1\left(k_1^3 + m_1^2 k_1 + k_3\right)}{k_1}, \\ a_1 = a_4 = a_5 = a_8 = k_2 = m_2 = 0, \end{cases} \tag{4.109}$$

where $k_1 \neq 0$. Inserting Eq. (4.109) along with Eq. (4.105) into the Eq. (4.86), we get the interaction solution of Eq. (4.85) as

$$u_5 = 2(\ln\phi)_x, \tag{4.110}$$

where, $\phi = \left(a_2 y - \frac{a_2\left(k_1^3 + k_3\right)}{3k_1} z\right)^2 + \left(a_6 x - \frac{a_6\left(k_1^3 + k_3\right)}{3k_1} y + a_9 t\right)^2 + a_9 + p\cosh(\xi_1) + q\cos(\xi_2)$,

$$\xi_1 = m_1 x + \frac{m_1\left(k_1^3 + m_1^2 k_1 + k_3\right)}{k_1} t,\ \xi_2 = k_1 x + k_3 t.$$

Case-3:

$$\begin{cases} a_4 = -\dfrac{3a_1 a_7}{a_6}, k_3 = -\dfrac{k_1\left(3a_7 + a_6 k_1^2\right)}{a_6}, m_3 = \dfrac{m_1\left(-3a_7 + a_6 m_1^2\right)}{a_6}, \\ a_2 = a_3 = a_5 = a_8 = k_2 = m_2 = 0, \end{cases} \tag{4.111}$$

where $a_6 \neq 0$. Inserting Eq. (4.111) along with Eq. (4.105) into the Eq. (4.86),we get the interaction solution of Eq. (4.85).

$$u_6 = \frac{2\left(2\left(-\dfrac{3ta_1 a_7}{a_6} + a_1 x\right)a_1 + pm_1 \sinh\left(t\chi_5 + m_1 x\right) + qk_1 \sin\left(t\chi_6 - k_1 x\right)\right)}{\left(-\dfrac{3ta_1 a_7}{a_6} + a_1 x\right)^2 + \left(ya_6 + za_7\right)^2 + a_9 + p\cosh\left(t\chi_5 + m_1 x\right) + q\cos\left(t\chi_6 - k_1 x\right)}, \tag{4.112}$$

where

$$\chi_5 = \frac{m_1\left(-3a_7 + a_6 m_1^2\right)}{a_6}, \chi_6 = -\frac{k_1\left(3a_7 + a_6 k_1^2\right)}{a_6}.$$

Chapter Five

Graphical representation

In this section we describe some features of the solutions that we obtained from Burger equation, Gardner equation (or combined KdV-mKdV), Hirota-Ramani equation, Breaking Soliton (BS), asymmetric Nizhnik-Novikov-Veselov (ANNV) and generalized B-type Kadomtsev-Petviashvili (gBKP) equations in different cases. We depicted these solutions graphically with the help of computational software Maple and explain their behaviors in details.

5.1 Graphical illustration of the solutions of Burger Equation

In this subsection, we explain different type of traveling wave solution of Burger equation graphically obtained by using Modified Double Sub-Equation (MDSE) method. By implementing MDSE method, we obtained Sixty four complexiton solutions of Burger equation and have different type periodic shape. Some of these solutions are stated for specific values of the arbitrary constants with graphical illustration.

The complexiton solutions to the Burger's equations consist with two traveling variables ξ and η expressed in-terms of $\tanh\xi$ and $\tan\eta, \sec\eta$; $\tanh\xi$ and $\cot\eta, \operatorname{cosec}\eta$ and $\coth\xi$ and $\tan\eta$, $\sec\eta$ gives the kinky –periodic wave. When coefficients of ξ is greater than that of the η gives solution with kinky dominate on periodicity (see Fig. 1.1) but when coefficients of ξ is smaller than that of the η gives solution with periodicity increases and dominate on kink type (see Fig. 1.2).

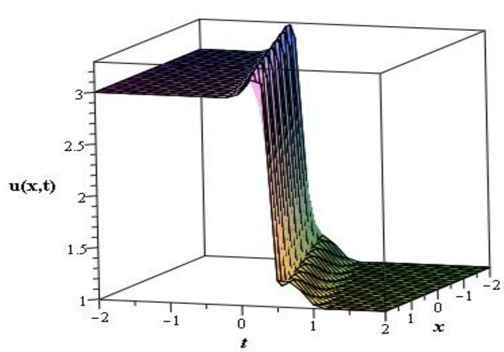


Fig-1.1: Kinky-periodic wave solution for $b_0 = 2$, $k_1 = 1, k_2 = -1.5$ of the real part of $u_{1,2}$.

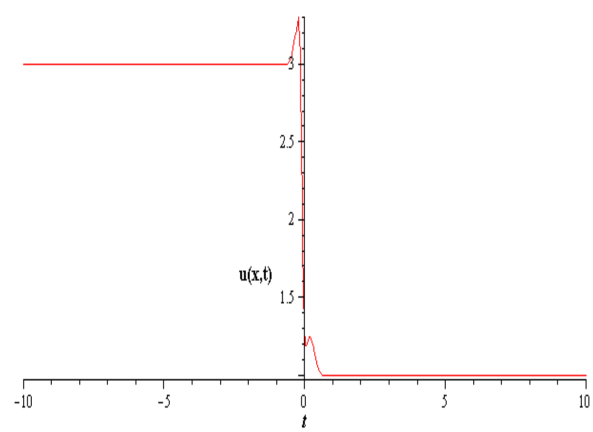


Fig-1.1(a): 2D plot shows the wave propagation pattern at $x = 0$.

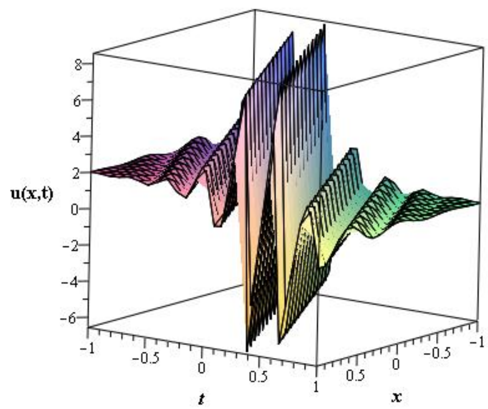


Fig-1.2: Kinky-periodic wave solution for $b_0 = 1, k_1 = 1, k_2 = 5$ of the real part of $u_{17,18}$.

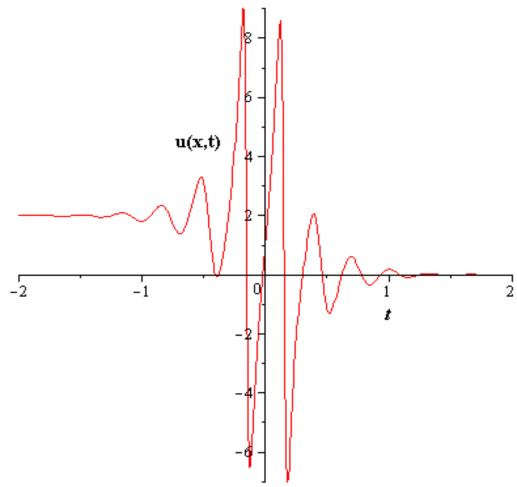


Fig-1.2(a): 2D plot shows the wave propagation pattern at $x = 0$.

On the other hand, the complexiton solutions consist with two traveling variables ξ and η expressed in-terms of $\coth\xi$ and $\cot\eta, \cos ec\eta$; $\coth\xi$ and $\tan\eta$ gives multi-soliton solutions like Fig. 1.3 of $u_{7,8}$.

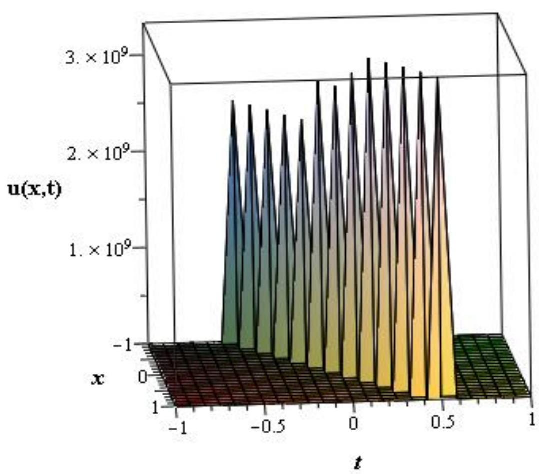
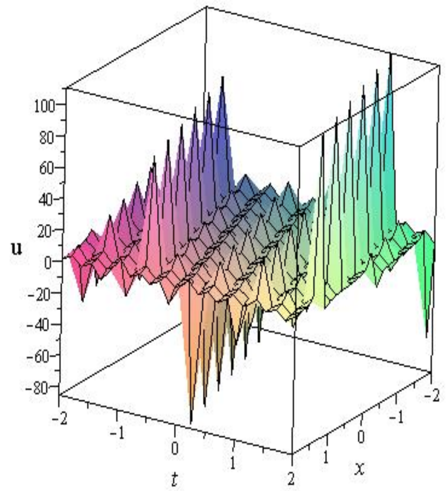


Fig-1.3: Multi-soliton solution for $b_0 = 1, k_1 = 1, k_2 = 2$ of the real part of $u_{7,8}$.

The complexiton solutions consist with two traveling variables ξ and η expressed in-terms of $\cot\xi$ and $\cot\eta, \cos ec\eta$; $\cot\xi$ and $\tan\eta$ gives double-periodic solutions like Fig. 1.4 of $u_{41,42}$.



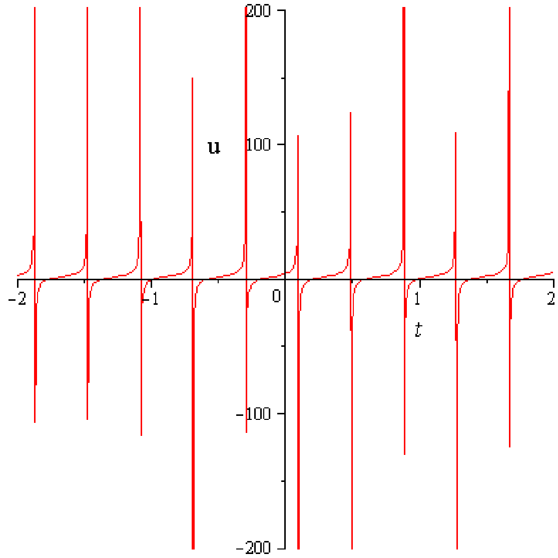
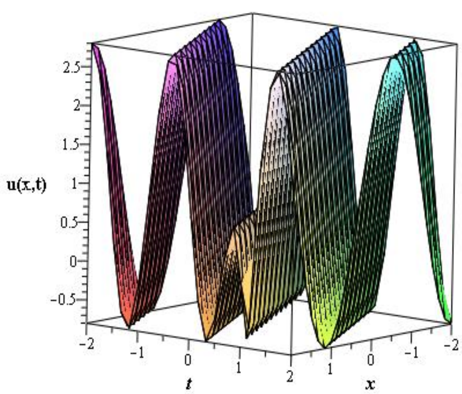


Fig-1.4: Doubly-periodic wave solution for $b_0 = 2, k_1 = 1, k_2 = 2$ of $u_{41,42}$.

Fig-1.4(a): 2D plot shows the wave propagation pattern at $x = 0$.

The complexiton solutions consist with two traveling variables ξ and η expressed in-terms of $\tan\xi$ and $\tanh\eta, \sec h\eta$; $\tan\xi$ and $\coth\eta, \cos ech\eta$ gives bell type-periodic solutions like Fig. 1.5 of $u_{57,58}$.



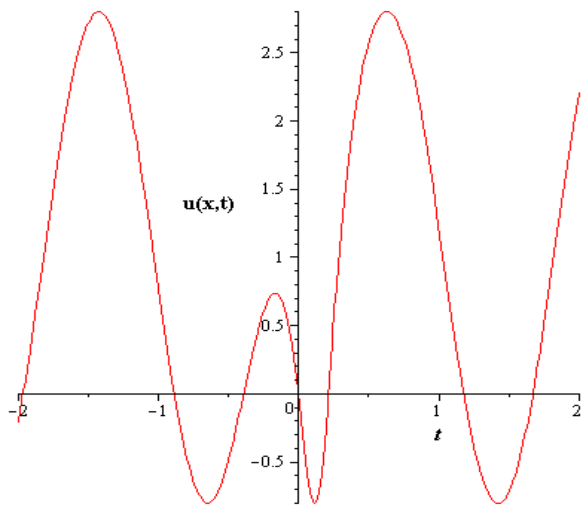


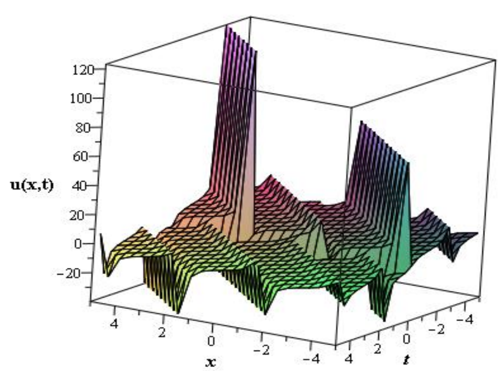
Fig-1.5: Bell-periodic wave solution for $b_0 = 1, k_1 = 1, k_2 = 3$ of the real part of $u_{57,58}$.

Fig-1.5(a): 2D plot shows the wave propagation pattern $x = 0$.

5.2 Graphical representation of the solutions of Gardner Equation

In this subsection, we explain different type of traveling wave solution of Gardner equation graphically obtained by using Modified Double Sub-Equation (MDSE) method. By implementing MDSE method, we obtained Forty complexiton solutions of Gardner equation and have different type periodic shape. The graphical demonstrations of the obtained solutions for specific values of the arbitrary constants are exposed in Fig. 2.1 to Fig. 2.4.

The complexiton solutions to the Gardner equations consist with two traveling variables ξ and η expressed in-terms of $\tanh\xi$ and $\tan\eta, \sec\eta$; $\coth\xi$ and $\tan\eta, \sec\eta$; $\tanh\xi$ and $\cot\eta, \cos ec\eta$; $\coth\xi$ and $\tan\eta, \sec\eta$ gives the kinky –periodic wave. The Fig. 2.1 gives this type of wave and it is plotted for the solution $u_{1,2}$. The solutions involving combinations of $\tanh\xi$ and $\tan\eta$; $\coth\xi$ and $\cot\eta$ gives kinky-periodic wave solutions like Fig. 2.2 and it is plotted for the solution $u_{17,18}$. The solutions involving combinations of $\coth\xi$ and $\coth\eta$, $\cos ech\eta$; $\tanh\xi$ and $\cot\eta$; $\coth\xi$ and $\tan\eta$; some times $\tanh\xi$ and $\cot\eta, \cos ec\eta$ gives single soliton solutions. The Fig. 2.3 gives this type of wave and it is plotted for the solution $u_{11,12}$. The solutions involving combinations of $\tanh\xi$ and $\tanh\eta, \sec h\eta$; $\coth\xi$ and $\tanh\eta, \sec h\eta$; $\coth\xi$ and $\coth\eta, \cos ech\eta$ gives collitions of three solitons (two kinks with one bell type wave) solutions like Fig. 2.4 and it is plotted for the solution $u_{27,28}$.



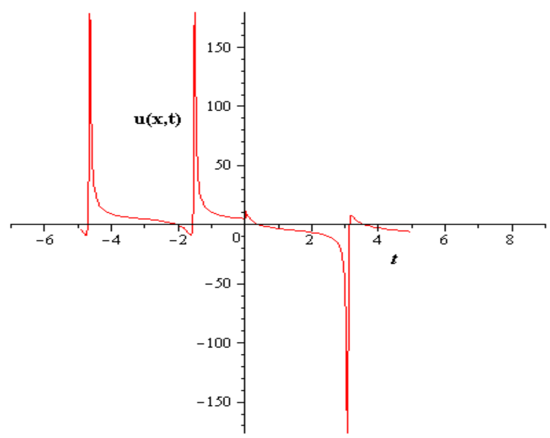
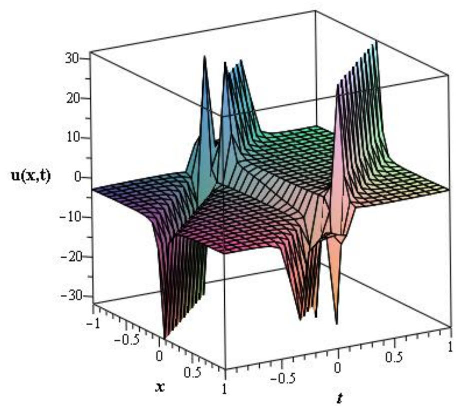


Fig-2.1: Cross Kinky-periodic wave solution for $a_4 = 1, b_1 = 0, b_2 = b_3 = w_2 = k_2 = 2,\ k_1 = 3$ of the real part of $u_{1,2}$.

Fig-2.1(a): 2D plot along $t = 0$ shows the wave propagation pattern.



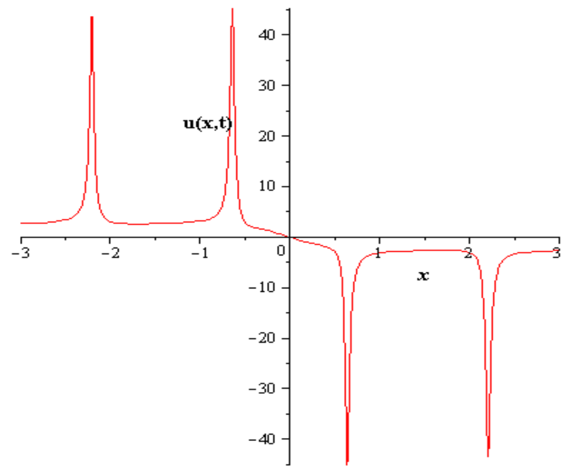
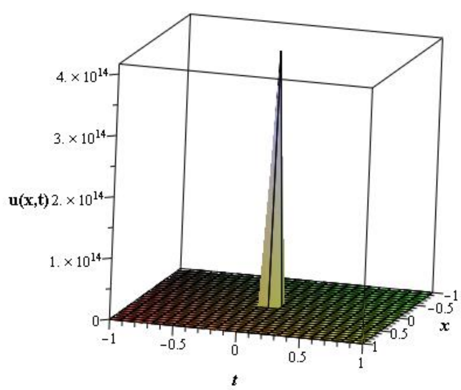


Fig-2.2: Kinky-periodic wave solution for $a_4 = 1, b_1 = 0, b_2 = b_3 = w_2 = k_2 = 2, k_1 = 3$ of the real part of $u_{17,18}$.

Fig-2.2(a): 2D plot along $t = 0$ shows the wave propagation pattern.



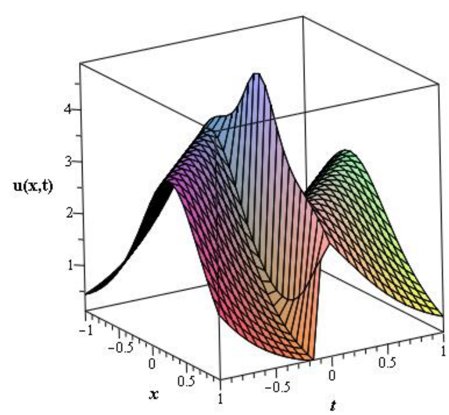


Fig-2.3: Single soliton wave solution for $a_4 = 1, b_1 = 0, b_2 = b_3 = w_2 = k_2 = 2, k_1 = 3$ of the real part of $u_{11,12}$.

Fig-2.4: Collision of two kink with a bell shaped soliton solution for $b_2 = b_3 = w_2 = k_2 = 2,$ $a_4 = 1,\ b_1 = 0,\ k_1 = 3$ of the real part of $u_{27,28}$.

5.3 Graphical representation of the solutions of Hirota-Ramani Equation

In this subsection, we explain different type of traveling wave solution of Hirota-Ramani equation graphically obtained by using Modified Double Sub-Equation (MDSE) method. By implementing MDSE method, we obtained thirty-two traveling wave solutions and have different type periodic shape. The graphical demonstrations of some obtained complexiton solutions for choosing suitable values of the arbitrary constants are exposed in Fig. 3.1 to Fig. 3.6.

The complexiton solutions contain with two traveling variables ξ and η expressed in-terms of $\tanh\xi$ and $\tan\eta, \sec\eta$; $\tanh\xi$ and $\cot\eta$ gives soliton solutions like Fig. 3.1 and Fig. 3.2 respectively.

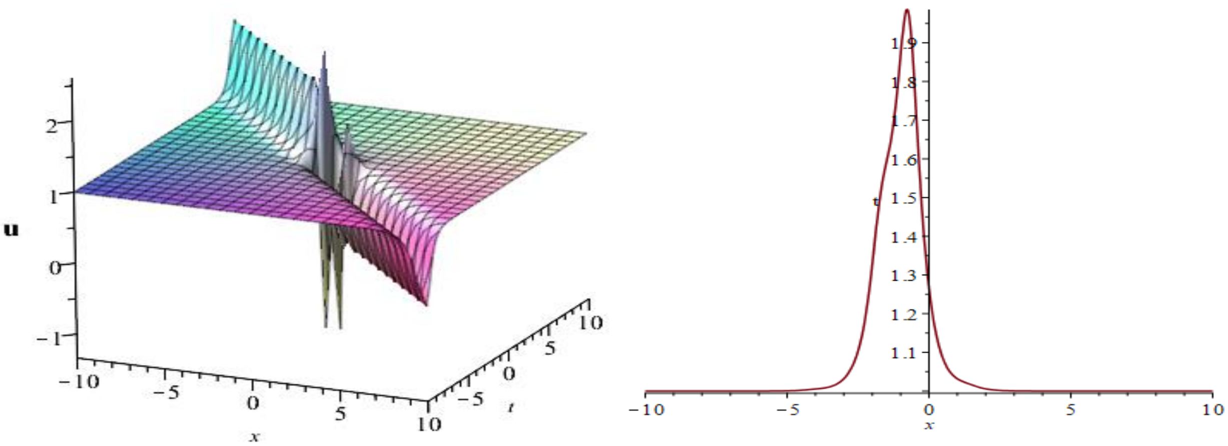


Fig-3.1: Profile of u_1 for $a_0 = 1, k_1 = 1, k_2 = 2.5$.

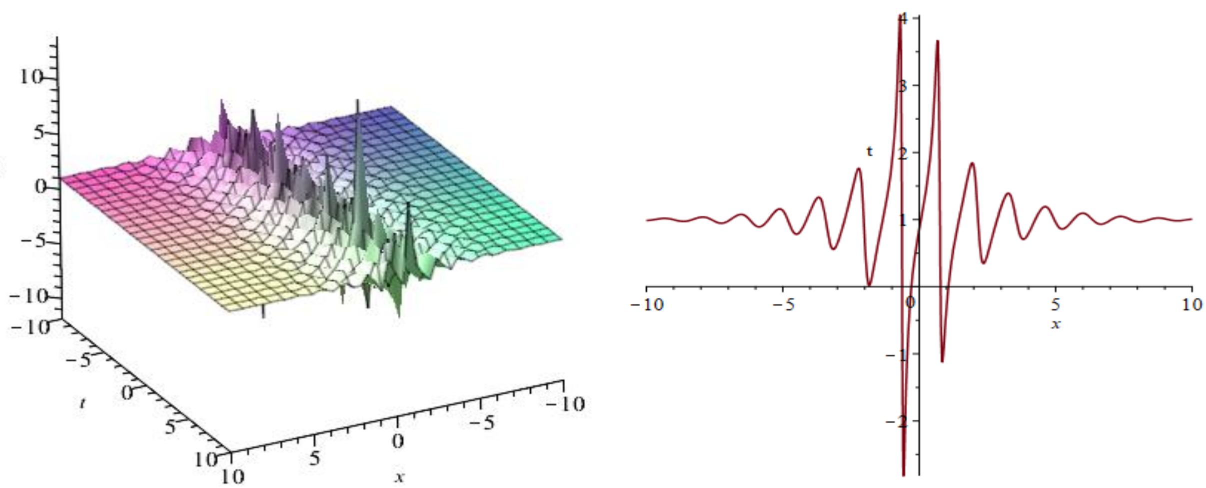


Fig-3.2: Profile of u_5 for $a_0 = 1, k_1 = 0.25, k_2 = 2.25$.

On the otherhand, the complexiton solutions consist with two traveling variables ξ and η expressed in-terms of $\tan\xi$ and $\sec\eta$ gives soliton solutions like Fig. 3.3 and $\sec\xi, \tan\xi$ and $\cot\eta$ gives doubly-periodic wave solution like Fig. 3.4 of u_{15} and $\sec\xi, \tan\xi$ and $\tanh\eta, \sec h\eta$ gives breather wave solutions like Fig. 3.5 of u_{19} and $\sec\xi, \tan\xi$ and $\tanh\eta, \sec h\eta$ gives bell shaped periodic solution like Fig. 3.6 of u_{22}.

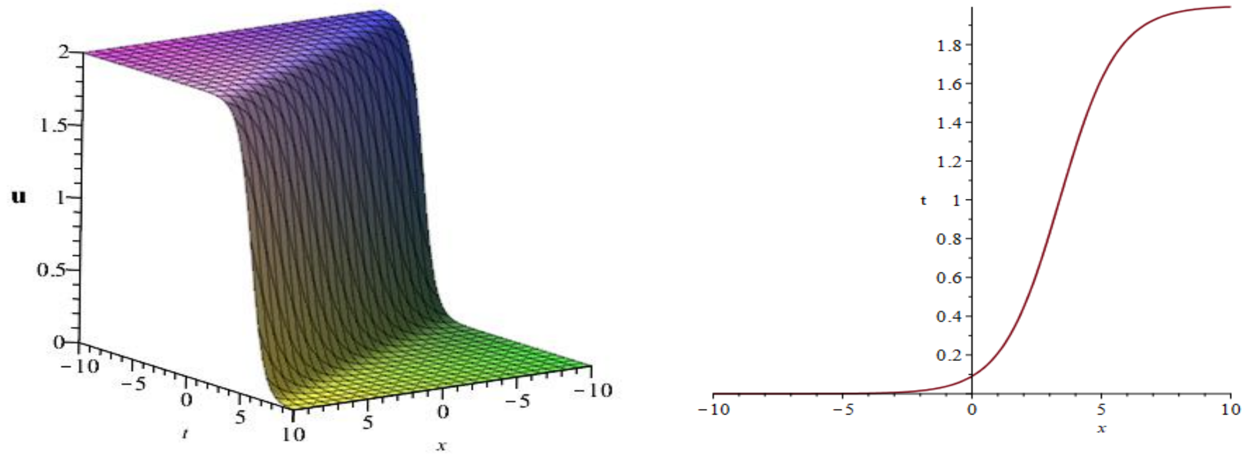


Fig-3.3: Soliton Profile of u_8 for $a_0 = 1, k_1 = 0.025, k_2 = 0.5$.

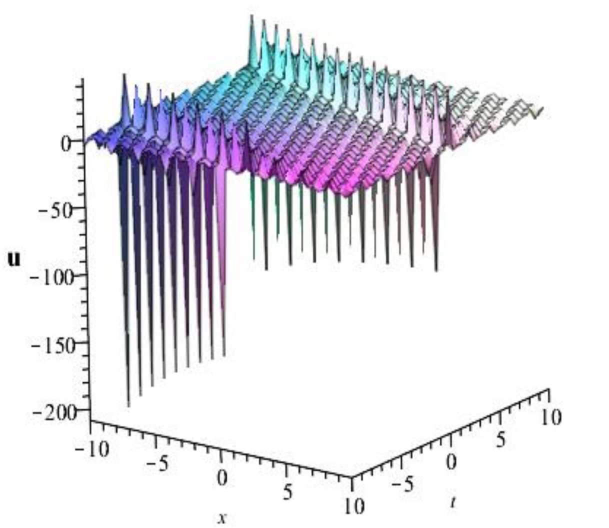


Fig-3.4: Doubly-periodic wave solution for $a_0 = 1, k_1 = 1, k_2 = 2.5$ of u_{15}.

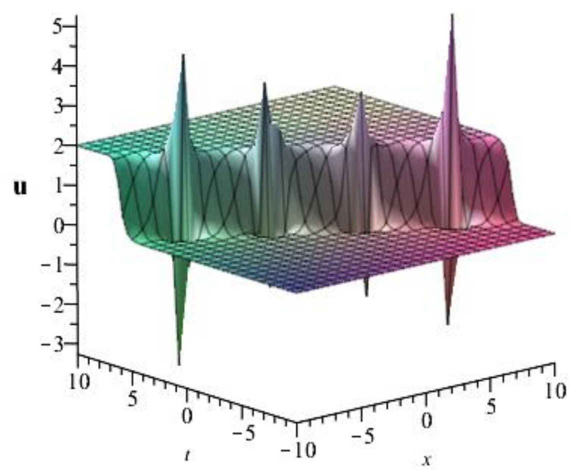


Fig-3.5: Kinky periodic lump wave solution for $a_0 = 1,\ \ k_1 = 1, k_2 = 2$ of u_{19}.

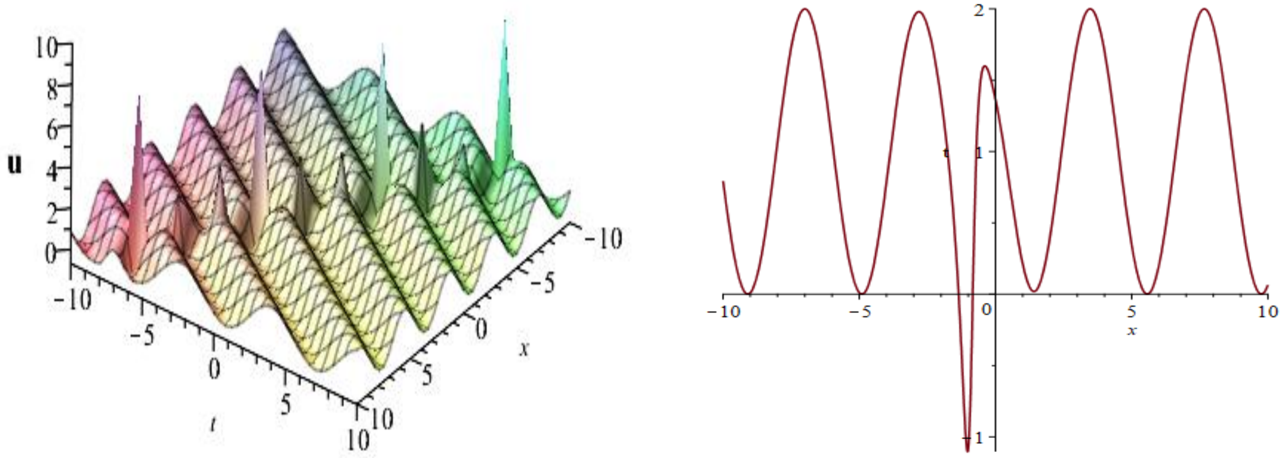


Fig-3.6: Interaction of Bell and periodic wave solution for $a_0 = 1, k_1 = -1.5, k_2 = 2$ of u_{22}.

5.4 Graphical representation of the solutions of BS Equation

In this subsection, we explain different type of traveling wave solution of Breaking Soliton equation graphically obtained by Hirota's bilinear method. Using this method, we obtained some traveling wave solutions which are denoted as Eq. 4.30, Eq. 4.34, Eq. 4.37, Eq. 4.40, and Eq. 4.47. The graphical demonstrations of some obtained complexiton solutions for choosing suitable values of the arbitrary constants are exposed in Fig. 4.1 to Fig. 4.7.

Fig.4.1 shows the sketch of rogue waves for dissimilar values a_1, a_4, a_5, a_6, a_8, (a) gives 3D views from which one can reveal the standard rogue wave feathers. It is also clear that the Fig.4.1 of Eq. 4.30 is the recognized eye-shaped rogue wave solution which has one local hump and two valleys (clears from the views (b)). Besides this, we discover that rogue wave has the uppermost peak in its surrounding waves and it can be forms in a short time and also can be realized from the perspective view of (c).

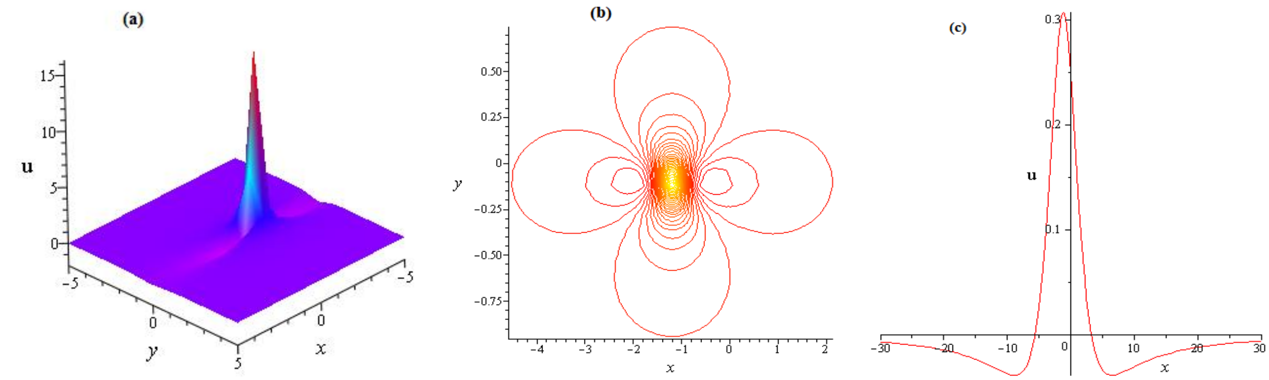


Fig-4.1: Rogue wave solution (4.30) for Eq. (4.21) by choosing suitable parameters: $a_1 = -1,\ a_4 = -2,$ $a_5 = 2, a_6 = -4,$ and $a_8 = 2$. (a) 3-D plot of the wave at $t = 10$. (b) Corresponding contour plot of the wave. (c) 2-D plot of the wave along the x axis.

From Fig. 4.2, it is clear that the amplitude, velocity and width of the one-soliton keep constant during the wave propagation. One can show that the amplitudes of anxious position are limited and almost same in different spaces. In Fig. 4.3, the collision into the couple of bell-shaped soliton has elastic characteristics. When they fully meet, the amplitude changed and the changed amplitude is more than two times than the real amplitude of the two waves. The two waves converted to one eave direction after the collision with their original amplitude and shape. All the phenomena indicate that there is no energy loss during collision.

Now we will show the wave propagation situations of solitary wave by two figures. Fig. 4.2 and Fig. 4.3 show the one-soliton (4.37) and two-soliton solution (4.34), respectively, by choosing suitable parameters.

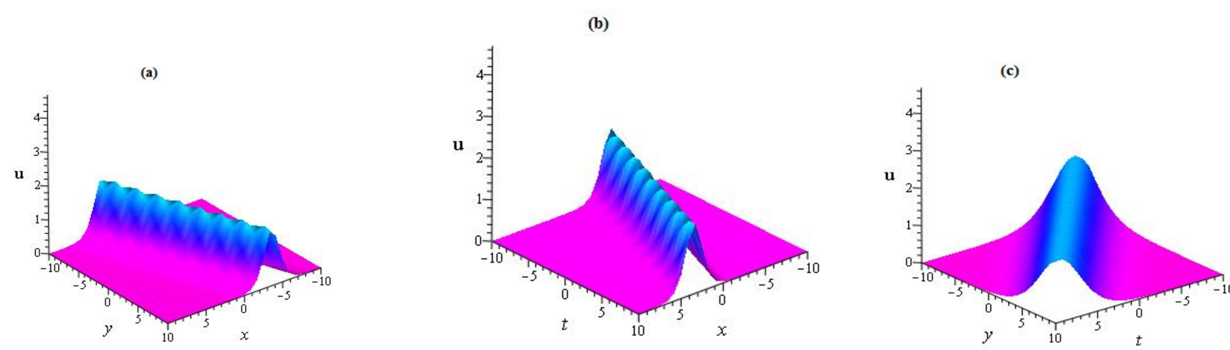


Fig-4.2: One-soliton solution Eq. (4.37) for Eq. (4.21) in the $(y,x),(t,x)$ and (y,t) three different planes with suitable parameters: $k_1 = 1.5,\ k_2 = 0.5, \lambda = 1.5, \mu = 0.5$.

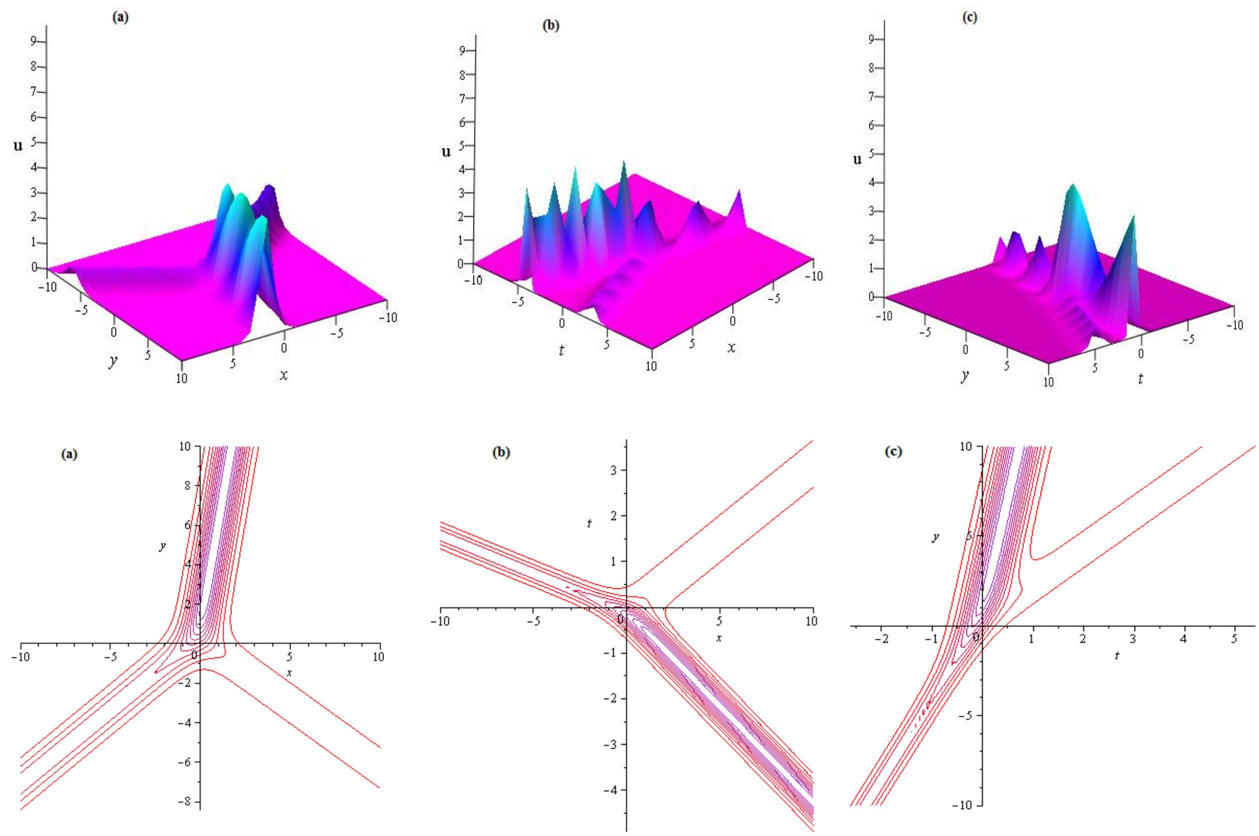
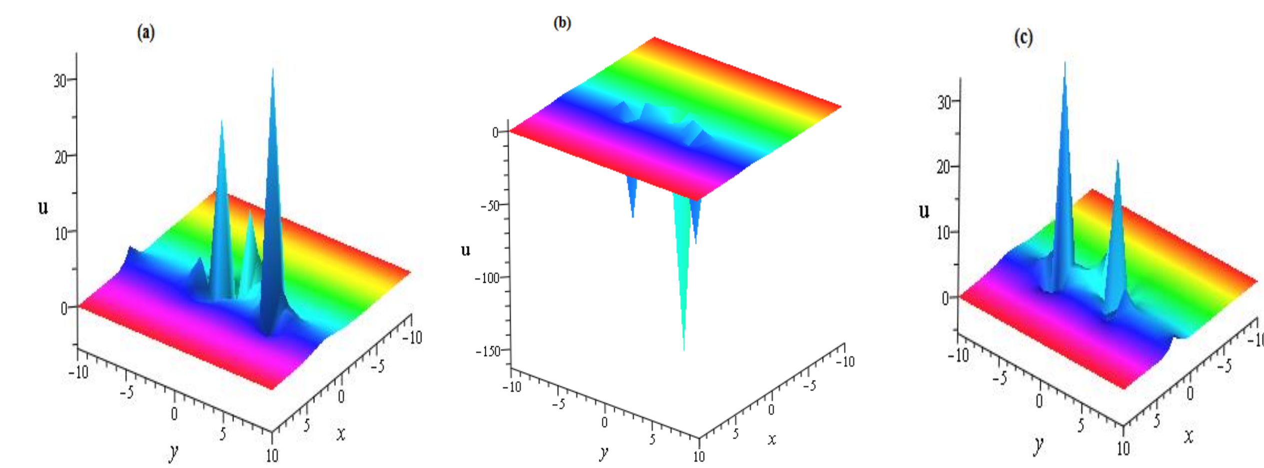


Fig-4.3: Two-soliton solution Eq. (4.34) for Eq. (4.21) with: $k_1 = 2.5,\ k_4 = 1.5,\ k_5 = -2,\ \lambda = 1.5,\ \mu = 2$ in the $(y,x),(t,x)$ and (y,t) three different planes and corresponding contour plots (bottom) respectively.

In what follows, Fig. 4.4 and Fig. 4.5 appeared exact solution (4.40) by taking the suitable parameters.



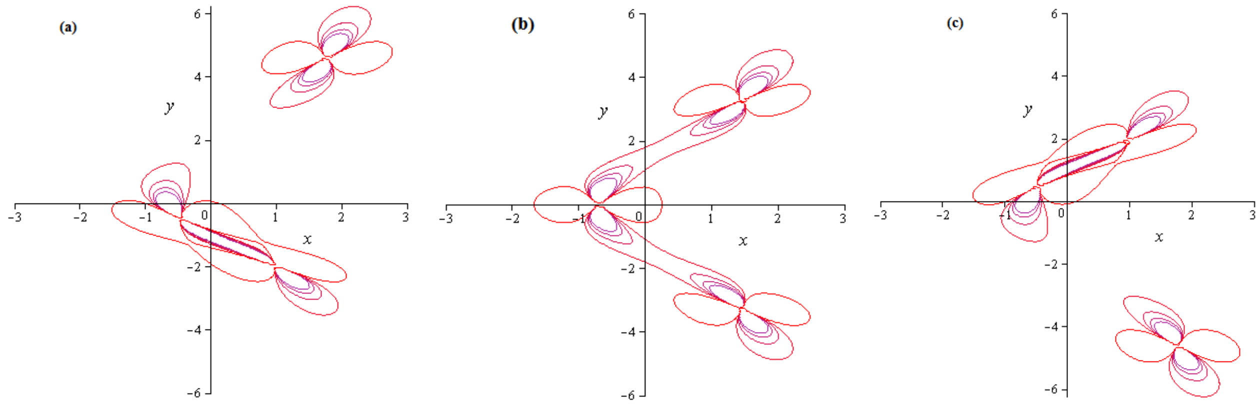


Fig-4.4: Interaction phenomena between rogue wave and solitary wave solution (4.40) for Eq. (4.21) by choosing suitable parameters: $a_1 = 0.5, a_4 = 0.1\ a_6 = -0.6, a_9 = 1, \lambda = 1, \mu = 2$ with three-dimensional plots for different times (a) $t = -15,$ (b) $t = 0,$ and (c) $t = 15$ and corresponding contour plots (bottom) respectively.

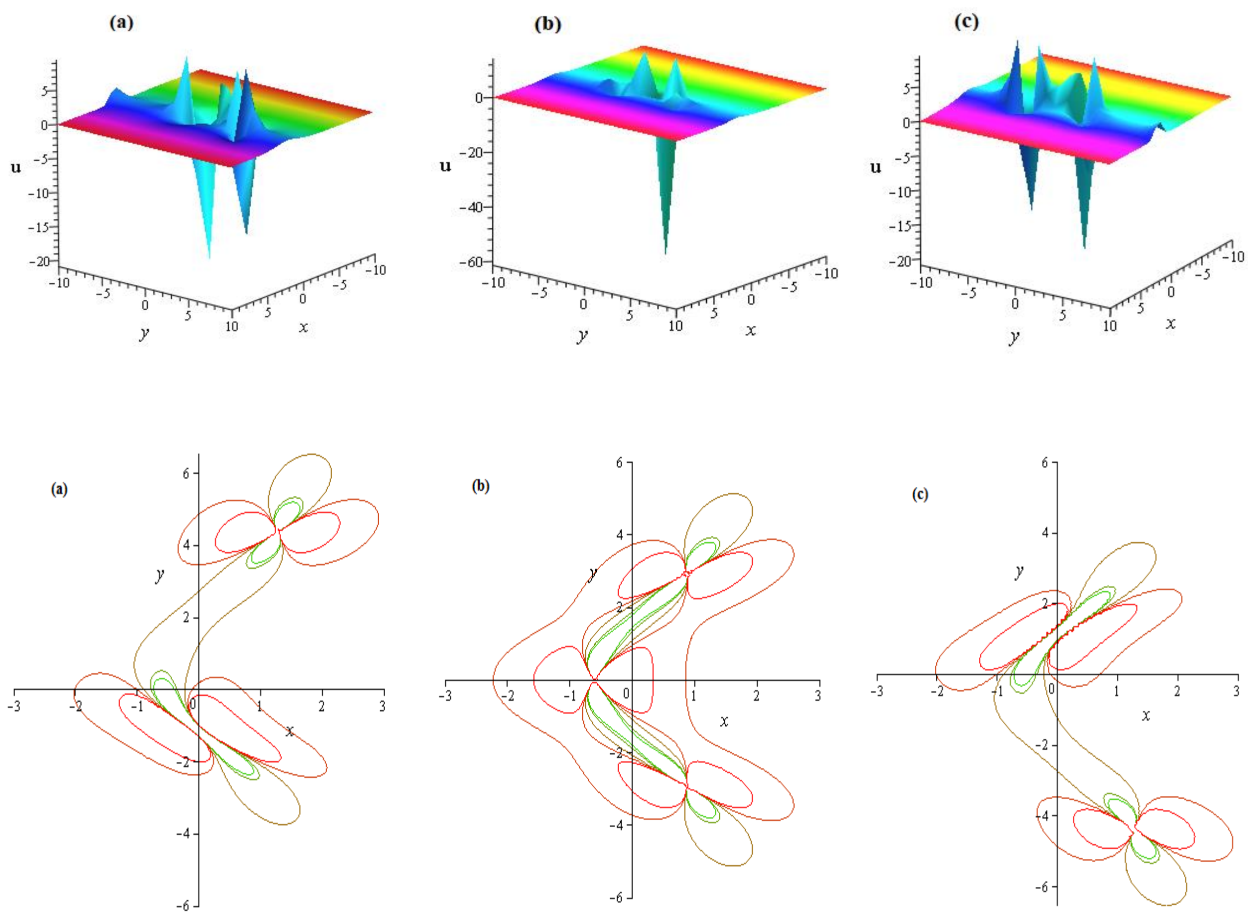


Fig-4.5: Interaction phenomena between rogue wave and solitary wave solution (4.40) for Eq. (4.21) by choosing suitable parameters: $a_1 = 0.5, a_4 = -0.1\ a_6 = -0.6, a_9 = 1, \lambda = 1.5, \mu = 2$ with three-dimensional plots for different times (a) $t = -15,$ (b) $t = 0,$ and (c) $t = 15$ and corresponding contour plots (bottom) respectively.

In what follows, Fig. 4.6 and Fig. 4.7 appeared exact solution (4.47) by taking the suitable parameters.

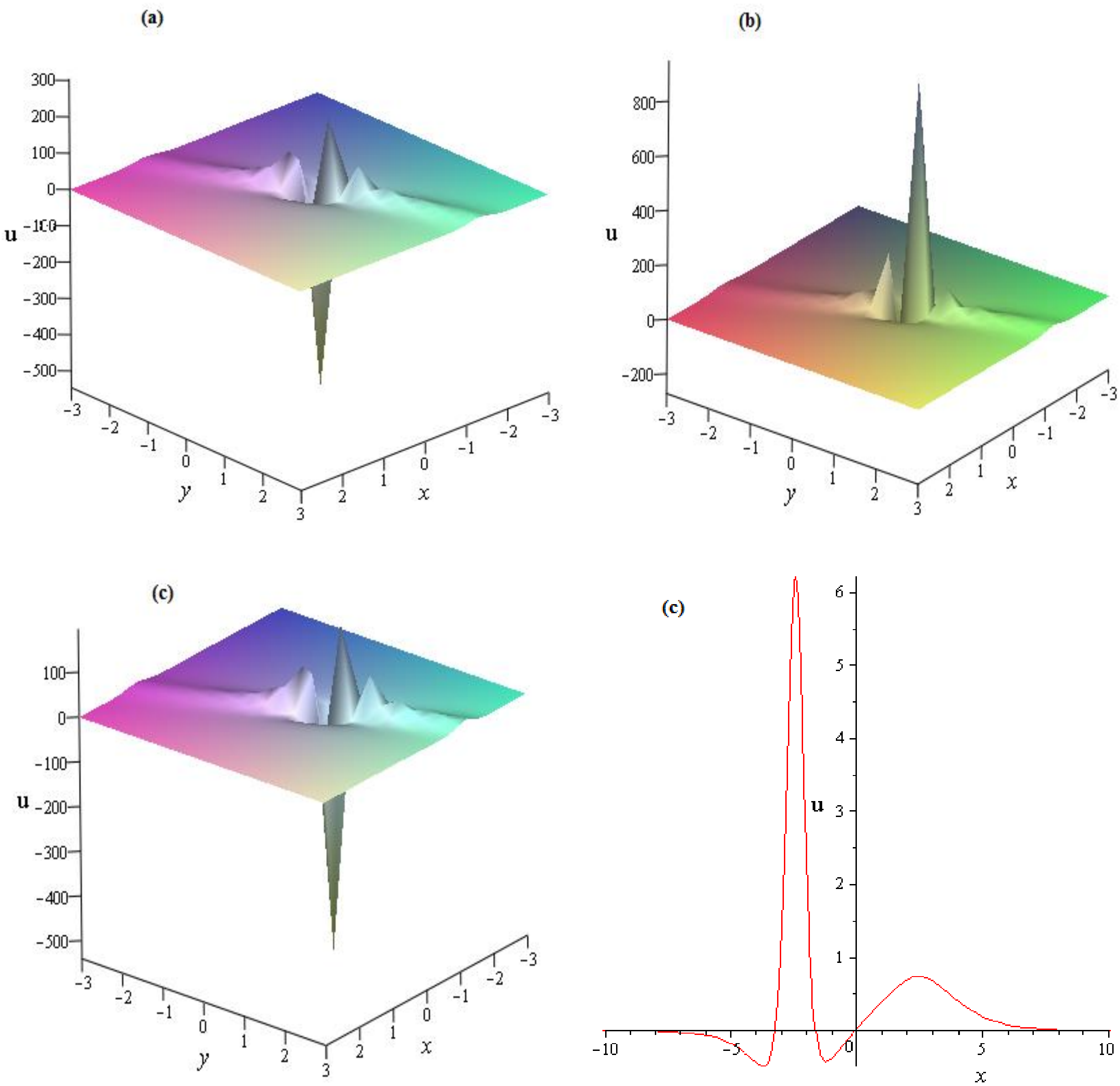


Fig-4.6: Profile of interaction between rogue wave and hyperbolic solution (4.47) for Eq. (4.21) by choosing suitable parameters: $a_4 = 0.54, a_5 = -0.5, a_7 = 0.3, a_8 = -0.8, a_9 = 1, \lambda = 10, \ \mu = 0.1,$ with 3D plots for different times (a) $t = 0,$ (b) $t = 3,$ and (c) $t = 5$ respectively; (d) 2D plot (c).

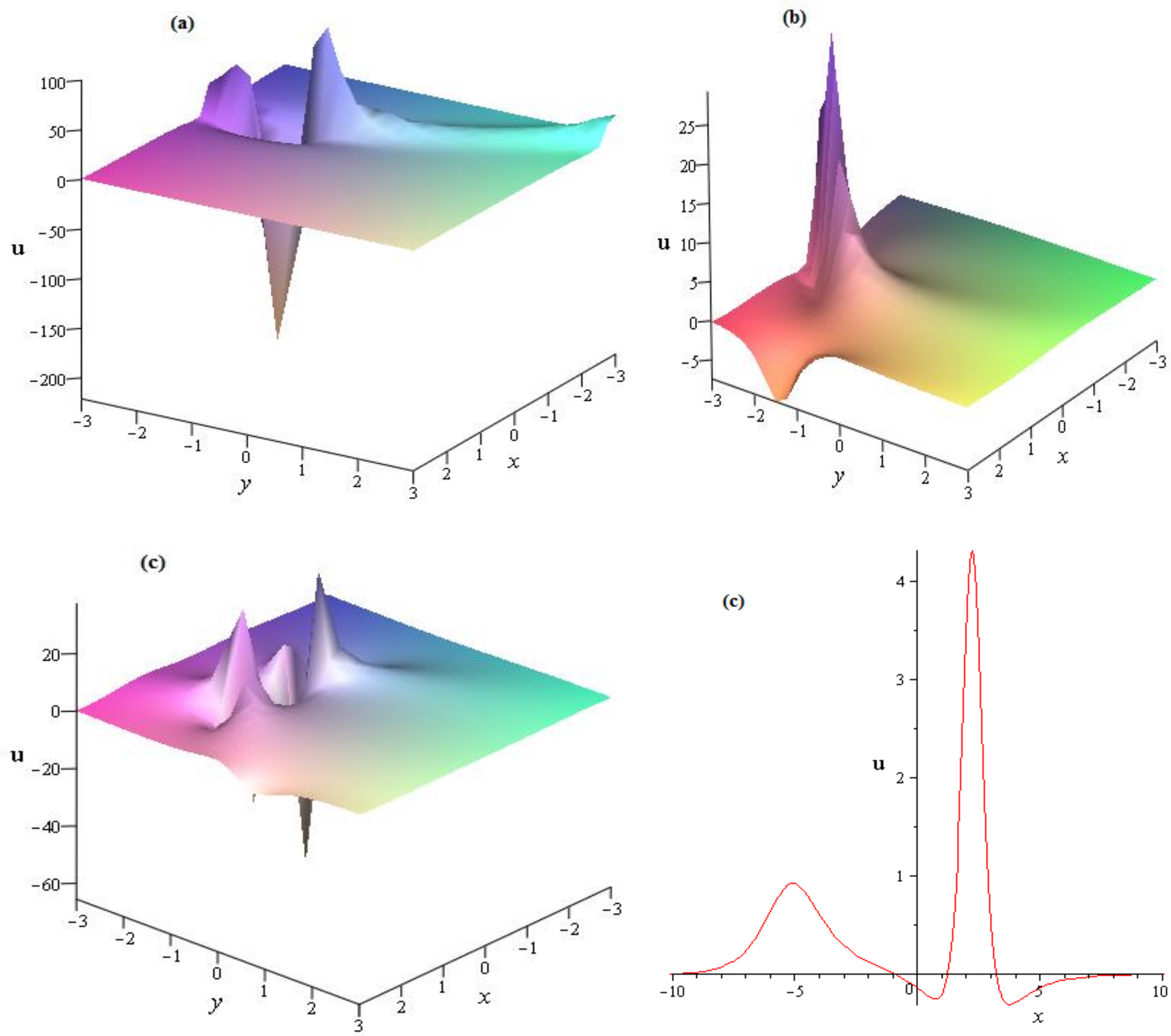


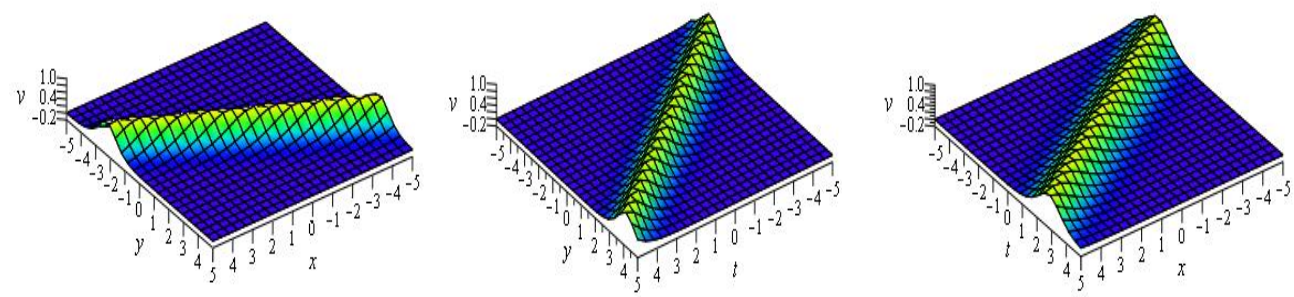
Fig-4.7: Interaction between rogue wave and hyperbolic solution (4.47) for Eq. (4.21) by choosing suitable parameters: $a_4 = 0.54, a_5 = -0.5, a_7 = 0.3, a_8 = -0.8, a_9 = 1, \lambda = 0.5, \mu = 2$ with 3D plots for different times (a) $t = 0$, (b) $t = 3$, and (c) $t = 5$ respectively; (d) 2D plot (c).

5.5 Graphical representation of the solutions of (2+1)-D ANNV Equation

In this subsection, we explain different type of traveling wave solution of asymmetric Nizhnik-Novikov-Veselov (ANNV) equation graphically obtained by using Direct method called Hirota's bilinear method. Using this method, we obtained some traveling wave solutions which are denoted as Eq. 4.55, Eq. 4.57, Eq. 4.60, Eq. 4.68, Eq. 4.77, Eq. 4.82 and Eq. 4.83. The graphical illustrations of some obtained solutions for choosing suitable values of the arbitrary constants are exposed in Fig. 5.1 to Fig. 5.10.

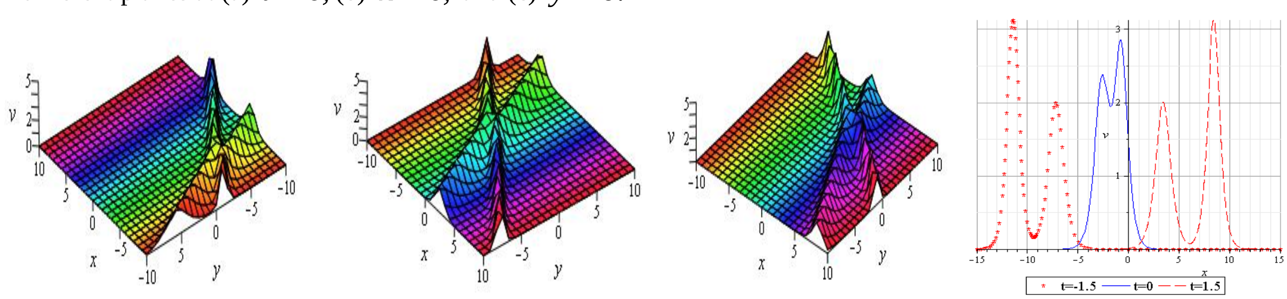
During the wave propagation, we see that the amplitude, velocity and envelop shape of the one-soliton keep constant (see Fig. 5.1). One can confirm that the amplitudes of impatient position are limited and around same in different spaces.

As depicted from Fig. 5.2, the collision is elastic between two bell-shaped solitons, because the velocities, amplitudes and envelop shapes of a moving soliton always keep fixed their shapes after the interaction. All the phenomena concludes that energy will remain unchanged during collision. Whereas we see that from Fig. 5.3, the interaction between two bell-shaped solitons is completely non-elastic. That is the soliton velocity, amplitude and wave shape are changed after collision. Now we will illustrate the wave pattern situations of solitary wave by three figures. Fig. 5.1 highlight the one-soliton (4.55), Fig. 5.2 and Fig. 5.3 demonstrates the two-soliton solution (4.57), Fig. 5.4, special type solution of Eq. (4.57) called breather solution and Fig. 5.5 demonstrates the three-soliton solution (4.60), by selecting appropriate parameters.



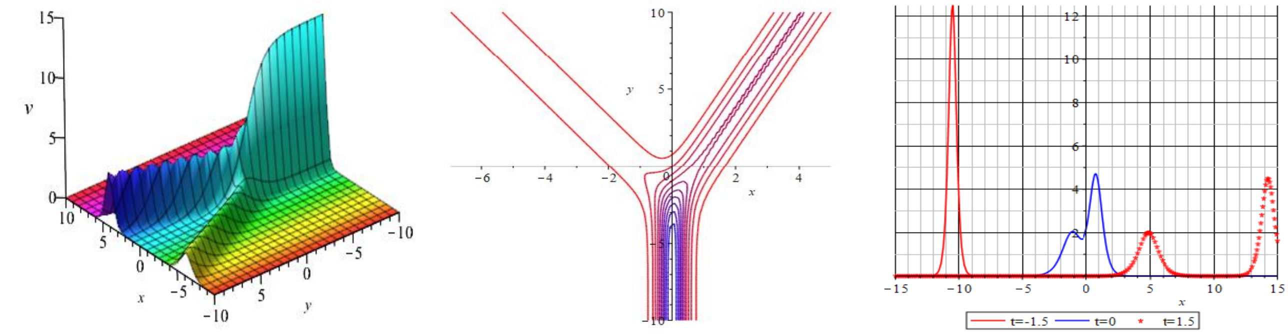
(a) (b) (c)

Fig-5.1: The one-stripe soliton solution for Eq. (4.49) with $a_0 = 2,\ a_1 = 1,\ l_1 = 1.25,\ m_1 = 2.5,$ 3D shape in different planes at (a) $t = 0$; (b) $x = 0$; and (c) $y = 0$.



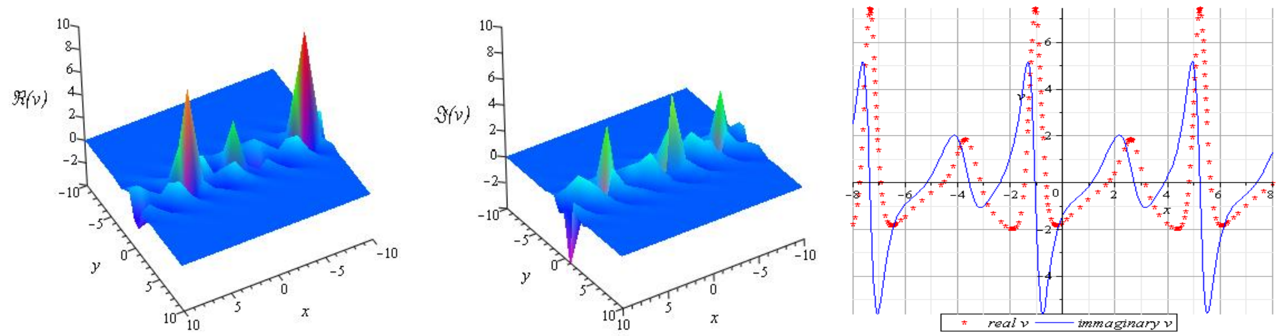
(a) (b) (c) (d)

Fig-5.2: The two-stripe soliton solution for Eq. (4.49) with $a_1 = 1,\ a_2 = 1, a_3 = 10,\ l_1 = 2, l_2 = 2.5,\ m_1 = 1,$ $m_2 = 3.5,$ with 3D plots for different times (a) $t = -1.5$; (b) $t = 0$; and (c) $t = 1.5$ respectively, (d) Corresponding 2D plot.



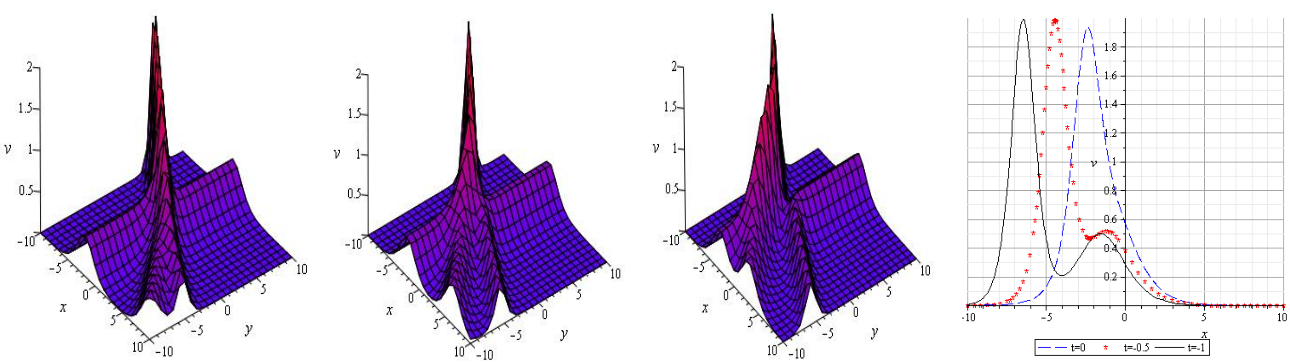
(a) (b) (c)

Fig-5.3: The two-stripe soliton solution (non-elastic) for Eq. (4.49) with $a_1 = 1, a_2 = 1, a_3 = 10,\ l_1 = 2,\ l_2 = -3,$ $m_1 = m_2 = 1,$ at time $t = 0$ (a) 3D plot (b) Contour plot and (c) Corresponding 2D plot for different time.



(a) (b) (c)

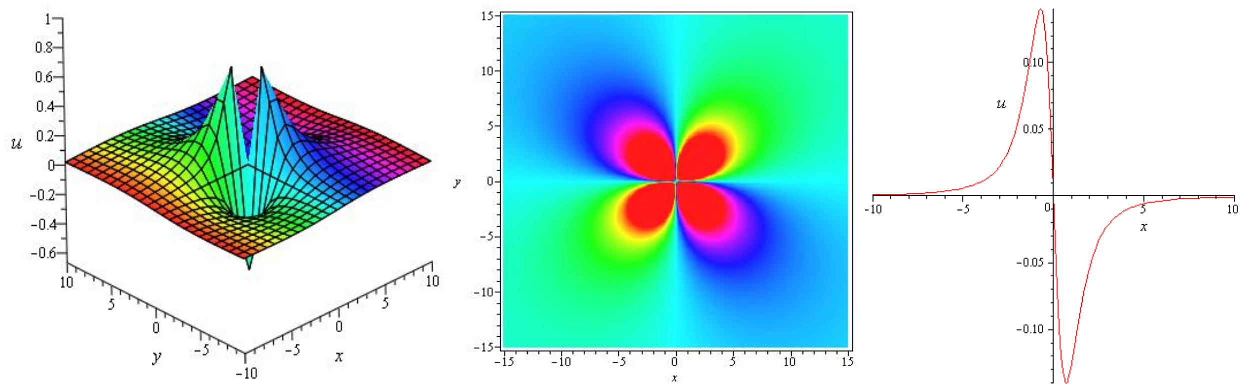
Fig-5.4: The breather solution for Eq. (4.49) with $a_1 = 1,\ a_2 = 1,\ a_3 = 10,\ l_1 = 2, l_2 = 2.5,\ m_1 = 1,\ m_2 = 3.5$ with $t = 0$: 3D plots (a), (b) and (c) Corresponding 2D plot.



(a) (b) (c) (d)

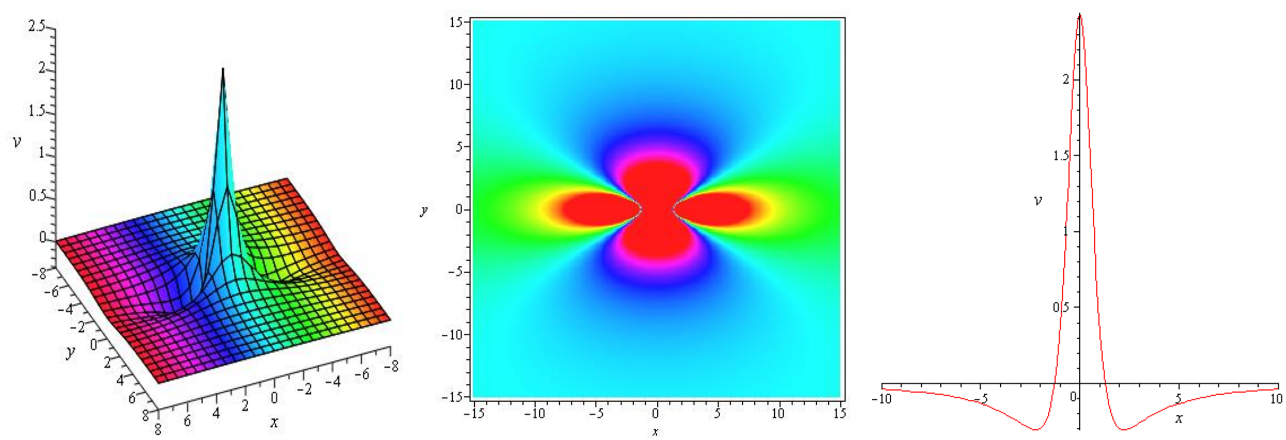
Fig-5.5: The three-stripe soliton solution for Eq. (4.49) by choosing parameters: $a_{13} = 2,\ l_1 = 1,\ l_2 = 2,\ l_3 = 2,$ $m_2 = 1, m_3 = 3,$ with 3D plots at (a) $t = 0,$ (b) $t = -0.5,$ and (c) $t = -1$ respectively, (d) 2D plot at $t = 0, -0.5$ and $t = -1$ respectively.

Fig. 5.6 shows the sketch the lump solution u in Eq. (4.68) whereas Fig. 5.7 shows the sketch lump of v in Eq. (4.68) called rogue waves for some values $p_1 = 2,\ l_2 = 1,\ m_1 = 2,$ and $m_2 = 1$, (a) gives 3D views from which can expose the standard rogue wave features. It is also clear that the Fig. 5.7(a) is the well-known eye-shaped rogue wave solution which has two valleys and one local hump. Moreover, we notice that rogue wave has the highest peak in its surrounding waves and forms in a tiny time, which is clear from Fig. 5.7(c). For fixed t, the variables can determine the rogue wave is symmetric about the x axis (see Fig. 5.7(b)).



(a) (b) (c)

Fig-5.6: Lump solution u in (4.68) for Eq. (4.49) by choosing suitable parameters: $p_1 = 1.2,\ l_2 = 0.8,$ $m_1 = -0.8,$ and $m_2 = 0.4$. (a) 3-D plot of u (b) density plot of u (c) 2-D plot of u.



(a) (b) (c)

Fig-5.7: Lump solution v in (4.68) for Eq. (4.49) by choosing suitable parameters: $p_1 = 2,\ l_2 = 1,\ m_1 = 2,$ and $m_2 = 1$. (a) 3-D plot of v (b) density plot of v (c) 2-D plot of v.

In what follows, Fig. 5.8 presents exact solution of Eq. (4.77) by choosing the suitable parameters, which can show the interaction phenomena between solitary wave and lump waves.

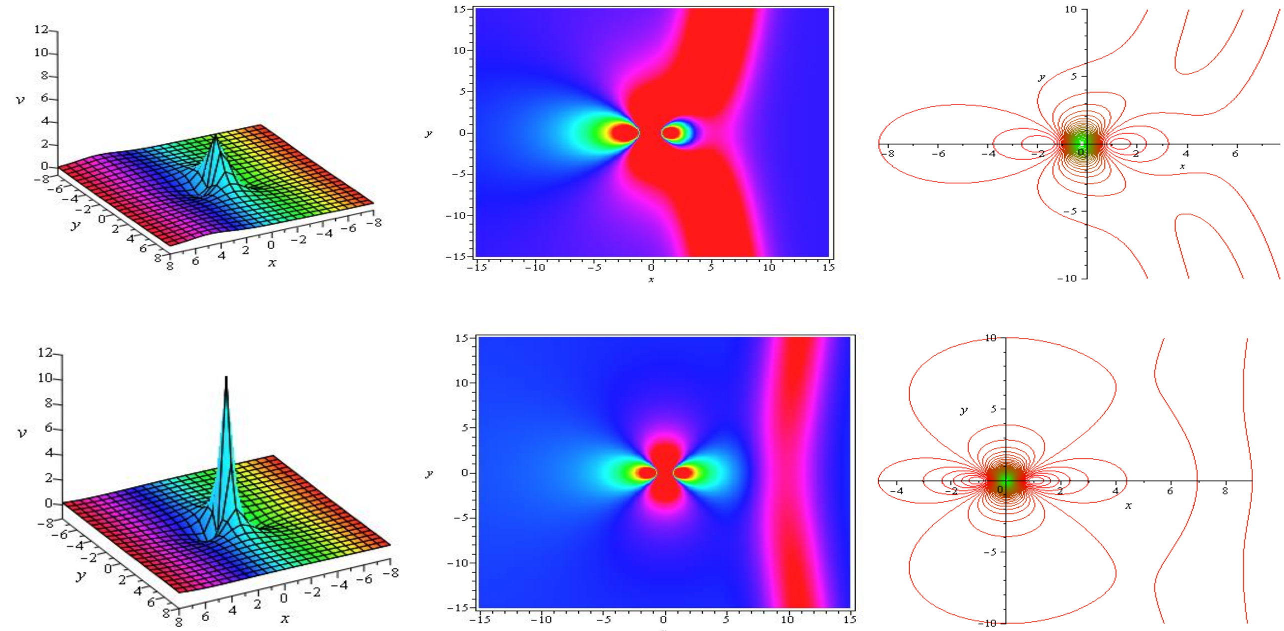
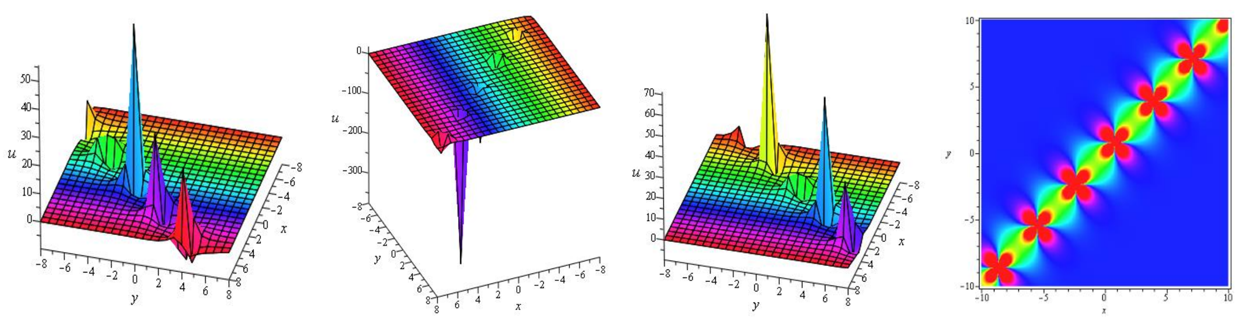


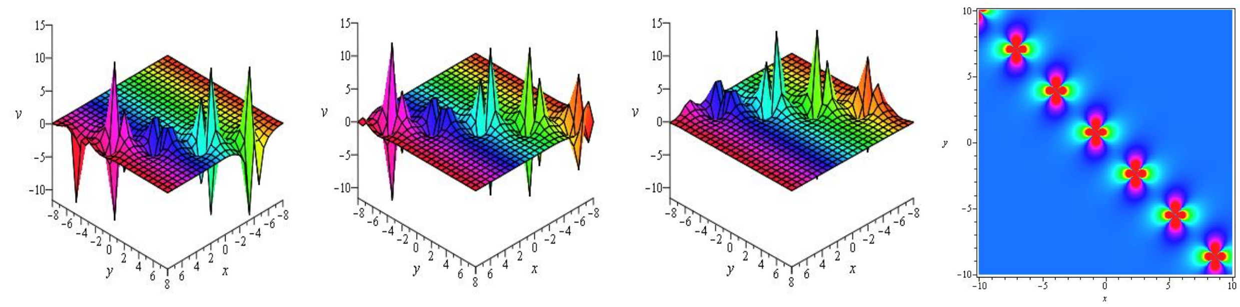
Fig-5.8: Profiles of v in (4.77) with $t=0$: 3d plots, density plot and contour plot (top for $a_2=5$) and bottom for $a_2=0.05$ with $a_1=2.5,\ m_1=2.3,\ m_2=1,\ l_1=1.5,$ and $l_3=1.$

In what follows, Fig. 5.9 and Fig. 5.10 present exact solution of Eq. (4.82) and Eq. (4.83) respectively by choosing the suitable parameters, which can demonstrate the interaction phenomena among multi lump solution.



(a) (b) (c) (d)

Fig-5.9: Profiles of u in (4.82) with $t=-1.5,0,1.5$: 3d plots (a), (b), (c) respectively and (d) corresponding density plot (b) with $h_1=1,\ h_2=2,\ \ p_1=1, p_2=-1$ and $n_2=1$.



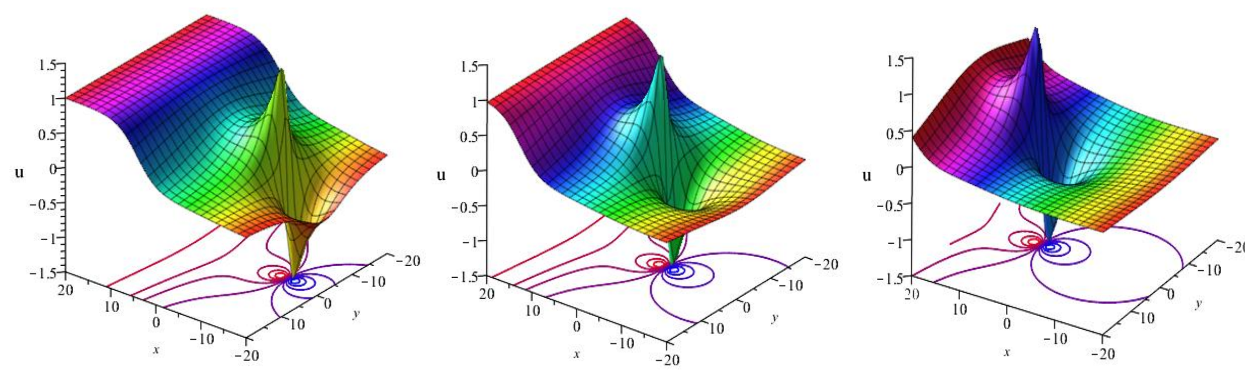
(a) (b) (c) (d)

Fig-5.10: Profiles of v in (4.83) with $t=-2,0,2$: 3d plots (a), (b), (c) respectively and (d) corresponding density plot (b) with $h_1=1,\ h_2=2,\ \ p_1=1,\ \ p_2=-1$ and $n_2=-1$.

5.6 Graphical representation of the solutions of (3+1)-D gBKP Equation

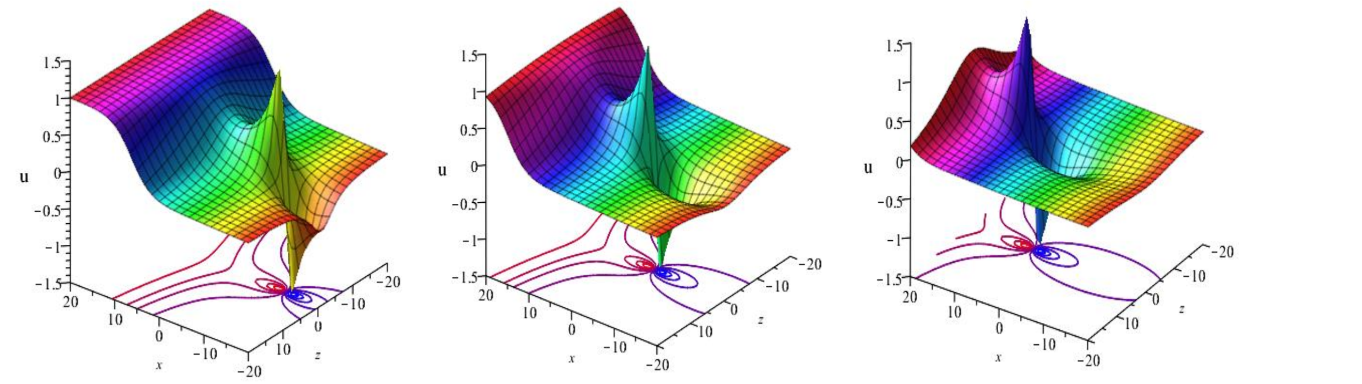
In this subsection, we explain different type of traveling wave solution of generalized B-type Kadomtsev-Petviashvili (gBKP) equation graphically obtained by using Direct method called Hirota's bilinear method. Using this method, we obtained some traveling wave solutions which are denoted as Eq. 4.93, Eq. 4.98, Eq. (4.100), Eq. 4.104, Eq. 4.108 and Eq. 4.110. The graphical illustrations of some obtained solutions for choosing suitable values of the arbitrary constants are exposed in Fig. 6.1 to Fig. 6.7.

In what follows, Fig. 6.1 present particular solution of Eq. (4.93) in $xy-$plane and Fig. 6.2 present this in the $xz-$plane at dissimilar times by appropriate parameters selection. Curved lines strained in the bottom of the 3D figures are its corresponding contour plots.



(a) **(b)** **(c)**

Fig-6.1: Profile of solution (4.93) for Eq. (4.85) with $a_1=a_7=1, a_5=3,\ a_8=a_{10}=2,\ a_{11}=1.5,\ \lambda=1.25$ (a) $t=-1.5,$ (b) $t=0,$ and (c) $t=1.5$ respectively for $z=0.$



(a) **(b)** **(c)**

Fig-6.2: Profile of solution (4.93) for Eq. (4.85) with $a_1 = a_7 = 1, a_5 = 3,\ a_8 = a_{10} = 2,\ a_{11} = 1.5,\ \lambda = 1.25$ (a) $t = -1.5,$ (b) $t = 0,$ and c) $t = 1.5$ respectively for $y = 0.$

In what follows, Fig. 6.3 present exact solution via the Eq. (4.98) by selecting the appropriate values of constants, that illustrate the solitonic interaction between lump and periodic waves produce a breather waves solution. Curved lines strained at the bottom of these figures are corresponding contours. While Fig. 6.3(b) produces the shape of single lump wave degenerated from the solution (4.98) via parametric limit approach.

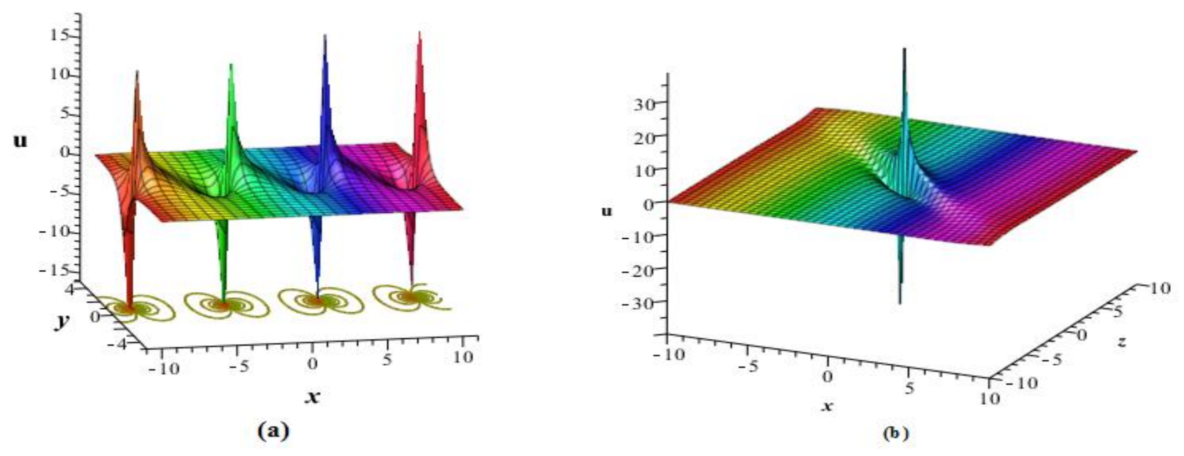
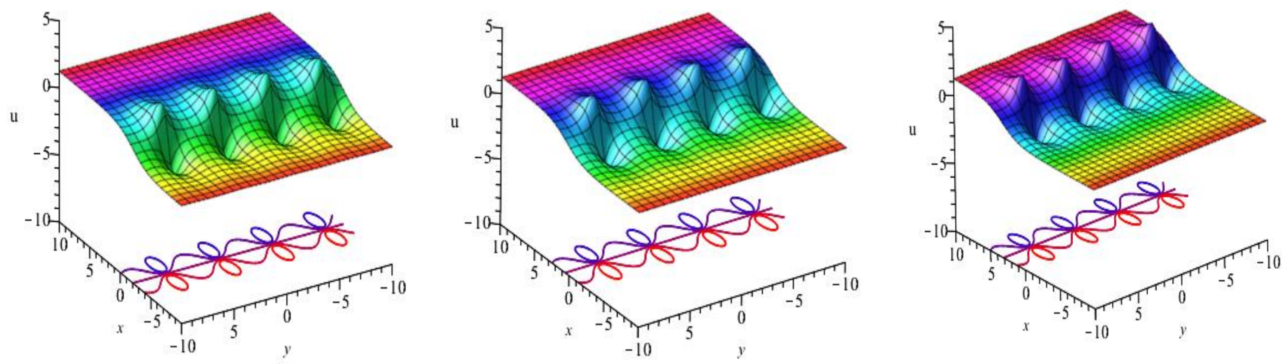


Fig-6.3: Profiles of solution (4.98) for Eq. (4.85) with $a_1 = b_2 = c_2 = p_1 = r_2 = p_1 = h_2 = 1$ **at** $t = 0$: (a) Perspective view of the wave for $z = 0,$ and (b) degeneration of (4.98) by parametric limit of the wave Eq. (4.100) when and $y = 0$.

The Fig. 6.4 interprets the wave shapes of the solution Eq. (4.104) with totally different parameters and consequently the curve exhausted below of the figure is the shape line.

From the figures we see that the desired lump wave is $y-$periodic and propagate along $x-$direction as time goes.



(a) **(b)** **(c)**

Fig-6.4: Profile of solution (4.104) for Eq. (4.85) with $a_2 = b_2 = 1, r_1 = 2.5,\ h_1 = 0.5,\ \ h_2 = 1.2,\ p_1 = 0.25,$ $p_2 = 1.25$ (a) $t = -1,$ (b) $t = 0,$ and (c) $t = 1$ respectively taking $z = 2$.

Different conditions on the parameters p and q, u_4 (i,e., Eq. (4.108)) offers four different interaction solutions among the kinky, lumps and periodic waves. On the condition $p = 0$ and $q = 0,\ u_4$ exhibits a single lump solution (see Fig. 6.5(a)). It is known that a lump wave has one valley and one peak (see Fig. 6.5(a)). But for the parametric condition $p = 0$ and $q \neq 0,\ u_4$ (i,e., Eq. (4.108)) displays an interaction between a lump and a periodic wave (see Figs. 6.5(b)–6.5(c)). In such case, the interaction between a lump and a periodic wave delivers one valley and one peak serially which split into two valleys and two peaks by fission (i.e. a fission phenomenon occurs for lump wave) as q gradually increases depicted in the Figs. 6.5(b)–6.5(c). Fission of lump is cleared from the comparison of Fig. 6.5(b) and Fig. 6.5(c), as in Fig. 6.5(b) has one lump (one peak and one valley) and but in Fig. 6.5(c) has two lumps (two valleys and two peaks).

Due to the condition $p \neq 0$ and $q = 0,\ u_4$ (i,e., Eq. (4.108)) offers an interaction solitonic wave in which a lump get into a double kink waves (see Fig. 6.6(a)). Finally, on the condition $p \neq 0$ and $q \neq 0,\ u_4$ exposes an interaction among the lumps, double kinks and periodic waves. On observations of the Figs. 6.6(b)–6.6(c), It is obvious that one valley and one peak of the lump (in Fig. 6.6(b)) split into two valleys and two peaks (in Fig. 6.6(c)) by fission as q increases into a double kinky periodic waves.

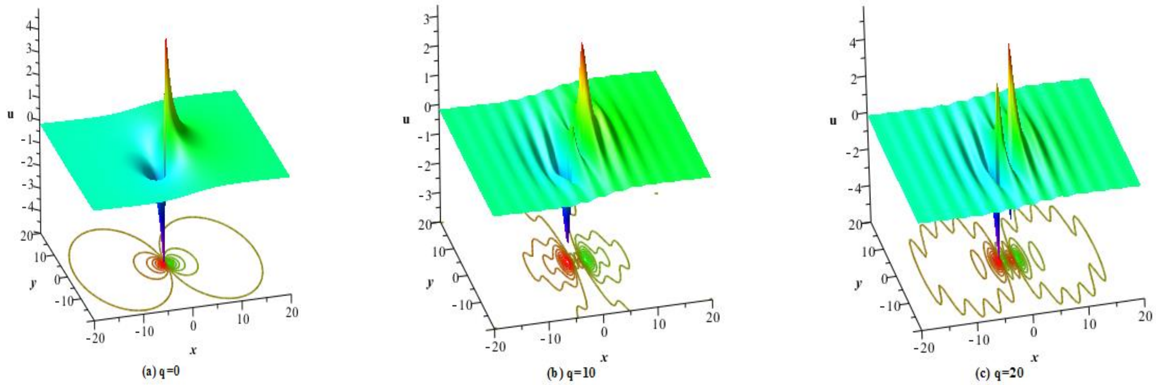


Fig-6.5: (Fission of lump wave) Profile of solution (4.108) for Eq. (4.85) with $a_2 = 1, a_5 = 2, a_6 = a_8 = 1.5,$ $a_9 = 2, m_1 = 1, k_1 = 2, p = 0, t = 0$ and $z = 0.$

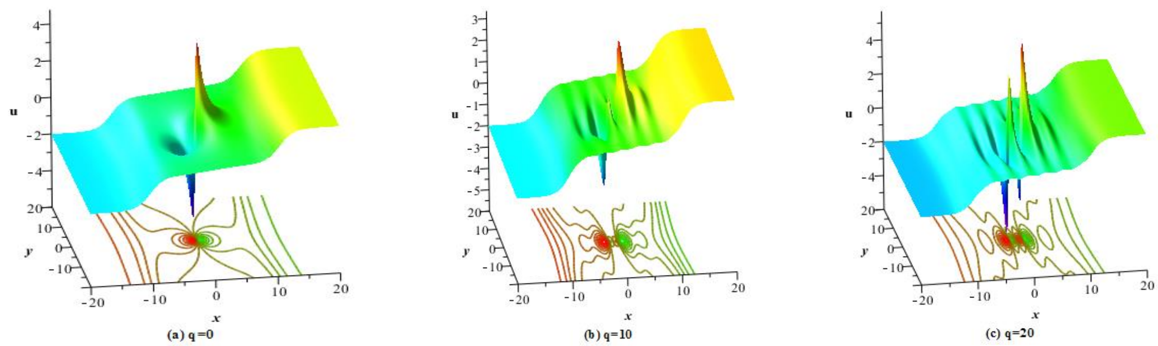


Fig-6.6: (Fission of lump wave) Profile of solution (4.108) for Eq. (4.85) with $a_2 = 1, a_5 = 2, a_6 = a_8 = 1.5,$ $a_9 = 2, m_1 = 1, k_1 = 2, p = \frac{1}{6}$ and $z = 0$: (a) $q = 0;$ (b) $q = 10$, (c) $q = 20.$

The solution Eqs. (4.110) and (4.112) have the similar four conditions like Eq. (4.108). The Fig. 6.7 present specific solution Eq. (4.110) by selecting the appropriate values of constants that illustrate the interaction phenomena. If we agreed with $p \neq 0$ and $q = 0$ in Eq. (4.110), then we experience with an interaction between a lump and double kinky waves of Eq. (4.85) (see Fig. 6.7(a)). But If we agreed with $p \neq 0$ and $q \neq 0$ in Eq. (4.110), then we experience a fission phenomenon as $q \neq 0$ increases, in which lump waves split into more than one lump waves get into a double kinky wave (see Figs. 6.7(b)–6.7(c)). On comparison between the Figs. 6.7(b) and 6.7(c), It is obvious that one valley and one peak of the lump (in Fig. 6.7(b)) split into two valleys and two peaks (in Fig. 6.7(c)) by fission as q increases that get into a double kinky periodic waves.

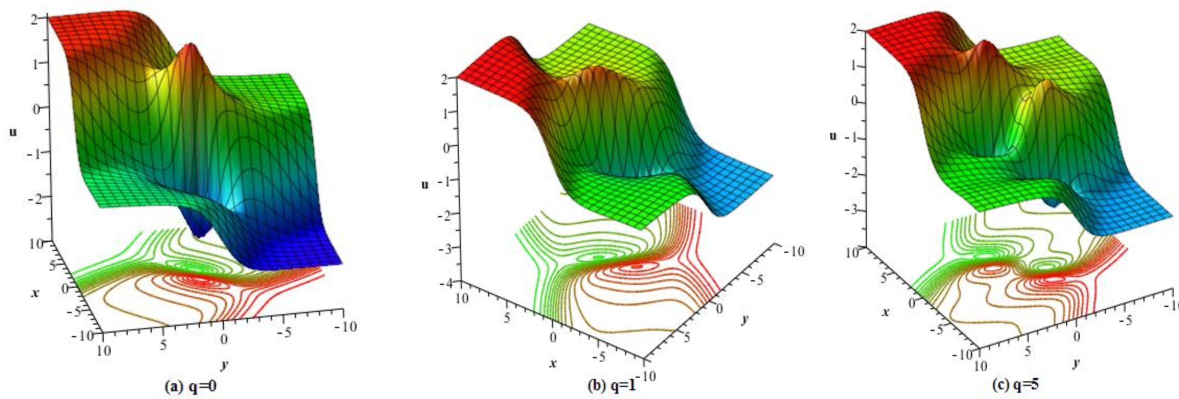


Fig-6.7: Profile of solution Eq. (4.110) for Eq. (4.85) with $a_2 = 1, a_5 = a_9 = 2, a_6 = a_8 = 1.5,\ m_1 = k_1 = 1,$ $k_3 = 1,\ p = 0.02$: (a) a lump get into a double kinky waves, (b) a lump going to fission that get into a double kinky waves and (c) a lump fission into two lump that get into a double kinky waves.

Chapter six

Conclusions

In this paper, modified version of double sub-equation method is proposed for solving non-linear evolution equation. As a concrete example, we consider the (1+1)-dimensional Burger's equation, the (1+1)-dimensional Gardner equation (or combined KdV-mKdV) and the (1+1)-dimensional Hirota-Ramani equation. Applying this method, we acquired novel some complexiton solutions in the combination of trigonometric and hyperbolic functions with different structures. It is hoped that the study of these complexiton solutions could further assist understanding, identifying and classifying nonlinear integrable and nonintegrable differential equations and their exact solutions. In fact, we naturally use two or more really different sub-equations to handle complexiton solution with two different traveling variables i.e., multi-variable Riccati equations. Thus we can obtain more prosperous complexiton solutions possessing a mixture of trigonometric periodic and hyperbolic functions.

Additionally, we have successfully implemented the direct method to the (2+1)-dimensional Breaking soliton equation, the (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov and the (3 + 1)-D gBKP model. Based on the Hirota bilinear formulation and by a symbolic computation Maple, We have derived soliton solution, breathers, lump solutions, mixed lump stripe solutions of the (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov equation. We have presented some interaction phenomena between rogue waves and other kinds of solutions to the (2+1)-dimensional Breaking soliton equation.

We have successfully determined three types of interaction solutions among the lump, kink and periodic waves for the (3 + 1)-D gBKP model. By exploitation of direct approach, we have acquired some interactions solutions such as the lump-kink wave solution Eq. (4.93), breather-waves solutions Eqs. (4.98) and (4.104) of the model. Also, we have presented some new interaction solutions among lump, kink and periodic waves solutions Eqs. (4.108), (4.110) and (4.112) via a different "rational-cosh-cos" type test functions. Moreover, we derive a single lump wave solution Eq. (4.100) by parametric limit approach that degenerate from the breather wave solution Eq. (4.98). Four different conditions on the exist parameters of the solutions Eqs. (4.108), (4.110) and (4.112) are given to illustrate fission properties of lump waves into kink waves.

Meanwhile, the performances of the mentioned techniques are substantially powerful and absolutely reliable to search new explicit solutions of other NPDEs.

References

[1] M.J. Ablowitz, P.A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scatting, Cambridge University Press, New York, 1991.

[2] V.B. Matveev, M.A. Salle, Darboux Transformation and Solitons, Springer, Berlin, 1991.

[3] Hirota, R., 2004. The direct method in soliton theory. Cambridge University Press, Cambridge.

[4] A.M. Wazwaz, New solutions of distinct physical structures to high-dimensional nonlinear evolution equations, Appl. Math. Comput. 196 (2008) 363–370.

[5] W.X. Ma, D.T. Zhou, Solitary wave solutions to a generalized KdV equation, Acta Phys. Sin. 42 (1993) 1731–1734.

[6] M.L. Wang, Y.B. Zhou, Z.B. Li, Applications of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, Phys. Lett. A 216 (1996) 67–75.

[7] Z. Zhu, Lax pair. Bäcklund transformation, solitary wave solution and finite conservation laws of the general KP equation and MKP equation with variable coefficients, Phys. Lett. A. 180 (1993) 409-412.

[8] W.X. Ma, B. Fuchssteiner, Explicit and exact solutions to a Kolmogorov–Petrovskii–Piskunov equation, Int. J. Non-Linear Mech. 31 (1996) 329–338.

[9] B. Tibor, L. Bèla, M. Csaba, U. Zsolt, The hyperbolic tangent distribution family, Powder Technol. 97 (1998) 100–108.

[10] C. Yan, A simple transformation for nonlinear waves, Phys. Lett. A 224 (1996) 77–84.

[11] M. Wang, X. Li, J. Zhang. The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A. 372 (2008) 417-423.

[12] H.O. Roshid, M.N. Alam, M.F. Hoque and M.A. Akbar, A new extended (G'/G)-Expansion method to find exact traveling wave solutions of nonlinear evolution equations, Mathematics and Statistics, 1 (2013) 162-166.

[13] W.X. Ma, Tingwen Huang, Yi Zhang. A multiple exp-function method for nonlinear differential equations and its application, Phys. Scripta. 82 (2010) 065003

[14] Z.T. Fu, S.K. Liu, S.D. Liu, Q. Zhao, New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations, Phys. Lett. A 290 (2001) 72–76.

[15] H.T. Chen, H.Q. Zhang, Improved Jacobin elliptic function method and its applications, Chaos, Solitons Fractals 15 (2003) 585–591.

[16] H. T. Chen, H. Q. Zhang, New multiple soliton-like solutions to the Generalized (2 + 1)-dimensional KP equation, Appl. Math. Comput. 157 (2004) 765-773.

[17] H. T. Chen, H. Q. Zhang, New double periodic and multiple soliton solutions of the generalized (2 + 1)-dimensional Boussinesq equation, Chaos, Solitons Fractals 20 (2004) 765-769.

[18] E. G. Fan, Extanded tanh-function method and its applications to nonlinear equations, Phys. Lett. A 277 (2000) 212-218.

[19] S. Zhang, A generalized auxiliary equation method and its application to (2+1)-dimensional Korteweg-de Vries equations, Comput. Math. Appl. 54 (2007) 1028-1038.

[20] W. X. Ma, Complexiton solutions to the Korteweg-de Vries equation, Phys. Lett. A. 301 (2002) 35-44.

[21] Y. Chen, Q. Wang, Multiple Riccati equations rational expansion method and complexiton solutions of the whitham-Broer-Kaup equation, Phys. Lett. A. 347 (2005) 215-227.

[22] H. T. Chen, S. H. Yang, W. X. Ma, Double sub-equation method for complexiton solutions of nonlinear partial differential equations, Appl. Math. Comput. 219(9) (2013) 4775-4781.

[23] W.X. Ma, A. Abdeljabbar, M.G. Asaad, Wronskian and Grammian solutions to a (3 +1)-dimensional generalized KP equation, Appl. Math. Comput. 217 (2011) 10016–10023.

[24] X.B. Hu, J.X. Zhao, H.W. Tam, Pfaffianization of the two-dimensional Toda lattice, J. Math. Anal. Appl. 296 (2004) 256–261.

[25] C.Z. Qu, Symmetries and solutions to the thin film equations, J. Math. Anal. Appl. 317 (2006) 381–397.

[26] R. Hirota, Exact solutions of the Korteweg–de Vries equation for multiple collisions of solitons, Phys. Rev. Lett. 27(18) (1971) 1192–1194.

[27] J. Manafian, M. Lakestani, A. Bekir, Study of the analytical treatment of the (2 +1)-dimensional Zoomeron, the Duffing and the SRLW equations via a new analytical approach, Int. J. Appl. Comput. Math. 2(2) (2016) 243–268.

[28] O. G. Gaxiola, R. B. Jaquez, Applying Adomian decomposition method to solve Burgess equation with a non-linear source, Int. J. Appl. Comput. Math. 3(1) (2017) 213–224.

[29] Z. Odibat, A Riccati equation approach and travelling wave solutions for non-linear evolution equations, Int. J. Appl. Comput. Math. 3(1) (2017) 1–13.

[30] B. Bira, T. R. Sekhar, Some exact solutions for sediment transport equations, Int. J. Appl. Comput. Math. 1(2) (2015) 251–255.

[31] M. Shakeel, S.T. Mohyud-Din, A novel $(G'/G)-$ expansion method and its application to the (3 +1)-dimensional Burger's equations, Int. J. Appl. Comput. Math. 2(1) (2016) 13–24.

[32] H.O. Roshid, W.X. Ma, Dynamics of mixed lump-solitary waves of an extended (2+1)-dimensional shallow water wave model, Phys. Lett. A 382 (45) (2018) 3262–3268.

[33] H.O. Roshid, M.M. Rashidi, Multi-soliton fusion phenomenon of Burgers equation and fission, fusion phenomenon of Sharma–Tasso–Olver equation, J. Ocean Eng. Sci. 2(2) (2017) 120–126.

[34] A.M. Wazwaz, (2 +1)-Dimensional Burgers equations BE(m +n +1): using the recursion operator, Appl. Math. Comput. 219 (2013) 9057–9068.

[35] H.O. Roshid, Novel solitary wave solution in shallow water and ion acoustic plasma waves in-terms of two nonlinear models via MSE method, J. Ocean Eng. Sci. 2(3) (2017) 196–202.

[36] H.O. Roshid, M.M. Roshid, N. Rahman, M.R. Pervin, New solitary wave in shallow water, plasma and ion acoustic plasma via the GZK-BBM equation and the RLW equation, Propuls. Power Res. 6(1) (2017) 49–57.

[37] C.Y. Qin, X.B. Wang, S.F. Tian, T.T. Zhang, Rogue waves, bright–dark solitons and traveling wave solutions of the -dimensional generalized Kadomtsev–Petviashvili equation, Comput. Math. Appl. 75 (2018) 4221–4231.

[38] M.B. Hossen, H.O. Roshid, M.Z. Ali, Characteristics of the solitary waves and rogue waves with interaction phenomena in a (2 + 1)-dimensional Breaking Soliton equation, Phys. Lett. A 382 (19) (2018) 1268–1274.

[39] Z.L. Zhao, Y. Chen, B. Han, Lump soliton, mixed lump stripe and periodic lump solutions of a (2 + 1)-dimensional asymmetrical Nizhnik–Novikov–Veselov equation, Mod. Phys. Lett. B 31 (2017) 1750157.

[40] S.Y. Lou, X.Y. Tang, Nonlinear mathematical physics method, Academic Press, Beijing, 2006.

[41] W.X. Ma, Lump solutions to the Kadomtsev–Petviashvili equation, Phys. Lett. A 379 (2015) 1975–1978.

[42] H.C. Ma, A.P. Deng, Lump Solution of (2+1)-Dimensional Boussinesq Equation, Commun. Theor. Phys. 65 (2016) 546–552.

[43] J.Y. Yang, W.X. Ma, Lump solutions to the BKP equation by symbolic computation, Int. J. Mod. Phys. B 30 (1-7) (2016) 1640028.

[44] W.Q. Peng, S.F. Tian, T.T. Zhang, Analysis on lump, lumpoff and rogue waves with predictability to the (2 + 1)-dimensional B-type Kadomtsev–Petviashvili equation, Phys. Lett. A 382 (38) (2018) 2701–2708.

[45] H. Wang, Lump and interaction solutions to the (2+1)-dimensional Burgers equation, Appl. Math. Lett. 85 (2018) 27–34.

[46] W.X. Ma, Lump-Type Solutions to the (3+1)-Dimensional Jimbo-Miwa Equation, Int. J. Nonlinear Sci. Numer. Stimul. 17 (2016) 355–359.

[47] Y. Yue, L. Huang, Y. Chen, N-solitons, breathers, lumps and rogue wave solutions to a (3+1)-dimensional nonlinear evolution equation, Comput. Math. Appl. 75 (7) (2018) 2538–2548.

[48] X. Li, Y. Wang, M. Chen, B. Li, Lump solutions and resonance stripe solitons to the (2+1)-dimensional Sawada-Kotera equation, Adv. Math. Phys. 2017 (2017) 6, 1743789.

[49] M.D. Chen, X. Li, Y. Wang, B. Li, A pair of resonance stripe solitons and lump solutions to a reduced (3+1)-dimensional nonlinear evolution equation, Commun, Theor. Phys. 67 (6) (2017) 595.

[50] L.L. Wen, H.Q. Zhang, Rogue wave solutions of the (2+1)-dimensional derivative nonlinear Schrodinger equation, Nonlinear Dyn. 86 (2) (2016) 877–889.

[51] N. Akhmediev, A. Ankiewicz, M. Taki, Waves that appear from nowhere and disappear without a trace, Phys. Lett. A 373 (2009) 675–678.

[52] S.Y. Lou, H.C. Hu, X.Y. Tang, Interactions among periodic waves and solitary waves of the (N+1)- dimensional sine-Gordon field, Phys. Rev. E 71 (2005) 036604.

[53] S. Kutluay, A.R. Bahadir, A. Ozdes, Numerical solution of one-dimensional Burgers equation: explicit and exact-explicit finite difference methods, J. Comput. Appl. Math. **103** (1999) 251–261.

[54] J. Biazar, H. Aminikhah, Exact and numerical solutions for non-linear Burger's equation by VIM, Math. Comput. Model. **49** (2009) 1394–1400.

[55] S. Kumar, A. Kumar, D. Baleanu, Two analytical methods for time-fractional nonlinear coupled Boussinesq–Burger's equations arise in propagation of shallow water waves, Nonlinear Dyn. 85(2) (2016) 699–715.

[56] Y. Çenesiz, D. Baleanu, A. Kurt, O. Tasbozan, New exact solutions of Burgers' type equations with conformable derivative, Waves Random Complex Media 27(1) (2017) 103–116.

[57] S.J. Johnston, H. Jafari, S.P. Moshokoa, V.M. Ariyan, D. Baleanu, Laplace homotopy perturbation method for Burgers equation with space-and time-fractional order, Open Phys. 14(1) (2016) 247–252.

[58] O.E. Hepson, Numerical solutions of the Gardner equation by extended form of the cubic B-splines. arXiv:1702.02776v1 [math.NA] 9 Feb 2017

[59] A.M. Wazwaz, New solitons and kink solutions for the Gardner equation, Commun. Nonlinear Sci. Numer. Simul. 12(8) (2007) 1395–1404.

[60] M.S. Ruderman, T. Talipova, E. Pelinovsky, Dynamics of modulationally unstable ion-acoustic wavepackets in plasmas with negative ions, J. Plasma Phys. 74(05) (2008) 639–656.

[61] A.M. Kamchatnov, Y.H. Kuo, T.C. Lin, T.L. Horng, S.C. Gou, R. Clift, G.A. El, R.H. Grimshaw, Undular bore theory for the Gardner equation, Phys. Rev. E 86(3) (2012) 036605.

[62] R. Hirota, A. Ramani, The Miura transformations of Kaup's equation and of Mikhailovs equation, Phys. Lett. A, 76(2) (1980) 95-96.

[63] J. Ji, A new expansion and new families of exact traveling solutions to Hirota equation, Appl. Math. Comput. 204(20) (2008) 881-883.

[64] S. Koonprasert, M. Punpocha, More Exact Solutions of Hirota–Ramani Partial Differential Equations by Applying F-Expansion Method and Symbolic Computation, Global J. Pure Appl. Math. 12(3) (2016) 1903-1920.

[65] A. Reza, A. Rasoul, Hyperbolic, Trigonometric, and Rational Function Solutions of Hirota-Ramani Equation via (G′/G)-Expansion Method, Math. Prob. Eng. Volume 2011, Article ID 424801, 11 pages. doi:10.1155/2011/424801.

[66] H. O. Roshid, M. N. Alam, Multi-Soliton Solutions to Nonlinear Hirota-Ramani Equation, Appl. Math. Inf. Sci. 11 (3) (2017) 723-727.

[67] Z. Zhao, Z. Dai, G. Mu, The breather-type and periodic-type soliton solutions for the (2 +1)-dimensional breaking soliton equation, Comput. Math. Appl. 68(8) (2011) 2048–2045.

[68] S. Zhang, New exact non-traveling wave and coefficient function solutions of the (2 +1)-dimensional breaking soliton equations, Phys. Lett. A 368 (2007) 470.

[69] J.Q. Mei, H.Q. Zhang, New types of exact solutions for a breaking soliton equa-tion, Chaos, Solitons Fractals 20 (2004) 771.

[70] Y.Z. Peng, E.V. Krishnan, Two classes of new exact solutions to (2 +1)-dimensional breaking soliton equation, Commun. Theor. Phys. 44 (2005) 807.

[71] C.Q. Dai, J.F. Zhang, Chaotic behaviors in the (2 +1)-dimensional breaking soli-ton system, Chaos Solitons Fractals 39 (2009) 889.

[72] G.F. Yu, H.W. Tam, A vector asymmetrical NNV equation: Soliton solutions, bilinear Backlund transformation and Lax pair, J. Math. Anal. Appl. 344 (2) (2008) 593–600.

[73] Z.L. Hua, L.X. Qiang, B.C. Lin, Symmetry, reductions and new exact solutions of ANNV equation through Lax pair, Commun. Theor. Phys. 50 (1) (2008) 1–6.

[74] P.G. Est_evez, S. Leble, A wave equation in 2þ1: Painleve analysis and solutions, Inverse Prob. 11 (4) (1995) 925–937.

[75] Y. Ren, H. Zhang, New generalized hyperbolic functions and auto-Backlund transformation to find new exact solutions of the -dimensional NNV equation, Phys. Lett. A 357 (2006) 438–448.

[76] M. Boiti, J. Leon, P. Pempinelli, Spectral transform for a two spatial dimension extension of the dispersive long wave equation, Inverse Prob. 3 (1987) 371.

[77] M.S. Osman, A.M. Wazwaz, Analyzing the combined multi-waves polynomial solutions in a two-layer-liquid medium, Comput. Math. Appl. 76 (2) (2018) 276–283.

[78] M.S. Osman, Multiwave solutions of time-fractional (2 + 1)-dimensional Nizhnik–Novikov–Veselov equations, Pramana - J. Phys. 88 (4) (2017) 67.

[79] M.S. Osman, H.I. Gawad, Multi-wave solutions of the (2+1)-dimensional Nizhnik- Novikov-Veselov equations with variable coefficients, Eur. Phys. J. Plus 130 (2015) 215.

[80] M.S. Osman, H.I. Gawad, M.A. Mahdy, Two-layer-atmospheric blocking in a medium with high nonlinearity and lateral dispersion, Results Phys. 8 (2018) 1054–1060.

[81] S.Y. Lou, X.B. Hu, Infinitely many Lax pairs and symmetry constraints of the KP equation, J. Math. Phys. 38 (12) (1997) 6401.

[82] X.B. Wang, S.F. Tian, M.J. Xu, T.T. Zhang, On integrability and quasi-periodic wave solutions to a (3+1)-dimensional generalized KdV-like model equation, Appl. Math. Comput. 283 (2016) 216–233.

[83] X.B. Wang, S.F. Tian, L.L. Feng, H. Yan, T.T. Zhang, Quasiperiodic waves, solitary waves and asymptotic properties for a generalized (3 + 1)-dimensional variable coefficient B-type Kadomtsev–Petviashvili equation, Nonlinear Dyn. 88 (2017) 2265–2279.

[84] C. Xifang, Adv. Math. Phys. Volume 2018, Article ID 7843498, 5. Xifang Cao, "Lump Solutions to the (3+1)-Dimensional Generalized B-Type Kadomtsev-Petviashvili Equation", Advances in Mathematical Physics, vol. 2018, Article ID 7843498, 5 pages, 2018. https://doi.org/10.1155/2018/7843498

[85] W. X. Ma, Z. N. Zhu, Solving the (3 + 1)-dimensional generalized KP and BKP equations by the multiple exp-function algorithm, Appl. Math. Comput. 218 (2012), 11871-11879.

[86] S. Liu, Y. Ding, J. G. Liu, A Class of Exact Solutions of (3+1)-Dimensional Generalized B-Type Kadomtsev–Petviashvili Equation, Int. J. Nonlinear Sci. Numer. Simul. 18(2) (2017) 137-144.

[87] W. X. Ma, E. G. Fan, Linear superposition principle applying to Hirota bilinear equations, Comput. Math. Appl. 61 (2011), 950-959.

[88] Y. Liu, X. Wen, Fission and fusion interaction phenomena of mixed lump kink solutions for a generalized (3+1)-dimensional B-type Kadomtsev–Petviashvili equation, Mod. Phys. Lett. B 32(15) (2018) 1850161.

[89] C. He, Y. Tang, J. Ma, New interaction solutions for the (3+1)-dimensional Jimbo-Miwa equation, Comput. Math. Appl. 76(9) (2018) 2141-2147.

[90] M.B. Hossen, H.O. Roshid, M.Z. Ali, Modified double sub-equation method for finding complexiton solutions to the (1+1)dimensional nonlinear evolution equations, Int. J. Appl. Comput. Math. 3(3) (2017) 1–19.

[91] M. B. Hossen, H. O. Roshid, M. Z. Ali, Multi-soliton, breathers, lumps and interaction solution to the (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov equation, Heliyon 5(10) (2019) e02548.